

## CONGRUENCES FOR THE RAMANUJAN FUNCTION AND GENERALIZED CLASS NUMBERS

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ABSTRACT. The Ramanujan  $\tau$ -function satisfies well-known congruences modulo the so-called exceptional prime numbers 2, 3, 5, 7, 23, 691. In this paper we prove new congruences related to the irregular primes 131 and 593, involving generalized class numbers. As an application we obtain distribution results. We obtain a new proof of the famous 691 congruence and congruences of the related Rankin L-funtion.

### 1. INTRODUCTION AND STATEMENT OF RESULTS

The Ramanujan numbers [11]  $\tau(1) = 1, \tau(2) = -24, \tau(3) = 252, \tau(4) = -1472\dots$  satisfy certain famous well-known congruences modulo powers of the primes 2, 3, 5, 7, 23 and 691, e.g.,

$$(1.1) \quad \tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}.$$

Here we prove that the Ramanujan numbers also satisfy congruences with generalized class numbers discovered by Cohen [1]:  $3H(11, 3) = -3694, 2H(11, 4) = -50521, H(11, 7) = -9006448, H(11, 8) = -36581523\dots$ . These congruences are modulo the irregular prime numbers 131 and 593. Our motivation for finding congruences of this new type was the hope of proving by congruence considerations Lehmer's conjecture that  $\tau(n)$  never vanishes. As it turns out, this approach is fruitful, but in an unexpected sense.

Fundamental for the modern theory of modular forms is Deligne's [2] proof of the Serre conjecture on the existence of  $l$ -adic Galois representations  $\rho_l$  attached to a modular form. In 1972 Swinnerton-Dyer determined the possible images of  $\tilde{\rho}_l$ , the reduction of  $\rho_l$  mod  $l$  for level one. This was generalized by Ono [9].

There exist only finitely many primes  $l$  such that the image of  $\tilde{\rho}_l$  does not contain  $SL_2(\mathbb{Z}/l\mathbb{Z})$ . These primes are *exceptional* and yield congruences. It is striking that the full set of exceptional primes for the modular form related to  $\tau(n)$  is just the set  $\{2, 3, 5, 7, 23, 691\}$ . This suggests finding congruences by looking at *small images* of the reduced Galois representation. For example, Ono [9] notes that: *The search for congruences depends on the classification of subgroups of  $GL_2(\mathbb{Z}/p\mathbb{Z})$* , where  $GL_2(\mathbb{Z}/p\mathbb{Z})$  is the full image of the representation if  $l$  is not exceptional. This is certainly correct. And there is another interesting issue. The prime 691 is special in the sense that it is also an irregular prime, which gives a connection to Bernoulli

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numbers, Kummer's approach to Fermat's Last Theorem, and the congruences observed here.

Our goals are threefold. We give a new proof of the 691 congruence (1.1), which as an application involves a congruence for the Rankin  $L$ -function  $\sum_{n=1}^{\infty} \tau(n)^2 n^{-s}$  at the critical value  $s = 22$  (Zagier ([13])).

**Theorem 1.1.** *Let  $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$  be the discriminant function with Fourier coefficients  $\tau(n)$  and  $\|\Delta\|$  the Petersson norm of  $\Delta$ . Then*

$$(1.2) \quad -\frac{\pi}{2 \cdot 11 \cdot \zeta(11) \cdot \zeta(22) \cdot \|\Delta\|^2} \sum_{n=1}^{\infty} \tau(n)^2 n^{-22}$$

is a quadratic residue mod 691 and

$$(1.3) \quad 12! \cdot B_{12} \frac{\sum_{n=1}^{\infty} \tau(n)^2 n^{-22}}{\pi^{11} \cdot \zeta(11) \cdot \|\Delta\|^2} \equiv 2^{26} \cdot 3^2 \cdot 11 \pmod{691}.$$

Also, new congruences for  $\tau(n)$  are presented, related to the irregular prime numbers 131 and 593, which involve the generalized class numbers  $H(t, N)$  of Cohen. Here  $t \in \mathbb{N}$  and  $-N$  is the discriminant of a binary positive semi-definite quadratic form. For  $t = 1$  these are the classical class numbers discussed by Kronecker, Hurwitz and others. Finally, numerical data illustrates the results related to  $H(t, N)$ .

The *Ramanujan  $\tau$ -function*  $\tau : \mathbb{N} \rightarrow \mathbb{Z}$  is defined by the Fourier coefficients of the discriminant function  $\Delta$  on  $\mathfrak{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ :

$$(1.4) \quad \Delta(z) := x \prod_{m=1}^{\infty} (1 - x^m)^{24} = \sum_{n=1}^{\infty} \tau(n) x^n \quad (x = e^{2\pi i z}).$$

It is imbedded in the family of power series of type

$$(1.5) \quad \left\{ (1-x) \cdot (1-x^2) \cdot (1-x^3) \cdot \dots \right\}^{\alpha},$$

where  $\alpha$  varies. For example  $\alpha = -1$  gives the partition function and  $\alpha = 24$  the discriminant  $\Delta$ . Many properties of  $\tau(n)$  are elaborated in Ramanujan writings. He made several conjectures which have profoundly influenced mathematics. For example, he conjectured that  $\tau(n \cdot m) = \tau(n) \cdot \tau(m)$  with  $m, n$  coprime and that  $\tau(p^l) = \tau(p^{l-1})\tau(p) - p^{11}\tau(p^{l-2})$  for all primes  $p$  and integers  $l \geq 2$ . This is equivalent to the assertian that the Dirichlet series attached to  $\Delta$  has an Euler product:

$$(1.6) \quad \sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_p \frac{1}{(1 - \tau(p)p^{-s} + p^{11-2s})}.$$

Here the product runs over all primes  $p$  and  $s \in \mathbb{C}$  with real part of  $s$  large enough. This was proven by Mordell and was the starting point of Hecke theory, a fundamental tool in the theory of automorphic forms. The point is that the values  $\tau(p)$  determine the discriminant function completely.

Furthermore, Ramanujan predicted that the roots  $\alpha_p$  and  $\beta_p$  of the polynomial in the denominator of (1.6) are complex conjugates, which is equivalent to the estimate  $|\tau(p)| \leq 2p^{\frac{11}{2}}$ . This was proven by Deligne as a consequence of his proof of the Weil conjectures. But there are still open questions, such as the Lehmer conjecture, that  $\tau(n) \neq 0$  for all  $n \in \mathbb{N}$ . The following table provides some information, at the level of examples:

$n$	$\tau(n)$	$n$	$\tau(n)$	$n$	$\tau(n)$
1	1	11	534612	...	...
2	-24	12	-370944	100	37534859200
3	252	13	-577738	101	81742959102
4	-1472	14	401856	102	41767088832
(1.7)	5	4830	15	1217160	103
	6	-6048	16	987136	104
	7	-16744	17	-6905934	...
	8	84480	18	2727432	1050
	9	-113643	19	10661420	1051
	10	-115920	20	-7109760	...
					...
					...
					...
					...

For example, if we list the  $\tau(n)$  modulo 23, then a certain pattern emerges:

$n$	$\tau(n) \pmod{23}$	$n$	$\tau(n) \pmod{23}$	$n$	$\tau(n) \pmod{23}$
1	1	11	0	...	...
2	-1	12	0	100	0
3	-1	13	-1	101	2
4	0	14	0	102	0
(1.8)	5	0	15	0	103
	6	1	16	-1	104
	7	0	17	0	...
	8	1	18	0	1050
	9	0	19	0	1051
	10	0	20	0	...
					...
					...
					...
					...

Examination of this data leads to the conjecture  $\tau(n) \equiv 0 \pmod{23}$  if  $n$  is coprime to 23 and is quadratic nonresidue modulo 23. This was proven by Ramanujan via an identity of Euler.

The following congruences mentioned earlier can also be found in his notes:

$$\begin{aligned}
 (1.9) \quad \tau(n) &\equiv \sigma_{11}(n) \pmod{2^8} \quad (\text{for } n \text{ odd}), \\
 \tau(n) &\equiv n^2 \sigma_7(n) \pmod{3^3}, \\
 \tau(n) &\equiv n \sigma_9(n) \pmod{5^2}, \\
 \tau(n) &\equiv n \sigma_3(n) \pmod{7}, \\
 \tau(n) &\equiv \sigma_{11}(n) \pmod{691}.
 \end{aligned}$$

To illustrate the 691 congruence we have calculated several coefficients modulo 691. Obviously, one has to have some other insight to guess the congruence of  $\tau(n)$  with  $\sigma_{11}(n)$ .

$n$	$\tau(n) \pmod{691}$	$n$	$\tau(n) \pmod{691}$	$n$	$\tau(n) \pmod{691}$
1	1	11	469	...	...
2	667	12	123	100	398
3	252	13	629	101	210
4	601	14	385	102	140
5	684	15	309	103	539
6	171	16	388	104	20
7	531	17	611	...	...
8	178	18	55	1050	67
9	372	19	672	1051	464
10	168	20	630	...	...

Let  $D \in \mathbb{Z}$  and

$$(1.11) \quad N_a(D) := \#\{x \pmod{2a} \mid x^2 \equiv D \pmod{4a}\}.$$

Then put

$$(1.12) \quad L_D(s) := \frac{\zeta(2s)}{\zeta(s)} \sum_{a=1}^{\infty} N_a(D) a^{-s}.$$

This function has a meromorphic continuation to the whole complex plane and can be evaluated at negative integers. The generalized class numbers of Cohen are

$$(1.13) \quad H(r, N) := L_{-N}(1 - r)$$

for  $r \in \mathbb{N}$  odd and  $N \in \mathbb{N}_0$ .

**Theorem 1.2.** *For all positive integers  $n$  we have congruences between the Ramanujan numbers  $\tau(n)$  and the generalized class numbers  $H(11, N)$  ( $0 < N \leq 4n$ ) with respect to the irregular prime numbers 131 and 593. Then we have:*

$$(1.14) \quad \tau(n) \equiv \frac{H(11, 4n) + 2 \sum_{\substack{0 < -D < 4n \\ D+4n=\square}} H(11, -D)}{H(11, 4) + 2H(11, 3)} \pmod{q}.$$

For example, from the data here,

$$\begin{aligned} \tau(3) &\equiv \frac{H(11, 12) + 2H(11, 11) + 2H(11, 8) + 2H(11, 3)}{H(11, 4) + 2H(11, 3)} \pmod{131} \\ &\equiv \frac{-7750665838/3 - 2 \cdot 1035686046 - 2 \cdot 36581523 - 2 \cdot 3694/3}{-50521/2 - 2 \cdot 3694/3} \pmod{131}. \end{aligned}$$

The first  $n \in \mathbb{N}$  such that  $\tau(n) \equiv 0 \pmod{131}$  is given by  $n = 1051$ . This we found with Pari. From the properties of  $\tau(n)$  already mentioned it follows that 1051 has to be a prime. But more is true, since 131 is not an exceptional prime. That is, Deligne's results on Galois representations attached to modular forms can be applied, leading to congruences of generalized class numbers. For the irregular primes 131 and 593 we have congruences satisfied for a set of primes of positive density and restricted to an arbitrary arithmetic progression.

**Theorem 1.3.** *Let  $a, b$  and  $l$  be positive integers and let  $a$  and  $l$  be coprime. Assume*

$$\mathbb{X}_l(a) := \{p \text{ prime} \mid p \equiv a \pmod{l}\}.$$

*Then the congruence*

$$(1.15) \quad H(11, 4p) + 2 \sum_{\substack{0 < -D < 4p \\ D+4p=\square}} H(11, -D) \equiv b \pmod{131 \cdot 593}$$

*is satisfied by a subset of  $\mathbb{X}_l(a)$  of positive density.*

## 2. GENERALIZED CLASS NUMBERS, GALOIS REPRESENTATION AND CONGRUENCES

It is well known that the discriminant  $\Delta$  is an elliptic cusp form. This makes applicable the theory of modular forms and properties of the associated Galois representation.

For a positive even integer  $k$ , let  $S_k$  be the modular forms of weight  $k$  with respect to  $SL_2(\mathbb{Z})$ . A primitive newform  $f \in S_k$  has  $a_1 = 1$  and is an eigenform for Hecke operators. Let  $x$  and  $y$  be the real and imaginary part of  $z$ . The Petersson norm of  $\Delta$  is given by

$$(2.1) \quad \sqrt{\int_{SL_2(\mathbb{Z}) \backslash \mathfrak{H}} \Delta(z) \overline{\Delta(z)} \operatorname{Im}(z)^{10} dx dy},$$

In the paper [1], H. Cohen introduced generalized class numbers  $H(t, N)$ , where  $t \in \mathbb{N}$  and  $-N$  is the discriminant of a binary quadratic form. Specifically, the numbers  $H(1, N)$  are the Hurwitz-Kronecker class numbers  $H(n)$ , defined for  $N > 0$ . Let  $t$  be odd and  $D := -N < 0$ . Let  $D = D_0 f^2$ , where  $D_0$  is a discriminant of a quadratic number field. Then

$$(2.2) \quad H(t, -D) := L(1-t, \chi_{D_0}) \sum_{d|f} \mu(d) \chi_{D_0}(d) d^{t-1} \sum_{l|(f/d)} \left(\frac{f}{d}\right)^{2t-1}.$$

Here  $L(s, \chi_{D_0})$  is the Dirichlet series  $\sum_{n=1}^{\infty} \chi_{D_0}(n) n^{-s}$  attached to the Dirichlet character  $(\frac{D}{*})$  and  $\mu$  the Moebius function. For  $D = 0$  put  $H(t, 0) = \zeta(1-2t)$ . The values of the Riemann zeta function  $\zeta(s)$  at negative odd negative integers  $s = 1-m$  are given by  $-\frac{B_m}{m}$  where  $B_m$  is the  $m$ -th Bernoulli number. It is not difficult to see that this coincides with the definition of  $H(t, N)$  given in the introduction.

Then  $H(t, N)$  is a rational number and for fixed odd  $t_0 > 0$  the denominator of  $H(t_0, N)$  is bounded. Cohen gave numerical data in the case  $t = 1, 2, 3, 4$ , extended in the book of Eichler and Zagier towards  $t = 5, 7$  (for Jacobi forms). Since the Ramanujan numbers  $\tau(n)$  are directly related to the case  $t = 11$  we have calculated

$H(11, N)$  for  $N = 3, 4 \dots, 35$ , and also the residues modulo 131 and 593:

$N$	$H(11, N)$	(mod 131)	(mod 593)
3	$-3694/3$	35	350
4	$-50521/2$	88	535
7	$-9006448$	64	36
8	$-36581523$	96	54
11	$-1035686046$	85	314
12	$-7750665838/3$	84	238
15	$-26914406416$	119	350
16	$-1059502667113/2$	70	100
19	$-3217455764142$	74	545
20	$-551609685150$	86	40
23	$-55067979610704/23$	27	392
24	$-3741386059246$	60	199
27	$-38640544735576/3$	4	471
28	$-18878676839792$	3	208
31	$-54994838246496$	30	114
32	$-76717050704019$	124	459
35	$-196480469755548$	67	527

*Proof of Theorem 1.3.* Let  $q = 131$  or  $593$ . From Table (2.3)

$$(2.4) \quad H(11, 4) + 2H(11, 3) \pmod{q}$$

is a unit  $\kappa_q$  in the field  $\mathbb{Z}/q\mathbb{Z}$ . This allows us to transfer a result of Swinnerton-Dyer [12] on elliptic modular forms, applied to  $\Delta$  and combined with Theorem 1.2, to the generalized class numbers. Let  $f \in S_k$  be a primitive newform with the property that all Fourier coefficients are rational integers. Then the following is satisfied: Let  $m_1$  and  $m_2$  be positive integers, and let  $c \in \mathbb{Z}/m_1\mathbb{Z}$  and  $d \in (\mathbb{Z}/m_2\mathbb{Z})^*$ . Suppose that any prime divisor of  $m_1$  is not exceptional for  $f$ . Then the set of prime numbers  $p$  such that  $p \equiv d \pmod{m_2}$  with the property  $a_p \equiv c \pmod{m_1}$  has positive density.

Since the discriminant function  $\Delta$  is a normalized cusp form of weight 12 and  $\dim S_{12} = 1$ , the function  $\Delta$  is a primitive newform. Moreover, it follows from the definition that all Fourier coefficients are integers. Since  $\{2, 3, 5, 7, 23, 691\}$  is the set of all exceptional primes related to  $l$ -adic Galois representations of  $\Delta$  we can apply the theorem for the case  $m_1 = 131$  and  $593$ .  $\square$

### 3. THE RAMANUJAN NUMBERS AND THE SYMMETRIC SQUARE $L$ -SERIES

The symmetric square  $L$ -function of  $\Delta$  is

$$(3.1) \quad D(\Delta, s) = \frac{\zeta(2s-22)}{\zeta(s-11)} \sum_{n=1}^{\infty} \tau(n)^2 n^{-s}.$$

It converges absolutely for  $\text{Re}(s) > 12$  and has a holomorphic continuation to the complex plane. Further,  $m = 22$  is a critical value in the sense of Deligne, and

$$D(\Delta, 22)/\pi^{11} = \| \Delta \|^2 Q$$

with  $Q \in \mathbb{Q}^*$  and  $\| \Delta \|$  the Petersson norm of  $\Delta$ .

**3.1. Proof of Theorem 1.1.** For  $g, k$  positive integers and  $k > g + 1$  denote by  $E_k^g(z)$  the holomorphic Siegel-Eisenstein series of weight  $k$  on the Siegel upper half-space  $\mathfrak{H}_g$ ,

$$(3.2) \quad E_k^g(z) := \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_g} \det(cz + d)^{-k},$$

where  $\Gamma_g = Sp_g(\mathbb{Z})$  is the Siegel modular group and  $\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_g \mid c = 0 \right\}$ . The Fourier coefficients of the Eisenstein series,

$$(3.3) \quad E_k^g(z) = \sum_{T \in X_g} A_k^g(T) e^{2\pi i \operatorname{tr}(Tz)}$$

( $X_g$  the set of half-integral positive semi-definite matrices) are rational integers with bounded denominator. We are interested in  $g = 1$  and  $g = 2$ . The classical Eisenstein series  $E_k(z) = E_k^1(z)$  of degree one has the Fourier expansion

$$(3.4) \quad E_k(z) = 1 + \frac{-2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad (q = e^{2\pi i z}),$$

$\sigma_m(n) := \sum_{d \mid n} d^m$ . The Bernoulli numbers can be defined by

$$(3.5) \quad \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

For example,

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \dots.$$

It is well known that  $B_m = 0$  for  $m > 1$  odd. From Kummer's study of Fermat's Last Theorem we have the concept of irregular primes, and all primes  $p$  dividing the numerator  $N_k$  of  $\frac{B_k}{-2k}$  are irregular. We will show that the Ramanujan numbers  $\tau(n)$  are related to  $N_{12} = 691$ . The arithmetic nature of the Fourier coefficients of the Eisenstein series of degree 2 is more complicated but we have still some control. With  $(n, r, m) := \left( \begin{smallmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{smallmatrix} \right) \in X_2$ , the denominator of the rational integer

$$(3.6) \quad N_k N_{2k-2} A_k^2(n, r, m)$$

is not divisible by  $p | N_k N_{2k-2}$ . Maass conjectured that such primes  $p$  are optimal, in the sense that, for  $p | N_k N_{2k-2}$  there exists a  $T \in X_2$  such that

$$(3.7) \quad N_k N_{2k-2} A_k^2(T) \in \mathbb{Z}_p^*.$$

To prove the 691 Ramanujan congruences for  $\tau(n)$  it is sufficient that

$$(3.8) \quad 691^2 A_{12}^2(T) \equiv 0 \pmod{691}.$$

But this is well known and follows from results of Maass [8]. Let  $D < 0$  be a discriminant. Then for all  $T = (n, r, 1) \in X_2$  with  $D = r^2 - 4n$ ,

$$(3.9) \quad A_k^2(n, r, 1) = 2 \frac{L_D(2-k)}{\zeta(1-k) \zeta(3-2k)}.$$

Further, we have  $A_k^2(0, 0, 0) = 0$  and  $A_k^2(n, 0, 0) = A_k^1(n)$ . The calculation of these rational integers is straightforward. For example,

$$(3.10) \quad A_{12}^2(1, 0, 1) = \frac{2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 23 \cdot 2659}{131 \cdot 593 \cdot 691},$$

$$(3.11) \quad A_{12}^2(1, 1, 1) = \frac{2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 23 \cdot 1847}{131 \cdot 593 \cdot 691},$$

$$(3.12) \quad A_{12}^2(1, 2, 1) = \frac{2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}{691}.$$

Put  $A_{12}^2(1, 1) := A_{12}^2(1, 0, 1) + 2A_{12}^2(1, 1, 1) + 2A_{12}^2(1, 2, 1)$ . Then

$$(3.13) \quad A_{12}^2(1, 1) = \frac{2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 1549}{131 \cdot 593 \cdot 691}.$$

Fixing the imbedding  $\mathfrak{H} \times \mathfrak{H}$  into  $\mathfrak{H}_2$ ,  $(z, w) \mapsto \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$ ,  $E_k^2(z, w) \in \text{Sym}^2 M_k$ , where  $M_k$  denotes the space of elliptic modular forms of weight  $k$  with respect to  $\Gamma = SL_2(\mathbb{Z})$ . Since  $M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$ ,

$$(3.14) \quad E_{12}^2|_{\mathfrak{H} \times \mathfrak{H}} = E_{12} \otimes E_{12} + \alpha(E_{12} \otimes \Delta + \Delta \otimes E_{12}) + \beta\Delta \otimes \Delta.$$

It follows from [7] that  $\alpha = 0$  and  $\beta \neq 0$ , since Siegel type Eisenstein series are elements of the Maass Spezialschar and there are no elliptic cusp forms of weight 14. Moreover, it is obvious that  $\beta$  is a rational number. Garrett's viewpoint [4], [5] on the doubling method and the triple  $L$ -function leads to

$$(3.15) \quad \beta = \frac{\pi}{2^9 \cdot 11 \cdot \zeta(12) \cdot \zeta(11) \|\Delta\|^2} \sum_{n=1}^{\infty} \tau(n)^2 n^{-22}.$$

The Fourier expansion and Fourier coefficients  $A_{12}^2(n, m)$  of  $E_{12}^2|_{\mathfrak{H} \times \mathfrak{H}}$  can be easily deduced from  $A_{12}^2(n, r, m)$ , since

$$(3.16) \quad A_{12}^2(n, m) = \sum_{r \in \mathbb{Z}, r \leq \sqrt{4nm}} A_{12}^2(n, r, m).$$

Let  $a(n)$  denote the Fourier coefficient of the elliptic Eisenstein series. Then we obtain from (3.14) the identity

$$(3.17) \quad A_{12}^2(1, 1) = a(1)^2 + \frac{\pi}{2^9 \cdot 11 \cdot \zeta(12) \cdot \zeta(11) \|\Delta\|^2} \sum_{n=1}^{\infty} \tau(n)^2 n^{-22}.$$

Then we have  $691^2\beta \in \mathbb{Z}_{691}^*$  and  $\frac{\beta}{a(1)^2} \equiv -1 \pmod{691}$ . This proves the first part of the theorem on the quadratic residue mod 691. We also have

$$(3.18) \quad A_{12}^2(n, 1) = a(n)a(1) + \beta\tau(n).$$

Hence  $0 \equiv \frac{a(n)}{a(1)} + (-1)\tau(n) \pmod{691}$ . This proves the Ramanujan congruence in the theorem since  $a(n)/a(1) = \sigma_{11}(n)$ .

*Remark.* Note that this proof is constructive. For example, it is easy to deduce that

$$(3.19) \quad \beta = \frac{2^{14} \cdot 3^7 \cdot 5^3 \cdot 7^3 \cdot 13}{131 \cdot 593 \cdot 691^2}.$$

**3.2. Congruences between the Ramanujan numbers and class numbers.** In this subsection we prove that  $\tau(n)$  satisfies certain congruences mod 131 and 593 related to the class numbers  $H(11, N)$ . Here  $N_{22} = 131 \cdot 593$ .

*Proof of Theorem 1.2.* We have already seen that  $A_{12}^2(1, 1) = a(1)^2 + \beta$ . From this we easily obtain for  $q = 131$  or  $593$  the congruence

$$2(H(11, 4) + 2H(11, 3)) \equiv \zeta(-11) \cdot \zeta(-21) \cdot \beta \pmod{q}.$$

Here we used the explicit formula of the Fourier coefficients of the Siegel type Eisenstein series and, twice, the fact that  $\zeta(-11)\zeta(-21)a(1) \equiv 0 \pmod{q}$ . More generally, let  $n \in \mathbb{N}$  and choose  $r \in \mathbb{N}$  such that  $4n - r^2 > 0$  and  $4n - r^2 \leq 0$ . Then

$$\begin{aligned} & 2\left(H(11, 4n) + 2H(11, 4n-1) + \dots + 2H(11, 4n-r^2)\right) \\ & \equiv \zeta(-11) \cdot \zeta(-21) \cdot \beta \cdot \tau(n) \pmod{q}. \end{aligned}$$

Since  $H(11, 4) + 2H(11, 3) \not\equiv 0 \pmod{q}$  the theorem is proven.  $\square$

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