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ANALYSIS FOR QUADRILATERAL MITC ELEMENTS FOR THE REISSNER-MINDLIN PLATE PROBLEM

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Abstract. The present paper is made up of two parts. In the first part, we study the mathematical stability and convergence of the quadrilateral MITC elements for the Reissner-Mindlin plate problem in an abstract setting. We generalize the Brezzi-Bathe-Fortin conditions to the quadrilateral MITC elements by weakening the second and fourth conditions. Under these conditions, we show the well-posedness of the discrete problem and establish an abstract error estimate in the energy norm. The conclusion of this part is sparsity in the mathematical research of the quadrilateral MITC elements in the sense that one only needs to check these five conditions.

In the second part, we extend four families of rectangular MITC elements of Stenberg and Süri to the quadrilateral meshes. We prove that these quadrilateral elements satisfy the generalized Brezzi-Bathe-Fortin conditions from the first part. We develop the h-p error estimates in both energy and L^2 norm for these quadrilateral elements. For the first three families of quadrilateral elements, the error estimates indicate that their convergent rates in both energy and L^2 norm depend on the mesh distortion parameter α. We can get optimal error estimates for them provided that $\alpha = 1$. In addition, we show the optimal convergence rates in energy norm uniformly in α for the fourth family of quadrilateral elements. Like their rectangular counterparts, these quadrilateral elements are locking-free.

1. Introduction

This paper is devoted to the finite element approximation for the Reissner-Mindlin plate problem (R-M hereinafter) which reads: Given $g \in L^2(\Omega)$ find $(\omega, \boldsymbol{\phi}) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$ with

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\boldsymbol{\gamma}, \nabla v - \boldsymbol{\psi}) = (g, v) \quad \text{for all } (v, \boldsymbol{\psi}) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 . \tag{1.1}$$

With the plate thickness t and the other notation defined in Section 2 below, the shear stress $\boldsymbol{\gamma} \in \boldsymbol{H}^{-1}(\mathrm{div}, \Omega)$ reads

$$\boldsymbol{\gamma} = \lambda t^{-2}(\nabla \omega - \boldsymbol{\phi}). \tag{1.2}$$

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This plate theory has become a popular plate bending model in the engineering community due to its simplicity and effectiveness. However, a direct finite element approximation usually yields poor numerical results, i.e. they are too small compared with the continuous solutions. Such a phenomenon is usually referred to as *shear-locking*. To weaken or even overcome the locking, many methods have been proposed, most of them can be regarded as reduction integration methods [2, 4, 9, 11, 10, 14, 16, 19, 22, 23, 29, 32, 26, 24]. Among others, the MITC method is an efficient and popular one [11, 9, 10, 14, 16, 19, 24]. So far, most of the MITC plate bending elements proposed and analyzed in literature are restricted to the affine meshes, namely, the triangular and parallelogram meshes. Needless to say, the quadrilateral meshes are more flexible than the rectangular or parallelogram ones, especially when a domain with a curved boundary is considered. Therefore, the quadrilateral elements are important in theory and applications and deserve a careful study.

In [11], Brezzi, Bathe and Fortin have proposed five conditions (hereafter Brezzi-Bathe-Fortin conditions) for the stability and convergence of the MITC elements defined on the affine meshes. Since the commutative property of the mixed finite element methods for the second order elliptic problem is invalid for the general quadrilateral meshes, the second and fourth conditions no longer hold for this case.

The first aim of this paper is to generalize the Brezzi-Bathe-Fortin conditions to the quadrilateral meshes by weakening the second and fourth conditions. Thanks to these general conditions and the discrete Helmholtz decomposition established, we obtain error estimates in an abstract setting for the limit problem with the plate thickness $t = 0$ and the general case with $t > 0$.

The second aim of this paper is to generalize four classes of rectangular MITC elements proposed in [29] to the quadrilateral meshes and develop their h-p error estimates in both energy and L^2 norm. We show that the convergence rates for three classes of these quadrilateral elements depend on the mesh distortion parameter α. Thus the loss of accuracy will be expected for these generalizations unless the quadrilateral meshes satisfy the bi-section mesh condition from [27]. Importantly, we prove that one class of these quadrilateral elements yields optimal error estimates with respect to the meshsize h uniformly in α in energy norm.

This paper is organized as follows. Next, we introduce the notation in Section 2. In Section 3, we list the quadrilateral version of the Brezzi-Bathe-Fortin conditions and present the discrete problem in an abstract setting. In Section 4, we check the stability and convergence of the discrete problem with $t = 0$. In Section 5, after showing the discrete Helmholtz decomposition, we examine the well-posedness of the discrete problem with $t > 0$, and prove their error estimates based on the generalized Brezzi-Bathe-Fortin conditions. In Sections 6, 7 and 8, we generalize the rectangular MITC elements proposed in [29] to the quadrilateral meshes, and address the h-p error estimates. For completeness and also compactness, we give the h-p error analysis of the reduction operator and other interpolation operators over the quadrilateral meshes in the appendix, namely, in Section A and Section B.

In this paper, the generic constant C is assumed to be independent of the plate thickness t, the mesh size h and the degree of polynomials k. However, it may depend on the regularity index m in general.

2. Notation

This section presents the definitions of notation. Let Ω denote the region occupied by the plate. Assume that Ω is a convex polygon with the boundary $\partial\Omega$. We use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ for $s \geq 0$ [1], the boldface $\boldsymbol{H}^s(\Omega)$ denotes the corresponding vector-valued function space; this rule is applicable to the others spaces, unknowns and operators. The standard associated inner product is denoted by $(\cdot,\cdot)_s$, and the norm by $\|\cdot\|_s$ with $|\cdot|_s$ the seminorm, respectively. For $s = 0$, $H^s(\Omega)$ coincides with $L^2(\Omega)$. In this case, the norm and inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$ respectively. As usual, $H_0^s(\Omega)$ is the subspace of $H^s(\Omega)$ with vanishing trace on $\partial\Omega$. Let $L_0^2(\Omega)$ be the set of all $L^2(\Omega)$ functions with zero integral mean. Denote $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ equipped with the norm $\|\cdot\|_{-1}$.

Throughout this paper, ω and $\boldsymbol{\phi} = (\phi_1, \phi_2)^t$ denote the transverse displacement and the rotation of the fiber normal to Ω, respectively. g is the scaled transverse loading function, $\lambda = Ek/2(1+\nu)$ is the shear modulus with E the Young modulus, ν the Poisson ratio, and κ the shear correction factor. The bilinear form $a(\cdot,\cdot)$ models the linear elastic energy and is defined by

$$a(\boldsymbol{\phi},\boldsymbol{\psi}) = \frac{E}{12(1-\nu^2)}\int_\Omega ((1-\nu)\mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\psi}) + \nu\nabla\cdot\boldsymbol{\phi}\nabla\cdot\boldsymbol{\psi})dxdy, \tag{2.1}$$

the linear Green strain $\mathcal{E}(\boldsymbol{\phi}) = \frac{1}{2}[\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\phi}^T]$ is the symmetric part of the gradient field $\nabla\boldsymbol{\phi}$.

We use the standard differential operators:

$$\nabla r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \qquad \mathbf{curl} p = \begin{pmatrix} \partial p/\partial y \\ -\partial p/\partial x \end{pmatrix}.$$

We denote the gradient operator on the reference element $\hat{K} = [-1,1]^2$ with respect to $(\xi,\eta) \in \hat{K}$ by $\hat{\nabla}$. For a vector function $\boldsymbol{\psi} = (\psi_1,\psi_2)$, define

$$\operatorname{div}\boldsymbol{\psi} = \partial\psi_1/\partial x + \partial\psi_2/\partial y, \quad \operatorname{rot}\boldsymbol{\psi} = \partial\psi_2/\partial x - \partial\psi_1/\partial y.$$

We also need the vector spaces

$$\boldsymbol{H}_0(\mathrm{rot},\Omega) = \{\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega), \operatorname{rot}\boldsymbol{q} \in L^2(\Omega), \boldsymbol{q}\cdot\boldsymbol{t} = 0 \text{ on } \partial\Omega \,\},$$

where $\boldsymbol{t}$ denotes as the unit tangent to $\partial\Omega$, and

$$\boldsymbol{H}(\mathrm{div},\Omega) = \{\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega), \operatorname{div}\boldsymbol{q} \in L^2(\Omega) \,\},$$

which are endowed with the norms, respectively,

$$\|\boldsymbol{q}\|_{\boldsymbol{H}(\mathrm{rot})} = (\|\boldsymbol{q}\|_0^2 + \|\operatorname{rot}\boldsymbol{q}\|_0^2)^{1/2}$$

and

$$\|\boldsymbol{\eta}\|_{\boldsymbol{H}(\mathrm{div})} = (\|\boldsymbol{\eta}\|_0^2 + \|\operatorname{div}\boldsymbol{\eta}\|_0^2)^{1/2}.$$

Let J^h be a partition of Ω into convex quadrilaterals. Define $h:= \max_{K\in J^h} h_K$ where h_K is the diameter of K for each $K \in J^h$. The usual regularity for J^h is assumed in the sense of Ciarlet and Raviart [15, pp. 247], the quasi-uniformity of J^h is also assumed. We denote the distance between the midpoints of two diagonals of K by d_K, and assume J^h to satisfy the $(1+\alpha)$-*section condition* [25], i.e., d_K is of order $\mathcal{O}(h_K^{1+\alpha})$ uniformly for all elements K as h tends to zero for $0 \leq \alpha \leq 1$. In particular, we recover the *bi-section condition* [27] if $\alpha = 1$.

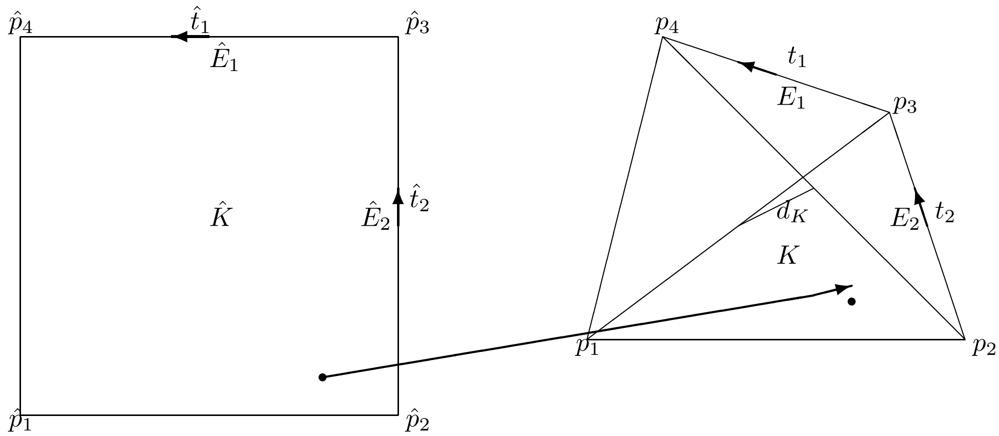

FIGURE 1. Quadrilateral K and the reference element $\hat{K}$.

Remark 2.1. In practice, α can be greater than one with $\alpha = \infty$ associated to the parallelogram meshes. Since the methods under consideration will give optimal error bounds in both energy and L^2 norm for the case with $\alpha > 1$ and the analysis below covers this case, we restrict ourselves to considering the case of $0 \leq \alpha \leq 1$.

Given element $K \in J^h$ with four nodes $p_i(x_i, y_i), i = 1, \ldots, 4$, let $\hat{K} = [-1, 1]^2$ denote the reference element with nodes $\hat{p}_i(\xi_i, \eta_i), i = 1, \ldots, 4$. Define the bilinear transformation $F_K : \hat{K} \to K$ by

$$x = \sum_{i=1}^{4} x_i N_i(\xi, \eta), \quad y = \sum_{i=1}^{4} y_i N_i(\xi, \eta), \quad (\xi, \eta) \in \hat{K},$$

with $N_i(\xi, \eta), i = 1, 2, 3, 4$ the bilinear basis functions, which read

$$\begin{aligned} N_1(\xi, \eta) &= \frac{1}{4}(1-\xi)(1-\eta), \quad N_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta), \\ N_3(\xi, \eta) &= \frac{1}{4}(1+\xi)(1+\eta), \quad N_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta). \end{aligned}$$

Define

$$\begin{pmatrix} c_0 & d_0 \\ c_1 & d_1 \\ c_2 & d_2 \\ c_{12} & d_{12} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}.$$

Then the Jacobian matrix of the bilinear transformation F_K can be expressed as

$$DF_K = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} c_1 + c_{12}\eta & c_2 + c_{12}\xi \\ d_1 + d_{12}\eta & d_2 + d_{12}\xi \end{pmatrix}$$

with the determinant $J_K(\xi, \eta) = J_{0,K} + J_{1,K}\xi + J_{2,K}\eta$, where $J_{0,K} = c_1 d_2 - c_2 d_1$, $J_{1,K} = c_1 d_{12} - c_{12} d_1$, $J_{2,K} = c_{12} d_2 - c_2 d_{12}$, and its inverse is

$$DF_K^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \frac{1}{J_K(\xi, \eta)} \begin{pmatrix} d_2 + d_{12}\xi & -c_2 - c_{12}\xi \\ -d_1 - d_{12}\eta & c_1 + c_{12}\eta \end{pmatrix}.$$

In terms of the aforementioned mesh parameters, $d_K = \mathcal{O}(h_K^{1+\alpha})$ implies

$$|c_{12}| + |d_{12}| \le Ch^{1+\alpha}, \tag{2.2}$$

$$|J_{1,K}| + |J_{2,K}| \le Ch^{2+\alpha}, \tag{2.3}$$

$$|\hat{\nabla} J_K| \le Ch^{1+\alpha}. \tag{2.4}$$

For an edge $\hat{E}_i$ of $\hat{K}$ we let $E_i = F_K(\hat{E}_i), i = 1, \dots, 4$, be the corresponding edge of K. The unit tangents of E_i and $\hat{E}_i$ are denoted by $\boldsymbol{t}_i$ and $\hat{\boldsymbol{t}}_i$, respectively.

3. Quadrilateral finite element approximations and sufficient conditions for the stability and convergence

This section defines the discrete problem in an abstract setting and presents the generalized Brezzi-Bathe-Fortin conditions.

Assume now that we are given finite element subspaces $\boldsymbol{\Theta}_h \subset \boldsymbol{H}_0^1(\Omega)$ and $W_h \subset H_0^1(\Omega)$ over the quadrilateral partition J^h. To overcome locking, a common procedure is to somehow reduce the influence of the shear energy. We consider here the case in which the reduction is carried out in the following way: we assume that we are given a third finite element space $\boldsymbol{\Gamma}_h$, and a linear operator $\boldsymbol{R}_h$ which is defined on the space $\boldsymbol{H}_0(\mathrm{rot}, \Omega) \cap \boldsymbol{H}^1(\Omega)$ and takes values in $\boldsymbol{\Gamma}_h$. Then we use $\boldsymbol{R}_h(\nabla\omega_h - \boldsymbol{\phi}_h)$ instead of $\nabla\omega_h - \boldsymbol{\phi}_h$. For simplicity we consider the case where

$$\boldsymbol{R}_h \nabla w_h = \nabla w_h, \quad \forall w_h \in W_h \tag{3.1}$$

and

$$\|\boldsymbol{R}_h \boldsymbol{\psi}\|_1 \le C\|\boldsymbol{\psi}\|_1, \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega). \tag{3.2}$$

For the limit problem with $t = 0$, the discrete problem reads

Problem 3.1. Find $(\omega_h, \boldsymbol{\phi}_h) \in W_h \times \boldsymbol{\Theta}_h$ such that

$$\begin{cases} a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + (\boldsymbol{\gamma}_h, \boldsymbol{R}_h\boldsymbol{\psi} - \nabla v) = (g, v), \quad \forall \boldsymbol{\psi} \in \boldsymbol{\Theta}_h, \forall v \in W_h, \\ \qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{R}_h \boldsymbol{\phi}_h = \nabla\omega_h. \end{cases} \tag{3.3}$$

For the general problem with $t > 0$, the discrete problem can be stated as

Problem 3.2. Find $(\omega_h, \boldsymbol{\phi}_h) \in W_h \times \boldsymbol{\Theta}_h$ such that

$$a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + (\boldsymbol{\gamma}_h, \nabla v - \boldsymbol{R}_h\boldsymbol{\psi}) = (g, v), \quad \forall (v, \boldsymbol{\psi}) \in W_h \times \boldsymbol{\Theta}_h, \tag{3.4}$$

where

$$\boldsymbol{\gamma}_h = \lambda t^{-2}(\nabla\omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h). \tag{3.5}$$

For the stability and convergence of the discrete problem, Brezzi, Bathe and Fortin have proposed five conditions in the case of the rectangular meshes [11]. In what follows, we generalize these Brezzi-Bathe-Fortin conditions to the general quadrilateral meshes by weakening the second and fourth conditions.

Condition 1. The gradient field of the discrete displacement space is included in the discrete shear force space, i.e.,

$$\nabla W_h \subset \boldsymbol{\Gamma}_h.$$

Condition 2. There exist two auxiliary spaces $\mathcal{Q}_h$ and $\mathcal{Q}_{1,h}$ which are related to each other in the following way:

$$\mathcal{Q}_h = \{q \in L_0^2(\Omega), q\,|_K = \hat{q}(F_K^{-1}(x)), \hat{q} \in \mathcal{Q}(\hat{K}), \quad \forall K \in J^h\}, \tag{3.6}$$

$$\mathcal{Q}_{1,h} = \{q \in L^2(\Omega), q\,|_K = \frac{J_{0,K}\hat{q}(F_K^{-1}(x))}{J_K}, \hat{q} \in \mathcal{Q}(\hat{K}), \quad \forall K \in J^h\}, \tag{3.7}$$

where $\mathcal{Q}(\hat{K})$ is some polynomial space over $\hat{K}$. The reduction operator $\boldsymbol{R}_h$ is defined in such a way that

$$(\mathrm{rot}(\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}), q) = 0, \quad \forall q \in \mathcal{Q}_h, \quad \forall \boldsymbol{\phi} \in \boldsymbol{H}(\mathrm{rot}, \Omega). \tag{3.8}$$

Moreover,the following inclusion relation holds for the rotation field of the discrete shear force space and the auxiliary space $\mathcal{Q}_{1,h}$,

$$\mathrm{rot}\, \boldsymbol{\Gamma}_h \subset \mathcal{Q}_{1,h}. \tag{3.9}$$

Condition 3. The space pair $(\boldsymbol{\Theta}_h,\ \mathcal{Q}_h)$ is stable for the Stokes problem in the sense that we have the discrete inf-sup condition

$$\inf_{q\in\mathcal{Q}_h} \sup_{\boldsymbol{\eta}\in\boldsymbol{\Theta}_h} \frac{(\mathrm{rot}\,\boldsymbol{\eta}, q)}{\|\boldsymbol{\eta}\|_1\|q\|_0} \geq \beta(h,k). \tag{3.10}$$

In addition, we assume there exists a constant $C(k) > 0$ which only depends on k such that

$$\lim_{h\to 0} \beta(h,k) = C(k), \tag{3.11}$$

with k the degree of the polynomials in consideration.

Remark 3.3. For example, the schemes in Section 6 admit $\beta(h,k) = \frac{Ck^{-1/2}}{1+h^{\alpha}k^{5/2}}$. Therefore, $\beta(h,k)$ tends to $Ck^{-1/2} = C(k)$ when h goes to zero and k is fixed.

Condition 4. The space pair $(\boldsymbol{\Gamma}_h, \mathcal{Q}_h)$ is stable for the second order elliptic problem in the sense that the following discrete problem admits a unique solution: Find $(\boldsymbol{\alpha}, p) \in \boldsymbol{\Gamma}_h \times \mathcal{Q}_h$ such that

$$(\boldsymbol{\alpha}, \boldsymbol{s}) - (\mathrm{rot}\,\boldsymbol{s}, p) = (f, \boldsymbol{s}), \quad \forall \boldsymbol{s} \in \boldsymbol{\Gamma}_h,$$
$$(\mathrm{rot}\,\boldsymbol{\alpha}, m) = (g, m), \quad \forall m \in \mathcal{Q}_h.$$

Condition 5. We have

$$\{\boldsymbol{\delta}_h \in \boldsymbol{\Gamma}_h, \mathrm{rot}\,\boldsymbol{\delta}_h = 0\} \subset \nabla W_h.$$

4. Abstract error analysis for the limit problem

This section presents the error analysis for the finite element methods with Conditions 1–5 for the limit problem with $t = 0$.

For our analysis, we first need the following result which is related to the discrete Helmholtz decomposition on the quadrilateral meshes.

Theorem 4.1. *For any $\boldsymbol{\sigma} \in \boldsymbol{\Gamma}_h$, if*

$$(\mathrm{rot}\,\boldsymbol{\sigma}, q_0) = 0, \quad \forall q_0 \in \mathcal{Q}_h, \tag{4.1}$$

then, it holds that

$$\mathrm{rot}\,\boldsymbol{\sigma} = 0. \tag{4.2}$$

Proof. With Condition 2, one can assume that $\operatorname{rot}\boldsymbol{\sigma}$ has the form

$$q\mid_K = \operatorname{rot}\boldsymbol{\sigma}\mid_K = \frac{q_K}{J_K}, \quad \forall K \in J^h,$$

where $q_K = \hat{q}_K(F_K^{-1}(x))$ with $\hat{q}_K \in \mathcal{Q}(\hat{K})$. Set

$$q_0\mid_K = \frac{q_K}{J_{0,K}} - C_0$$

with

$$C_0 = \frac{1}{\mid\Omega\mid}\sum_{K\in J^h}\int_K \frac{q_K}{J_{0,K}}dxdy.$$

By virtue of $q \in L_0^2(\Omega)$, we have

$$\begin{aligned} 0 &= (\operatorname{rot}\boldsymbol{\sigma}, q_0) = \sum_{K\in J^h}(\frac{q_K}{J_K}, \frac{q_K}{J_{0,K}} - C_0)_K \\ &= \sum_{K\in J^h}(\frac{q_K}{J_K}, \frac{q_K}{J_{0,K}})_K \geq \sum_{K\in J^h} C_K(\frac{q_K}{J_{0,K}}, \frac{q_K}{J_{0,K}})_K. \end{aligned}$$

Here we use the regularity $C_K J_K \leq J_{0,K}$ of the mesh with the positive constant C_K depending on the geometry of each element K. This leads to

$$q_K = 0, \forall K \in J^h,$$

which completes the proof. □

Let us go to the limit problem and its discrete counterpart which we state here for the convenience of the reader:

$$\text{(4.3)} \qquad \begin{cases} a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\boldsymbol{\gamma}, \boldsymbol{\psi} - \nabla v) = (g, v), \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega), \quad \forall v \in H_0^1(\Omega), \\ \hfill \boldsymbol{\phi} = \nabla\omega, \end{cases}$$

$$\text{(4.4)} \qquad \begin{cases} a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + (\boldsymbol{\gamma}_h, \boldsymbol{R}_h\boldsymbol{\psi} - \nabla v) = (g, v), \quad \forall \boldsymbol{\psi} \in \boldsymbol{\Theta}_h, \quad \forall v \in W_h, \\ \hfill \boldsymbol{R}_h\boldsymbol{\phi}_h = \nabla\omega_h. \end{cases}$$

A direct consequence of Theorem 4.1 is the following error estimates for the limit problem.

Theorem 4.2. *Let the finite element method satisfy Conditions* 1–5*, and let* $(\boldsymbol{\phi}, \omega)$ *and* $(\boldsymbol{\phi}_h, \omega_h)$ *be the solutions to Problem* (4.3) *and Problem* (4.4)*, respectively. Then*

$$\text{(4.5)} \qquad \|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_1 \leq C\left(\sup_{\boldsymbol{\psi}\in\boldsymbol{\Theta}_h}\frac{(\boldsymbol{\gamma}, \boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_1} + \inf_{\boldsymbol{\psi}\in\boldsymbol{\Theta}_h}\|\boldsymbol{\phi} - \boldsymbol{\psi}\|_1\right),$$

$$\text{(4.6)} \qquad \|\nabla\omega - \nabla\omega_h\|_0 \leq C\big(\|\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}\|_0 + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1\big).$$

Proof. We first recall by the Korn inequality that there exists a constant C such that

$$a(\boldsymbol{\psi}, \boldsymbol{\psi}) \geq C\|\boldsymbol{\psi}\|_1^2, \quad \forall \boldsymbol{\psi} \in \boldsymbol{\Theta}_h.$$

Now consider the following problem: Find $(\boldsymbol{\phi}^I, p^I) \in \boldsymbol{\Theta}_h \times \mathcal{Q}_h$ with

$$\begin{aligned} (\nabla\boldsymbol{\phi}^I, \nabla\boldsymbol{\psi}) - (p^I, \operatorname{rot}\boldsymbol{\psi}) &= (\nabla\boldsymbol{\phi}, \nabla\boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{\Theta}_h, \\ (q, \operatorname{rot}\boldsymbol{\phi}^I) &= 0, \quad \forall q \in \mathcal{Q}_h. \end{aligned}$$

Thanks to Condition 3 in the previous section and the mixed finite element theory [13], there exist a unique solution to this problem with

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}^I\|_1 \leq C \inf_{\boldsymbol{\psi} \in \boldsymbol{\Theta}_h} \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_1. \tag{4.7}$$

We use Condition 2 to obtain

$$(\operatorname{rot} \boldsymbol{R}_h \boldsymbol{\phi}^I, q) = (\operatorname{rot} \boldsymbol{\phi}^I, q) = 0, \quad \forall q \in \mathcal{Q}_h.$$

It follows from Theorem 4.1 that

$$\operatorname{rot} \boldsymbol{R}_h \boldsymbol{\phi}^I = 0.$$

Since $\boldsymbol{R}_h \boldsymbol{\phi}^I \in \boldsymbol{\Gamma}_h$, this identity and Condition 5 assert there exists a unique $\omega^I \in W_h$ such that

$$\nabla \omega^I = \boldsymbol{R}_h \boldsymbol{\phi}^I.$$

We set $\varepsilon_{\boldsymbol{\phi}} = \boldsymbol{\phi}_h - \boldsymbol{\phi}^I$ and $\varepsilon_\omega = \omega_h - \omega^I$ to get

$$\boldsymbol{R}_h \varepsilon_{\boldsymbol{\phi}} = \nabla \varepsilon_\omega. \tag{4.8}$$

Take, in both the limit problem and its discrete problem, $\boldsymbol{\psi} = \varepsilon_{\boldsymbol{\phi}}$, $v = 0$ and $\boldsymbol{\psi} = 0$, $v = \varepsilon_\omega$, respectively, we then come to

$$\begin{aligned} a(\boldsymbol{\phi}_h - \boldsymbol{\phi}, \varepsilon_{\boldsymbol{\phi}}) + (\boldsymbol{\gamma}_h, \boldsymbol{R}_h \varepsilon_{\boldsymbol{\phi}}) - (\boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\phi}}) &= 0, \\ (\boldsymbol{\gamma}_h - \boldsymbol{\gamma}, \nabla \varepsilon_\omega) &= 0. \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) with Problem (4.3) and Problem (4.4), we derive that

$$\begin{aligned} C\|\varepsilon_{\boldsymbol{\phi}}\|_1^2 &\leq a(\varepsilon_{\boldsymbol{\phi}}, \varepsilon_{\boldsymbol{\phi}}) = a(\boldsymbol{\phi}_h - \boldsymbol{\phi}^I, \varepsilon_{\boldsymbol{\phi}}) \\ &= a(\boldsymbol{\phi}_h - \boldsymbol{\phi}, \varepsilon_{\boldsymbol{\phi}}) + a(\boldsymbol{\phi} - \boldsymbol{\phi}^I, \varepsilon_{\boldsymbol{\phi}}) \\ &= (\boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\phi}}) - (\boldsymbol{\gamma}_h, \boldsymbol{R}_h \varepsilon_{\boldsymbol{\phi}}) + a(\boldsymbol{\phi} - \boldsymbol{\phi}^I, \varepsilon_{\boldsymbol{\phi}}) \\ &= (\boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\phi}} - \boldsymbol{R}_h \varepsilon_{\boldsymbol{\phi}}) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{R}_h \varepsilon_{\boldsymbol{\phi}}) + a(\boldsymbol{\phi} - \boldsymbol{\phi}^I, \varepsilon_{\boldsymbol{\phi}}) \\ &= (\boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\phi}} - \boldsymbol{R}_h \varepsilon_{\boldsymbol{\phi}}) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla \varepsilon_\omega) + a(\boldsymbol{\phi} - \boldsymbol{\phi}^I, \varepsilon_{\boldsymbol{\phi}}) \\ &= (\boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\phi}} - \boldsymbol{R}_h \varepsilon_{\boldsymbol{\phi}}) + a(\boldsymbol{\phi} - \boldsymbol{\phi}^I, \varepsilon_{\boldsymbol{\phi}}), \end{aligned}$$

which implies that

$$\|\varepsilon_{\boldsymbol{\phi}}\|_1 \leq C\|\boldsymbol{\phi} - \boldsymbol{\phi}^I\|_1 + C \sup_{\boldsymbol{\psi} \in \boldsymbol{\Theta}_h} \frac{(\boldsymbol{\gamma}, \boldsymbol{\psi} - \boldsymbol{R}_h \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_1}.$$

Then (4.5) follows from (4.7) and the triangle inequality. (4.6) is a direct consequence of (4.5), the following inequality and the boundedness of the operator $\boldsymbol{R}_h$.

$$\|\nabla(\omega - \omega_h)\|_0 = \|\boldsymbol{\phi} - \boldsymbol{R}_h \boldsymbol{\phi}_h\|_0 \leq \|\boldsymbol{\phi} - \boldsymbol{R}_h \boldsymbol{\phi}\|_0 + \|\boldsymbol{R}_h \boldsymbol{\phi} - \boldsymbol{R}_h \boldsymbol{\phi}_h\|_0. \qquad \square$$

Remark 4.3. This lemma and its proof are actually the quadrilateral version of those from [11].

5. Abstract error estimate for the general problem

In this section, we establish, in an abstract setting, error estimates for the finite element methods satisfying Conditions 1–5 proposed in Section 3 for the general problem with $t > 0$. Our analysis is based on the discrete Helmholtz decomposition on the quadrilateral meshes (see Lemma 5.2 below). Therefore, throughout this section, we assume that these five conditions are met by the finite element methods used to discretize the Reissner-Mindlin plate problem.

5.1. **An equivalent formulation of the R-M plate problem.** For our analysis we shall make use of an equivalent formulation of the Reissner-Mindlin plate equations proposed by Brezzi and Fortin in [12]. This formulation is derived from Problem 1.1 by using the Helmholtz Theorem of decomposition of the shear stress vector

$$\boldsymbol{\gamma} = \nabla r + \mathbf{curl}\, p, \tag{5.1}$$

with $(r, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$. Moreover, if $\boldsymbol{\gamma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$, then $r \in H_0^1(\Omega) \bigcap H^2(\Omega)$ and $p \in \hat{H}^1(\Omega) := H^1(\Omega) \cap L_0^2(\Omega)$, which admits the following estimate:

$$\|r\|_2 + \|p\|_1 \le C\|\boldsymbol{\gamma}\|_{\boldsymbol{H}(\mathrm{div})}. \tag{5.2}$$

Following [14], we introduce the auxiliary variable $\boldsymbol{\alpha} = \mathbf{curl}\, p$, then Brezzi and Fortin's formulation for the Reissner-Mindlin plate can be rewritten as

Problem 5.1. Find $(r, \boldsymbol{\phi}, p, \boldsymbol{\alpha}, \omega) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega) \times H_0^1(\Omega)$, such that

$$(\nabla r, \nabla v) = (g, v), \quad \forall v \in H_0^1(\Omega), \tag{5.3}$$

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) - (p, \mathrm{rot}\, \boldsymbol{\psi}) = (\nabla r, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega), \tag{5.4}$$

$$-(\mathrm{rot}\, \boldsymbol{\phi}, q) - \lambda^{-1} t^2 (\mathrm{rot}\, \boldsymbol{\alpha}, q) = 0, \quad \forall q \in L_0^2(\Omega), \tag{5.5}$$

$$(\boldsymbol{\alpha}, \boldsymbol{\delta}) - (p, \mathrm{rot}\, \boldsymbol{\delta}) = 0, \quad \forall \boldsymbol{\delta} \in \boldsymbol{H}_0(\mathrm{rot}, \Omega), \tag{5.6}$$

$$(\nabla \omega, \nabla s) = (\boldsymbol{\phi} + \lambda^{-1} t^2 \nabla r, \nabla s), \quad \forall s \in H_0^1(\Omega). \tag{5.7}$$

It is classic to show that Problem 5.1 is equivalent to Problem 1.1. The existence and uniqueness of the solution to Problem 5.1 can be found, for instance, in [12, 2].

Note that, for Problem 5.1, two Poisson problems (5.3) and (5.7) are decoupled from the system, and the remaining part is a Stokes-like problem. To analyze it, we define the two bilinear forms $\mathcal{A} : (\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega)) \times (\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega)) \to R$ and $\mathcal{B} : L_0^2(\Omega) \times (\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega)) \to R$ by

$$\mathcal{A}(\boldsymbol{\phi}, \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) = a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda^{-1} t^2 (\boldsymbol{\alpha}, \boldsymbol{\delta}),$$

$$\mathcal{B}(q; \boldsymbol{\psi}, \boldsymbol{\delta}) = -(\mathrm{rot}\, \boldsymbol{\psi}, q) - \lambda^{-1} t^2 (\mathrm{rot}\, \boldsymbol{\delta}, q).$$

Define the following norm:

$$|||\boldsymbol{\psi}, \boldsymbol{\delta}|||^2 = \|\boldsymbol{\psi}\|_1^2 + t^2 \|\boldsymbol{\delta}\|_0^2 + t^4 \|\mathrm{rot}\, \boldsymbol{\delta}\|_0^2.$$

With this norm these two bilinear forms are bounded in the sense that

$$\mathcal{A}(\boldsymbol{\phi}, \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) \le C |||\boldsymbol{\phi}, \boldsymbol{\alpha}||| \, |||\boldsymbol{\psi}, \boldsymbol{\delta}|||,$$

$$\mathcal{B}(q; \boldsymbol{\psi}, \boldsymbol{\delta}) \le C \|q\|_0 |||\boldsymbol{\psi}, \boldsymbol{\delta}|||,$$

with C independent of t. Then (5.4)-(5.6) can be rewritten as: Find $(\boldsymbol{\phi}, \boldsymbol{\alpha}, p) \in \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega) \times L_0^2(\Omega)$ such that

$$\mathcal{A}(\boldsymbol{\phi}, \boldsymbol{\alpha}; \boldsymbol{\psi}, \boldsymbol{\delta}) + \mathcal{B}(p; \boldsymbol{\psi}, \boldsymbol{\delta}) = (\nabla r, \boldsymbol{\psi}), \quad \forall (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega), \tag{5.8}$$

$$\mathcal{B}(q; \boldsymbol{\phi}, \boldsymbol{\alpha}) = 0, \quad \forall q \in L_0^2(\Omega). \tag{5.9}$$

The existence and uniqueness of the solution to this saddle problem can be easily shown by the classic theory.

5.2. **An equivalent formulation of the discrete problem.** In this subsection, we derive an equivalent formulation for the discrete problem under Conditions 1–5 from Section 3.

First we use Condition 4, Condition 5 and Theorem 4.1 to prove the following discrete Helmholtz decomposition, on which our analysis is based.

Lemma 5.2. *For any $\boldsymbol{q} \in \boldsymbol{\Gamma}_h$, there exist unique $r \in W_h$, $p \in \mathcal{Q}_h$ and $\boldsymbol{\alpha} \in \boldsymbol{\Gamma}_h$ such that*

$$\boldsymbol{q} = \nabla r + \boldsymbol{\alpha}, \ \textit{and} \ (\boldsymbol{\alpha}, \boldsymbol{\sigma}) = (\mathrm{rot}\, \boldsymbol{\sigma}, p), \quad \forall \boldsymbol{\sigma} \in \boldsymbol{\Gamma}_h. \tag{5.10}$$

Proof. Consider the following mixed problem: Find $(\boldsymbol{\alpha}, p) \in \boldsymbol{\Gamma}_h \times \mathcal{Q}_h$ such that

$$(\boldsymbol{\alpha}, \boldsymbol{\sigma}) - (\mathrm{rot}\, \boldsymbol{\sigma}, p) = 0, \quad \forall \boldsymbol{\sigma} \in \boldsymbol{\Gamma}_h,$$
$$(\mathrm{rot}\, \boldsymbol{\alpha}, m) = (\mathrm{rot}\, \boldsymbol{q}, m), \quad \forall m \in \mathcal{Q}_h.$$

By virtue of Condition 4, this problem admits a unique solution, and it follows from Theorem 4.1 that

$$\mathrm{rot}(\boldsymbol{q} - \boldsymbol{\alpha}) = 0.$$

Then Condition 5 concludes that there exists a unique $r \in W_h$ such that $\boldsymbol{q} - \boldsymbol{\alpha} = \nabla r$, which completes the proof. □

Using Condition 2 and this discrete Helmholtz decomposition and following the line of [14], we can rewrite Problem 3.2 as the following equivalent form

Problem 5.3. Find $(r_h, \boldsymbol{\phi}_h, p_h, \boldsymbol{\alpha}_h, \omega_h) \in W_h \times \boldsymbol{\Theta}_h \times \mathcal{Q}_h \times \boldsymbol{\Gamma}_h \times W_h$ such that

$$(\nabla r_h, \nabla v) = (g, v), \quad \forall v \in W_h, \tag{5.11}$$

$$a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) - (p_h, \mathrm{rot}\, \boldsymbol{\psi}) = (\nabla r_h, \boldsymbol{R}_h \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{\Theta}_h, \tag{5.12}$$

$$-(\mathrm{rot}\, \boldsymbol{\phi}_h, q) - \lambda^{-1} t^2 (\mathrm{rot}\, \boldsymbol{\alpha}_h, q) = 0, \quad \forall q \in \mathcal{Q}_h, \tag{5.13}$$

$$(\boldsymbol{\alpha}_h, \boldsymbol{\delta}) - (p_h, \mathrm{rot}\, \boldsymbol{\delta}) = 0, \quad \forall \boldsymbol{\delta} \in \boldsymbol{\Gamma}_h, \tag{5.14}$$

$$(\nabla \omega_h, \nabla s) = (\boldsymbol{R}_h \boldsymbol{\phi}_h + \lambda^{-1} t^2 \nabla r_h, \nabla s), \quad \forall s \in W_h. \tag{5.15}$$

Similar to (5.4)-(5.6), for the discrete problem, (5.12)-(5.14) can be rewritten as: Find $(\boldsymbol{\phi}_h, \boldsymbol{\alpha}_h, p_h) \in \boldsymbol{\Theta}_h \times \boldsymbol{\Gamma}_h \times \mathcal{Q}_h$ such that

$$\mathcal{A}(\boldsymbol{\phi}_h, \boldsymbol{\alpha}_h; \boldsymbol{\psi}, \boldsymbol{\delta}) + \mathcal{B}(p_h; \boldsymbol{\psi}, \boldsymbol{\delta}) = (\nabla r_h, \boldsymbol{R}_h \boldsymbol{\psi}), \quad \forall (\boldsymbol{\psi}, \boldsymbol{\delta}) \in \boldsymbol{\Theta}_h \times \boldsymbol{\Gamma}_h, \tag{5.16}$$

$$\mathcal{B}(q; \boldsymbol{\phi}_h, \boldsymbol{\alpha}_h) = 0, \quad \forall q \in \mathcal{Q}_h. \tag{5.17}$$

5.3. **The well-posedness of the discrete problem and error estimates.** In this subsection, we consider the well-posedness of the discrete problem and the error estimates. Since the two discrete Poisson equations (5.11) and (5.15) are decoupled from the system, we only need to check the well-posedness of the discrete problem (5.16)-(5.17). By the mixed finite element theory from [13], the well-posedness of (5.16)-(5.17) hangs on the following two assumptions which we will examine below.

(1) K-ellipticity. There exists a constant $C > 0$ such that

$$\mathcal{A}(\boldsymbol{\psi},\boldsymbol{\delta};\boldsymbol{\psi},\boldsymbol{\delta}) \geq C|||\boldsymbol{\psi},\boldsymbol{\delta}|||^2, \tag{5.18}$$

for all

$$\begin{aligned}(\boldsymbol{\psi},\boldsymbol{\delta}) \in \boldsymbol{Z}_h &= \{(\boldsymbol{\psi},\boldsymbol{\delta}) \in \boldsymbol{\Theta}_h \times \boldsymbol{\Gamma}_h, \mathcal{B}(q;\boldsymbol{\psi},\boldsymbol{\delta}) = 0, \quad \forall q \in \mathcal{Q}_h\} \\ &= \{(\boldsymbol{\psi},\boldsymbol{\delta}) \in \boldsymbol{\Theta}_h \times \boldsymbol{\Gamma}_h, \operatorname{rot} \boldsymbol{R}_h\boldsymbol{\psi} = -\lambda^{-1}t^2 \operatorname{rot} \boldsymbol{\delta}\}.\end{aligned}$$

(2) B-B condition. There exists a constant $\beta(h,k)$ such that

$$\sup_{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{\Theta}_h\times\boldsymbol{\Gamma}_h} \frac{\mathcal{B}(q;\boldsymbol{\psi},\boldsymbol{\delta})}{|||\boldsymbol{\psi},\boldsymbol{\delta}|||} \geq \beta(h,k)\|q\|_0, \quad \forall q \in \mathcal{Q}_h. \tag{5.19}$$

In order to prove the K-ellipticity condition (5.18), we need the following result

Lemma 5.4. *There exists a constant C independent of h and k such that*

$$\|\operatorname{rot} \boldsymbol{R}_h\boldsymbol{\psi}\|_0 \leq C\|\operatorname{rot}\boldsymbol{\psi}\|_0, \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega). \tag{5.20}$$

Proof. With Condition 2 in Section 3, we have for the reduction operator $\boldsymbol{R}_h$,

$$(\operatorname{rot} \boldsymbol{R}_h\boldsymbol{\psi}, q_0) = (\operatorname{rot}\boldsymbol{\psi}, q_0), \quad \forall q_0 \in \mathcal{Q}_h.$$

Again from Condition 2, one can assume that

$$q\,|_K = \operatorname{rot} \boldsymbol{R}_h\boldsymbol{\psi}|_K = \frac{q_K}{J_K}, \quad \forall K \in J^h,$$

with $q_K = \hat{q}_K(F_K^{-1}(x))$ and $\hat{q}_K \in \mathcal{Q}(\hat{K})$. Similar to the proof of Theorem 4.1, we set

$$q_0\,|_K = \frac{q_K}{J_{0,K}} - C_0, \quad \text{with} \quad C_0 = \frac{1}{|\,\Omega\,|}\sum_{K\in J^h}\int_K \frac{q_K}{J_{0,K}}dxdy$$

and

$$q_1\,|_K = \frac{q_K}{J_{0,K}}.$$

Applying the Cauchy and Holder inequalities,

$$\begin{aligned}\|q_0\|_0^2 &\leq 2[\int_\Omega q_1^2 dxdy + C_0^2\,|\,\Omega\,|] \\ &\leq 2[C\int_\Omega q^2 dxdy + C\int_\Omega q^2 dxdy] \\ &\leq C\|\operatorname{rot}\boldsymbol{R}_h\boldsymbol{\psi}\|_0^2.\end{aligned}$$

By virtue of $\operatorname{rot} \boldsymbol{R}_h \boldsymbol{\psi} \in L_0^2(\Omega)$ and the regularity $C_K J_{0,K} \le J_K$ of the mesh with the positive constant C_K only depending on the geometry of each element K, we have

$$\begin{aligned}(\operatorname{rot} \boldsymbol{\psi}, q_0) &= (\operatorname{rot} \boldsymbol{R}_h \boldsymbol{\psi}, q_0) = (\operatorname{rot} \boldsymbol{R}_h \boldsymbol{\psi}, q_1) \\ &\ge \sum_{K \in J^h} C_K \left(\frac{q_K}{J_K}, \frac{q_K}{J_K}\right)_K \\ &\ge \min_{K \in J^h} (C_K) \| \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\psi} \|_0^2 = C \| \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\psi} \|_0^2,\end{aligned}$$

which, together with the bound of $\|q_0\|_0$, implies that

$$\| \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\psi} \|_0 \le C \| \operatorname{rot} \boldsymbol{\psi} \|_0,$$

which completes the proof. □

Then, we have the following result about the existence and uniqueness of the solution to the discrete problem.

Theorem 5.5. *Problem* 5.3 *admits a unique solution* $(r_h, \boldsymbol{\phi}_h, p_h, \boldsymbol{\alpha}_h, \omega_h) \in W_h \times \boldsymbol{\Theta}_h \times Q_h \times \boldsymbol{\Gamma}_h \times W_h$.

Proof. Since $\boldsymbol{\Theta}_h \subset \boldsymbol{H}_0^1(\Omega)$, the Korn inequality holds, then for all $(\boldsymbol{\psi}, \boldsymbol{\delta}) \in \boldsymbol{Z}_h$, it follows from the definition of the bilinear form $\mathcal{A}(\cdot, \cdot; \cdot, \cdot)$ (see Subsection 5.1) and the kernel space $\boldsymbol{Z}_h$ that

$$\begin{aligned}\mathcal{A}(\boldsymbol{\psi}, \boldsymbol{\delta}; \boldsymbol{\psi}, \boldsymbol{\delta}) &\ge C\|\boldsymbol{\psi}\|_1^2 + \lambda^{-1} t^2 \|\boldsymbol{\delta}\|_0^2 \\ &\ge C\|\boldsymbol{\psi}\|_1^2 + \lambda^{-1} t^2 \|\boldsymbol{\delta}\|_0^2 + C\| \operatorname{rot} \boldsymbol{\psi}\|_0^2 \\ &\ge C(\|\boldsymbol{\psi}\|_1^2 + t^2 \|\boldsymbol{\delta}\|_0^2 + \|t^2 \operatorname{rot} \boldsymbol{\delta}\|_0^2) \\ &= C|||\boldsymbol{\psi}, \boldsymbol{\delta}|||^2.\end{aligned} \tag{5.21}$$

Let $\boldsymbol{\delta} = 0$. We obtain with Condition 3 in the form of (3.10) and (3.11) that

$$\sup_{(\boldsymbol{\psi}, \boldsymbol{\delta}) \in \boldsymbol{\Theta}_h \times \boldsymbol{\Gamma}_h} \frac{\mathcal{B}(q; \boldsymbol{\psi}, \boldsymbol{\delta})}{|||\boldsymbol{\psi}, \boldsymbol{\delta}|||} \ge \beta(h, k) \|q\|_0. \tag{5.22}$$

Then one can use the mixed finite element theory from [13] to end the proof. □

We are now ready to prove the following abstract error bounds.

Theorem 5.6. *Let* $(r,\boldsymbol{\phi},p,\boldsymbol{\alpha},\omega)\in H_0^1(\Omega)\times \boldsymbol{H}_0^1(\Omega)\times L_0^2(\Omega)\times \boldsymbol{H}_0(\mathrm{rot},\Omega)\times H_0^1(\Omega)$ *and* $(r_h,\boldsymbol{\phi}_h,p_h,\boldsymbol{\alpha}_h,\omega_h)\in W_h\times\boldsymbol{\Theta}_h\times\mathcal{Q}_h\times\boldsymbol{\Gamma}_h\times W_h$ *be the solutions to Problem* 5.1 *and Problem* 5.3*, respectively. Let the shear force* $\boldsymbol{\gamma}$ *and its discrete counterpart* $\boldsymbol{\gamma}_h$ *be defined in* (1.2) *and* (3.5)*, respectively. Then,*

$$\|\nabla r-\nabla r_h\|_0\le C\inf_{v\in W_h}\|\nabla r-\nabla v\|_0, \tag{5.23}$$

$$\begin{aligned}&|||\boldsymbol{\phi}_h-\boldsymbol{\phi},\boldsymbol{\alpha}_h-\boldsymbol{\alpha}|||+\beta(h,k)\|p-p_h\|_0\\ &\le C\frac{1}{\beta(h,k)}\left(\inf_{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{\Theta}_h\times\boldsymbol{\Gamma}_h}|||\boldsymbol{\phi}-\boldsymbol{\psi},\boldsymbol{\alpha}-\boldsymbol{\delta}|||+\inf_{q\in\mathcal{Q}_h}\|p-q\|_0\right.\\ &\qquad\left.+\sup_{\boldsymbol{\eta}\in\boldsymbol{\Theta}_h}\frac{|\,(\nabla r_h,\boldsymbol{R}_h\boldsymbol{\eta})-(\nabla r,\boldsymbol{\eta})\,|}{\|\boldsymbol{\eta}\|_1}\right),\end{aligned} \tag{5.24}$$

$$\begin{aligned}\|\nabla\omega_h-\nabla\omega\|_0\le C(&\inf_{v\in W_h}\|\nabla\omega-\nabla v\|_0+\|\boldsymbol{\phi}-\boldsymbol{R}_h\boldsymbol{\phi}_h\|_0\\ &+\lambda^{-1}t^2\|\nabla r-\nabla r_h\|_0),\end{aligned} \tag{5.25}$$

$$\|\boldsymbol{\alpha}-\boldsymbol{\alpha}_h\|_{-1}\le C\left(h\inf_{\boldsymbol{\delta}\in\boldsymbol{\Gamma}_h}\|\boldsymbol{\alpha}-\boldsymbol{\delta}\|_0+\|p-p_h\|_0\right), \tag{5.26}$$

$$\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{-1}\le C(\|r-r_h\|_0+\|\boldsymbol{\alpha}-\boldsymbol{\alpha}_h\|_{-1}), \tag{5.27}$$

$$\begin{aligned}&\|\boldsymbol{\phi}_h-\boldsymbol{\phi}\|_1+t\|\boldsymbol{\alpha}_h-\boldsymbol{\alpha}\|_0+\beta(h,k)\|p-p_h\|_0\\ &\le\frac{C}{\beta(h,k)}\inf_{\boldsymbol{\psi}_h\in\boldsymbol{\Theta}_h}\|\boldsymbol{\phi}-\boldsymbol{\psi}_h\|_1+Ct\|\boldsymbol{\alpha}-\boldsymbol{R}_h\boldsymbol{\alpha}\|_0\\ &\quad+C\left(\inf_{q\in\mathcal{Q}_h}\|p-q\|_0+\sup_{\boldsymbol{\eta}\in\boldsymbol{\Theta}_h}\frac{|\,(\nabla r_h,\boldsymbol{R}_h\boldsymbol{\eta})-(\nabla r,\boldsymbol{\eta})\,|}{\|\boldsymbol{\eta}\|_1}\right).\end{aligned} \tag{5.28}$$

Proof. Thanks to the equivalent formulations for the R-M plate problem and its discrete problem, the well-posedness of the discrete problem in the form of (5.21) and (5.22), we can prove (5.23)-(5.27) by the standard arguments; we refer the readers to [13, 14] for further details.

Next, we only show the inequality (5.28). Given $(\boldsymbol{v},\boldsymbol{s})\in\boldsymbol{Z}_h$ and $q\in\mathcal{Q}_h$, it follows from (5.8), (5.16) and $\mathcal{B}(p_h-q;\boldsymbol{v},\boldsymbol{s})=0$ that

$$\begin{aligned}&\mathcal{A}(\boldsymbol{\phi}_h-\boldsymbol{\phi},\boldsymbol{\alpha}_h-\boldsymbol{\alpha};\boldsymbol{v},\boldsymbol{s})\\ &\quad=\mathcal{B}(p-p_h;\boldsymbol{v},\boldsymbol{s})+(\nabla r_h,\boldsymbol{R}_h\boldsymbol{v})-(\nabla r,\boldsymbol{v})\\ &\quad=\mathcal{B}(p-q;\boldsymbol{v},\boldsymbol{s})+(\nabla r_h,\boldsymbol{R}_h\boldsymbol{v})-(\nabla r,\boldsymbol{v})\\ &\quad=-(\mathrm{rot}\,\boldsymbol{v},p-q)-\lambda^{-1}t^2(\mathrm{rot}\,\boldsymbol{s},p-q)\\ &\qquad+(\nabla r_h,\boldsymbol{R}_h\boldsymbol{v})-(\nabla r,\boldsymbol{v}).\end{aligned} \tag{5.29}$$

With this identity, we use the fact that $(\boldsymbol{\phi}_h,\boldsymbol{\alpha}_h)\in\boldsymbol{Z}_h$ and the K-ellipticity (5.21) to derive as

$$
\begin{aligned}
&\|\boldsymbol{\phi}_h-\boldsymbol{\psi}\|_1+t\|\boldsymbol{\alpha}_h-\boldsymbol{\delta}\|_0\\
&\le \sqrt{2}\big(\|\boldsymbol{\phi}_h-\boldsymbol{\psi}\|_1^2+t^2\|\boldsymbol{\alpha}_h-\boldsymbol{\delta}\|_0^2+\|t^2\operatorname{rot}(\boldsymbol{\alpha}_h-\boldsymbol{\delta})\|_0^2\big)^{1/2}\\
&=\sqrt{2}|||\boldsymbol{\phi}_h-\boldsymbol{\psi},\boldsymbol{\alpha}_h-\boldsymbol{\delta}|||\\
&\le C\sup_{(\boldsymbol{v},\boldsymbol{s})\in\boldsymbol{Z}_h}\frac{\mathcal{A}(\boldsymbol{\phi}_h-\boldsymbol{\psi},\boldsymbol{\alpha}_h-\boldsymbol{\delta};\boldsymbol{v},\boldsymbol{s})}{|||\boldsymbol{v},\boldsymbol{s}|||}\\
&\le C\sup_{(\boldsymbol{v},\boldsymbol{s})\in\boldsymbol{Z}_h}\frac{\mathcal{A}(\boldsymbol{\phi}_h-\boldsymbol{\phi},\boldsymbol{\alpha}_h-\boldsymbol{\alpha};\boldsymbol{v},\boldsymbol{s})}{|||\boldsymbol{v},\boldsymbol{s}|||}\\
&\quad+C\sup_{(\boldsymbol{v},\boldsymbol{s})\in\boldsymbol{Z}_h}\frac{\mathcal{A}(\boldsymbol{\phi}-\boldsymbol{\psi},\boldsymbol{\alpha}-\boldsymbol{\delta};\boldsymbol{v},\boldsymbol{s})}{|||\boldsymbol{v},\boldsymbol{s}|||}\\
&\le C\inf_{q\in\mathcal{Q}_h}\|p-q\|_0+C\sup_{\boldsymbol{\eta}\in\boldsymbol{\Theta}_h}\frac{|\,(\nabla r_h,\boldsymbol{R}_h\boldsymbol{\eta})-(\nabla r,\boldsymbol{\eta})\,|}{\|\boldsymbol{\eta}\|_1}\\
&\quad+C(\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_1+t\|\boldsymbol{\alpha}-\boldsymbol{\delta}\|_0).
\end{aligned}
\tag{5.30}
$$

This and the triangle inequality lead to

(5.31)
$$
\begin{aligned}
\|\boldsymbol{\phi}_h-\boldsymbol{\phi}\|_1+t\|\boldsymbol{\alpha}_h-\boldsymbol{\alpha}\|_0 &\le C\inf_{q\in\mathcal{Q}_h}\|p-q\|_0+C\sup_{\boldsymbol{\eta}\in\boldsymbol{\Theta}_h}\frac{|\,(\nabla r_h,\boldsymbol{R}_h\boldsymbol{\eta})-(\nabla r,\boldsymbol{\eta})\,|}{\|\boldsymbol{\eta}\|_1}\\
&\quad+C\inf_{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{Z}_h}(\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_1+t\|\boldsymbol{\alpha}-\boldsymbol{\delta}\|_0).
\end{aligned}
$$

We remain to estimate the last term in the above inequality. Owing to the discrete B-B condition, for any $\boldsymbol{v}\in\boldsymbol{\Theta}_h$ and $\boldsymbol{R}_h\boldsymbol{\alpha}\in\boldsymbol{\Gamma}_h$, there exists $(\boldsymbol{\eta},\boldsymbol{s})\in\boldsymbol{\Theta}_h\times\boldsymbol{\Gamma}_h$ such that

$$
\mathcal{B}(q;\boldsymbol{\eta},\boldsymbol{s})=\mathcal{B}(q;\boldsymbol{\phi}-\boldsymbol{v},\boldsymbol{\alpha}-\boldsymbol{R}_h\boldsymbol{\alpha}),\quad\forall q\in\mathcal{Q}_h.
$$

Since $\mathcal{B}(q;\boldsymbol{\phi},\boldsymbol{\alpha})=0$, this implies that $(\boldsymbol{\eta}+\boldsymbol{v},\boldsymbol{s}+\boldsymbol{R}_h\boldsymbol{\alpha})\in\boldsymbol{Z}_h$. Taking into account Condition 2, we get $(\operatorname{rot}(\boldsymbol{\alpha}-\boldsymbol{R}_h\boldsymbol{\alpha}),q)=0$. In view of the definition of $\mathcal{B}(\cdot;\cdot,\cdot)$, we obtain

$$
\mathcal{B}(q;\boldsymbol{\eta},\boldsymbol{s})=-(\operatorname{rot}(\boldsymbol{\phi}-\boldsymbol{v}),q).
$$

This and the discrete B-B condition (5.19) yield

$$
|||\boldsymbol{\eta},\boldsymbol{s}|||\le\frac{C}{\beta(h,k)}\|\boldsymbol{\phi}-\boldsymbol{v}\|_1.
$$

Taking $\boldsymbol{\psi}=\boldsymbol{\eta}+\boldsymbol{v}$ and $\boldsymbol{\delta}=\boldsymbol{s}+\boldsymbol{R}_h\boldsymbol{\alpha}$ in (5.31), we apply the triangle inequality and the above estimate to get

$$
\begin{aligned}
\|\boldsymbol{\phi}_h-\boldsymbol{\phi}\|_1+t\|\boldsymbol{\alpha}_h-\boldsymbol{\alpha}\|_0 \quad\le\quad & C\inf_{q\in\mathcal{Q}_h}\|p-q\|_0+C\sup_{\boldsymbol{\eta}\in\boldsymbol{\Theta}_h}\frac{|\,(\nabla r_h,\boldsymbol{R}_h\boldsymbol{\eta})-(\nabla r,\boldsymbol{\eta})\,|}{\|\boldsymbol{\eta}\|_1}\\
&+\frac{C}{\beta(h,k)}\inf_{\boldsymbol{v}\in\boldsymbol{\Theta}_h}\|\boldsymbol{\phi}-\boldsymbol{v}\|_1+Ct\|\boldsymbol{\alpha}-\boldsymbol{R}_h\boldsymbol{\alpha}\|_0.
\end{aligned}
$$

Given $q \in \mathcal{Q}_h$, it follows from the discrete B-B condition (5.22) that

$$\begin{aligned}
\|p_h - q\|_0 &\leq \frac{C}{\beta(h,k)} \sup_{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{\Theta}_h\times\boldsymbol{\Gamma}_h} \frac{\mathcal{B}(p_h - q;\boldsymbol{\psi},\boldsymbol{\delta})}{|||\boldsymbol{\psi},\boldsymbol{\delta}|||} \\
&\leq \frac{C}{\beta(h,k)} \sup_{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{\Theta}_h\times\boldsymbol{\Gamma}_h} \frac{\mathcal{B}(p_h - p;\boldsymbol{\psi},\boldsymbol{\delta})}{|||\boldsymbol{\psi},\boldsymbol{\delta}|||} \\
&\quad + \frac{C}{\beta(h,k)} \sup_{(\boldsymbol{\psi},\boldsymbol{\delta})\in\boldsymbol{\Theta}_h\times\boldsymbol{\Gamma}_h} \frac{\mathcal{B}(p - q;\boldsymbol{\psi},\boldsymbol{\delta})}{|||\boldsymbol{\psi},\boldsymbol{\delta}|||} \\
&\leq \frac{C}{\beta(h,k)}(\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_1 + t\|\boldsymbol{\alpha}_h - \boldsymbol{\alpha}\|_0) \\
&\quad + \frac{C}{\beta(h,k)} \left(\sup_{\boldsymbol{\eta}\in\boldsymbol{\Theta}_h} \frac{|\,(\nabla r_h, \boldsymbol{R}_h\boldsymbol{\eta}) - (\nabla r, \boldsymbol{\eta})\,|}{\|\boldsymbol{\eta}\|_1} + \|p - q\|_0 \right),
\end{aligned}$$

in the last inequality, we use equations (5.8) and (5.16). An application of the triangle inequality yields the estimate for the pressure, which ends the proof of (5.28). □

Remark 5.7. The importance of estimate (5.28) lies in the fact that term $t^2 \inf_{\boldsymbol{\delta}\in\boldsymbol{\Gamma}_h} \|\,\mathrm{rot}(\boldsymbol{\alpha}-\boldsymbol{\delta})\|_0$ in (5.24) is dropped, which is crucial to obtain error bounds independent of α in energy norm for Method 4 in Section 6, since the error bound of $t^2 \inf_{\boldsymbol{\delta}\in\boldsymbol{\Gamma}_h} \|\,\mathrm{rot}(\boldsymbol{\alpha} - \boldsymbol{\delta})\|_0$ depends on α and cannot be improved [34, 5, 25, 18]; see also Lemma 7.8 (see, [18], for counterexamples).

Remark 5.8. It follows immediately from Theorem 4.2 and Theorem 5.6 that MITC elements converge and are locking-free as long as these five conditions from Section 3 are met.

Remark 5.9. The framework can be used to analyze the first order quadrilateral element proposed in [16]. With a corresponding modification due to the nonconformity, it can be easily extended to the first order nonconforming quadrilateral elements [19, 24].

6. Four families of quadrilateral MITC elements

In this section, we generalize the rectangular MITC plate bending elements proposed in [29] to the quadrilateral meshes.

We introduce some notation. As usual, for $S \subset R^2$, we let $P_k(S)$ denote the set of polynomials of total degree $\leq k$ and $Q_k(S)$ the set of polynomials of degree $\leq k$ in each variable. Moreover, $Q_k^{'}(S)$ will denote the "trunk" or "serendipity" space of polynomials [15].

The spaces W_h, $\boldsymbol{\Theta}_h$ and $\mathcal{Q}_h$ are defined as

$$W_h = \{v \in H_0^1(\Omega), v\,|_K = \hat{v} \circ F_K^{-1}, \hat{v} \in W_k(\hat{K}), \quad \forall K \in J^h\},$$

$$\boldsymbol{\Theta}_h = \{\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega), \boldsymbol{\psi}\,|_K = \hat{\boldsymbol{\psi}} \circ F_K^{-1}, \hat{\boldsymbol{\psi}} \in \boldsymbol{\Theta}_k(\hat{K}), \quad \forall K \in J^h\},$$

$$\mathcal{Q}_h = \{q \in L_0^2(\Omega), q\,|_K = \hat{q} \circ F_K^{-1}, \hat{q} \in \mathcal{Q}_k(\hat{K}), \quad \forall K \in J^h\},$$

where $W_k(\hat{K})$, $\boldsymbol{\Theta}_k(\hat{K})$ and $\mathcal{Q}_k(\hat{K})$, which shall be specified in the sequel, are polynomial spaces on the reference element $\hat{K}$.

The space $\boldsymbol{\Gamma}_h$ is defined in a slightly different way. Let

$$\boldsymbol{\Gamma}_h = \{\boldsymbol{q} \in \boldsymbol{H}_0(\mathrm{rot},\Omega), \boldsymbol{q}\,|_K \in \boldsymbol{\Gamma}_k(K), \forall K \in J^h\},$$

where the space $\mathbf{\Gamma}_k(K)$ is now defined from the space $\mathbf{\Gamma}_k(\hat{K})$ on the reference square through the following "Piola transformation" for the operator "rot":

$$\mathbf{\Gamma}_k(K) = \{\boldsymbol{\sigma}, \boldsymbol{\sigma} = DF_K^{-T}\bar{\boldsymbol{\sigma}} \circ F_K^{-1}, \bar{\boldsymbol{\sigma}} \in \mathbf{\Gamma}_k(\hat{K})\}.$$

The reduction operator $\boldsymbol{R}_h$ is also defined locally on each element from the reduction operator $\widehat{\boldsymbol{R}}_{\hat{K}}$ defined on the reference element with the same transformation,

$$\boldsymbol{R}_h\boldsymbol{\sigma}\mid_K = DF_K^{-T}\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} \circ F_K^{-1}, \text{ for any } \boldsymbol{\sigma} \in \boldsymbol{H}^1(\Omega), \tag{6.1}$$

where $\bar{\boldsymbol{\sigma}} = DF_K^T\hat{\boldsymbol{\sigma}} = DF_K^T\boldsymbol{\sigma} \circ F_K$.

Remark 6.1. For ease of presentation, here we assume that $\boldsymbol{\sigma} \in \boldsymbol{H}^1(\Omega)$ in the definition of the reduction operator $\boldsymbol{R}_h$, this restriction can be relaxed to $\boldsymbol{\sigma} \in L^p(\Omega)^2 \cap \boldsymbol{H}(\mathrm{rot}, \Omega)$ with $p > 2$; we refer readers to Section III 3.3 of [13] for further details. Similarly, we assume the domain for the operator $\widehat{\boldsymbol{R}}_{\hat{K}}$ is $\boldsymbol{H}^1(\hat{K})$.

The properties of the "Piola transformation" for the operator "rot" are summarized in the following Lemma [20, 18, 29].

Lemma 6.2. *Let K be a convex quadrilateral of J^h and $\boldsymbol{\sigma} \in \boldsymbol{H}^1(K)$ be any vector-valued function. Then*

$$\mathrm{rot}\,\boldsymbol{\sigma} = \frac{\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}}{J_K}, \tag{6.2}$$

$$\int_K \mathrm{rot}\,\boldsymbol{\sigma} w dxdy = \int_{\hat{K}} \widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}\hat{w} d\xi d\eta, \forall w \in L^2(K), \tag{6.3}$$

$$\int_E \boldsymbol{\sigma} \cdot \boldsymbol{t} ds = \int_{\hat{E}} \bar{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{t}} d\hat{s}, \tag{6.4}$$

with $\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}} = \partial\bar{\sigma}_2/\partial\xi - \partial\bar{\sigma}_1/\partial\eta$.

Four classes of rectangular plate bending elements have been proposed in [29], we next present their quadrilateral versions, and prove that they satisfy those five conditions proposed in Section 3. For brevity, we only give the full details of the proof for Method 1 below since the others can be proved similarly.

Method 1. In this element, the pressure finite element space is chosen as

$$\mathcal{Q}_k(\hat{K}) = P_{k-1}(\hat{K}), \tag{6.5}$$

and the rotation space reads

$$\mathbf{\Theta}_k(\hat{K}) = (Q_k(\hat{K}) \cap P_{k+2}(\hat{K}))^2. \tag{6.6}$$

In this case, we choose $\mathbf{\Gamma}_k(\hat{K})$ as the following BDFM space [13],

$$\mathbf{\Gamma}_k(\hat{K}) = \{\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}} \in P_k(\hat{K})\backslash \mathrm{span}\{\xi^k\} \times P_k(\hat{K})\backslash \mathrm{span}\{\eta^k\}\}, \tag{6.7}$$

and the reduction operator $\widehat{\boldsymbol{R}}_{\hat{K}}$ is defined as

$$\begin{aligned} &\int_{\hat{E}} (\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{t}}\hat{w} d\hat{s} = 0, \quad \forall \hat{w} \in P_{k-1}(\hat{E}) \text{ for every edge } \hat{E} \text{ of } \hat{K}, \\ &\int_{\hat{K}} (\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{v}} d\xi d\eta = 0, \quad \forall \hat{\boldsymbol{v}} \in P_{k-2}(\hat{K})^2, \end{aligned} \tag{6.8}$$

for any $\bar{\boldsymbol{\sigma}} \in \boldsymbol{H}^1(\hat{K})$. It remains to select the deflection space, which can be chosen as

$$W_k(\hat{K}) = Q_k(\hat{K}) \cap P_{k+1}(\hat{K}). \tag{6.9}$$

In what follows, we will show Conditions 1–5 proposed in Section 3 for this class of quadrilateral elements.

First, it is easy to see that Condition 1 and Condition 2 hold. Moreover, it is proved in [18] that there exists a positive constant C such that the inf-sup condition is valid with $\beta(h,k) = C\frac{k^{-\frac{1}{2}}}{1+h^{\alpha}k^{\frac{5}{2}}}$ for $k \geq 2$, which implies Condition 3.

We now prove Condition 4. Given $q \in \mathcal{Q}_h$, We use Lemma 6.2 and the definition of $\boldsymbol{R}_h$ in (6.1) to deduce

$$
\begin{aligned}
(\mathrm{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}), q) &= \sum_{K\in J^h}\int_K \mathrm{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_K\boldsymbol{\sigma})q\,dxdy \\
&= \sum_{K\in J^h}\int_{\hat{K}} \widehat{\mathrm{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\hat{q}\,d\xi d\eta \\
&= -\sum_{K\in J^h}\sum_{\hat{E}\subset\partial\hat{K}}\int_{\hat{E}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\cdot\hat{\boldsymbol{t}}\hat{q}\,d\hat{s} \\
&\quad + \sum_{K\in J^h}\int_{\hat{K}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\cdot\widehat{\mathbf{curl}}\hat{q}\,d\xi d\eta = 0.
\end{aligned}
\tag{6.10}
$$

Based on a scaling argument, one can prove that $\boldsymbol{R}_h$ is a bounded operator in the sense that

$$
\|\boldsymbol{R}_h\boldsymbol{\sigma}\|_{H(\mathrm{rot})} \leq C\|\boldsymbol{\sigma}\|_1, \text{ for any } \boldsymbol{\sigma} \in \boldsymbol{H}^1(\Omega). \tag{6.11}
$$

(6.10) and (6.11) are essentially two conditions for the Fortin technique [13]. With these two conditions, one can show that the discrete B-B condition holds uniformly for the space pair $(\boldsymbol{\Gamma}_h, \mathcal{Q}_h)$. In fact, for any $q \in \mathcal{Q}_h$, there exists $\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega)$ with

$$
\mathrm{rot}\,\boldsymbol{\psi} = q, \text{ and } \|\boldsymbol{\psi}\|_1 \leq C\|q\|_0. \tag{6.12}
$$

This, together with (6.10) and (6.11), leads to

$$
\begin{aligned}
\|q\|_0 &= \frac{(\mathrm{rot}\,\boldsymbol{\psi}, q)}{\|q\|_0} = \frac{(\mathrm{rot}\,\boldsymbol{R}_h\boldsymbol{\psi}, q)}{\|q\|_0} \leq C\frac{(\mathrm{rot}\,\boldsymbol{R}_h\boldsymbol{\psi}, q)}{\|\boldsymbol{\psi}\|_1} \\
&\leq C\frac{(\mathrm{rot}\,\boldsymbol{R}_h\boldsymbol{\psi}, q)}{\|\boldsymbol{R}_h\boldsymbol{\psi}\|_{H(\mathrm{rot})}} \leq C\sup_{\boldsymbol{\sigma}\in\boldsymbol{\Gamma}_h}\frac{(\mathrm{rot}\,\boldsymbol{\sigma}, q)}{\|\boldsymbol{\sigma}\|_{H(\mathrm{rot})}}.
\end{aligned}
\tag{6.13}
$$

On the other hand, it follows from Theorem 4.1 that

$$
\|\boldsymbol{\sigma}\|_0 = \|\boldsymbol{\sigma}\|_{H(\mathrm{rot})} \tag{6.14}
$$

for any

$$
\boldsymbol{\sigma} \in \{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h, (\mathrm{rot}\,\boldsymbol{\delta}, q) = 0, \forall\boldsymbol{\delta} \in \mathcal{Q}_h\}.
$$

This completes the proof of Condition 4.

It remains to show that Condition 5 holds. In fact, the condition $\mathrm{rot}\,\boldsymbol{\sigma} = 0$ readily implies that

$$
\boldsymbol{\sigma} = \nabla w, \text{ for some } w \in H_0^1(\Omega).
$$

In particular, the condition $w = 0$ on $\partial\Omega$ comes from the property $\boldsymbol{\sigma}\cdot\boldsymbol{t} = 0$ on $\partial\Omega$. What we have to check is that such a w actually belongs to W_h defined by (6.9). In fact, on the reference element $\hat{K}$, $\bar{\boldsymbol{\sigma}}$ belongs to $\boldsymbol{\Gamma}_k(\hat{K})$, therefore $\hat{w}$ has to belong to $W_k(\hat{K})$ on $\hat{K}$.

Method 2. In this method, W_h, $\mathcal{Q}_h$ and $\mathbf{\Gamma}_h$ are the same as in Method 1 with a different choice of the rotation space, which reads as

$$\mathbf{\Theta}_k(\hat{K}) = Q_k(\hat{K})^2. \tag{6.15}$$

As it was pointed out in [29], that compared with the first method this choice will lead to $\mathcal{O}(k^2)$ more degrees of freedom. Since the two methods have the same order of convergence, the first appears to be preferable.

Method 3. The spaces for the rotation and the auxiliary pressure are chosen as in Method 1. However, we take the following BDM space [13],

$$\mathbf{\Gamma}_k(\hat{K}) = \{\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}} \in P_k(\hat{K})^2 \oplus \operatorname{span}\{\nabla(\xi\eta^{k+1})\} \oplus \operatorname{span}\{\nabla(\xi^{k+1}\eta)\}\}, \tag{6.16}$$

as the shear force space with the reduction operator defined by

$$\begin{aligned} &\int_{\hat{E}} (\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{t}} \hat{w} d\hat{s} = 0, \quad \forall \hat{w} \in P_k(\hat{E}) \text{ for every edge } \hat{E} \text{ of } \hat{K}, \\ &\int_{\hat{K}} (\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{v}} d\xi d\eta = 0, \quad \forall \hat{\boldsymbol{v}} \in P_{k-2}(\hat{K})^2. \end{aligned} \tag{6.17}$$

Therefore, the deflection space has to be selected as

$$W_k(\hat{K}) = Q'_{k+1}(\hat{K}). \tag{6.18}$$

Here and throughout this paper, $Q'_{k+1}(\hat{K})$ denotes the "trunk" or "serendipity" space of polynomials over $\hat{K}$ [15].

The last method proposed in [29] is tailored to be a real quadrilateral method, for which we shall show that the convergence rate is independent of the distortion parameter α of the mesh.

Method 4. Here, we take the pressure and the rotation spaces as

$$\mathcal{Q}_k(\hat{K}) = Q_{k-1}(\hat{K}), \tag{6.19}$$

$$\mathbf{\Theta}_k(\hat{K}) = \{\boldsymbol{\psi} \in Q_{k+1}(\hat{K})^2, \boldsymbol{\psi}\,|_{\hat{E}} \in P_k(\hat{E})^2 \quad \text{for every edge } \hat{E} \text{ of } \hat{K}\}. \tag{6.20}$$

It is easy to see that $Q_k(\hat{K})^2 \subset \mathbf{\Theta}_k(\hat{K})$. The corresponding space for the shear force is the following R-T space:

$$\mathbf{\Gamma}_k(\hat{K}) = \{\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}} \in Q_{k-1,k}(\hat{K}) \times Q_{k,k-1}(\hat{K})\}, \tag{6.21}$$

with the reduction operator defined by

$$\begin{aligned} &\int_{\hat{E}} (\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{t}} \hat{w} d\hat{s} = 0, \quad \forall \hat{w} \in P_{k-1}(\hat{E}) \text{ for every edge } \hat{E} \text{ of } \hat{K}, \\ &\int_{\hat{K}} (\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{v}} d\xi d\eta = 0, \quad \forall \hat{\boldsymbol{v}} \in Q_{k-1,k-2}(\hat{K}) \times Q_{k-2,k-1}(\hat{K}). \end{aligned} \tag{6.22}$$

The space for the deflection is selected as

$$W_k(\hat{K}) = Q_k(\hat{K}). \tag{6.23}$$

7. Energy norm error estimates

In this section, we present an error analysis in energy norm for the quadrilateral MITC elements defined in the previous section.

Throughout this section, we assume that the following regularity holds for the solution

$$\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{H}^m(\Omega)\,, r, \omega \in H_0^1(\Omega) \cap H^m(\Omega)\,, p \in H^{m-1}(\Omega)\,, \boldsymbol{\alpha} \in \boldsymbol{H}^{m-1}(\mathrm{rot}, \Omega).$$

Let $\mathbf{\Pi}_h$ the usual Lagrangian interpolation operator from $\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega)$ to $\mathbf{\Theta}_h$, let Π_h be the L^2 projection operator from $L^2(\Omega)$ to $\mathcal{Q}_h$ and let Π_h^ω be the Lagrangian interpolation operator from $H^2(\Omega)\cap H_0^1(\Omega)$ to W_h.

7.1. **Error estimates for some operators.** Here and in Section 8, we will frequently use the error estimates for the reduction operator $\boldsymbol{R}_h$, the H^1-projection operators $\mathbf{\Pi}_h$ and Π_h^ω, and the L^2-projection operator Π_h. This subsection presents these estimates without proof for compactness. For completeness, we will provide their corresponding error analysis in the appendix, namely, Sections A and B.

For the BDFM elements from Method 1 of Section 6, we have the following L^2 error estimates.

Lemma 7.1. *Let the reduction operator $\boldsymbol{R}_h$ be defined by* (6.1) *and* (6.8) *for the BDFM elements in Method 1 of Section 6. Assume $\boldsymbol{\sigma}\in\boldsymbol{H}^{m-1}(\Omega)$. Then,*

$$\begin{aligned}&\|\boldsymbol{\sigma}-\boldsymbol{R}_h\boldsymbol{\sigma}\|_0\le C_\varepsilon k^{-m+3/2+\varepsilon}h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)}\|\boldsymbol{\sigma}\|_{m-1},\quad \mu>2,\\&\|\boldsymbol{\sigma}-\boldsymbol{R}_h\boldsymbol{\sigma}\|_0\le C_\varepsilon k^{-m+3/2+\varepsilon}h^{\mu-1+\alpha}\|\boldsymbol{\sigma}\|_{m-1},\quad \mu=1,2,\end{aligned}\tag{7.1}$$

where $\mu=\min(m-1,k)$ and C_ε depends on $0<\varepsilon<m-3/2$.

Proof. The proof is given in Lemma A.4 below. □

Remark 7.2. In Lemma 7.1 and other places, C_ε indicates that the constant depends on the parameter ε. Given $\varepsilon>0$, this dependence is from the imbedding theorem,

$$\|u\|_{0,\partial\Omega}\le C_\varepsilon\|u\|_{\frac12+\varepsilon,\Omega},\ \text{ for any } u\in H^{\frac12+\varepsilon}(\Omega).$$

Similarly, one can prove the following L^2 error estimates for the BDM elements from Method 3 based on the analog to (A.4) from [29, Remark 5.2].

Lemma 7.3. *Let the reduction operator $\boldsymbol{R}_h$ be defined by* (6.1) *and* (6.17) *for the BDM elements in Method 3 of Section 6. Assume $\boldsymbol{\sigma}\in\boldsymbol{H}^{m-1}(\Omega)$. Then,*

$$\begin{aligned}&\|\boldsymbol{\sigma}-\boldsymbol{R}_h\boldsymbol{\sigma}\|_0\le C_\varepsilon k^{-m+3/2+\varepsilon}h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)}\|\boldsymbol{\sigma}\|_{m-1},\quad \mu>2,\\&\|\boldsymbol{\sigma}-\boldsymbol{R}_h\boldsymbol{\sigma}\|_0\le C_\varepsilon k^{-m+3/2+\varepsilon}h^{\mu-1+\alpha}\|\boldsymbol{\sigma}\|_{m-1},\quad \mu=1,2,\end{aligned}\tag{7.2}$$

where $\mu=\min(m-1,k)$ and C_ε depends on $0<\varepsilon<m-3/2$.

We have the following L^2 error estimates of the reduction operator $\boldsymbol{R}_h$ for the R-T elements.

Lemma 7.4. *Let $\boldsymbol{\sigma}\in\boldsymbol{H}^{m-1}(\Omega)$ and the reduction operator $\boldsymbol{R}_h$ be defined by* (6.1) *and* (6.22) *for the R-T elements in Method 4 of Section 6. Then,*

$$\begin{aligned}&\|\boldsymbol{\sigma}-\boldsymbol{R}_h\boldsymbol{\sigma}\|_0\le Ck^{-m+3/2}h^{\mu}\|\boldsymbol{\sigma}\|_{m-1},\qquad \text{if } m-1<\mu+1,\\&\|\boldsymbol{\sigma}-\boldsymbol{R}_h\boldsymbol{\sigma}\|_0\le Ck^{-m+3/2}h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)}\|\boldsymbol{\sigma}\|_{m-1},\qquad \text{if } m-1\ge\mu+1,\end{aligned}\tag{7.3}$$

with $\mu=\min(m-1,k)$.

Proof. We refer the interested readers to Lemma A.6 for the proof. □

Remark 7.5. Notice from Lemma 7.4 that if $m-1=k$, we get optimal error bounds for the interpolation operator of the R-T elements with respect to the meshsize h.

We now consider the $H(\mathrm{rot})$ error estimates for the reduction operators $\boldsymbol{R}_h$. For the BDFM elements in Method 1 of Section 6, we have

Lemma 7.6. *Let the reduction operator* $\boldsymbol{R}_h$ *be defined by* (6.1) *and* (6.8)*, and* $\boldsymbol{\sigma} \in \boldsymbol{H}^{m-1}(\mathrm{rot}, \Omega)$*. Then,*

$$\|\operatorname{rot} \boldsymbol{\sigma} - \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\sigma}\|_0 \leq C k^{-m+1} h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)} \|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot})}, \quad \mu \geq 3, \tag{7.4}$$

$$\|\operatorname{rot} \boldsymbol{\sigma} - \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\sigma}\|_0 \leq C k^{-m+1} h^{\mu-1+\alpha} \|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot})}, \quad \mu = 1, 2, \tag{7.5}$$

where $\mu = \min(m-1, k)$*.*

Proof. The proof can be found in Lemma B.2 □

Similarly, we have the following result for the BDM elements in Method 3 of Section 6.

Lemma 7.7. *Let the reduction operator* $\boldsymbol{R}_h$ *be defined by* (6.1) *and* (6.17)*, and* $\boldsymbol{\sigma} \in \boldsymbol{H}^{m-1}(\mathrm{rot}, \Omega)$*. Then,*

$$\begin{aligned} &\|\operatorname{rot} \boldsymbol{\sigma} - \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\sigma}\|_0 \leq C k^{-m+1} h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)} \|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot})}, \quad \mu \geq 3, \\ &\|\operatorname{rot} \boldsymbol{\sigma} - \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\sigma}\|_0 \leq C k^{-m+1} h^{\mu-1+\alpha} \|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot})}, \quad \mu = 1, 2, \end{aligned} \tag{7.6}$$

where $\mu = \min(m-1, k)$*.*

For the R-T elements, we have the following $H(\mathrm{rot})$ error estimates.

Lemma 7.8. *Let* $\operatorname{rot} \boldsymbol{\sigma} \in H^{m-1}(\Omega)$ *and* $\boldsymbol{R}_h$ *be defined by* (6.1) *and* (6.22) *for the R-T elements in Method 4 of Section 6. Then,*

$$\begin{aligned} &\|\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h \boldsymbol{\sigma})\|_0 \leq C k^{-m+1} h^{\mu+\alpha-1} \|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot})}, m-1 < \mu+1, \\ &\|\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h \boldsymbol{\sigma})\|_0 \leq C k^{-m+1} h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)} \|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot})}, m-1 \geq \mu+1, \end{aligned} \tag{7.7}$$

with $\mu = \min(m-1, k)$*.*

Proof. See Lemma B.4 for the details of the proof. □

Remark 7.9. Note from Lemma 7.8 that the $H(\mathrm{rot})$-norm error estimates of interpolations of R-T elements depend on α. The counterexamples from [18] show that this result cannot be improved.

In the rest of this subsection, we are concerned with the error estimates of the H^1-projection operators $\boldsymbol{\Pi}_h$ and Π_h^{ω}, and the L^2-projection operator Π_h. The proof of the following two lemmas can be found in Lemma B.5 and Lemma B.6, respectively.

Lemma 7.10. *Let the discrete rotation space* $\boldsymbol{\Theta}_h$ *be defined in Method 1, or in Method 2, or in Method 3, of Section 6, and let* $\boldsymbol{\Pi}_h$ *be the usual* H^1*-projection operator from* $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ *to* $\boldsymbol{\Theta}_h$*. Then,*

$$\|\boldsymbol{\Pi}_h \boldsymbol{\psi} - \boldsymbol{\psi}\|_s \leq C k^{-m+s} h^{\mu+[\frac{1+\mu}{2}](\alpha-1)-s} \|\boldsymbol{\psi}\|_m, \tag{7.8}$$

for any $\boldsymbol{\psi} \in \boldsymbol{H}^m(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ *with* $m \geq 2$*. Where* $s = 0, 1$*,* $\mu = \min(k+1, m)$*.*

Lemma 7.11. *Let the discrete rotation space* $\boldsymbol{\Theta}_h$ *be defined in Method 4 of Section 6, and let* $\boldsymbol{\Pi}_h$ *be the usual* H^1*-projection operator from* $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ *to* $\boldsymbol{\Theta}_h$*. Then,*

$$\begin{aligned} &\|\boldsymbol{\Pi}_h \boldsymbol{\psi} - \boldsymbol{\psi}\|_s \leq C k^{-m+s} h^{\mu-s} \|\boldsymbol{\psi}\|_m, \ \textit{if } m < \mu+1, \\ &\|\boldsymbol{\Pi}_h \boldsymbol{\psi} - \boldsymbol{\psi}\|_s \leq C k^{-m+s} h^{\mu+[\frac{1+\mu}{2}](\alpha-1)-s} \|\boldsymbol{\psi}\|_m, \ \textit{if } m \geq \mu+1, \end{aligned} \tag{7.9}$$

for any $\boldsymbol{\psi} \in \boldsymbol{H}^m(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ *with* $m \geq 2$*,* $\mu = \min(m, k+1)$ *and* $s = 0, 1$*.*

Remark 7.12. One can prove the analogous results of Lemma 7.10 and Lemma 7.11 for operators Π_h^ω and Π_h:

(1) Let the space W_h and $\mathcal{Q}_h$ be defined in Method 1, or in Method 2, or in Method 3 of Section 6. Let Π_h^ω be the usual H^1-projection operator from $H^2(\Omega) \cap H_0^1(\Omega)$ to W_h, and let Π_h be the L^2 projection operator from $L^2(\Omega)$ onto $\mathcal{Q}_h$. Then,

$$\|\Pi_h^\omega \psi - \psi\|_s \le Ck^{-m+s}h^{\mu+[\frac{1+\mu}{2}](\alpha-1)-s}\|\psi\|_m, \tag{7.10}$$

$$\|\Pi_h q - q\|_0 \le Ck^{-m+1}h^{\mu-1+[\frac{\mu}{2}](\alpha-1)}\|q\|_{m-1}, \tag{7.11}$$

for any $\psi \in H^m(\Omega) \cap H_0^1(\Omega)$ and $q \in L_0^2(\Omega) \cap H^{m-1}(\Omega)$ with $m \ge 2$, $s = 0, 1$, $\mu = \min(k+1, m)$.

(2) Let Π_h^ω and Π_h be the corresponding operators with the spaces W_h and $\mathcal{Q}_h$ from Method 4. Then,

$$\begin{aligned} &\|\Pi_h^\omega \psi - \psi\|_s \le Ck^{-m+s}h^{\mu-s}\|\psi\|_m, \text{ if } m < \mu+1, \\ &\|\Pi_h^\omega \psi - \psi\|_s \le Ck^{-m+s}h^{\mu+[\frac{1+\mu}{2}](\alpha-1)-s}\|\psi\|_m, \text{ if } m \ge \mu+1, \end{aligned} \tag{7.12}$$

$$\begin{aligned} &\|\Pi_h q - q\|_0 \le Ck^{-m+1}h^{\mu-1}\|q\|_{m-1}, \text{ if } m < \mu+1, \\ &\|\Pi_h q - q\|_q \le Ck^{-m+1}h^{\mu-1+[\frac{\mu}{2}](\alpha-1)}\|q\|_{m-1}, \text{ if } m \ge \mu+1, \end{aligned} \tag{7.13}$$

for any $\psi \in H^m(\Omega) \cap H_0^1(\Omega)$ and $q \in L_0^2(\Omega) \cap H^{m-1}(\Omega)$ with $m \ge 2$, $\mu = \min(m, k+1)$, $s = 0, 1$.

7.2. **The estimate of $(\nabla r, \mathbf{R}_h \boldsymbol{\eta} - \boldsymbol{\eta})$.** This subsection presents the analysis for the consistency term $(\nabla r, \boldsymbol{R}_h \boldsymbol{\eta} - \boldsymbol{\eta})$.

For the BDFM elements, we have

Lemma 7.13. *Let $r \in H^m(\Omega)$, $\boldsymbol{\eta} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathrm{rot}, \Omega)$, and $\boldsymbol{R}_h$ be defined by* (6.1) *and* (6.8) *for the BDFM elements. Then,*

$$| (\nabla r, \boldsymbol{R}_h \boldsymbol{\eta} - \boldsymbol{\eta}) | \le C_\varepsilon k^{-m+3/2+\varepsilon} h^{\mu+[\frac{\mu+1}{2}](\alpha-1)+\alpha}\|r\|_m \|\boldsymbol{\eta}\|_1, \tag{7.14}$$

where $\mu = \min(m-2, k-1)$ and C_ε depends on $0 < \varepsilon < 1/2$.

Proof. In this case, we define

$$\boldsymbol{\Theta}_{k-2}^P(J^h) := \{\boldsymbol{\psi} \in L^2(\Omega)^2, \boldsymbol{\psi}|_K = \widehat{\boldsymbol{\psi}} \circ F_K^{-1}, \widehat{\boldsymbol{\psi}} \in P_{k-2}(\hat{K})^2, \forall K \in J^h\}.$$

Let $\boldsymbol{\Pi}_{k-2}^P$ denote the piecewise L^2 projection operator from $L^2(\Omega)^2$ onto $\boldsymbol{\Theta}_{k-2}^P(J^h)$, we have the following decomposition:

$$(\nabla r, \boldsymbol{R}_h \boldsymbol{\eta} - \boldsymbol{\eta}) \;=\; (\nabla r - \boldsymbol{\Pi}_{k-2}^P \nabla r, \boldsymbol{R}_h \boldsymbol{\eta} - \boldsymbol{\eta}) + (\boldsymbol{\Pi}_{k-2}^P \nabla r, \boldsymbol{R}_h \boldsymbol{\eta} - \boldsymbol{\eta}) = I_1 + I_2.$$

Proceeding along the same line of Lemma 7.10 (see, Lemma B.5), one can prove

$$\|\nabla r - \boldsymbol{\Pi}_{k-2}^P \nabla r\|_0 \le Ck^{-m+2}h^{\mu+[\frac{\mu+1}{2}](\alpha-1)}\|\nabla r\|_{m-2},$$

with $\mu = \min(m-2, k-1)$. An application of Lemma 7.1 yields

$$|I_1| \le C_\varepsilon k^{-m+3/2+\varepsilon} h^{\mu+[\frac{\mu+1}{2}](\alpha-1)+\alpha}\|r\|_m \|\boldsymbol{\eta}\|_1,$$

with C_ε depending on $0 < \varepsilon < 1/2$.

We now turn to the term I_2. Let $M_K(\xi,\eta) = J_K DF_K^{-T}$ such that we have

$$\begin{aligned} I_2 &= \sum_{K\in J^h}\int_K \mathbf{\Pi}_{k-2}^P\nabla r\cdot(\boldsymbol{R}_h\boldsymbol{\eta}-\boldsymbol{\eta})dxdy \\ &= \sum_{K\in J^h}\int_{\hat K}\widehat{\mathbf{\Pi}_{k-2}^P\nabla r}\cdot M_K(\xi,\eta)(\widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\eta}}-\bar{\boldsymbol{\eta}})d\xi d\eta \\ &= \sum_{K\in J^h}\int_{\hat K}\widehat{\mathbf{\Pi}_{k-2}^P\nabla r}\cdot M_K(0,0)(\widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\eta}}-\bar{\boldsymbol{\eta}})d\xi d\eta \\ &\qquad \int_{\hat K}+\widehat{\mathbf{\Pi}_{k-2}^P\nabla r}\cdot(M_K(\xi,\eta)-M_K(0,0))(\widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\eta}}-\bar{\boldsymbol{\eta}})d\xi d\eta = I_3+I_4. \end{aligned}$$

It follows from the definition of $\widehat{\boldsymbol{R}}_{\hat K}$ that $I_3 = 0$. A straightforward investigation gives

$$\|[M_K(\xi,\eta)-M_K(0,0)]\|_F \le Ch^{1+\alpha},$$

where $\|\cdot\|_F$ is the Frobenius matrix norm.

Let $\widehat{\mathbf{\Pi}}_{k-3}^P$ denote the L^2 projection operator from $L^2(\hat K)^2$ onto the space $P_{k-3}(\hat K)^2$. With the definition of $M_k(\xi,\eta)$, we have $M_k^T(\xi,\eta)\hat{\boldsymbol{v}} \in P_{k-2}(\hat K)^2$ for any $\hat{\boldsymbol{v}} \in P_{k-3}(\hat K)^2$. This and the definition of $\widehat{\boldsymbol{R}}_{\hat K}$ (6.8) lead to

$$\begin{aligned} |I_4| &= |\sum_{K\in J^h}\int_{\hat K}\widehat{\mathbf{\Pi}_{k-2}^P\nabla r}\cdot(M(\xi,\eta)-M_K(0,0))(\widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\eta}}-\bar{\boldsymbol{\eta}})d\xi d\eta| \\ &= |\sum_{K\in J^h}\int_{\hat K}(\widehat{\mathbf{\Pi}_{k-2}^P\nabla r}-\widehat{\mathbf{\Pi}}_{k-3}^P\widehat{\mathbf{\Pi}_{k-2}^P\nabla r})\cdot(M_K(\xi,\eta)-M_K(0,0))(\widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\eta}}-\bar{\boldsymbol{\eta}})d\xi d\eta| \\ &\le Ch^{1+\alpha}\sum_{K\in J^h}\|\widehat{\mathbf{\Pi}_{k-2}^P\nabla r}-\widehat{\mathbf{\Pi}}_{k-3}^P\widehat{\mathbf{\Pi}_{k-2}^P\nabla r}\|_{0,\hat K}\|\widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\eta}}-\bar{\boldsymbol{\eta}}\|_{0,\hat K}. \end{aligned}$$

Based on Lemma 7.1, a similar argument in Lemma 7.10 (see Lemma B.5) proves

$$\begin{aligned} |I_4| &\le Ch^{1+\alpha}k^{-m+2}\sum_{K\in J^h}\inf_{\hat{\boldsymbol{v}}\in P_{k-3}\times P_{k-3}}\|\widehat{\mathbf{\Pi}_{k-2}^P\nabla r}-\hat{\boldsymbol{v}}\|_{m-2,\hat K}\|\widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\eta}}-\bar{\boldsymbol{\eta}}\|_{0,\hat K} \\ &\le C_\varepsilon k^{-m+3/2+\varepsilon}h^{\mu_1+[\frac{\mu_1+1}{2}](\alpha-1)+1+2\alpha}\|r\|_m\|\boldsymbol{\eta}\|_1, \end{aligned}$$

with $\mu_1 = \min(m-2, k-2)$. This completes the proof. □

Similarly, we have the following result for the BDM elements.

Lemma 7.14. *Let $r \in H^m(\Omega)$, $\boldsymbol{\eta} \in \boldsymbol{H}^1(\Omega)\cap\boldsymbol{H}_0(\mathrm{rot},\Omega)$, and let $\boldsymbol{R}_h$ be defined by* (6.1) *and* (6.17) *for the BDFM elements. Then,*

$$| (\nabla r, \boldsymbol{R}_h\boldsymbol{\eta}-\boldsymbol{\eta}) | \le C_\varepsilon k^{-m+3/2+\varepsilon}h^{\mu+[\frac{\mu+1}{2}](\alpha-1)+\alpha}\|r\|_m\|\boldsymbol{\eta}\|_1, \tag{7.15}$$

where $\mu = \min(m-2, k-1)$, and C_ε depending on $0<\varepsilon<1/2$.

For the R-T elements, we have

Lemma 7.15. *Let $r \in H^m(\Omega)$, $\boldsymbol{\eta} \in \boldsymbol{H}^1(\Omega)\cap\boldsymbol{H}_0(\mathrm{rot},\Omega)$, and let $\boldsymbol{R}_h$ be defined by* (6.1) *and* (6.22) *for the R-T elements. Then,*

(7.16)

$$| (\nabla r, \boldsymbol{R}_h\boldsymbol{\eta}-\boldsymbol{\eta}) | \le Ck^{-m+3/2}h^{\mu+1}\|r\|_m\|\boldsymbol{\eta}\|_1, \ \textit{if } \mu = m-2 \le k-1,$$

$$| (\nabla r, \boldsymbol{R}_h\boldsymbol{\eta}-\boldsymbol{\eta}) | \le Ck^{-m+3/2}h^{\mu+1+[\frac{\mu+1}{2}](\alpha-1)}\|r\|_m\|\boldsymbol{\eta}\|_1, \ \textit{if } \mu = k-1 < m-2.$$

Proof. Denote by $\mathbf{\Pi}_{k-2}$ the piecewise L^2 projection operator from $L^2(\Omega)^2$ onto $\mathbf{\Theta}_{k-2}(J^h)$, where

$$\mathbf{\Theta}_{k-2}(J^h) := \{\boldsymbol{\psi} \in L^2(\Omega)^2, \boldsymbol{\psi}|_K = \widehat{\boldsymbol{\psi}} \circ F_K^{-1}, \widehat{\boldsymbol{\psi}} \in Q_{k-2}(\hat{K})^2, \quad \forall K \in J^h\}.$$

Then, we have the decomposition

$$(\nabla r, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta}) = (\nabla r - \mathbf{\Pi}_{k-2}\nabla r, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta}) + (\mathbf{\Pi}_{k-2}(\nabla r), \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta}) = I_1 + I_2.$$

A similar argument of Lemma 7.11 (see Lemma B.6) proves

(7.17)
$$\|\nabla r - \mathbf{\Pi}_{k-2}\nabla r\|_0 \leq Ck^{-m+2}h^{\mu}\|\nabla r\|_{m-2}, \text{ if } \mu = m-2 \leq k-1,$$
$$\|\nabla r - \mathbf{\Pi}_{k-2}\nabla r\|_0 \leq Ck^{-m+2}h^{\mu+[\frac{\mu+1}{2}](\alpha-1)}\|\nabla r\|_{m-2}, \text{ if } \mu = k-1 < m-2.$$

This and Lemma 7.4 leads to

(7.18)
$$|(\nabla r - \mathbf{\Pi}_{k-2}\nabla r, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})| \leq Ck^{-m+3/2}h^{\mu+1}\|r\|_m\|\boldsymbol{\eta}\|_1, \text{ if } \mu = m-2 \leq k-1,$$
$$|(\nabla r - \mathbf{\Pi}_{k-2}\nabla r, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})| \leq Ck^{-m+3/2}h^{\mu+1+[\frac{\mu+1}{2}](\alpha-1)}\|r\|_m\|\boldsymbol{\eta}\|_1, \quad \text{if } \mu = k-1 < m-2.$$

It remains to take care of the second term I_2,

$$\begin{aligned} I_2 &= (\mathbf{\Pi}_{k-2}\nabla r, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta}) \\ &= \sum_{K \in J^h} \int_K \mathbf{\Pi}_{k-2}\nabla r \cdot (\boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})dxdy \\ &= \sum_{K \in J^h} \int_{\hat{K}} \widehat{\mathbf{\Pi}_{k-2}\nabla} r \cdot M_K(\xi,\eta)(\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\eta}} - \bar{\boldsymbol{\eta}})d\xi d\eta, \end{aligned}$$

where

$$M_K(\xi,\eta) = J_K DF_K^{-T} = \begin{pmatrix} d_2 + d_{12}\xi & -d_1 - d_{12}\eta \\ -c_2 - c_{12}\xi & c_1 + c_{12}\eta \end{pmatrix}.$$

Therefore, $M_K^T(\xi,\eta)\widehat{\mathbf{\Pi}_{k-2}\nabla} r \in Q_{k-1,k-2}(\hat{K}) \times \in Q_{k-2,k-1}(\hat{K})$. This, together with the definition of the reduction operator $\widehat{\boldsymbol{R}}_{\hat{K}}$ in (6.22), implies

$$I_2 = 0,$$

which completes the proof. □

7.3. **Energy norm error estimates.** For the sake of brevity, we introduce a new notation

$$|||\,(r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega)\,||| = \|\boldsymbol{\phi}\|_m + \|r\|_m + t\|\boldsymbol{\alpha}\|_{m-1} + t^2\|\operatorname{rot}\boldsymbol{\alpha}\|_{m-1} + \|p\|_{m-1} + \|\omega\|_m.$$

Then we have the following energy norm error estimates for Method 1–Method 3.

Theorem 7.16. *Let* $(r, \boldsymbol{\phi}, p, \boldsymbol{\alpha}, \omega) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega) \times H_0^1(\Omega)$ *and let* $(r_h, \boldsymbol{\phi}_h, p_h, \boldsymbol{\alpha}_h, \omega_h) \in W_h \times \mathbf{\Theta}_h \times \mathcal{Q}_h \times \mathbf{\Gamma}_h \times W_h$ *be the solutions of Problem* 5.1 *and Problem* 5.3, *respectively. Let the shear* $\boldsymbol{\gamma}$ *and the discrete shear force* $\boldsymbol{\gamma}_h$

be defined in (1.2) *and* (3.5)*, respectively. Then, it holds, for Method 1–Method 3 of Section 6,*

$$\|r - r_h\|_1 \le Ck^{-m+1}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)}\|r\|_m, \tag{7.19}$$

$$\begin{aligned}&|||\boldsymbol{\phi}_h - \boldsymbol{\phi}, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h||| + \|\omega - \omega_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + t^2\|\operatorname{rot}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0\\ &+\frac{k^{-1/2}}{1+h^\alpha k^{\frac{5}{2}}}\|p - p_h\|_0 + \frac{k^{-1}}{1+h^\alpha k^{\frac{5}{2}}}(\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{-1} + \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_{-1})\\ &\le C_{\varepsilon,h,k}k^{-m+3/2+\varepsilon}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)} \;||||\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;||||\;.\end{aligned} \tag{7.20}$$

Where $C_{\varepsilon,h,k} = C_\varepsilon(1 + h^\alpha k^{5/2})$ *with* C_ε *depending on* $0 < \varepsilon < 1/2$ *and* $\mu = \min(m-1, k)$.

Proof. First, one can use (5.24) and Remark 7.12 for Π_h^ω to show

$$\|r - r_h\|_1 \le Ck^{-m+1}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)}\|r\|_m. \tag{7.21}$$

We only show (7.20) for Method 1 by bounding the terms on the right-hand of (5.28).

In view of Lemma 7.10, Lemma 7.1, Remark 7.12 for Π_h, and $\beta(h,k) = C\frac{k^{-1/2}}{1+h^\alpha k^{5/2}}$ from Method 1 of Section 6 (see [18]), we obtain

$$\begin{aligned}&\frac{C}{\beta(h,k)}\inf_{\boldsymbol{\psi}\in\boldsymbol{\Theta}_h}\|\boldsymbol{\phi} - \boldsymbol{\psi}\|_1 + t\|\boldsymbol{\alpha} - \boldsymbol{R}_h\boldsymbol{\alpha}\|_0 + \inf_{q\in\mathcal{Q}_h}\|p - q\|_0\\ &\quad\le C_\varepsilon(1 + h^\alpha k^{5/2})k^{-m+3/2+\varepsilon}h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)}(\|\boldsymbol{\phi}\|_m + t\|\boldsymbol{\alpha}\|_{m-1} + \|p\|_{m-1}),\end{aligned}$$

with $\mu = \min(m-1, k)$ and C_ε depending on $0 < \varepsilon < 1/2$.

The consistency error term $(\nabla r_h, \boldsymbol{R}_h\boldsymbol{\eta}) - (\nabla r, \boldsymbol{\eta})$ can be decomposed as

$$(\nabla r_h, \boldsymbol{R}_h\boldsymbol{\eta}) - (\nabla r, \boldsymbol{\eta}) = (\nabla r_h - \nabla r, \boldsymbol{R}_h\boldsymbol{\eta}) + (\nabla r, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta}).$$

Taking into account (7.21) and Lemma 7.13, we deduce as

$$\begin{aligned}|(\nabla r_h, \boldsymbol{R}_h\boldsymbol{\eta}) - (\nabla r, \boldsymbol{\eta})| &\le |(\nabla r_h - \nabla r, \boldsymbol{R}_h\boldsymbol{\eta})| + |(\nabla r, \boldsymbol{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})|\\ &\le C_\varepsilon k^{-m+3/2+\varepsilon}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)}\|r\|_m\|\boldsymbol{\eta}\|_1,\end{aligned}$$

with μ defined as above.

We substitute these two estimates in (5.28) to prove

$$\begin{aligned}&\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + t\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_0 + \frac{k^{-1/2}}{1+h^\alpha k^{5/2}}\|p - p_h\|_0\\ &\quad\le C_\varepsilon(1 + h^\alpha k^{5/2})k^{-m+3/2+\varepsilon}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)} \;||||\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;||||\;.\end{aligned}$$

Applying this inequality, (5.25), (7.21), and Remark 7.12 for Π_h^ω, we proceed as follows:

$$\begin{aligned}|\omega - \omega_h|_1 &\le \inf_{v\in W_h}\|\nabla\omega - \nabla v\|_0 + \|\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}_h\|_0 + \lambda^{-1}t^2\|\nabla r - \nabla r_h\|_0\\ &\le \|\nabla\omega - \nabla\Pi_h^\omega\omega\|_0 + \|\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}\|_0 + C\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \lambda^{-1}t^2\|\nabla r - \nabla r_h\|_0\\ &\le C_\varepsilon(1 + h^\alpha k^{5/2})k^{-m+3/2+\varepsilon}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)} \;||||\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;||||\;.\end{aligned}$$

It follows from the decomposition of the shear forces $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_h$ that

$$\begin{aligned}t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 &\le C\|\nabla r - \nabla r_h\|_0 + t\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_0\\ &\le C_\varepsilon(1 + h^\alpha k^{5/2})k^{-m+3/2+\varepsilon}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)} \;||||\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;||||\;.\end{aligned}$$

Thanks to the definition of the (resp. discrete) shear force and the (resp. discrete) Helmholtz decomposition, we have

$$\boldsymbol{\alpha} - \boldsymbol{\alpha}_h = \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}) - \lambda t^{-2}(\nabla\omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h) + \nabla r_h - \nabla r.$$

This and Lemma 7.6 lead to

$$\begin{aligned} t^2\|\operatorname{rot}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_h)\|_0 &= \lambda\|\operatorname{rot}(\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}_h)\|_0 \\ &\le \lambda\|\operatorname{rot}\boldsymbol{\phi} - \operatorname{rot}\boldsymbol{R}_h\boldsymbol{\phi}\|_0 + \lambda\|\operatorname{rot}\boldsymbol{R}_h(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0 \\ &\le Ck^{-m+1}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)}\|\boldsymbol{\phi}\|_m + C\|\operatorname{rot}(\boldsymbol{\phi} - \boldsymbol{\phi}_h)\|_0 \\ &\le C_\varepsilon(1 + h^\alpha k^{5/2})k^{-m+3/2+\varepsilon}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)} \; \|\|\|\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;\|\|\|\; . \end{aligned}$$

Due to $\boldsymbol{\alpha} = \mathbf{curl}\, p$, we get $\|\boldsymbol{\alpha}\|_{m-2} \le C\|p\|_{m-1}$. Then, it follows from (5.26), the estimate of $\|p - p_h\|_0$, and Lemma 7.1 with $m-2$ that

$$\begin{aligned} \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_{-1} &\le C(h \inf_{\boldsymbol{\delta}\in\boldsymbol{\Gamma}_h} \|\alpha - \boldsymbol{\delta}\|_0 + \|p - p_h\|) \\ &\le C(h\|\alpha - \boldsymbol{R}_h\boldsymbol{\alpha}\|_0 + \|p - p_h\|) \\ &\le C_\varepsilon k^{-m+\frac{5}{2}+\varepsilon}h^{\mu_1+1+(1+[\frac{\mu_1}{2}])(\alpha-1)}\|\boldsymbol{\alpha}\|_{m-2} + \|p - p_h\|_0 \\ &\le C_\varepsilon(1 + h^\alpha k^{5/2})^2 k^{-m+5/2+\varepsilon}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)} \;\|\|\|\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;\|\|\|, \end{aligned}$$

where $\mu_1 = \min(m-2, k)$. With (5.27), we get from this estimate and the Poincaré inequality for $\|r - r_h\|_0$ that

$$\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{-1} \le C_\varepsilon(1 + h^\alpha k^{5/2})^2 k^{-m+5/2+\varepsilon}h^{\mu+[\frac{\mu+2}{2}](\alpha-1)} \;\|\|\|\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;\|\|\|\; .$$

A summary of these estimates shows the inequality (7.20). □

Similarly, we have the following error estimates for Method 4 of Section 6.

Theorem 7.17. *Let* $(r, \boldsymbol{\phi}, p, \boldsymbol{\alpha}, \omega) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega) \times H_0^1(\Omega)$ *and* $(r_h, \boldsymbol{\phi}_h, p_h, \boldsymbol{\alpha}_h, \omega_h) \in W_h \times \boldsymbol{\Theta}_h \times \mathcal{Q}_h \times \boldsymbol{\Gamma}_h \times W_h$ *be the solutions of Problem* 5.1 *and Problem* 5.3, *respectively. Let the shear* $\boldsymbol{\gamma}$ *and the discrete shear force* $\boldsymbol{\gamma}_h$ *be defined in* (1.2) *and* (3.5), *respectively. Then, for Method 4:*

(I) *If* $\mu = m \le k+1$, *we have*

$$\|r - r_h\|_1 \le Ck^{-m+1}h^{\mu-1}\|r\|_m, \tag{7.22}$$

$$\begin{aligned} &\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_1 + \|\omega - \omega_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + t\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_0 \\ &+ \frac{k^{-1/2}}{1 + h^\alpha k^{\frac{5}{2}}}\|p - p_h\|_0 + \frac{k^{-1}}{1 + h^\alpha k^{\frac{5}{2}}}(\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{-1} + \|\boldsymbol{\alpha} - \boldsymbol{\alpha}\|_{-1}) \\ &\le C_{h,k}k^{-m+3/2}h^{\mu-1} \;\|\|\|\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;\|\|\|, \end{aligned} \tag{7.23}$$

$$\begin{aligned} &t^2(\|\operatorname{rot}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 + \|\operatorname{rot}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_h)\|_0) \\ &\le C_{h,k}k^{-m+3/2}h^{\mu+\alpha-2} \;\|\|\|\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;\|\|\|\; . \end{aligned} \tag{7.24}$$

(II) *If* $\mu = k+1 < m$, *we have*

$$\|r - r_h\|_1 \le Ck^{-m+1}h^s\|r\|_m, \tag{7.25}$$

$$\begin{aligned} &\|\|\boldsymbol{\phi}_h - \boldsymbol{\phi}, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|\| + \|\omega - \omega_h\|_1 + t\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_0 + t^2\|\operatorname{rot}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 \\ &+ \frac{k^{-1/2}}{1 + h^\alpha k^{\frac{5}{2}}}\|p - p_h\|_0 + \frac{k^{-1}}{1 + h^\alpha k^{\frac{5}{2}}}(\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{-1} + \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_{-1}) \\ &\le C_{h,k}k^{-m+3/2}h^s \;\|\|\|\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;\|\|\|, \end{aligned} \tag{7.26}$$

where $C_{h,k} = C(1 + h^\alpha k^{5/2})$ *and* $s = \mu + [\frac{\mu+1}{2}](\alpha - 1) - 1$.

Proof. A similar argument of Theorem 7.16 and using Lemma 7.15, Lemma 7.4, Lemma 7.8, and Lemma 7.11 instead can prove this result. For brevity, we omit the details. □

Remark 7.18. The estimate given in Theorem 7.17 is not optimal with respect to the degree k. Applying the technique of [29] and using the well-known *K-Method* [21], we can slightly improve these results, i.e., from $k^{-m+3/2}$ to $k^{-m+1+\varepsilon}$.

Remark 7.19. Note from Theorem 7.17 we can obtain optimal error estimates in energy norm with respect to the meshsize h provided that $m = k+1$ for Method 4.

Remark 7.20. If the mesh is mildly distorted in the sense $\alpha = 1$, then all of these methods can yield optimal convergence rates in energy norm with respect to the meshsize h. Usually, the meshes used in the practical computations satisfy this condition.

8. L^2 ERROR ESTIMATES

This section presents the L^2 error analysis for the methods proposed in Section 6. For brevity, we only give the details for Method 4. For completeness, we list the corresponding result for Method 1–Method 3, which can be established with a similar argument.

In order to obtain optimal L^2 error estimate for Method 4, we need the following result from [18].

Lemma 8.1. *Let $\boldsymbol{R}_h$ be defined by* (6.1) *and* (6.22) *for the R-T elements in Method 4. Assume that* $\operatorname{rot}\boldsymbol{\sigma} \in H^1(\Omega)$ *and* $k \geq 2$. *Then,*

$$\|\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})\|_0 \leq Chk^{-1}\|\operatorname{rot}\boldsymbol{\sigma}\|_1. \tag{8.1}$$

Proof. From the proof of Lemma 7.8 (see Lemma B.4), we can see that for any $K \in J^h$,

$$\int_K |\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})|^2\, dxdy \leq Ch^{-2}\int_{\hat K} |\widehat{\operatorname{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\sigma}})|^2\, d\xi d\eta,$$
$$\|\widehat{\operatorname{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat K}\bar{\boldsymbol{\sigma}})\|_{0,\hat K} \leq Ck^{-1}\inf_{\hat v\in Q_{k-1}(\hat K)}\|\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}} - \hat v\|_{1,\hat K}.$$

By Lemma 6.2, one has $\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}} = J_K\widehat{\operatorname{rot}\boldsymbol{\sigma}}$. Given $\hat v \in Q_{k-2}(\hat K)$, we have $J_K\hat v \in Q_{k-1}(\hat K)$. This observation leads to

$$\begin{aligned}
\inf_{\hat v\in Q_{k-1}(\hat K)}\|\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}} - \hat v\|_{1,\hat K} &\leq \inf_{\hat v\in Q_{k-2}(\hat K)}\|J_K(\widehat{\operatorname{rot}\boldsymbol{\sigma}} - \hat v)\|_{1,\hat K}\\
&\leq Ch^2\inf_{\hat v\in Q_{k-2}(\hat K)}\|\widehat{\operatorname{rot}\boldsymbol{\sigma}} - \hat v\|_{1,\hat K}\\
&\leq Ch^2|\widehat{\operatorname{rot}\boldsymbol{\sigma}}|_{1,\hat K} \leq Ch^2\|\operatorname{rot}\boldsymbol{\sigma}\|_{1,K}.
\end{aligned}$$

Substituting this inequality into the previous one and summing over all the elements completes the proof. □

To use the Aubin-Nitsche dual argument to derive the L^2 error estimate, we define an auxiliary problem:

Problem 8.2. Given $\boldsymbol{d} \in \boldsymbol{L}^2(\Omega)$ find $(\boldsymbol{\phi}_d, p_d, \boldsymbol{\alpha}_d) \in \boldsymbol{H}_0^1 \times L^2(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega)$ such that

$$\mathcal{A}(\boldsymbol{\psi}, \boldsymbol{\delta}; \boldsymbol{\phi}_d, \boldsymbol{\alpha}_d) + \mathcal{B}(p_d; \boldsymbol{\psi}, \boldsymbol{\delta}) = (\boldsymbol{d}, \boldsymbol{\psi}), \quad \forall(\boldsymbol{\psi}, \boldsymbol{\delta}) \in \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega), \tag{8.2}$$

$$\mathcal{B}(q; \boldsymbol{\phi}_d, \boldsymbol{\alpha}_d) = 0, \quad \forall q \in L^2(\Omega). \tag{8.3}$$

The solution to this problem admits the following regularity:

$$\|\boldsymbol{\phi}_d\|_2 + \|p_d\|_2 + \|\boldsymbol{\alpha}_d\|_0 + t\|\boldsymbol{\alpha}_d\|_1 + t^2\|\mathrm{rot}\,\boldsymbol{\alpha}_d\|_1 \le C\|\boldsymbol{d}\|_0. \tag{8.4}$$

Theorem 8.3. *Let* $(r, \boldsymbol{\phi}, p, \boldsymbol{\alpha}, \omega) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega) \times H_0^1(\Omega)$ *and* $(r_h, \boldsymbol{\phi}_h, p_h, \boldsymbol{\alpha}_h, \omega_h) \in W_h \times \boldsymbol{\Theta}_h \times \mathcal{Q}_h \times \boldsymbol{\Gamma}_h \times W_h$ *be the solutions of Problem* 5.1 *and Problem* 5.3*, respectively, then for Method 4, we have the error bounds:*

(I) *If* $\mu = m \le k+1$*, then,*

$$\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_0 + \|\omega - \omega_h\|_0 \le C_{h,k}^2 k^{-m+1} h^{\mu+\alpha-1} \, |||| \, (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \, |||| \, . \tag{8.5}$$

(II) *If* $\mu = k+1 < m$*, then,*

$$\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_0 + \|\omega - \omega_h\|_0 \le C_{h,k}^2 k^{-m+1} h^{s} \, |||| \, (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \, |||| \, . \tag{8.6}$$

Where $s = \mu + [\frac{\mu+1}{2}](\alpha - 1)$ *and* $C_{h,k} = C(1 + h^\alpha k^{5/2})$.

Proof. First, it is easy to show

$$\|r - r_h\|_0 \le Ck^{-m} h^{\mu} \|r\|_m, \text{ if } \mu = m \le k+1, \tag{8.7}$$

$$\|r - r_h\|_0 \le Ck^{-m} h^{\mu + [\frac{\mu+1}{2}](\alpha-1)} \|r\|_m, \text{ if } \mu = k+1 < m. \tag{8.8}$$

Taking $\boldsymbol{\psi} = \boldsymbol{\phi} - \boldsymbol{\phi}_h$ in (8.2) yields

$$\begin{aligned}
(\boldsymbol{d}, \boldsymbol{\phi} - \boldsymbol{\phi}_h) &= \mathcal{A}(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h; \boldsymbol{\phi}_d, \boldsymbol{\alpha}_d) + \mathcal{B}(p_d; \boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h) \\
&= \mathcal{A}(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h; \boldsymbol{\phi}_d - \boldsymbol{\Pi}_h \boldsymbol{\phi}_d, \boldsymbol{\alpha}_d - \boldsymbol{R}_h \boldsymbol{\alpha}_d) \\
&\quad + \mathcal{B}(p_d - \Pi_h p_d; \boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h) \\
&\quad + \mathcal{A}(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h; \boldsymbol{\Pi}_h \boldsymbol{\phi}_d, \boldsymbol{R}_h \boldsymbol{\alpha}_d) \\
&\quad + \mathcal{B}(\Pi p_d; \boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h).
\end{aligned} \tag{8.9}$$

We use (5.8), (5.16), $\mathcal{B}(p; \boldsymbol{\phi}_d, \boldsymbol{\alpha}_d) = 0$, and $\mathcal{B}(p_h; \boldsymbol{\phi}_d, \boldsymbol{\alpha}_d) = 0$, to derive as

$$\begin{aligned}
&\mathcal{A}(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h; \boldsymbol{\Pi}_h \boldsymbol{\phi}_d, \boldsymbol{R}_h \boldsymbol{\alpha}_d) \\
&\quad = (\nabla r, \boldsymbol{\Pi}_h \boldsymbol{\phi}_d) - (\nabla r_h, \boldsymbol{R}_h \boldsymbol{\Pi}_h \boldsymbol{\phi}_d) \\
&\qquad + \mathcal{B}(p_h - p; \boldsymbol{\Pi}_h \boldsymbol{\phi}_d, \boldsymbol{R}_h \boldsymbol{\alpha}_d) \\
&\quad = (\nabla r, \boldsymbol{\Pi}_h \boldsymbol{\phi}_d) - (\nabla r_h, \boldsymbol{R}_h \boldsymbol{\Pi}_h \boldsymbol{\phi}_d) \\
&\qquad + (\mathrm{rot}(\boldsymbol{\Pi}_h \boldsymbol{\phi}_d - \boldsymbol{\phi}_d), p - p_h) \\
&\qquad + \lambda^{-1} t^2 (\mathrm{rot}(\boldsymbol{R}_h \boldsymbol{\alpha}_d - \boldsymbol{\alpha}_d), p - p_h).
\end{aligned}$$

Using $\mathcal{B}(\Pi p_d;\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)=0$ and inserting this identity into (8.9) and applying a further decomposition for the term $(\nabla r,\boldsymbol{\Pi}_h\boldsymbol{\phi}_d)-(\nabla r_h,\boldsymbol{R}_h\boldsymbol{\Pi}_h\boldsymbol{\phi}_d)$, we obtain

(8.10)

$$\begin{aligned}(\boldsymbol{d},\boldsymbol{\phi}-\boldsymbol{\phi}_h) &= \mathcal{A}(\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h;\boldsymbol{\phi}_d-\boldsymbol{\Pi}_h\boldsymbol{\phi}_d,\boldsymbol{\alpha}_d-\boldsymbol{R}_h\boldsymbol{\alpha}_d)\\ &\quad+\mathcal{B}(p_d-\Pi_h p_d;\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)\\ &\quad+(\nabla(r-r_h),\boldsymbol{\Pi}_h\boldsymbol{\phi}_d)-(\nabla r,(I-\boldsymbol{R}_h)(I-\boldsymbol{\Pi}_h)\boldsymbol{\phi}_d)\\ &\quad+(\nabla r,(I-\boldsymbol{R}_h)\boldsymbol{\phi}_d)+(\nabla(r_h-r),(I-\boldsymbol{R}_h)\boldsymbol{\Pi}_h\boldsymbol{\phi}_d)\\ &\quad+(\operatorname{rot}(\boldsymbol{\Pi}_h\boldsymbol{\phi}_d-\boldsymbol{\phi}_d),p-p_h)+\lambda^{-1}t^2(\operatorname{rot}(\boldsymbol{R}_h\boldsymbol{\alpha}_d-\boldsymbol{\alpha}_d),p-p_h)\\ &= I_1+I_2+\cdots+I_8.\end{aligned}$$

We now bound the eight terms on the right-hand side of (8.10). There are two cases of which we need to take care.

(I) If $\mu=m\le k+1$. We use Theorem 7.17 and Lemma 7.4 and Lemma 7.11 and Remark 7.12 to obtain

$$\begin{aligned}|I_1| &= |\mathcal{A}(\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h;\boldsymbol{\phi}_d-\boldsymbol{\Pi}_h\boldsymbol{\phi}_d,\boldsymbol{\alpha}_d-\boldsymbol{R}_h\boldsymbol{\alpha}_d)|\\ &\le C(\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_1+t\|\boldsymbol{\alpha}-\boldsymbol{\alpha}_h\|_0)\\ &\qquad\times(\|\boldsymbol{\phi}_d-\boldsymbol{\Pi}_h\boldsymbol{\phi}_d\|_1+t\|\boldsymbol{\alpha}_d-\boldsymbol{R}_h\boldsymbol{\alpha}_d\|_0)\\ &\le C_{h,k}k^{-m+1}h^{\mu}\,|||\,(r,\boldsymbol{\phi},\boldsymbol{\alpha},p,\omega)\,|||\,\|d\|_0,\\ |I_2| &= |\mathcal{B}(p_d-\Pi_h p_d;\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)|\\ &\le C(\|\operatorname{rot}(\boldsymbol{\phi}-\boldsymbol{\phi}_h)\|_0+\|t^2\operatorname{rot}(\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)\|_0)\times\|p_d-\Pi_h p_d\|_0\\ &\le C_{h,k}k^{-m+1/2}h^{\mu+\alpha-1}\,|||\,(r,\boldsymbol{\phi},\boldsymbol{\alpha},p,\omega)\,|||\,\|d\|_0.\end{aligned}$$

Integrating by parts and using (8.7) leads to

$$\begin{aligned}|I_3| &= |(\nabla(r-r_h),\boldsymbol{\Pi}_h\boldsymbol{\phi}_d)| = (r-r_h,\operatorname{div}\boldsymbol{\Pi}_h\boldsymbol{\phi}_d)\\ &\le\|r-r_h\|_0\|\operatorname{div}\boldsymbol{\Pi}_h\boldsymbol{\phi}_d\|_0\le Ck^{-m}h^{\mu}\|r\|_m\|\boldsymbol{\phi}_d\|_2.\end{aligned}$$

Thanks to Lemma 7.15 and Lemma 7.11, we have

$$\begin{aligned}|I_4| = |(\nabla r,(I-\boldsymbol{R}_h)(I-\boldsymbol{\Pi}_h)\boldsymbol{\phi}_d)| &\le Ck^{-m+3/2}h^{\mu-1}\|r\|_m\|(I-\boldsymbol{\Pi}_h)\boldsymbol{\phi}_d\|_1\\ &\le Ck^{-m+1/2}h^{\mu}\|r\|_m\|\boldsymbol{\phi}_d\|_2.\end{aligned}$$

With the projection operator $\boldsymbol{\Pi}_{k-2}$ from Lemma 7.15, we have

$$(\nabla r,(I-\boldsymbol{R}_h)\boldsymbol{\phi}_d)=(\nabla r-\boldsymbol{\Pi}_{k-2}\nabla r,(I-\boldsymbol{R}_h)\boldsymbol{\phi}_d).$$

With a similar argument of Lemma 7.15, this and Lemma 7.4 give

$$\begin{aligned}|I_5| = |(\nabla r,(I-\boldsymbol{R}_h)\boldsymbol{\phi}_d)| &\le Ck^{-m+2}h^{\mu-2}\|\nabla r\|_{m-2}\|(I-\boldsymbol{R}_h)\boldsymbol{\phi}_d\|_0\\ &\le Ck^{-m+1/2}h^{\mu}\|\nabla r\|_{m-2}\|\boldsymbol{\phi}_d\|_2.\end{aligned}$$

Using (7.21) and Lemma 7.4, we get

$$|I_6| = |(\nabla(r_h-r),(I-\boldsymbol{R}_h)\boldsymbol{\Pi}_h\boldsymbol{\phi}_d)|\le Ck^{-m+1/2}h^{\mu}\|r\|_m\|\boldsymbol{\phi}_d\|_2.$$

The last two terms are bounded as, respectively,

$$\begin{aligned}|I_7| &= |(\operatorname{rot}(\boldsymbol{\Pi}_h\boldsymbol{\phi}_d-\boldsymbol{\phi}_d),p-p_h)|\\ &\le C_{h,k}^2k^{-m+2}h^{\mu-1}\,|||\,(r,\boldsymbol{\phi},\boldsymbol{\alpha},p,\omega)\,|||\,\|\operatorname{rot}(\boldsymbol{\Pi}_h\boldsymbol{\phi}_d-\boldsymbol{\phi}_d)\|_0\\ &\le C_{h,k}^2k^{-m+1}h^{\mu}\,|||\,(r,\boldsymbol{\phi},\boldsymbol{\alpha},p,\omega)\,|||\,\|\boldsymbol{\phi}_d\|_2\end{aligned}$$

and

$$\begin{aligned}|I_8| &= |\lambda^{-1}t^2(\operatorname{rot}(\boldsymbol{R}_h\boldsymbol{\alpha}_d - \boldsymbol{\alpha}_d), p - p_h)| \\ &\le C_{h,k}^2 k^{-m+1}h^{\mu}t^2 \ ||||\ (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega)\ ||||\ \|\operatorname{rot}\boldsymbol{\alpha}_d\|_1.\end{aligned}$$

Then, a summary of these inequalities shows

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 \le C_{h,k}^2 k^{-m+1}h^{\mu+\alpha-1}\ ||||\ (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega)\ ||||\ . \tag{8.11}$$

To get the L^2 estimate for the deflection ω, we need a new auxiliary problem. Let $z \in H_0^1(\Omega)$ and $z_h \in W_h$ be the solutions of

$$(\nabla z, \nabla s) = (\omega - \omega_h, s), s \in H_0^1(\Omega),$$

$$(\nabla z_h, \nabla s) = (\omega - \omega_h, s), s \in W_h,$$

respectively. Then, the standard error estimate and the regularity property give

$$\|z - z_h\|_s \le Ck^{-2+s}h^{2-s}\|\omega - \omega_h\|_0, s = 0, 1.$$

In addition, we have

$$\begin{aligned}(\omega - \omega_h, \omega - \omega_h) &= (\nabla z, \nabla(\omega - \omega_h)) \\ &= (\nabla(z - z_h), \nabla(\omega - \omega_h)) + (\nabla z_h, \nabla(\omega - \omega_h)).\end{aligned}$$

In view of (5.7) and (5.15), we use (5.3) and (5.11) to deduce

$$\begin{aligned}(\nabla z_h, \nabla(\omega - \omega_h)) &= (\boldsymbol{\phi} + \lambda^{-1}t^2\nabla r, \nabla z_h) - (\boldsymbol{R}_h\boldsymbol{\phi}_h + \lambda^{-1}t^2\nabla r_h, \nabla z_h) \\ &= (\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}_h, \nabla z_h) = (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \nabla z_h) + ((I - \boldsymbol{R}_h)\boldsymbol{\phi}_h, \nabla z_h) \\ &= (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \nabla z_h) + ((I - \boldsymbol{R}_h)(\boldsymbol{\phi}_h - \boldsymbol{\phi}), \nabla z_h) \\ &\quad + ((I - \boldsymbol{R}_h)\boldsymbol{\phi}, \nabla z_h - \nabla z) + ((I - \boldsymbol{R}_h)\boldsymbol{\phi}, \nabla z).\end{aligned}$$

For the different terms we obtain

$$\begin{aligned}|(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \nabla z_h)| &\le C_{h,k}^2 k^{-m+1}h^{\mu+\alpha-1}\ ||||\ (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega)\ ||||\ \|z\|_2, \\ |((I - \boldsymbol{R}_h)(\boldsymbol{\phi}_h - \boldsymbol{\phi}), \nabla z_h)| &\le Ck^{-1/2}h\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_1\|z\|_2 \\ &\le C_{h,k}k^{-m+1}h^{\mu}\ ||||\ (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega)\ ||||\ \|z\|_2, \\ |((I - \boldsymbol{R}_h)\boldsymbol{\phi}, \nabla z_h - \nabla z)| &\le Ck^{-m+3/2}h^{\mu-1}\|\boldsymbol{\phi}\|_{m-1}\|\nabla z_h - \nabla z\|_0 \\ &\le Ck^{-m+1/2}h^{\mu}\|\boldsymbol{\phi}\|_{m-1}\|z\|_2.\end{aligned}$$

With $\mathbf{\Pi}_{k-2}$ from Lemma 7.15, a similar argument of Lemma 7.15 shows

$$((I - \boldsymbol{R}_h)\boldsymbol{\phi}, \nabla z) = ((I - \boldsymbol{R}_h)\boldsymbol{\phi}, \nabla z - \mathbf{\Pi}_{k-2}\nabla z).$$

This yields

$$|((I - \boldsymbol{R}_h)\boldsymbol{\phi}, \nabla z)| \le Ck^{-m+1/2}h^{\mu}\|\boldsymbol{\phi}\|_{m-1}\|\nabla z\|_1. \tag{8.12}$$

A combination of these estimates gives

$$\|\omega - \omega_h\|_0 \le C_{h,k}^2 k^{-m+1}h^{\mu+\alpha-1}\ ||||\ (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega)\ ||||\ .$$

(II) If $\mu = k+1 < m$. The result for this case can be proved by proceeding along the line of the case with $\mu = m \le k + 1$. □

Similarly, we have the following L^2 error estimates for Method 1–Method 3.

Theorem 8.4. *Let* $(r, \boldsymbol{\phi}, p, \boldsymbol{\alpha}, \omega) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}, \Omega) \times H_0^1(\Omega)$ *and* $(r_h, \boldsymbol{\phi}_h, p_h, \boldsymbol{\alpha}_h, \omega_h) \in W_h \times \boldsymbol{\Theta}_h \times \mathcal{Q}_h \times \boldsymbol{\Gamma}_h \times W_h$ *be the solutions of Problem* 5.1 *and Problem* 5.3*, respectively, then for Method 1–Method 3, we have the error bounds:*

$$\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_0 + \|\omega - \omega_h\|_0 \leq C^2_{\varepsilon,h,k} k^{-m+1+\varepsilon} h^{s+\alpha} \;|||\; (r, \boldsymbol{\phi}, \boldsymbol{\alpha}, p, \omega) \;|||, \tag{8.13}$$

where $C_{\varepsilon,h,k} = C_\varepsilon(1 + h^\alpha k^{5/2})$, $s = \mu + [\frac{\mu+2}{2}](\alpha - 1)$ *with* $\mu = \min(m-1, k)$.

Remark 8.5. Note from the two theorems above that all of these methods converge optimally in L^2 norm with respect to the meshsize h provided that $\alpha = 1$.

Remark 8.6. For the limiting problem, a similar error estimate hold.

Appendix A. Error estimates of the reduction operator in L^2 norm

In this section, we will prove the L^2 error estimates of the reduction operator $\boldsymbol{R}_h$ presented in Lemma 7.1 and Lemma 7.4, respectively. For the readers' convenience, we will present them again in the following.

Throughout this section, we will use the following notation:

$$M = DF_K^{-T}, \tag{A.1}$$

$$[\hat{u}]_{m,\hat{K},\xi} = \|\frac{\partial^m \hat{u}}{\partial \xi^m}\|_{0,\hat{K}}, \quad [\hat{u}]_{m,\hat{K},\eta} = \|\frac{\partial^m \hat{u}}{\partial \eta^m}\|_{0,\hat{K}}.$$

With this notation, we have the following relation [17].

Lemma A.1. *Given* $K \in J^h$*, let* $v \in H^l(K)$ *and* $\hat{v} = v \circ F_K(\xi, \eta)$*. Then*

$$[\hat{v}]_{l,\hat{K},\xi} + [\hat{v}]_{l,\hat{K},\eta} \leq C h^{l-1} |v|_{l,K}. \tag{A.2}$$

with h *the diameter of* K.

Lemma A.2. *Given* $K \in J^h$*, let* $v \in H^l(K)$ *and* $\hat{v} = v \circ F_K(\xi, \eta)$*. Then,*

$$\begin{aligned} &|\,\hat{v}\,|^2_{l,\hat{K}} \leq C h^{-2} \sum_{j=0}^{[\frac{l+1}{2}]} h^{2l+2j\alpha-2j} \,|\, v\,|^2_{l-j,K}, \quad \textit{if } l \geq 2, \\ &|\,\hat{v}\,|^2_{l,\hat{K}} \leq C h^{2(l-1)} \|v\|^2_{l,K}, \quad \textit{if } l = 0, 1. \end{aligned} \tag{A.3}$$

h *is the diameter of* K*, and* α *is the mesh distortion parameter defined in Section 2. Given an integer* i*, we denote by* $[\frac{i+1}{2}]$ *the max integer not greater than* $\frac{i+1}{2}$.

Proof. Since F_K is a bilinear transformation, one can prove this result by a straightforward investigation. □

The following result is concerned with the error estimate of the interpolation operator $\widehat{\boldsymbol{R}}_{\hat{K}}$ for the BDFM elements in Method 1 from Section 6. Its proof can be found in Lemma 5.1 of [29].

Lemma A.3. *Let the reduction operator* $\widehat{\boldsymbol{R}}_{\hat{K}}$ *be defined in* (6.8) *with* $\bar{\boldsymbol{\sigma}} \in \boldsymbol{H}^{m-1}(\hat{K})$. *Then,*

$$\|\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}} \bar{\boldsymbol{\sigma}}\|_{0,\hat{K}} \leq C_\varepsilon k^{3/2+\varepsilon-m} \|\bar{\boldsymbol{\sigma}}\|_{m-1,\hat{K}}, \tag{A.4}$$

with C_ε *depending on* $0 < \varepsilon < m - 3/2$.

Lemma A.4 (Lemma 7.1 in Section 7)**.** *Let the reduction operator* $\boldsymbol{R}_h$ *be defined by* (6.1) *and* (6.8) *for the BDFM elements in Method 1 of Section 6. Assume* $\boldsymbol{\sigma} \in \boldsymbol{H}^{m-1}(\Omega)$*. Then,*

$$(A.5)\qquad \begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0 \le C_\varepsilon k^{-m+3/2+\varepsilon} h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)}\|\boldsymbol{\sigma}\|_{m-1}, \quad \mu > 2,\\ &\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0 \le C_\varepsilon k^{-m+3/2+\varepsilon} h^{\mu-1+\alpha}\|\boldsymbol{\sigma}\|_{m-1}, \quad \mu = 1, 2, \end{aligned}$$

where $\mu = (m-1, k)$ *and* C_ε *depends on* $0 < \varepsilon < m - 3/2$*.*

Proof. Thanks to the inequality (A.4), we have, for any $\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})$,

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_{0,K} \quad &\le \quad C\|\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}}\|_{0,\hat{K}} = C\|(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}) - \widehat{\boldsymbol{R}}_{\hat{K}}(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}})\|_{0,\hat{K}}\\ &\le \quad C_\varepsilon k^{-m+3/2+\varepsilon}\|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}}. \end{aligned}$$

This leads to

$$(A.6)\qquad \|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_{0,K} \le C_\varepsilon k^{-m+3/2+\varepsilon} \inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}}.$$

We now estimate the term on the right-hand side of (A.6). There are two cases of which we need to take care.

(I) We first consider the case where $m - 1 < \mu + 1$. For this case, one has $\mu = m - 1 \le k$. Then, it follows from the definition of the BDFM elements (see Method 1 of Section 6),

$$(A.7)\qquad \inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}} = \inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{\mu,\hat{K}} \le C|\bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}}.$$

Since $\bar{\boldsymbol{\sigma}} = M^{-1}\hat{\boldsymbol{\sigma}} = M^{-1}\boldsymbol{\sigma} \circ F_K(\xi, \eta)$, one can use the expression of M^{-1} (see Section 2, also Lemma A.6 below) and the estimates from (2.2) for c_1, c_{12}, d_1 and d_{12} to get

$$(A.8)\qquad |\bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}} \le C(h|\hat{\boldsymbol{\sigma}}|_{\mu,\hat{K}} + h^{1+\alpha}|\hat{\boldsymbol{\sigma}}|_{\mu-1,\hat{K}}).$$

Thanks to Lemma A.2, this leads to

$$(A.9)\qquad \begin{aligned} &|\bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}} \le Ch^{\mu+(1+[\frac{\mu}{2}])(\alpha-1))}\|\boldsymbol{\sigma}\|_{\mu,K}, \text{ if } \mu \ge 3,\\ &|\bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}} \le Ch^{\mu-1+\alpha}\|\boldsymbol{\sigma}\|_{\mu,K}, \text{ if } \mu = 1, 2. \end{aligned}$$

A combination of (A.6)-(A.9) proves the result for this case.

(II) We now study the case with $m - 1 \ge \mu + 1$. For this case, we have $\mu = k < m - 1$.

(A.10)

$$\inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}} \le \inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} (\|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{\mu,\hat{K}} + \sum_{i=\mu+1}^{m-1} |\ \bar{\boldsymbol{\sigma}}\ |_{i,\hat{K}}) \le C \sum_{i=\mu}^{m-1} |\ \bar{\boldsymbol{\sigma}}\ |_{i,\hat{K}}\ .$$

With $\bar{\boldsymbol{\sigma}} = M^{-1}\hat{\boldsymbol{\sigma}}$ and the estimates from (2.2) for c_1, c_{12}, d_1 and d_{12}, one can show that

$$(A.11)\qquad \sum_{i=\mu}^{m-1} |\ \bar{\boldsymbol{\sigma}}\ |_{i,\hat{K}} \le C(h \sum_{i=\mu}^{m-1} |\ \hat{\boldsymbol{\sigma}}\ |_{i,\hat{K}} + h^{1+\alpha} \sum_{i=\mu-1}^{m-2} |\ \hat{\boldsymbol{\sigma}}\ |_{i,\hat{K}}).$$

Thanks to Lemma A.2, we have

$$(A.12)\qquad \begin{aligned} &|\bar{\boldsymbol{\sigma}}|_{i,\hat{K}} \le Ch^{i+(1+[\frac{i}{2}])(\alpha-1)}\|\boldsymbol{\sigma}\|_{m-1,K}, \text{ if } i \ge 3,\\ &|\bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}} \le Ch^{i-1+\alpha}\|\boldsymbol{\sigma}\|_{m-1,K}, \text{ if } i = 1, 2. \end{aligned}$$

Substituting (A.12) and (A.10) into (A.6) completes the proof for this case. □

The following result is concerned with the p-type error estimate of the R-T elements on the reference element $\hat{K}$; see, for instance, [30, 31, Lemma 3.1].

Lemma A.5. *Let the reduction operator* $\widehat{\boldsymbol{R}}_{\hat{K}}$ *be defined in* (6.22) *with* $\bar{\boldsymbol{\sigma}} \in \boldsymbol{H}^{m-1}(\hat{K})$. *Then,*

$$\|\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}}\|_{0,\hat{K}} \leq Ck^{3/2-m}\|\bar{\boldsymbol{\sigma}}\|_{m-1,\hat{K}}, \tag{A.13}$$

with k the degree of polynomials.

We have the following error estimates of the reduction operator $\boldsymbol{R}_h$ for the R-T elements.

Lemma A.6 (Lemma 7.4 in Section 7)**.** *Let* $\boldsymbol{\sigma} \in \boldsymbol{H}^{m-1}(\Omega)$. *The reduction operator* $\boldsymbol{R}_h$ *is defined by* (6.1) *and* (6.22) *for the R-T elements in Method 4 of Section 6. Then,*

$$\begin{aligned} &\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0 \leq Ck^{-m+3/2}h^{\mu}\|\boldsymbol{\sigma}\|_{m-1}, \quad \text{if } m-1 < \mu+1, \\ &\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0 \leq Ck^{-m+3/2}h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)}\|\boldsymbol{\sigma}\|_{m-1}, \quad \text{if } m-1 \geq \mu+1, \end{aligned} \tag{A.14}$$

with $\mu = \min(m-1, k)$.

Proof. Thanks to the inequality (A.13), we have, for any $\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})$,

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_{0,K} &\leq C\|\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_h\bar{\boldsymbol{\sigma}}\|_{0,\hat{K}} = C\|(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}) - \widehat{\boldsymbol{R}}_h(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}})\|_{0,\hat{K}} \\ &\leq Ck^{-m+3/2}\|(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}})\|_{m-1,\hat{K}}. \end{aligned}$$

This leads to

$$\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_{0,K} \leq Ck^{-m+3/2} \inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|(\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}})\|_{m-1,\hat{K}}. \tag{A.15}$$

We now estimate the term on the right-hand side of (A.15). There are two cases of which we need to take care.

(I) We first consider the case where $m-1 < \mu+1$. For this case, one has $\mu = m-1 \leq k$. Then, it follows from the definition of the R-T elements (see Method 4 of Section 6),

$$\begin{aligned} \inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}} &= \inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{\mu,\hat{K}} \\ &\leq C([\bar{\boldsymbol{\sigma}}_1]_{\mu,\hat{K},\xi} + [\bar{\boldsymbol{\sigma}}_2]_{\mu,\hat{K},\eta}), \end{aligned} \tag{A.16}$$

where $\bar{\boldsymbol{\sigma}}_1$ and $\bar{\boldsymbol{\sigma}}_2$ are two components of $\bar{\boldsymbol{\sigma}} = M^{-1}\hat{\boldsymbol{\sigma}}$ with $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ F_K(\xi, \eta)$. Since two components in the first row of matrix

$$M^{-1} = \begin{pmatrix} c_1 + c_{12}\eta & d_1 + d_{12}\eta \\ c_2 + c_{12}\xi & d_2 + d_{12}\xi \end{pmatrix}$$

are linear functions with respect to η and constant functions of ξ, and two components in the second row are linear functions with respect to ξ and constant functions of η, we can obtain, by the estimates (2.2) for the mesh parameters c_1, c_{12}, d_1 and d_{12}, that

$$[\bar{\boldsymbol{\sigma}}_1]_{\mu,\hat{K},\xi} + [\bar{\boldsymbol{\sigma}}_2]_{\mu,\hat{K},\eta} \leq Ch([\hat{\boldsymbol{\sigma}}]_{\mu,\hat{K},\xi} + [\hat{\boldsymbol{\sigma}}]_{\mu,\hat{K},\eta}) \leq Ch^{\mu}\|\boldsymbol{\sigma}\|_{\mu,K}.$$

In the last inequality, we use Lemma A.1 with $l = \mu$. Inserting this inequality into (A.16), we get

$$\inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}} \leq Ch^{\mu} \|\boldsymbol{\sigma}\|_{m-1,K}. \tag{A.17}$$

With (A.15), this proves the first inequality in (A.14).

(II) We now study the case with $m - 1 \geq \mu + 1$. For this case, we have $\mu = k < m - 1$,

$$\inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}} \leq \inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} (\|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{\mu,\hat{K}} + \sum_{i=\mu+1}^{m-1} |\, \bar{\boldsymbol{\sigma}} \,|_{i,\hat{K}} + \sum_{i=\mu+1}^{m-1} |\, \bar{\boldsymbol{v}} \,|_{i,\hat{K}}). \tag{A.18}$$

We take $\bar{\boldsymbol{v}}$ as the L^2 projection of $\bar{\boldsymbol{\sigma}}$ onto the space $P_{\mu-1}(\hat{K})^2 \subset \boldsymbol{\Gamma}_k(\hat{K})$ in (A.18). This leads to

$$\inf_{\bar{\boldsymbol{v}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}} \leq C \sum_{i=\mu}^{m-1} |\, \bar{\boldsymbol{\sigma}} \,|_{i,\hat{K}} \,. \tag{A.19}$$

Using the fact that the four components of the matrix M^{-1} are linear functions of ξ and η as in the first part of the proof, and the estimates for c_1, c_{12}, d_1 and d_{12}, one can prove

$$\sum_{i=\mu}^{m-1} |\, \bar{\boldsymbol{\sigma}} \,|_{i,\hat{K}} \leq C(h \sum_{i=\mu}^{m-1} |\, \hat{\boldsymbol{\sigma}} \,|_{i,\hat{K}} + h^{1+\alpha} \sum_{i=\mu-1}^{m-2} |\, \hat{\boldsymbol{\sigma}} \,|_{i,\hat{K}}). \tag{A.20}$$

Combining inequalities (A.18)-(A.20) with (A.3) leads to

$$\inf_{\bar{\boldsymbol{\sigma}} \in \boldsymbol{\Gamma}_k(\hat{K})} \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{v}}\|_{m-1,\hat{K}} \leq Ch^{\mu + (1 + [\frac{\mu}{2}])(\alpha - 1)} \|\boldsymbol{\sigma}\|_{m-1,K}. \tag{A.21}$$

Substituting (A.21) into (A.15) proves the second inequality of (A.14). □

Appendix B. Error estimates of the reduction operator in $H(\mathrm{rot})$ norm

This section presents the proof of the $H(\mathrm{rot})$ error estimates given in Lemma 7.6 and Lemma 7.8 for the reduction operator $\boldsymbol{R}_h$. They will be presented in Lemma B.2 and Lemma B.4, respectively. We also provide the proof of Lemma 7.10 and Lemma 7.11 for the H^1-projection operators $\boldsymbol{\Pi}_h$. For the readers' convenience, we will recall them in Lemma B.5 and Lemma B.6, respectively.

For the error estimates for the BDFM elements and BDM elements, we need the following result [8, 7].

Lemma B.1. *We have*

$$\inf_{\hat{v}_k \in P_{k-1}(\hat{K})} \|\hat{u} - \hat{v}_k\|_{s,\hat{K}} \leq Ck^{-m+1+s} \|\hat{u}\|_{m-1,\hat{K}}, \quad s = 0, 1. \tag{B.1}$$

For the BDFM elements in Method 1 of Section 6, we have

Lemma B.2 (Lemma 7.6 in Section 7)**.** *Let the reduction operator $\boldsymbol{R}_h$ be defined by* (6.1) *and* (6.8)*, and $\boldsymbol{\sigma} \in \boldsymbol{H}^{m-1}(\mathrm{rot}, \Omega)$. Then,*

$$\|\operatorname{rot} \boldsymbol{\sigma} - \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\sigma}\|_0 \leq Ck^{-m+1} h^{\mu + (1 + [\frac{\mu}{2}])(\alpha - 1)} \|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot})}, \quad \mu \geq 3, \tag{B.2}$$

$$\|\operatorname{rot} \boldsymbol{\sigma} - \operatorname{rot} \boldsymbol{R}_h \boldsymbol{\sigma}\|_0 \leq Ck^{-m+1} h^{\mu - 1 + \alpha} \|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot})}, \quad \mu = 1, 2, \tag{B.3}$$

where $\mu = \min(m - 1, k)$.

Proof. Set $w = \operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})$. Given $K \in J^h$, it follows from Lemma 6.2 that

$$\begin{aligned}\int_K |\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})|^2\, dxdy &= \int_K \operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}) w dxdy \\ &= \int_{\hat{K}} \widehat{\operatorname{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\hat{w} d\xi d\eta \leq \|\widehat{\operatorname{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\|_{0,\hat{K}} \|\hat{w}\|_{0,\hat{K}}.\end{aligned} \tag{B.4}$$

Since $\|\hat{w}\|_{0,\hat{K}} \leq Ch^{-1}\|w\|_{0,K}$, this yields

$$\int_K |\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})|^2\, dxdy \leq Ch^{-2}\int_{\hat{K}} |\widehat{\operatorname{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})|^2\, d\xi d\eta. \tag{B.5}$$

To bound the right-hand side of (B.5), we note from the definition of $\widehat{\boldsymbol{R}}_{\hat{K}}$ (6.8) that

$$\int_{\hat{K}} \widehat{\operatorname{rot}}(\hat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}})\hat{w} d\xi d\eta = 0, \quad \forall \hat{w} \in P_{k-1}(\hat{K}). \tag{B.6}$$

Since $\widehat{\operatorname{rot}}\, \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} \in P_{k-1}(\hat{K})$, this implies that $\widehat{\operatorname{rot}}\, \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}}$ is the projection of $\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}}$ onto $P_{k-1}(\hat{K})$. Let $\hat{\Pi}_{k-1}$ denote this projection operator. This means

$$\widehat{\operatorname{rot}}\, \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} = \hat{\Pi}_{k-1}\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}}.$$

Given $\hat{v} \in P_{k-1}(\hat{K})$ and $\hat{w} \in P_{k-1}(\hat{K})$, we have

$$\|\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}} - \hat{\Pi}_{k-1}\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}}\|_0 = \|\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}} - \hat{v} - \hat{w} - \hat{\Pi}_{k-1}(\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}} - \hat{v} - \hat{w})\|_0.$$

This observation and (B.1) lead to

$$\|\widehat{\operatorname{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\|_{0,\hat{K}} \leq Ck^{-m+1} \inf_{\hat{v} \in P_{k-1}(\hat{K})} \|\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}} - \hat{v}\|_{m-1,\hat{K}}. \tag{B.7}$$

We remain to estimate the right-hand side of (B.7). We need to take care of two cases.

(I) We first consider the case $\mu = m - 1 \leq k$. Then,

$$\inf_{\hat{v} \in P_{k-1}(\hat{K})} \|\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}} - \hat{v}\|_{m-1,\hat{K}} \leq C|\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}}. \tag{B.8}$$

In view of $\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}} = J_K\widehat{\operatorname{rot}\boldsymbol{\sigma}}$ from Lemma 6.2 and the fact $J_K = J_{0,K} + J_{1,K}\xi + J_{2,K}\eta$, we get, for any $i + j \geq 1$,

$$\frac{\partial^{i+j}\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}}}{\partial\xi^i\partial\eta^j} = iJ_{1,K}\frac{\partial^{i-1+j}\widehat{\operatorname{rot}\boldsymbol{\sigma}}}{\partial\xi^{i-1}\partial\eta^j} + jJ_{2,K}\frac{\partial^{i+j-1}\widehat{\operatorname{rot}\boldsymbol{\sigma}}}{\partial\xi^i\partial\eta^{j-1}} + J_K\frac{\partial^{i+j}\widehat{\operatorname{rot}\boldsymbol{\sigma}}}{\partial\xi^i\partial\eta^j}. \tag{B.9}$$

This and the estimates for J_K, $J_{1,K}$ and $J_{2,K}$ from (2.3) yield

$$|\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}} \leq Ch^{2+\alpha}|\widehat{\operatorname{rot}\boldsymbol{\sigma}}|_{\mu-1,\hat{K}} + Ch^2|\widehat{\operatorname{rot}\boldsymbol{\sigma}}|_{\mu,\hat{K}}. \tag{B.10}$$

An application of Lemma A.2 leads to

$$\begin{aligned}&|\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}} \leq Ch^{\mu+\alpha+[\frac{\mu}{2}](\alpha-1)}\|\operatorname{rot}\boldsymbol{\sigma}\|_{\mu,K}, \text{ if } \mu \geq 3, \\ &|\widehat{\operatorname{rot}}\, \bar{\boldsymbol{\sigma}}|_{\mu,\hat{K}} \leq Ch^{\mu+\alpha}, \text{ if } \mu = 1, 2.\end{aligned} \tag{B.11}$$

Combining (B.5), (B.7), (B.8), and (B.11) ends the proof for this case.

(II) We now study the case $\mu = k < m-1$. For this case, we have

(B.12)
$$\inf_{\hat{v}\in P_{k-1}(\hat{K})} \|\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}} - \hat{v}\|_{m-1,\hat{K}} \le \inf_{\hat{v}\in P_{k-1}(\hat{K})} (\|\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}} - \hat{v}\|_{\mu,\hat{K}} + \sum_{i=\mu+1}^{m-1} |\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}}|_{i,\hat{K}}) \le C \sum_{i=\mu}^{m-1} |\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}}|_{i,\hat{K}}.$$

Applying (B.9) and (2.3) and Lemma A.2 leads to

(B.13)
$$\sum_{i=\mu}^{m-1} |\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}}|_{i,\hat{K}} \le Ch^{\mu+\alpha+[\frac{\mu}{2}](\alpha-1)} \|\operatorname{rot}\boldsymbol{\sigma}\|_{m-1,K}, \text{ if } \mu \ge 3,$$
$$\sum_{i=\mu}^{m-1} |\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}}|_{i,\hat{K}} \le Ch^{\mu+\alpha} \|\operatorname{rot}\boldsymbol{\sigma}\|_{m-1,K}, \text{ if } \mu = 1,2.$$

Inserting (B.12) and (B.13) into (B.5) then completes the proof. □

Next, we have the following result [8, 7].

Lemma B.3. *We have*

(B.14)
$$\inf_{\hat{v}_k\in Q_{k-1}(\hat{K})} \|\hat{u} - \hat{v}_k\|_{s,\hat{K}} \le Ck^{-m+1+s} \|\hat{u}\|_{m-1,\hat{K}}, s = 0, 1.$$

For the R-T elements, we have the following error estimates.

Lemma B.4 (Lemma 7.8 in Section 7)**.** *Let* $\operatorname{rot}\boldsymbol{\sigma} \in H^{m-1}(\Omega)$ *and* $\boldsymbol{R}_h$ *be defined by* (6.1) *and* (6.22) *for the R-T elements. Then,*

(B.15)
$$\|\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})\|_0 \le Ck^{-m+1}h^{\mu+\alpha-1}\|\boldsymbol{\sigma}\|_{H^{m-1}(\operatorname{rot})}, m-1 < \mu+1,$$
$$\|\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})\|_0 \le Ck^{-m+1}h^{\mu+(1+[\frac{\mu}{2}])(\alpha-1)}\|\boldsymbol{\sigma}\|_{H^{m-1}(\operatorname{rot})}, m-1 \ge \mu+1,$$

with $\mu = \min(m-1, k)$.

Proof. First, a similar argument at the very beginning of Lemma 7.6 proves

(B.16)
$$\int_K |\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})|^2\, dxdy \le Ch^{-2} \int_{\hat{K}} |\widehat{\operatorname{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})|^2\, d\xi d\eta.$$

In what follows, we will bound the right-hand side in (B.16), there are two cases of which we need to take care.

(I) We first consider the case $m-1 < \mu+1$, which implies $\mu = m-1 \le k$. By the definition of $\hat{\boldsymbol{R}}_{\hat{K}}$ defined in (6.22), it is easy to see that

(B.17)
$$\int_{\hat{K}} \widehat{\operatorname{rot}}(\hat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}})\hat{w}\, d\xi d\eta = 0, \quad \forall \hat{w} \in Q_{k-1}(\hat{K}).$$

Since $\widehat{\operatorname{rot}}\,\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}} \in Q_{k-1}(\hat{K})$, this implies that $\widehat{\operatorname{rot}}\,\widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}}$ is the projection of $\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}}$ onto $Q_{k-1}(\hat{K})$. Similar arguments in Lemma 7.6 and (B.14) yield

(B.18)
$$\|\widehat{\operatorname{rot}}\,(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\|_{0,\hat{K}} \le Ck^{-m+1} \inf_{\hat{v}\in Q_{k-1}(\hat{K})} \|\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}} - \hat{v}\|_{m-1,\hat{K}}$$
$$= Ck^{-m+1} \inf_{\hat{v}\in Q_{k-1}(\hat{K})} \|\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}} - \hat{v}\|_{\mu,\hat{K}}$$
$$\le Ck^{-m+1}([\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}}]_{\mu,\hat{K},\xi} + [\widehat{\operatorname{rot}}\,\bar{\boldsymbol{\sigma}}]_{\mu,\hat{K},\eta}).$$

Thanks to Lemma 6.2, we have

$$\widehat{\mathrm{rot}},\bar{\boldsymbol{\sigma}} = J_K\widehat{\mathrm{rot}\,\boldsymbol{\sigma}}.$$

An application of (B.9) with the estimates of $J_{1,K}$ and $J_{2,K}$ from (2.3) leads to

$$[\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}]_{\mu,\hat{K},\xi} \leq Ch^{2+\alpha}[\widehat{\mathrm{rot}\,\boldsymbol{\sigma}}]_{\mu-1,\hat{K},\xi} + Ch^2[\widehat{\mathrm{rot}\,\boldsymbol{\sigma}}]_{\mu,\hat{K},\xi}.$$

$$[\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}]_{\mu,\hat{K},\eta} \leq Ch^{2+\alpha}[\widehat{\mathrm{rot}\,\boldsymbol{\sigma}}]_{\mu-1,\hat{K},\eta} + Ch^2[\widehat{\mathrm{rot}\,\boldsymbol{\sigma}}]_{\mu,\hat{K},\eta}.$$

Using Lemma A.1 with $l = \mu - 1$ and $l = \mu$, respectively, we get from the above two inequalities that

$$\text{(B.19)} \qquad [\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}]_{\mu,\hat{K},\xi} + [\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}]_{\mu,\hat{K},\eta} \leq Ch^{\mu+\alpha}\|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot},K)}.$$

Inserting (B.19) and (B.18) into (B.16) proves the first inequality.

(II) We now turn to the case $m-1 \geq \mu+1$. This means $\mu = k < m-1$. It follows from (B.14) and (B.17) that

(B.20)

$$\|\widehat{\mathrm{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\|_{0,\hat{K}} \leq Ck^{-m+1} \inf_{\hat{v}\in Q_{k-1}(\hat{K})} \|\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}} - \hat{v}\|_{m-1,\hat{K}}$$

$$\leq Ck^{-m+1} \inf_{\hat{v}\in Q_{k-1}(\hat{K})} (\|\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}} - \hat{v}\|_{\mu,\hat{K}} + \sum_{i=\mu+1}^{m-1} |\,\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}\,|_{i,\hat{K}} + \sum_{i=\mu+1}^{m-1} |\,\widehat{\mathrm{rot}}\,\hat{v}\,|_{i,\hat{K}}).$$

Let $\hat{v}$ be the projection of $\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}$ onto the space $P_{k-1}(\hat{K}) \subset Q_{k-1}(\hat{K})$ such that we get

$$\text{(B.21)} \qquad \|\widehat{\mathrm{rot}}(\bar{\boldsymbol{\sigma}} - \widehat{\boldsymbol{R}}_{\hat{K}}\bar{\boldsymbol{\sigma}})\|_{0,\hat{K}} \leq Ck^{-m+1}\sum_{i=\mu}^{m-1} |\,\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}\,|_{i,\hat{K}}\ .$$

Using $\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}} = J_K\widehat{\mathrm{rot}\,\boldsymbol{\sigma}}$, (2.3) and (B.9) again, one can prove

$$\text{(B.22)} \qquad \sum_{i=\mu}^{m-1} |\,\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}\,|_{i,\hat{K}} \quad \leq \quad Ch^2(\sum_{i=\mu}^{m-1} |\,\widehat{\mathrm{rot}\,\boldsymbol{\sigma}}\,|_{i,\hat{K}} + h^\alpha \sum_{i=\mu-1}^{m-2} |\,\widehat{\mathrm{rot}\,\boldsymbol{\sigma}}\,|_{i,\hat{K}}).$$

Thanks to Lemma A.2, this leads to

$$\text{(B.23)} \qquad \sum_{i=\mu}^{m-1} |\,\widehat{\mathrm{rot}}\,\bar{\boldsymbol{\sigma}}\,|_{i,\hat{K}} \leq Ch^{\mu+1+(1+[\frac{\mu}{2}])(\alpha-1)}\|\boldsymbol{\sigma}\|_{H^{m-1}(\mathrm{rot},K)}.$$

Substituting (B.21) and (B.23) into (B.16) completes the proof of the second inequality of (B.15) □

In the following, we present the error analysis for the H^1-projection operators $\boldsymbol{\Pi}_h$.

Lemma B.5. *Let the discrete rotation space $\boldsymbol{\Theta}_h$ be defined in Method 1, or in Method 2, or in Method 3, of Section 6, and let $\boldsymbol{\Pi}_h$ be the usual H^1-projection operator from $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ to $\boldsymbol{\Theta}_h$. Then,*

$$\text{(B.24)} \qquad \|\boldsymbol{\Pi}_h\boldsymbol{\psi} - \boldsymbol{\psi}\|_s \leq Ck^{-m+s}h^{\mu+[\frac{1+\mu}{2}](\alpha-1)-s}\|\boldsymbol{\psi}\|_m,$$

for any $\boldsymbol{\psi} \in \boldsymbol{H}^m(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ with $m \geq 2$. Where $s = 0, 1$, $\mu = \min(k+1, m)$.

Proof. Since the case with $s = 0$ can be proved in a similar way, we only consider the case of $s = 1$ for Method 1. Let $\hat{\mathbf{\Pi}}$ denote the corresponding interpolation operator on the reference element $\hat{K}$ [7], [8]. Then, we have

$$\begin{aligned} |\mathbf{\Pi}_h\boldsymbol{\psi} - \boldsymbol{\psi}|_1^2 &= \sum_{K\in J^h}\int_K |\nabla(\mathbf{\Pi}_h\boldsymbol{\psi} - \boldsymbol{\psi})|^2 dxdy \\ &\le C\sum_{K\in J^h} |\hat{\mathbf{\Pi}}\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{\psi}}|_{1,\hat{K}}^2. \end{aligned} \tag{B.25}$$

Given any $\hat{\boldsymbol{v}} \in \mathbf{\Theta}_k(\hat{K})$ and $\hat{\boldsymbol{w}} \in \mathbf{\Theta}_k(\hat{K})$, one has,

$$|\hat{\mathbf{\Pi}}\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{\psi}}|_{1,\hat{K}} = |\hat{\mathbf{\Pi}}(\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}) - (\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}})|_{1,\hat{K}} = |\hat{\mathbf{\Pi}}(\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}) - (\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})|_{1,\hat{K}}. \tag{B.26}$$

This, together with (B.1) and the construction of the space $\mathbf{\Theta}_k(\hat{K})$, leads to

$$\begin{aligned} &|\hat{\mathbf{\Pi}}\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{\psi}}|_{1,\hat{K}} \\ &\quad \le Ck^{-m+1}\inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{m,\hat{K}}, \quad \text{for any } \hat{\boldsymbol{v}} \in \mathbf{\Theta}_k(\hat{K}). \end{aligned} \tag{B.27}$$

(I) If $\mu = m \le k+1$, then

$$\inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{m,\hat{K}} = \inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} (\|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{\mu,\hat{K}}) \le C|\hat{\boldsymbol{\psi}}|_{\mu,\hat{K}}. \tag{B.28}$$

An application of Lemma A.2 yields

$$\inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{m,\hat{K}} \le Ch^{\mu-1+[\frac{\mu+1}{2}](\alpha-1)}\|\boldsymbol{\psi}\|_{\mu,K}. \tag{B.29}$$

Inserting (B.27) and(B.29) into (B.25) proves the result for this case.

(II) If $\mu = k+1 < m$. For this case, we have

$$\inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{m,\hat{K}} \le \inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} (\|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{\mu,\hat{K}} + \sum_{i=\mu+1}^{m} |\hat{\boldsymbol{\psi}}|_{i,\hat{K}} + \sum_{i=\mu+1}^{m} |\hat{\boldsymbol{v}}|_{i,\hat{K}}). \tag{B.30}$$

We take $\hat{\boldsymbol{v}}$ as the projection of $\hat{\boldsymbol{v}}$ onto the space $(P_{\mu-1}(\hat{K}))^2 \subset \mathbf{\Theta}_k(\hat{K})$. This gives

$$\inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{m,\hat{K}} \le C\sum_{i=\mu}^{m} |\hat{\boldsymbol{\psi}}|_{i,\hat{K}}. \tag{B.31}$$

Thanks to Lemma A.2, we get

$$\inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{m,\hat{K}} \le Ch^{\mu-1+[\frac{\mu+1}{2}](\alpha-1)}\|\boldsymbol{\psi}\|_{m,K}. \tag{B.32}$$

A combination of (B.25), (B.27) and (B.32) completes the proof. □

Lemma B.6. *Let the discrete rotation space $\mathbf{\Theta}_h$ be defined in Method 4 of Section 6, and let $\mathbf{\Pi}_h$ be the usual H^1-projection operator from $\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ to $\mathbf{\Theta}_h$. Then,*

$$\begin{aligned} &\|\mathbf{\Pi}_h\boldsymbol{\psi} - \boldsymbol{\psi}\|_s \le Ck^{-m+s}h^{\mu-s}\|\boldsymbol{\psi}\|_m, \ \text{if } m < \mu+1, \\ &\|\mathbf{\Pi}_h\boldsymbol{\psi} - \boldsymbol{\psi}\|_s \le Ck^{-m+s}h^{\mu+[\frac{1+\mu}{2}](\alpha-1)-s}\|\boldsymbol{\psi}\|_m, \ \text{if } m \ge \mu+1, \end{aligned} \tag{B.33}$$

for any $\boldsymbol{\psi} \in \boldsymbol{H}^m(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ with $m \ge 2$, $\mu = \min(m, k+1)$ and $s = 0, 1$.

Proof. We only give the proof for the case of $s = 1$. First, proceeding along the same line at the beginning of Lemma 7.10, we can show

$$\begin{aligned} |\mathbf{\Pi}_h\boldsymbol{\psi} - \boldsymbol{\psi}|_1^2 &= \sum_{K\in J^h}\int_K |\nabla(\mathbf{\Pi}_h\boldsymbol{\psi} - \boldsymbol{\psi})|^2 dxdy \\ &\leq C\sum_{K\in J^h} |\hat{\mathbf{\Pi}}\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{\psi}}|_{1,\hat{K}}^2 \\ &\leq Ck^{-2m+2}\sum_{K\in J^h}\inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{m,\hat{K}}^2. \end{aligned} \tag{B.34}$$

(I) If $\mu = m < \mu + 1 \leq k + 2$. Since $Q_k(\hat{K})^2 \subset \mathbf{\Theta}_k(\hat{K})$, we apply Lemma A.1 to deduce

$$\begin{aligned} \inf_{\hat{\boldsymbol{v}}\in\mathbf{\Theta}_k(\hat{K})} \|\hat{\boldsymbol{\psi}} - \hat{\boldsymbol{v}}\|_{m,\hat{K}} &\leq C([\hat{\boldsymbol{\psi}}]_{\mu,\hat{K},\xi} + [\hat{\boldsymbol{\psi}}]_{\mu,\hat{K},\eta}) \\ &\leq Ch^{\mu-1}|\boldsymbol{\psi}|_{\mu,K}. \end{aligned} \tag{B.35}$$

A substitution of (B.35) into (B.34) proves the result for this case.

(II) If $\mu = k + 1 < m$. For this case, one can prove the result by using a similar argument as in the second part of Lemma 7.10.

A summary of the two parts shows the assertion. □

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