MATHEMATICS OF COMPUTATION
Volume 78, Number 266, April 2009, Pages 1119–1125
S 0025-5718(08)02120-0
Article electronically published on September 8, 2008

TRACE OF TOTALLY POSITIVE ALGEBRAIC INTEGERS AND INTEGER TRANSFINITE DIAMETER

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Abstract. Explicit auxiliary functions can be used in the "Schur-Siegel-Smyth trace problem". In the previous works, these functions were constructed only with polynomials having all their roots positive. Here, we use several polynomials with complex roots, which are found with Wu's algorithm, and we improve the known lower bounds for the absolute trace of totally positive algebraic integers. This improvement has a consequence for the search of Salem numbers that have a negative trace. The same method also gives a small improvement of the upper bound for the integer transfinite diameter of [0,1].

1. Introduction

1.1. **The trace of totally positive algebraic integers.** Let α be a totally positive algebraic integer of degree $d \geq 2$ (i.e. its conjugates $\alpha_1 = \alpha, ..., \alpha_d$ are all positive real numbers) and let P be its minimal polynomial. We define *the absolute trace of* α as

$$\mathrm{Trace}(\alpha) = \frac{1}{d}\mathrm{trace}(\alpha) = \frac{1}{d}\sum_{i=1}^{d} \alpha_i$$

and denote by $\mathcal{T}$ the set of all such Trace(α).

The "Schur-Siegel-Smyth trace problem" (so called by P. Borwein in his book [B]) is the following: Fix $\rho < 2$. Then show that all but finitely many totally positive algebraic integers α have $\mathrm{Trace}(\alpha) > \rho$.

Remark. Solving this problem is equivalent to proving that 2 is the smallest limit point of $\mathcal{T}$.

The problem was solved in 1918 by I. Schur for $\rho < \sqrt{e}$ [Sc]; in 1943 by C. L. Siegel for $\rho < 1.7337$ [Si] and in 1984 by C. J. Smyth for $\rho < 1.7719$ [Sm2]. More recently, it was solved in 2004 by J. F. McKee and C. J. Smyth for $\rho < 1.7783786$ [MS], and in 2006 by J. Aguirre and J. C. Peral for $\rho < 1.784109$ [AP]. The method of proof, as in [Sm1], uses an explicit auxiliary function of the following type:

$$\text{for } x > 0,\ f(x) = x - \sum_{1 \leq j \leq J} c_j \log|Q_j(x)| \geq m, \tag{1.1}$$

Received by the editor April 2, 2007 and, in revised form, June 22, 2007.
2000 *Mathematics Subject Classification.* Primary 11R04, 11Y40.

where the c_j are positive real numbers and the polynomials Q_j are nonzero elements of $\mathbb{Z}[X]$. Then

$$\sum_{i=1}^{d} f(\alpha_i) \geq md,$$

i.e.

$$\text{trace}(\alpha) \geq md + \sum_{1\leq j\leq J} c_j \log|\prod_{i=1}^{d} Q_j(\alpha_i)|.$$

We assume that P does not divide any Q_j, then $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j. Therefore, if α is not a root of Q_j, we have

$$\text{Trace}(\alpha) \geq m.$$

On the other hand, J. P. Serre (see Appendix B in [AP]) showed that this method does not give such an inequality for any ρ larger than 1.8983021.... Therefore, this method cannot be used to prove that 2 is the smallest limit point of $\mathcal{T}$. Nevertheless, it is interesting to try to get lower bounds for Trace(α). For instance, this was used for the search of Salem numbers of smallest degree with trace equal to -2 by J. F. McKee and C. J. Smyth [MS]. We will explain in Section 1.2 the consequence of our bound for the degree of Salem numbers of trace -3.

In this paper, we prove the following:

Theorem 1. *If α is a totally positive algebraic integer, then, with a finite set of explicit exceptions (see Table 1), we have*

$$\frac{1}{d}\text{trace}(\alpha) \geq 1.78702. \tag{1.2}$$

The main point is to find a good list of polynomials Q_j which gives a value of m as large as possible. In [Sm2] and [AP] the authors used an extended heuristic search to get polynomials of the following type: all their roots are positive and they have small absolute trace. Here we use a new approach relying on the auxiliary function (1.1) to a generalization of the integer transfinite diameter. Our polynomials are found by Wu's algorithm [Wu]. Surprisingly we get a lot of polynomials with some complex roots. This phenomenon has already been encountered by Habsieger and Salvy [HS] when they were studying the integer transfinite diameter of [0,1]. Their exceptional polynomial (after a natural transformation) will appear during our search but will not be used in the final result (1.2). The complete list of polynomials Q_j and coefficients c_j is given in Table 2.

1.2. **Salem numbers of trace -3.** A *Salem number* is a real algebraic integer greater than 1 whose conjugates all lie in the closed disc $|z| \leq 1$, with at least one on the unit circle. Its minimal polynomial is a reciprocal polynomial of degree $2d \geq 4$. Finding all Salem numbers of degree $2d$ and trace -3 is equivalent to finding all totally positive algebraic integers α of degree d and trace $2d-3$ such that $\alpha > 4$ and all other conjugates of α are in the interval]0,4[. In fact, let

$$P(x) = x^d - (2d-3)x^{d-1} + \dots.$$

be the minimal polynomial of such a totally positive algebraic integer. The transformation $x = z + \frac{1}{z} + 2$ produces a reciprocal polynomial

$$Q(z) = z^{2d} + 3z^{2d-1} + \cdots + 3z + 1$$

which is the minimal polynomial of a Salem number of degree $2d$ and trace -3 because the roots of P in the interval]0,4[give pairs of roots of Q on the unit circle while the root of P in the interval $]4,+\infty[$ gives a pair of reciprocal real positive roots of Q. We prove the following:

Theorem 2. *If a Salem number has trace* -3, *then its degree is at least* 30.

It is an easy consequence of Theorem 1. As $1.78702 > \frac{25}{14}$, then there exists no totally positive irreducible polynomial of degree 14 and trace 25 (corresponding or not to a Salem number of trace -3). Thus, the next largest possible degree for such a polynomial is 15 and so at least degree 30 for the corresponding Salem number. The previous bound for the absolute trace of a totally positive algebraic integer was 1.784109 [AP]. This proved that the minimal degree for a Salem number of trace -3 was at least 28.

1.3. **The integer transfinite diameter.** For a polynomial $P \in \mathbb{Z}[X]$ and a real interval $I = [a,b]$ we denote $|P|_{\infty,I} = \sup_{z\in I} |P(z)|$. We define the integer transfinite diameter of I by

$$t_{\mathbb{Z}}(I) = \liminf_{\substack{n\geq 1 \\ n\to+\infty}} \inf_{\substack{P\in\mathbb{Z}[X] \\ \deg(P)=n}} |P|_{\infty,I}^{\frac{1}{n}}. \tag{1.3}$$

Many authors have given lower and upper bounds of $t_{\mathbb{Z}}(I)$, especially when $I = [0,1]$ (see [AP] and [Pr] for an account of this). It is known that $t_{\mathbb{Z}}([0,1]) = t_{\mathbb{Z}}([0,1/4])^{1/2}$. To get an upper bound for $t_{\mathbb{Z}}([0,1/4])$, it is sufficient to get an explicit polynomial $Q \in \mathbb{Z}[X]$ and then to use the sequence of the successive powers of Q. So we search a polynomial Q of degree r such that $|Q|_{\infty,I}^{1/r} \leq e^{-m}$. By the change of variable $x \mapsto \frac{1}{x+4}$ and taking the logarithm we get

$$\text{for } x>0,\ f(x) = \log(x+4) - \sum_{1\leq j\leq J} c_j \log|Q_j(x)| \geq m, \tag{1.4}$$

where the numbers c_j are positive rational numbers satisfying the condition $\sum_{1\leq j\leq J} c_j \deg Q_j \leq 1$ and the polynomials Q_j are obtained from the irreducible factors of Q by the transformation above. It is plain that the algorithm given in Section 2 for the auxiliary function (1.1) is also convenient for the function (1.4). We get

Result 1.

$$t_{\mathbb{Z}}([0,1]) < 0.42291334. \tag{1.5}$$

This is a small improvement on the upper bound 0.42305209 given by Aguirre and Peral [AP]. The best known lower bound 0.4213 was given by Pritsker [Pr]. Here we also have several factors of Q, including the polynomial of Habsieger and Salvy, which has some complex roots.

In Section 2 we explain how to construct the auxiliary function (1.1) (the same method is used to construct the function (1.4)). The numerical results are given in Section 3. All the computations are done on an iBook Macintosh with the languages Pascal, Maple and Pari.

2. Construction of the explicit auxiliary function

2.1. Rewriting the auxiliary function. Inside the auxiliary function (1.1) we replace the numbers c_j by rational numbers.

So, we may write:

$$\text{for } x > 0,\ f(x) = x - \frac{t}{r} \log |Q(x)| \geq m$$

where $Q \in \mathbb{Z}[X]$ is of degree r and t is a positive real number. We want to get a function f whose minimum m on $(0, \infty)$ is as large as possible. Thus we search a polynomial $Q \in \mathbb{Z}[X]$ such that

$$\sup_{x>0} |Q(x)|^{t/r} e^{-x} \leq e^{-m}.$$

If we suppose that t is fixed, we need to get an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}([0,\infty)) = \liminf_{\substack{r \geq 1 \\ r \to +\infty}} \inf_{\substack{P \in \mathbb{Z}[X] \\ \deg(P)=r}} \sup_{x>0} \left(|P(x)|^{\frac{t}{r}}\ \varphi(x) \right)$$

in which we use the weight $\varphi(x) = e^{-x}$.

It is clear that this quantity is closely related to the usual integer transfinite diameter of an interval given by (1.3).

2.2. How to find the polynomials Q_j. We first take an initial value of t, say $t_0 = 1$ and a set E_0 of 50 control points uniformly distributed on $[0, 2.5]$. With Wu's algorithm [Wu], we compute a polynomial Q of degree at most 10 which is small on E_0. We define Q_1 as the irreducible factor of Q of smallest degree and we take the best value of c_1 to get the best auxiliary function f_1. We deduce from this the value $t_1 = c_1 \deg(Q_1)$. We add to the set E_0 the points of $[0, +\infty)$ where f_1 has a local minimum (including those greater than 2.5) to get a new set E_1 of control points. With Wu's algorithm we compute a polynomial Q of degree $10 + \deg Q_1$ which is a multiple of Q_1 of small norm on E_1 and take Q_2 as another irreducible factor of Q. We optimize (c_1, c_2) to get the best function f_2. This gives t_2. We get the set E_2 from E_1 adding the local minima of f_2. Then we search a polynomial Q which is a multiple of Q_1Q_2 of degree $10 + \deg Q_1 + \deg Q_2$ and we continue this process until two consecutive steps produce no new polynomial.

Remark. The previous algorithm is used repeatedly with the constant 10 replaced successively by 11 to 20.

We give a numerical example of search for degree 13.

We start with E_0 and t_0 defined as above. We search a polynomial Q of degree 13 such that $xQ(x)$ is small on E_0. LLL produces the polynomial $Q(x) = x^7(x-1)^4(2x-1)(3x-1)$. We optimize the auxiliary function

$$f_1 = x - c_1 \log x - c_2 \log |x-1| - c_3 \log |2x-1| - c_4 \log |3x-1|.$$

Only c_1 and c_2 are nonzero. We deduce the value of $t_1 = c_1 \deg(x) + c_2 \deg(x-1) = 1.606$. We add to the set E_0 the two points where f_1 has a local minimum and we get a new set E_1 of control points. We repeat LLL and now search for a polynomial Q such that $x^8(x-1)^4Q(x)$ is small on E_1. At the 8th step, LLL produces the polynomial

$$Q(x) = (x-1)(x^2-3x+1)(x^5-9x^4+27x^3-31x^2+12x-1)\\ \underline{(x^5-8x^4+22x^3-25x^2+10x-1)}.$$

The underlined polynomial is related to the exceptional polynomial (i.e. with some complex roots) obtained by Habsieger and Salvy as explained in Section 3. We test the two new polynomials in the auxiliary function. Both have nonzero exponents. At this point, we have $\frac{1}{d}\text{trace}(\alpha) \geq 1.762$. The polynomials used in the auxiliary function are the polynomials 1, 2, 3, 4, 6, 7, 11, 12, 14 in Table 1 and the underlined polynomial above.

2.3. **Optimization of the c_j.** For the optimization of the auxiliary function we use the semi-infinite linear programming method introduced into number theory by Smyth [Sm1]. We recall it briefly. We define by induction a sequence of finite sets X_n, $n \geq 0$, with $X_n \subset [0, +\infty)$. We start with an arbitrary set of points X_0 of cardinal greater than J. At each step $n \geq 0$, we compute the best values for c_j by linear programming on the set X_n. We get a function f_n whose minimum $m_n = \min_{x \in X_n} f_n(x)$ is greater than $m'_n = \min_{x>0} f_n(x)$. We add to X_n the points of $[0, +\infty)$ where f_n has a local minimum smaller than $m_n + \epsilon_n$, where $(\epsilon_n)_{n \geq 0}$ is a decreasing sequence of positive numbers tending to 0 when n is increasing and chosen such that the set X_n does not increase too quickly. We stop, for instance, when $m_n - m'_n < 10^{-6}$. If k steps are necessary, we take $m = m'_k$.

3. Numerical results

The method described above gives, among others, the 35 polynomials listed in Table 1 and the 28 polynomials listed in Table 2. Among these polynomials, 17 are new.

Four of them are minimal polynomials of totally positive algebraic integers.

The last thirteen polynomials have at least two complex roots. They are of the same type: their real parts all lie in $[0, 5.2]$ and their imaginary parts are small, i.e $|\mathsf{Im}(z)| < 0.6325$.

The first new polynomial of this family that appeared was $x^5 - 8x^4 + 22x^3 - 25x^2 + 10x - 1$. It has two complex roots, which has reminded us of the exceptional polynomial of Habsieger and Salvy. We transform this polynomial by $x \mapsto \frac{1}{x} - 4$ in order to get a polynomial whose real roots are in $[0, \frac{1}{4}]$. We find the polynomial $4921x^5 - 4594x^4 + 1697x^3 - 310x^2 + 28x - 1$ which is the polynomial A_8 in [HS]. But this polynomial does not belong to the final list of our polynomials.

TABLE 1. List of the polynomials Q_j that occur in the explicit auxiliary function (1.1) to obtain the inequality (1.2). Those marked with a sharp are the exceptions mentioned in Theorem 1. Those marked with an asterisk are only used for the lower bound of $t_{\mathbb{Z}}([0,1])$. ν is the number of complex roots of the new polynomials Q_j.

j	ν	Q_j
$1^{\#}$		x
$2^{\#}$		$-1+x$
3		$-2+x$
$4^{\#}$		$1-3x+x^2$
5		$1-4x+x^2$
6		$2-4x+x^2$
$7^{\#}$		$-1+6x-5x^2+x^3$
8		$-1+8x-6x^2+x^3$
9		$-1+9x-6x^2+x^3$
10		$-3+9x-6x^2+x^3$
$11^{\#}$		$1-7x+13x^2-7x^3+x^4$
$12^{\#}$		$1-8x+14x^2-7x^3+x^4$
13		$-1+11x-29x^2+26x^3-9x^4+x^5$
14		$-1+12x-31x^2+27x^3-9x^4+x^5$
15		$-1+13x-32x^2+27x^3-9x^4+x^5$
16		$-1+15x-35x^2+28x^3-9x^4+x^5$
17	2	$1-9x+33x^2-52x^3+35x^4-10x^5+x^6$
	2	$1-12x+43x^2-64x^3+41x^4-11x^5+x^6$ $(*)$
18		$1-13x+47x^2-68x^3+42x^4-11x^5+x^6$
19		$1-15x+53x^2-73x^3+43x^4-11x^5+x^6$
20		$1-15x+59x^2-78x^3+44x^4-11x^5+x^6$
	2	$1-16x+78x^2-155x^3+142x^4-63x^5+13x^6-x^7$ $(*)$
21		$-1+16x-78x^2+157x^3-143x^4+63x^5-13x^6+x^7$
	2	$1-15x+79x^2-202x^3+273x^4-197x^5+75x^6-14x^7+x^8$ $(*)$
22	2	$1-19x+111x^2-277x^3+339x^4-221x^5+78x^6-14x^7+x^8$
23	2	$1-21x+120x^2-289x^3+345x^4-222x^5+78x^6-14x^7+x^8$
24		$3-40x+187x^2-402x^3+445x^4-269x^5+89x^6-15x^7+x^8$
25		$3-42x+200x^2-428x^3+467x^4-277x^5+90x^6-15x^7+x^8$
26	2	$-1+20x-135x^2+424x^3-703x^4+651x^5-345x^6+103x^7-16x^8+x^9$
27		$1-24x+194x^2-743x^3+1526x^4-1798x^5+1265x^6-537x^7+134x^8-18x^9+x^{10}$
28		$1-24x+200x^2-766x^3+1560x^4-1822x^5+1273x^6-538x^7+134x^8-18x^9+x^{10}$
29		$1-24x+206x^2-813x^3+1662x^4-1920x^5+1320x^6-549x^7+135x^8-18x^9+x^{10}$
30	2	$1-32x+256x^2-916x^3+1760x^4-1967x^5+1331x^6-550x^7+135x^8-18x^9+x^{10}$
31	4	$1-26x+267x^2-1389x^3+4097x^4-7341x^5+8352x^6-6196x^7+3023x^8-958x^9+189x^{10}$ $-21x^{11}+x^{12}$
32	4	$1-27x+281x^2-1470x^3+4336x^4-7742x^5+8750x^6-6430x^7+3102x^8-972x^9+190x^{10}$ $-21x^{11}+x^{12}$
33	4	$1-27x+283x^2-1483x^3+4372x^4-7789x^5+8780x^6-6439x^7+3103x^8-972x^9+190x^{10}$ $-21x^{11}+x^{12}$
34	2	$-1+28x-313x^2+1837x^3-6338x^4+13689x^5-19217x^6+17929x^7-11240x^8+4730x^9$ $-1313x^{10}+230x^{11}-23x^{12}+x^{13}$
35	2	$-1+30x-358x^2+2246x^3-8359x^4+19715x^5-30607x^6+31950x^7-22636x^8+10851x^9$ $-3451x^{10}+695x^{11}-80x^{12}+4x^{13}$

TABLE 2. List of the coefficients c_j used in the explicit auxiliary function (1.1)

j	c_j	j	c_j	j	c_j	j	c_j
1	0.5584957222	10	0.003856972798	19	0.001108761904	28	0.004813507609
2	0.4916967610	11	0.02993703111	20	0.0008466162920	29	0.004620732100
3	0.07630565169	12	0.02755084495	21	0.0008291782706	30	0.003043369627
4	0.1761207656	13	0.005156588163	22	0.001195618507	31	0.001938859038
5	0.01072483141	14	0.007670086583	23	0.002200027138	32	0.003180059759
6	0.01038787553	15	0.003542904754	24	0.001352021285	33	0.006514951360
7	0.07277644358	16	0.004673480798	25	0.004580671504	34	0.002183415705
8	0.002809414882	17	0.001557114885	26	0.0008445959086	35	0.001482155949
9	0.005946815966	18	0.003429925309	27	0.004268551700		

Acknowledgement

This work would not have been possible without the precious help of Professor Georges Rhin in the computation programs. I am very much indebted to him.

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