

LOCAL SAMPLING THEOREMS FOR SPACES GENERATED BY SPLINES WITH ARBITRARY KNOTS

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ABSTRACT. Most of the known results on sampling theorems, e.g., regular and irregular sampling theorems for band-limited functions, are concerned with global sampling. That is, to recover a function at a point or on an interval, we have to know all the samples, which are usually infinitely many. On the other hand, local sampling, which invokes only finitely many samples to reconstruct a function on a bounded interval, is practically useful since we only need to consider a function on a bounded interval in many cases and hardware can process only finitely many samples. In this paper, we give a characterization of local sampling sequences for spaces generated by B-splines with arbitrary knots.

1. INTRODUCTION AND MAIN RESULTS

One of the main applications of sampling theorems is to reconstruct functions from sampled values. Specifically, for a certain function space V and a sequence of sampling points $\{x_k : k \in \mathbb{Z}\}$, there is a sequence of functions $\{S_k(x) : k \in \mathbb{Z}\} \subset V$ such that for any $f \in V$,

$$f(x) = \sum_{k \in \mathbb{Z}} f(x_k) S_k(x).$$

The sampling theorem is one of the most powerful tools in signal analysis and it is therefore very useful to characterize sampling sequences for a given function space. For the space of band-limited functions, the theory is well understood by the work of Beurling [5], Landau [15], Jaffard [13], Ortega-Cerdà and Seip [18], and several others. For the sampling on general shift invariant subspaces, however, many questions remain.

Note that in many cases (e.g., in the Shannon sampling theorem for band-limited functions [20] and many other sampling theorems for shift invariant subspaces [1, 2, 3, 4, 6, 10, 12, 16]), S_k is not compactly supported. That means, to recover the function at a point or on a finite interval, we have to know all the (infinitely many) samples. On the other hand, since hardware can handle only finitely many

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data, it is practically useful to consider local sampling. That is, given a function space V and an interval $[a, b]$, we want to find conditions on the sampling sequence $\{x_k : 0 \leq k \leq K - 1\} \subset [a, b]$ such that there is a sequence of functions $\{S_k : 0 \leq k \leq K - 1\}$ with

$$f(x) = \sum_{k=0}^{K-1} f(x_k)S_k(x), \quad \forall f \in V, x \in [a, b].$$

As far as we know, the only result in this aspect is the periodic nonuniform sampling theorem for cardinal B-spline subspaces. Specifically, let W_m be the shift invariant subspace generated by the cardinal B-spline $\chi_{[0,1]} * \cdots * \chi_{[0,1]}$ ($m + 1$ terms) and let $0 \leq x_0 < x_1 < \cdots < x_m < 1$ be fixed. Then we can find compactly supported $S_0, S_1, \dots, S_m \in W_m$ such that

$$f(x) = \sum_{k \in \mathbb{Z}} \sum_{0 \leq p \leq m} f(x_p + k(m+1))S_p(x - k(m+1)), \quad \forall f \in W_m.$$

In this case, $f(x)$ is determined by finitely many samples near x . However, the distribution of sampling points is bad. There are $m + 1$ sampling points in every interval such as $[k(m+1), k(m+1) + 1)$ while there is no sampling point in the subsequent intervals of length m .

In [24], the authors studied local sampling problems on spaces generated by cardinal B-splines and gave a characterization of local sampling sequences for these spaces. In this paper, we consider the same problem but with more general settings. Specifically, we consider spline spaces with arbitrary knot sequences.

Let $\Gamma = \{t_k : k \in \mathbb{Z}\}$ be a sequence of real numbers such that

$$(1.1) \quad t_k \leq t_{k+1} \quad \text{and} \quad t_k < t_{k+m}, \quad k \in \mathbb{Z},$$

where $m \geq 1$ is a fixed integer. Note that the above inequalities show that each point can appear at most m times in the sequence. Let φ_n be the m -degree B-spline with knots $(t_n, t_{n+1}, \dots, t_{n+m+1})$ and

$$V_m = \left\{ \sum_{n \in \mathbb{Z}} c_n \varphi_n : c_n \in \mathbb{C} \right\}.$$

Observe that V_m stands for a large class of function spaces. For example, for $\Gamma = \mathbb{Z}$, φ_n is exactly the m -degree cardinal B-spline, i.e., $\varphi_n = B_m(\cdot - n)$, where $B_m = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ ($m + 1$ terms).

Another interesting example is $\Gamma = \{[n/r] : n \in \mathbb{Z}\}$, where $1 \leq r \leq m$ and we use the notation $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$. In this case,

$$(1.2) \quad V_m^{(r)} = \left\{ \sum_{n \in \mathbb{Z}, 1 \leq l \leq r} c_{n,l} \psi_l(\cdot - n) : c_{n,l} \in \mathbb{C} \right\},$$

where ψ_l is the normalized B-spline with knots $([l/r], [(l+1)/r], \dots, [(l+m+1)/r])$. These splines are investigated in the study of wavelets of multiplicity r . We refer to [11, 17] for details.

For convenience, we introduce the following definitions. Given a knot sequence $\{t_k : k \in \mathbb{Z}\}$, we define

$$\begin{aligned} L(k) &:= L_\Gamma(k) = \min\{k' \leq k : t_{k'} = t_k\}, \\ R(k) &:= R_\Gamma(k) = \max\{k' \geq k : t_{k'} = t_k\}. \end{aligned}$$

For simplicity, we write $L(k)$ and $R(k)$ instead of $L_\Gamma(k)$ and $R_\Gamma(k)$, respectively. It is easy to see that t_k appears exactly $R(k) - L(k) + 1$ times in Γ . By (1.1), we have

$$(1.3) \quad 0 \leq R(k) - L(k) \leq m - 1, \quad \forall k \in \mathbb{Z}.$$

Let $\#E$ denote the cardinality of a sequence E .

Definition 1.1. We call a sequence E an m -spread sequence on $[t_{N_1}, t_{N_2}]$ with respect to $\{t_k : k \in \mathbb{Z}\}$ if $E \subset [t_{N_1}, t_{N_2}]$ consists of distinct points and satisfies the following:

$$(1.4) \quad \#E \geq L(N_2) - R(N_1) + m,$$

$$(1.5) \quad \#(E \cap [t_{N_1}, t_n]) \geq R(n) - R(N_1), \quad N_1 \leq n \leq N_2,$$

$$(1.6) \quad \#(E \cap (t_n, t_{N_2}]) \geq L(N_2) - L(n), \quad N_1 \leq n \leq N_2,$$

$$(1.7) \quad \#(E \cap (t_{n_1}, t_{n_2})) \geq R(n_2) - L(n_1) - m, \quad N_1 \leq n_1 \leq n_2 \leq N_2.$$

Definition 1.2. We call a sequence $E := \{x_k : 0 \leq k \leq K - 1\}$ a local sampling sequence for V_m on $[t_{N_1}, t_{N_2}]$ if $E \subset [t_{N_1}, t_{N_2}]$, and there is a sequence of functions $\{S_k : 0 \leq k \leq K - 1\}$ such that

$$(1.8) \quad f(x) = \sum_{k=0}^{K-1} f(x_k)S_k(x), \quad \forall f \in V_m, \quad x \in [t_{N_1}, t_{N_2}].$$

We only need to consider local sampling sequences consisting of distinct points. In fact, if E is a local sampling sequence, then the largest subsequence of E which contains only distinct points is also a local sampling sequence and vice visa.

Note that there is not a local sampling sequence for band-limited functions. In fact, since a band-limited function is the restriction of an analytical function on the real line, if $f(x) = \sum_{k=0}^{K-1} f(x_k)S_k(x)$ holds on some interval, then f is determined on the whole real line, which is impossible since a band-limited function is not determined by finitely many samples in general.

For the case of spaces generated by B-splines, however, such a local sampling sequence does exist. In fact, we have the following.

Theorem 1.1. *A sequence of distinct points is a local sampling sequence for V_m on $[t_{N_1}, t_{N_2}]$ if and only if it is an m -spread sequence with respect to $\{t_k : k \in \mathbb{Z}\}$ on the same interval.*

By setting $t_n = \lfloor n/r \rfloor$, we get a characterization of local sampling sequences for the space $V_m^{(r)}$ defined by (1.2).

Corollary 1.2. *A sequence of distinct points is a local sampling sequence for $V_m^{(r)}$ on $[N'_1, N'_2]$ if and only if it satisfies the following:*

$$(1.9) \quad \#E \geq r(N'_2 - N'_1 - 1) + m + 1,$$

$$(1.10) \quad \#(E \cap [N'_1, N'_1 + k]) \geq rk, \quad 0 \leq k \leq N'_2 - N'_1,$$

$$(1.11) \quad \#(E \cap (N'_2 - k, N'_2]) \geq rk, \quad 0 \leq k \leq N'_2 - N'_1,$$

$$(1.12) \quad \#(E \cap (n'_1, n'_2)) \geq r(n'_2 - n'_1 + 1) - m - 1, \quad N'_1 \leq n'_1 \leq n'_2 \leq N'_2.$$

By setting $r = 1$, we get [24, Theorem 1.1].

We give the proof of Theorem 1.1 in Section 2. As applications, we give several local sampling theorems for $V_m^{(r)}$ in Section 3.

2. PROOF OF MAIN RESULTS

We begin with a result on m -spread sequences.

Lemma 2.1. *Let E be an m -spread sequence on $[t_{N_1}, t_{N_2}]$ with respect to $\{t_k : k \in \mathbb{Z}\}$. Then there is a subsequence $E' \subset E$ such that $\#E' = L(N_2) - R(N_1) + m$ and E' is also an m -spread sequence on $[t_{N_1}, t_{N_2}]$.*

Proof. (i) Assume that $K := \#E \geq L(N_2) - R(N_1) + m + 1$. Let

$$\begin{aligned} a_n &= \#(E \cap [t_{N_1}, t_n]) - (R(n) - R(N_1)), \\ b_n &= \#(E \cap (t_n, t_{N_2}]) - (L(N_2) - L(n)), \quad N_1 \leq n \leq N_2. \end{aligned}$$

Then we see from (1.5) and (1.6) that

$$(2.1) \quad a_n, b_n \geq 0, \quad N_1 \leq n \leq N_2.$$

Let

$$(2.2) \quad n_a = \min\{n : N_1 \leq n \leq N_2 \text{ and } a_l \geq 1, n \leq l \leq N_2\},$$

$$(2.3) \quad n_b = \max\{n : N_1 \leq n \leq N_2 \text{ and } b_l \geq 1, N_1 \leq l \leq n\}.$$

(ii) Prove that $N_1 + 1 \leq n_a \leq n_b + 1$.

Since $a_{N_1} = 0$ and $a_{N_2} \geq K - 1 - (R(N_2) - R(N_1)) \geq L(N_2) - R(N_2) + m \geq 1$, thanks to (1.3), we have $N_1 + 1 \leq n_a \leq N_2$. By (2.2), we have $a_{n_a-1} = 0$. Consequently,

$$\#(E \cap [t_{N_1}, t_{n_a-1}]) = R(n_a - 1) - R(N_1).$$

Therefore,

$$(2.4) \quad \#(E \cap [t_{n_a-1}, t_{N_2}]) = K - (R(n_a - 1) - R(N_1)).$$

For $N_1 \leq n < L(n_a - 1)$, we have $t_n < t_{n_a-1}$. By (1.7) and (2.4),

$$\begin{aligned} & \#(E \cap (t_n, t_{N_2}]) \\ &= \#(E \cap (t_n, t_{n_a-1})) + \#(E \cap [t_{n_a-1}, t_{N_2}]) \\ &\geq (R(n_a - 1) - L(n) - m) + (K - (R(n_a - 1) - R(N_1))) \\ &= K - L(n) + R(N_1) - m. \end{aligned}$$

Hence

$$\begin{aligned} b_n &= \#(E \cap (t_n, t_{N_2}]) - (L(N_2) - L(n)) \\ &\geq (K - L(n) + R(N_1) - m) - (L(N_2) - L(n)) \\ &= K - L(N_2) + R(N_1) - m \\ &\geq 1, \quad N_1 \leq n < L(n_a - 1). \end{aligned}$$

For $L(n_a - 1) \leq n \leq n_a - 1$, we see from (2.4) that

$$\begin{aligned} b_n &= \#(E \cap (t_n, t_{N_2}]) - (L(N_2) - L(n)) \\ &= \#(E \cap (t_{n_a-1}, t_{N_2}]) - (L(N_2) - L(n_a - 1)) \\ &\geq K - (R(n_a - 1) - R(N_1)) - 1 - (L(N_2) - L(n_a - 1)) \\ &\geq m - R(n_a - 1) + L(n_a - 1) \geq 1. \end{aligned}$$

By (2.3), $n_b \geq n_a - 1$.

(iii) Prove that $\#(E \cap (t_{n_a-1}, t_{n_a})) \geq 1$.

Observe that

$$\begin{aligned} \#(E \cap [t_{N_1}, t_{n_a})) &\geq R(n_a) - R(N_1) + 1 \geq R(n_a - 1) - R(N_1) + 1 \\ &= \#(E \cap [t_{N_1}, t_{n_a-1})) + 1. \end{aligned}$$

We have $t_{n_a-1} < t_{n_a}$. Hence

$$(2.5) \quad R(n_a - 1) = n_a - 1 < n_a = L(n_a) \leq R(n_a).$$

By (2.2), $a_{n_a-1} = 0$ and $a_n \geq 1$ for $n_a \leq n \leq N_2$. Hence

$$\begin{aligned} (2.6) \quad \#(E \cap [t_{n_a-1}, t_n)) &= \#(E \cap [t_{N_1}, t_n)) - \#(E \cap [t_{N_1}, t_{n_a-1})) \\ &\geq (R(n) - R(N_1) + 1) - (R(n_a - 1) - R(N_1)) \\ &= R(n) - R(n_a - 1) + 1, \quad n_a \leq n \leq N_2. \end{aligned}$$

Therefore,

$$(2.7) \quad \#(E \cap (t_{n_a-1}, t_n)) \geq R(n) - R(n_a - 1), \quad n_a \leq n \leq N_2.$$

By setting $n = n_a$, we get

$$\#(E \cap (t_{n_a-1}, t_{n_a})) \geq R(n_a) - R(n_a - 1) \geq 1,$$

thanks to (2.5).

(iv) Take some $y' \in E \cap (t_{n_a-1}, t_{n_a})$ and let $E' = E \setminus \{y'\}$. Then E' is an m -spread sequence on $[t_{N_1}, t_{N_2}]$.

Since $\#E' = K - 1 \geq L(N_2) - R(N_1) + m$, we only need to show that E' satisfies (1.5)–(1.7).

For $N_1 \leq n \leq n_a - 1$, we have

$$\#(E' \cap [t_{N_1}, t_n)) = \#(E \cap [t_{N_1}, t_n)) = a_n + (R(n) - R(N_1)) \geq R(n) - R(N_1).$$

For $n_a \leq n \leq N_2$, we have

$$\#(E' \cap [t_{N_1}, t_n)) = \#(E \cap [t_{N_1}, t_n)) - 1 = a_n + (R(n) - R(N_1)) - 1 \geq R(n) - R(N_1).$$

On the other hand, for $N_1 \leq n \leq n_a - 1 \leq n_b$,

$$\begin{aligned} \#(E' \cap (t_n, t_{N_2}]) &= \#(E \cap (t_n, t_{N_2}]) - 1 \\ &= b_n + (L(N_2) - L(n)) - 1 \geq L(N_2) - L(n). \end{aligned}$$

For $n_a \leq n \leq N_2$, we have

$$\begin{aligned} \#(E' \cap (t_n, t_{N_2}]) &= \#(E \cap (t_n, t_{N_2}]) \\ &= b_n + (L(N_2) - L(n)) \geq L(N_2) - L(n). \end{aligned}$$

Hence (1.5) and (1.6) hold. Now it remains to prove that

$$\#(E' \cap (t_{n_1}, t_{n_2})) \geq R(n_2) - L(n_1) - m, \quad \forall N_1 \leq n_1 \leq n_2 \leq N_2.$$

There are three cases.

Case 1. $n_1 \geq n_a$ or $n_2 \leq n_a - 1$. In this case, we have

$$\#(E' \cap (t_{n_1}, t_{n_2})) = \#(E \cap (t_{n_1}, t_{n_2})) \geq R(n_2) - L(n_1) - m.$$

Case 2. $n_1 < L(n_a - 1)$ and $n_2 \geq n_a$. In this case,

$$\begin{aligned} \#(E' \cap (t_{n_1}, t_{n_2})) &= \#(E' \cap (t_{n_1}, t_{n_a-1})) + \#(E' \cap [t_{n_a-1}, t_{n_2})) \\ &= \#(E \cap (t_{n_1}, t_{n_a-1})) + \#(E \cap [t_{n_a-1}, t_{n_2})) - 1 \\ &\geq (R(n_a - 1) - L(n_1) - m) + (R(n_2) - R(n_a - 1)) \\ &= R(n_2) - L(n_1) - m, \end{aligned}$$

thanks to (2.6).

Case 3. $L(n_a - 1) \leq n_1 \leq n_a - 1$ and $n_2 \geq n_a$. By (2.7), we have

$$\begin{aligned} \#(E' \cap (t_{n_1}, t_{n_2})) &= \#(E' \cap (t_{n_a-1}, t_{n_2})) \\ &= \#(E \cap (t_{n_a-1}, t_{n_2})) - 1 \\ &\geq R(n_2) - R(n_a - 1) - 1 \\ &= R(n_2) - R(n_1) - 1 \\ &\geq R(n_2) - R(n_1) - m. \end{aligned}$$

Now, we get that E' is an m -spread sequence on $[t_{N_1}, t_{N_2}]$.

(v) Using the above procedures repeatedly, we obtain some $E' \subset E$ such that $\#E' = L(N_2) - R(N_1) + m$ and E' is an m -spread sequence on $[t_{N_1}, t_{N_2}]$. \square

Let $x_1 < x_2 < \dots < x_n$ be real numbers and f_1, f_2, \dots, f_n be functions. Define

$$M \begin{pmatrix} x_1, & x_2, & \dots, & x_n \\ f_1, & f_2, & \dots, & f_n \end{pmatrix} = \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{bmatrix}.$$

Lemma 2.2. *Let n, K be integers and $K > 0$. For any real numbers $x_0 < x_1 < \dots < x_{K-1}$, the matrix $M \begin{pmatrix} x_0 & x_1 & \cdots & x_{K-1} \\ \varphi_n & \varphi_{n+1} & \cdots & \varphi_{n+K-1} \end{pmatrix}$ is invertible if and only if*

$$\varphi_{n+k}(x_k) \neq 0, \quad 0 \leq k \leq K-1.$$

Proof. This is a consequence of [19, Theorem 4.61]. \square

Lemma 2.3. *Suppose that $E := \{x_k : 0 \leq k \leq K-1\}$ is an m -spread sequence on $[t_{N_1}, t_{N_2}]$ with respect to $\{t_k : k \in \mathbb{Z}\}$, where $K = L(N_2) - R(N_1) + m$. Suppose that $x_{k-1} < x_k$, $1 \leq k \leq K-1$. Then the matrix*

$$M \begin{pmatrix} x_0 & x_1 & \cdots & x_{K-1} \\ \varphi_{R(N_1)-m} & \varphi_{R(N_1)-m+1} & \cdots & \varphi_{L(N_2)-1} \end{pmatrix}$$

is invertible.

Proof. By Lemma 2.2, we only need to show that

$$\varphi_{R(N_1)-m+k}(x_k) \neq 0, \quad 0 \leq k \leq K-1.$$

Assume that there is some $0 \leq k_0 \leq K-1$ such that $\varphi_{R(N_1)-m+k_0}(x_{k_0}) = 0$. Since $\varphi_{R(N_1)-m+k_0}(x) > 0$ in $(t_{R(N_1)-m+k_0}, t_{R(N_1)+k_0+1})$, there are only two cases.

Case 1. $x_{k_0} \leq t_{R(N_1)-m+k_0}$. Then we have

$$t_{N_1} \leq x_k \leq x_{k_0} \leq t_{R(N_1)-m+k_0}, \quad 0 \leq k \leq k_0.$$

Hence $\#(E \cap [t_{N_1}, t_{R(N_1)-m+k_0}]) \geq k_0 + 1$. On the other hand, since

$$R(N_1) - m + k_0 \leq R(N_1) - m + K - 1 = L(N_2) - 1 < N_2,$$

we see from (1.6) that

$$\begin{aligned} \#(E \cap (t_{R(N_1)-m+k_0}, t_{N_2})) &\geq L(N_2) - L(R(N_1) - m + k_0) \\ &\geq L(N_2) - (R(N_1) - m + k_0) \\ &= K - k_0. \end{aligned}$$

Hence $\#(E \cap [t_{N_1}, t_{N_2}]) \geq K + 1$, which contradicts $\#E = K$.

Case 2. $x_{k_0} \geq t_{R(N_1)+k_0+1}$. Then we have

$$t_{R(N_1)+k_0+1} \leq x_{k_0} \leq x_k \leq t_{N_2}, \quad k_0 \leq k \leq K - 1.$$

Hence $\#(E \cap [t_{R(N_1)+k_0+1}, t_{N_2}]) \geq K - k_0$. But

$$\#(E \cap [t_{N_1}, t_{R(N_1)+k_0+1}]) \geq R(R(N_1) + k_0 + 1) - R(N_1) \geq k_0 + 1,$$

thanks to (1.5). Hence $\#(E \cap [t_{N_1}, t_{N_2}]) \geq K + 1$. Again, this contradicts $\#E = K$. \square

Proof of the sufficiency of Theorem 1.1. Assume that $E := \{x_k : 0 \leq k \leq K - 1\}$ is an m -spread sequence on $[t_{N_1}, t_{N_2}]$, where $x_0 < x_1 < \cdots < x_{K-1}$ and $K \geq K' := L(N_2) - R(N_1) + m$. Let

$$(2.8) \quad A = \left[\varphi_n(x_k) \right]_{0 \leq k \leq K-1, R(N_1)-m \leq n \leq L(N_2)-1}$$

be a $K \times K'$ matrix. By Lemma 2.1, there is some subsequence $E' \subset E$ which is also an m -spread sequence on $[t_{N_1}, t_{N_2}]$ such that $\#E' = K'$. Denote $E' = \{y_k : 0 \leq k \leq K' - 1\}$, where $y_0 < y_1 < \cdots < y_{K'-1}$. By Lemma 2.3, we have

$$A' := M \begin{pmatrix} y_0 & y_1 & \cdots & y_{K'-1} \\ \varphi_{R(N_1)-m} & \varphi_{R(N_1)-m+1} & \cdots & \varphi_{L(N_2)-1} \end{pmatrix}$$

is invertible. Hence, there is some constant $\alpha > 0$ such that

$$\|A'c\|_2 \geq \alpha \|c\|_2, \quad \forall c \in \mathbb{C}^{K'}.$$

Since A' consists of K' rows of A , we have

$$(2.9) \quad c^T A^T A c = \|Ac\|_2^2 \geq \|A'c\|_2^2 \geq \alpha^2 \|c\|_2^2, \quad \forall c \in \mathbb{C}^{K'}.$$

Hence $A^T A$ is invertible.

For any $f \in V_m$, there are some $c_n \in \mathbb{C}$ such that $f(x) = \sum_{n \in \mathbb{Z}} c_n \varphi_n(x)$. Observe that $\text{supp } \varphi_n = (t_n, t_{n+m+1})$. We have $\text{supp } \varphi_n \subset (-\infty, t_{N_1})$ for $n \leq R(N_1) - m - 1$ and $\text{supp } \varphi_n \subset (t_{N_2}, +\infty)$ for $n \geq L(N_2)$. Hence $\varphi_n(x) = 0$ for $x \in [t_{N_1}, t_{N_2}]$ and $n \notin [R(N_1) - m, L(N_2) - 1]$. Therefore,

$$f(x) = \sum_{n \in \mathbb{Z}} c_n \varphi_n(x) = \sum_{n=R(N_1)-m}^{L(N_2)-1} c_n \varphi_n(x), \quad \forall x \in [t_{N_1}, t_{N_2}].$$

It follows that

$$(2.10) \quad F = Ac,$$

where

$$F = [f(x_0), f(x_1), \dots, f(x_{K-1})]^T$$

and

$$c = [c_{R(N_1)-m}, c_{R(N_1)-m+1}, \dots, c_{L(N_2)-1}]^T.$$

Hence

$$(2.11) \quad c = (A^T A)^{-1} A^T F.$$

Let

$$(2.12) \quad \Phi = [\varphi_{R(N_1)-m}, \varphi_{R(N_1)-m+1}, \dots, \varphi_{L(N_2)-1}]^T$$

and

$$(2.13) \quad [S_0, S_1, \dots, S_{K-1}]^T = A(A^T A)^{-1} \Phi.$$

Then we have

$$(2.14) \quad f(x) = c^T \Phi(x) = \sum_{k=0}^{K-1} f(x_k) S_k(x), \quad \forall x \in [t_{N_1}, t_{N_2}].$$

Obviously, $\text{supp } S_k \subset [t_{R(N_1)-m}, t_{L(N_2)+m}]$ since $\text{supp } \varphi_n = (t_n, t_{n+m+1})$. \square

Let $T_{[t_{N_1}, t_{N_2}]} f$ be the restriction of a function f on $[t_{N_1}, t_{N_2}]$, i.e., $(T_{[t_{N_1}, t_{N_2}]} f)(x) = f(x) \chi_{[t_{N_1}, t_{N_2}]}(x)$. Set

$$V_{N_1, N_2} = T_{[t_{N_1}, t_{N_2}]} V_m := \{T_{[t_{N_1}, t_{N_2}]} f : f \in V_m\}.$$

Lemma 2.4. $\{T_{[t_{N_1}, t_{N_2}]} \varphi_n : R(N_2) - m \leq n \leq L(N_2) - 1\}$ is a basis for V_{N_1, N_2} .

Proof. First, for any $f \in V_{N_1, N_2}$, there are constants $c_{R(N_1)-m}, c_{R(N_1)-m+1}, \dots, c_{L(N_2)-1}$ such that

$$f(x) = \sum_{n=R(N_1)-m}^{L(N_2)-1} c_n \varphi_n(x) = \sum_{n=R(N_1)-m}^{L(N_2)-1} c_n T_{[t_{N_1}, t_{N_2}]} \varphi_n(x), \quad x \in [t_{N_1}, t_{N_2}].$$

Hence it suffices to show that $\{T_{[t_{N_1}, t_{N_2}]} \varphi_n : R(N_1) - m \leq n \leq L(N_2) - 1\}$ is linearly independent, or equivalently, $f(x) = 0$ on $[t_{N_1}, t_{N_2}]$ implies that $c_n = 0, R(N_1) - m \leq n \leq L(N_2) - 1$.

To see this, take $2m$ points in each interval of the form $(t_{L(n)}, t_{R(n+1)})$, $n \geq N_1$. Let E be the collection of all these points. It is easy to see that E is a local sampling sequence on $[t_{N_1}, t_{N_2}]$ with respect to $\{t_k : k \in \mathbb{Z}\}$. Let A be defined as in (2.8). Then we see from (2.11) that

$$c_n = 0, \quad R(N_1) - m \leq n \leq L(N_2) - 1,$$

since $f(x_k) = 0, 0 \leq k \leq K - 1$. \square

Proof of the necessity of Theorem 1.1. Suppose that $E = \{x_k : 0 \leq k \leq K - 1\}$ is a local sampling sequence for V_m on $[t_{N_1}, t_{N_2}]$, where $x_{k-1} < x_k, 1 \leq k \leq K - 1$. Then there is some sequence of functions $\{S_k : 0 \leq k \leq K - 1\}$ such that (1.8) holds.

For any $f \in V_{N_1, N_2}$, $f(x) = \sum_{k=0}^{K-1} f(x_k) S_k(x)$ on $[t_{N_1}, t_{N_2}]$. Hence $\dim V_{N_1, N_2} \leq K$. On the other hand, we see from Lemma 2.4 that $\dim V_{N_1, N_2} = L(N_2) - R(N_1) + m$. Hence $K \geq L(N_2) - R(N_1) + m$.

Next we prove that (1.5) holds. Otherwise, there is some $N_1 + 1 \leq n_0 \leq N_2$ such that $\#(E \cap [t_{N_1}, t_{n_0}]) < R(n_0) - R(N_1)$. Then we have $R(n_0) \geq R(N_1) + 1$ and $x_{R(n_0)-R(N_1)-1} \geq t_{n_0}$. Consequently,

$$\varphi_n(x_{R(n_0)-R(N_1)-1}) = 0, \quad \forall R(N_1) - m \leq n \leq R(n_0) - m - 1,$$

thanks to $\text{supp } \varphi_n = (t_n, t_{n+m+1})$. Therefore, the last row of the matrix

$$M \begin{pmatrix} x_0 & x_1 & \cdots & x_{R(n_0)-R(N_1)-1} \\ \varphi_{R(N_1)-m} & \varphi_{R(N_1)-m+1} & \cdots & \varphi_{R(n_0)-m-1} \end{pmatrix}$$

is 0. Hence its rank is no greater than $R(n_0) - R(N_1) - 1 < R(n_0) - R(N_1)$. Therefore, we can find some $c_{R(N_1)-m}, c_{R(N_1)-m+1}, \dots, c_{R(n_0)-m-1}$, not all of which are zeros, such that

$$\sum_{l=R(N_1)-m}^{R(n_0)-m-1} c_l \varphi_l(x_k) = 0, \quad 0 \leq k \leq R(n_0) - R(N_1) - 1.$$

Let

$$f(x) = \sum_{l=R(N_1)-m}^{R(n_0)-m-1} c_l \varphi_l(x).$$

Then we have $f(x_k) = 0$, $0 \leq k \leq R(n_0) - R(N_1) - 1$. On the other hand, since $\text{supp } f \subset (-\infty, t_{R(n_0)}) = (-\infty, t_{n_0})$ and $x_k \geq x_{R(n_0)-R(N_1)-1} \geq t_{n_0}$ for $k \geq R(n_0) - R(N_1) - 1$, we also have $f(x_k) = 0$ for $k \geq R(n_0) - R(N_1) - 1$. Hence

$$f(x_k) = 0, \quad 0 \leq k \leq K - 1.$$

It follows that we cannot reconstruct f on $[t_{N_1}, t_{N_2}]$ from $\{f(x_k) : 0 \leq k \leq K - 1\}$ since, by virtue of Lemma 2.4, f is not identically 0 on $[t_{N_1}, t_{N_2}]$. This contradicts the hypothesis.

Similarly, we can prove that (1.6) holds.

Now, it remains to establish (1.7). If (1.7) does not hold, then there are some $N_1 \leq n_1 < n_2 \leq N_2$ such that $\#(E \cap (t_{n_1}, t_{n_2})) < R(n_2) - L(n_1) - m$. Then we have $R(n_2) - L(n_1) - m \geq 1$. Therefore,

$$R(n_2) - L(n_1) \geq m + 1.$$

There are two cases.

Case 1. $\#(E \cap (t_{n_1}, t_{n_2})) = 0$. In this case, we have

$$(2.15) \quad x_k \leq t_{n_1} \quad \text{or} \quad x_k \geq t_{n_2}, \quad 0 \leq k \leq K - 1.$$

Let $f = \varphi_{L(n_1)}$. Then we have $\text{supp } f = (t_{L(n_1)}, t_{L(n_1)+m+1}) \subset (t_{L(n_1)}, t_{R(n_2)}) = (t_{n_1}, t_{n_2})$. By (2.15), $f(x_k) = 0$, $0 \leq k \leq K - 1$. Since E is a local sampling sequence, we have $f = 0$ on $[t_{N_1}, t_{N_2}]$, which is impossible since $t_{L(n_1)+m+1} - t_{L(n_1)} > 0$ and $f > 0$ in $(t_{L(n_1)}, t_{L(n_1)+m+1})$.

Case 2. $\#(E \cap (t_{n_1}, t_{n_2})) \geq 1$.

Let

$$\begin{aligned} k_1 &= \min\{k : 0 \leq k \leq K - 1, x_k \in E \cap (t_{n_1}, t_{n_2})\}, \\ k_2 &= \max\{k : 0 \leq k \leq K - 1, x_k \in E \cap (t_{n_1}, t_{n_2})\}. \end{aligned}$$

Denote the $(K - k_1) \times (L(N_2) - L(n_1))$ matrix $[\varphi_l(x_k)]_{k_1 \leq k \leq K-1, L(n_1) \leq l \leq L(N_2)-1}$

by $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, where

$$A_1 = [\varphi_l(x_k)]_{k_1 \leq k \leq k_2, L(n_1) \leq l \leq L(N_2)-1}$$

and

$$A_2 = \left[\varphi_l(x_k) \right]_{k_2+1 \leq k \leq K-1, L(n_1) \leq l \leq L(N_2)-1}.$$

Since there are $k_2 - k_1 + 1 \leq R(n_2) - L(n_1) - m - 1$ rows in A_1 , we have

$$\text{rank}(A_1) \leq R(n_2) - L(n_1) - m - 1.$$

Observe that

$$\text{supp } \varphi_l \subset (-\infty, t_{n_2}), \quad l \leq R(n_2) - m - 1.$$

We have $\varphi_l(x_k) = 0$ for $l \leq R(n_2) - m - 1$ and $x_k \geq t_{n_2}$. Hence the first $R(n_2) - L(n_1) - m$ columns of A_2 are 0. Thus

$$\text{rank}(A_2) \leq (L(N_2) - L(n_1)) - (R(n_2) - L(n_1) - m) = L(N_2) - R(n_2) + m.$$

Therefore,

$$\text{rank}(A) \leq \text{rank}(A_1) + \text{rank}(A_2) \leq L(N_2) - L(n_1) - 1 < L(N_2) - L(n_1).$$

Thus, we can find some $c_{L(n_1)}, c_{L(n_1)+1}, \dots, c_{L(N_2)-1}$, not all of which are zeros, such that

$$\sum_{l=L(n_1)}^{L(N_2)-1} c_l \varphi_l(x_k) = 0, \quad k_1 \leq k \leq K-1.$$

Let

$$f(x) = \sum_{l=L(n_1)}^{L(N_2)-1} c_l \varphi_l(x).$$

Then we have $f(x_k) = 0$, $k_1 \leq k \leq K-1$. On the other hand, since $x_k \leq t_{n_1}$ for $0 \leq k \leq k_1 - 1$ and $\text{supp } f \subset (t_{n_1}, +\infty)$, we also have $f(x_k) = 0$ for $0 \leq k \leq k_1 - 1$. Now we get

$$f(x_k) = 0, \quad 0 \leq k \leq K-1.$$

Hence, we cannot reconstruct f on $[t_{N_1}, t_{N_2}]$ from $\{f(x_k) : 0 \leq k \leq K-1\}$ since $f \neq 0$, thanks to Lemma 2.4. Again, we get a contradiction. This completes the proof. \square

Proof of Corollary 1.2. It suffices to show that, under the assumption $t_n = \lfloor n/r \rfloor$, conditions (1.9)–(1.12) are equivalent to (1.4)–(1.7).

Observe that for $n = rN + l$, $0 \leq l \leq r-1$,

$$L(n) = rN \quad \text{and} \quad R(n) = rN + r - 1.$$

First, we assume that (1.4)–(1.7) hold. Set $N_1 = rN'_1$ and $N_2 = rN'_2$. Then we have $t_{N_1} = N'_1$ and $t_{N_2} = N'_2$. By (1.4),

$$\#E \geq L(N_2) - R(N_1) + m = r(N'_2 - N'_1 - 1) + m + 1.$$

For $0 \leq k \leq N'_2 - N'_1$, we have

$$\begin{aligned} \#(E \cap [N'_1, N'_1 + k]) &= \#(E \cap [t_{rN'_1}, t_{rN'_2 + rk}]) \\ &\geq R(rN'_2 + rk) - R(rN'_1) = rk. \end{aligned}$$

Similarly, we can prove that

$$\#(E \cap (N'_2 - k, N'_2]) = \#(E \cap [t_{rN'_2 - rk}, t_{rN'_2 + r - 1}]) \geq rk$$

and

$$\#(E \cap (n'_1, n'_2)) = \#(E \cap (t_{rn'_1}, t_{rn'_2})) \geq r(n'_2 - n'_1 + 1) - m - 1.$$

Next we prove the converse. Fix some $N'_1 < N'_2$. Observe that

$$t_{rn-1} < t_{rn} = t_{rn+1} = \dots = t_{rn+r-1} < t_{r(n+1)}, \quad \forall n \in \mathbb{Z}.$$

We prove that (1.4)–(1.7) hold for $N_1 = rN'_1 + l_1$ and $N_2 = rN'_2 + l_2$, $0 \leq l_1, l_2 \leq r - 1$. By (1.9),

$$\begin{aligned} \#E &\geq r(N'_2 - N'_1 - 1) + m + 1 = rN'_2 - (rN'_1 + r - 1) + m \\ &= L(N_2) - R(N_1) + m. \end{aligned}$$

On the other hand, for $n = rN'_1 + rk + l$, $0 \leq k \leq N'_2 - N'_1$ and $0 \leq l \leq r - 1$, we have

$$\#(E \cap [t_{N_1}, t_n]) = \#(E \cap [N'_1, N'_1 + k]) \geq rk = R(n) - R(N_1).$$

Similarly, we can show that for $n = rN'_2 - rk + l$, $0 \leq k \leq N'_2 - N'_1$ and $0 \leq l \leq r - 1$,

$$\#(E \cap (t_n, t_{N_2}]) \geq L(N_2) - L(n).$$

Finally, for $n_1 = rN'_1 + rn'_1 + l'_1$ and $n_2 = rN'_2 + rn'_2 + l'_2$, where $0 \leq l'_1, l'_2 \leq r - 1$, we have

$$\begin{aligned} \#(E \cap [t_{n_1}, t_{n_2}]) &= \#(E \cap (N'_1 + n'_1, N'_1 + n'_2)) \\ &\geq rn'_2 - rn'_1 + r - m - 1 = R(n_2) - L(n_1) - m. \end{aligned}$$

This completes the proof. \square

3. LOCAL RECONSTRUCTION OF FUNCTIONS FROM FINITELY MANY SAMPLES

In this section, we give some applications of Theorem 1.1. We show that under certain conditions, we can reconstruct a function on an interval using only a finite number of samples. For simplicity, we consider the function space $V_m^{(r)}$ generated by m -degree normalized B-splines ψ_l with knots $(\lfloor l/r \rfloor, \lfloor (l+1)/r \rfloor, \dots, \lfloor (l+m+1)/r \rfloor)$, $l \in \mathbb{Z}$, where $1 \leq r \leq m$ are fixed integers.

3.1. Irregular sampling. Irregular sampling is widely studied in literature. For functions in the space $V_m^{(1)}$ generated by the m -degree cardinal B-splines, Aldroubi and Gröchening [2] proved that if the sampling sequence $\{x_k : k \in \mathbb{Z}\}$ satisfies

$$(3.1) \quad 0 < \alpha \leq x_{k+1} - x_k \leq \beta < 1, \quad k \in \mathbb{Z},$$

then there are constants $C_1, C_2 > 0$ such that

$$(3.2) \quad C_1 \|f\|_p^p \leq \sum_{k \in \mathbb{Z}} |f(x_k)|^p \leq C_2 \|f\|_p^p, \quad \forall f \in V_m^{(1)} \cap L^p(\mathbb{R})$$

and every $f \in V_m^{(1)} \cap L^p(\mathbb{R})$ can be reconstructed from $\{f(x_k) : k \in \mathbb{Z}\}$. The standard procedure to reconstruct a function from sampled values is carried out with the frame algorithm [2, 27].

In this subsection, we give a similar result for $V_m^{(r)}$. Moreover, our approach allows us to reconstruct a function locally using only finitely many samples.

Theorem 3.1. *Let $\{x_k : k \in \mathbb{Z}\} \subset \mathbb{R}$ be a sequence such that*

$$x_{k+1} - x_k > 0 \text{ and } x_{k+r} - x_k \leq \beta < 1, \quad k \in \mathbb{Z}.$$

Then for any integers $N_1 < N_2$, there are finitely many functions S_k , $k_1 \leq k \leq k_2$, such that

$$f(x) = \sum_{k_1 \leq k \leq k_2} f(x_k) S_k(x), \quad \forall f \in V_m^{(r)}, x \in [N_1, N_2].$$

Proof. Denote $E = \{x_k : k \in \mathbb{Z}\}$. Take some integer N'_2 such that $N'_2 - N_1 \geq (m + 1)\beta/(r(1 - \beta))$. Let $k_1 = \min\{k \in \mathbb{Z} : x_k \geq N_1\}$ and $k_2 = \max\{k \in \mathbb{Z} : x_k \leq N'_2\}$.

Since $x_{k+r} - x_k \leq \beta$, there are at least r points of E in every interval of length β . It follows that

$$\#(E \cap [N_1, N'_2]) \geq \left\lfloor \frac{N'_2 - N_1}{\beta} \right\rfloor \cdot r \geq \left(\frac{N'_2 - N_1}{\beta} - 1 \right) \cdot r.$$

Since $N'_2 - N_1 \geq (m + 1)\beta/(r(1 - \beta))$, we have $\#(E \cap [N_1, N'_2]) \geq r(N'_2 - N_1 - 1) + m + 1$. On the other hand, it is easy to see that E satisfies (1.10), (1.11) and (1.12). By Corollary 1.2, E is a local sampling sequence for $V_m^{(r)}$ on $[N_1, N'_2]$. Similarly to (2.13) we can find functions $S_k, k_1 \leq k \leq k_2$, such that

$$f(x) = \sum_{k_1 \leq k \leq k_2} f(x_k)S_k(x), \quad \forall f \in V_m^{(r)}, x \in [N_1, N_2]. \quad \square$$

Remark 3.1. If $E := \{x_k : k \in \mathbb{Z}\}$ is a sampling sequence for $V_m^{(r)}$, i.e., there are functions S_k such that

$$f(x) = \sum_{k \in \mathbb{Z}} f(x_k)S_k(x), \quad \forall f \in V_m^{(r)},$$

then we see from the proof of the necessity of Theorem 1.1 that for any integers $n_1 < n_2$,

$$\#(E \cap [n_1, n_2]) \geq r(n_2 - n_1 + 1) - m - 1.$$

Hence

$$\limsup_{(n_2 - n_1) \rightarrow \infty} \frac{\#(E \cap [n_1, n_2])}{n_2 - n_1} \geq r.$$

That is, the average density of sampling points is r . Therefore, the hypothesis $x_{k+r} - x_k \leq \beta < 1$ in Theorem 3.1 is reasonable.

3.2. Average sampling. Average sampling theorems for shift invariant subspaces have been established recently, e.g., see [1, 2, 21, 22, 23, 27] and references therein. Again, all of these results consider global sampling. That is, to recover a function on a bounded interval, we have to know all the averages near every sampling points.

In [10, Theorem 8.15], Feichtinger and Gröchenig proved that if $\delta := \sup_{k \in \mathbb{Z}}(x_{k+1} - x_k) < \frac{\pi}{\Omega}$, then every f with band limited to $[-\Omega, \Omega]$ is uniquely determined by averages $\frac{1}{y_k - y_{k-1}} \int_{y_{k-1}}^{y_k} f(x)dx$, where $y_k = \frac{x_k + x_{k+1}}{2}, k \in \mathbb{Z}$.

In this paper, we give a similar result for the space $V_m^{(r)}$. Moreover, we prove that the standard averaging function can be replaced by any other one which is supported in $[y_{k-1}, y_k]$. Again, our approach allows us to reconstruct a function on a finite interval using only finitely many samples.

Definition 3.1. We call $u(x)$ an averaging function if $u(x) \geq 0$ and $\int_{\mathbb{R}} u(x)dx = 1$.

Theorem 3.2. Suppose that $N_1 = x_0 < x_1 < \dots < x_K = N_2$ and

$$x_{k+r} - x_k < 1, \quad 0 \leq k \leq K - r,$$

where $K \geq r(N_2 - N_1 - 1) + m + 1$. Let $\{u_k : 0 \leq k \leq K - 1\}$ be a sequence of averaging functions such that $\text{supp } u_k \subset [x_k, x_{k+1}), 0 \leq k \leq K - 1$. Then there

exist functions $S_k \in V_m^{(r)}$ such that

$$f(x) = \sum_{k=0}^{K-1} \langle f, u_k \rangle S_k(x), \quad \forall f \in V_m^{(r)}, \quad x \in [N_1, N_2].$$

Proof. Fix some real sequence $\{c_l : l \in \mathbb{Z}\}$ and let $f(x) = \sum_{l \in \mathbb{Z}} c_l \psi_l(x)$.

Using the mean value theorem, for any $0 \leq k \leq K-1$, we can find some $y_k \in \text{supp } u_k \subset [x_k, x_{k+1})$ such that

$$\langle f, u_k \rangle = f(y_k).$$

Using the hypotheses, it is easy to check that the point sequence $\{y_k : 0 \leq k \leq K-1\}$ meets Corollary 1.2. Let

$$(3.3) \quad A = \left[\psi_l(y_k) \right]_{0 \leq k \leq K-1, \quad r+rN_1-m-1 \leq l \leq rN_2-1}$$

be a $K \times (r(N_2 - N_1 - 1) + m)$ matrix. We see from the proof of the sufficiency of Theorem 1.1 that $A^T A$ is invertible. Observe that

$$f(y_k) = \sum_{r+rN_1-m-1 \leq l \leq rN_2-1} c_l \psi_l(y_k) = \sum_{r+rN_1-m-1 \leq l \leq rN_2-1} c_l \langle \psi_l, u_k \rangle.$$

We have

$$(3.4) \quad \sum_{0 \leq k \leq K-1} |f(y_k)|^2 = c^T A^T A c = c^T B^T B c,$$

where $c = [c_{r+rN_1-m-1}, c_{r+rN_1-m}, \dots, c_{rN_2-1}]^T$ and

$$(3.5) \quad B = \left[\langle \psi_l, u_k \rangle \right]_{0 \leq k \leq K-1, \quad r+rN_1-m-1 \leq l \leq rN_2-1}.$$

Note that y_k , and therefore A depends on the coefficients c_l . Nevertheless, since $A^T A$ is invertible, we see from (3.4) that

$$c^T B^T B c > 0, \quad \forall c \in \mathbb{R}^{r(N_2-N_1-1)+m} \setminus \{0\}.$$

Hence $B^T B$ is also invertible.

Let $F = [\langle f, u_0 \rangle, \langle f, u_1 \rangle, \dots, \langle f, u_{K-1} \rangle]^T$. Then we have $F = Bc$ and

$$c = (B^T B)^{-1} B^T F.$$

Define $[S_0, S_1, \dots, S_{K-1}]^T = B(B^T B)^{-1} \Psi$, where

$$\Psi = \left[\psi_{r+rN_1-m-1}, \psi_{r+rN_1-m}, \dots, \psi_{rN_2-1} \right]^T.$$

Then we have

$$f(x) = c^T \Psi(x) = \sum_{k=0}^{K-1} \langle f, u_k \rangle S_k(x), \quad \forall x \in [N_1, N_2].$$

This completes the proof. \square

The following is an immediate consequence.

Corollary 3.3. *Suppose that $N_1 = x_0 < x_1 < \cdots < x_K = N_2$ and*

$$x_{k+r} - x_k < 1, \quad 0 \leq k \leq K - r,$$

where $K \geq r(N_2 - N_1 - 1) + m + 1$. Then there exist functions $S_k \in V_m^{(r)}$ such that

$$f(x) = \sum_{k=0}^{K-1} \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f(t) dt S_k(x), \quad \forall f \in V_m^{(r)}, \quad x \in [N_1, N_2].$$

Remark 3.2. The point sequence $x_k = N_1 + (N_2 - N_1)k/K$, $0 \leq k \leq K$, satisfies the hypotheses of Theorem 3.2 and Corollary 3.3.

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