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EVALUATING JACQUET'S GL(n) WHITTAKER FUNCTION

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ABSTRACT. Algorithms for the explicit symbolic and numeric evaluation of Jacquet's Whittaker function for the $GL(n, \mathbb{R})$ based generalized upper half-plane for $n \geq 2$, and an implementation for symbolic evaluation in the Mathematica package GL(n)pack, are described. This requires a comparison of the different definitions of Whittaker function which have appeared in the literature.

1. INTRODUCTION

Classical Whittaker functions have been known for many years as solutions to Whittaker's ordinary differential equation. Generalized Whittaker functions were introduced by Jacquet in his thesis [7], defined on Chevalley groups. Piatetski-Shapiro [10] and Shalika [11] derived the Fourier expansion of a Maass form for $SL(n, \mathbb{Z})$ in terms of these generalized Whittaker functions. Hence their importance. Stade, in a ground-breaking series of papers [13, 14, 15], showed how the integral representations for these functions could be better manipulated by deriving a recursive representation, expressing each function in terms of functions of lower dimension. It is this form which was implemented in GL(n)pack .

Goldfeld in [4] describes the theory and applications of Whittaker functions. Because there are differences in definition, it is essential that a "unification" of the different definitions be established, and that, in the main, is the goal of this paper.

Section 1 sets out the underlying definitions which are taken from [4], Section 2 establishes the connection between Stade's and Goldfeld's definitions, Section 4 gives the classical Whittaker function, Section 5 unifies the classical definition and Jacquet's function, Section 6 is an explicit evaluation of the constant in the standard decomposition of Jacquet's function, and Section 7 gives some symbolic and numeric evaluations of the generalized functions in dimensions 2, 3, 4 and 5.

This paper is consistent with [4] but is self-contained in that all essential definitions are included.

2. DEFINITIONS

Let $n \geq 2$ unless otherwise noted. The real general linear and orthogonal groups are $GL(n, \mathbb{R})$ and $O(n, \mathbb{R})$, respectively. The subgroup of diagonal matrices with non-zero constant value is $\mathbb{R}^{\times}$. Each $g \in GL(n, \mathbb{R})$ can be expressed uniquely as $g = x.y.o.d$, the so-called **Iwasawa form**, where x is upper triangular unipotent, y

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is positive diagonal with 1 in the bottom entry, o orthogonal and d is in $\mathbb{R}^\times$. Then x and y are unique. Explicitly,

$$x.y = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & & x_{1,n} \\ & 1 & x_{2,3} & & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}.$$

Then $O(n,\mathbb{R}) \times \mathbb{R}^\times$ acts on the right on $GL(n,\mathbb{R})$ by matrix multiplication, and we set

$$\mathfrak{h}^n = GL(n,\mathbb{R})/O(n,\mathbb{R}) \times \mathbb{R}^\times.$$

From the Iwasawa form we can express each element of $\mathfrak{h}^n$as the matrix $x.y$. It is convenient to use the coordinates $y_1, \cdots, y_{n-1}$ for the matrix y where the 1,1 entry is $y_1 \cdots y_{n-1}$, the (i,i)th is $y_1 \cdots y_{n-i}$ so the $(n-1,n-1)$th is y_1. These $n-1$ variables (y_i), together with the $n(n-1)/2$ variables $(x_{i,j})$ from the above diagonal terms of the unipotent matrix x, constitute the so-called **Iwasawa coordinates** for $\mathfrak{h}^n$.

From [4, Definition 2.4.1], let $b_{i,j} := ij$ when $i+j \leq n$ and $(n-i)(n-j)$ otherwise, and $\nu = (\nu_1, \cdots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, then the so-called **power function** $I_\nu : \mathfrak{h}^n \to \mathbb{C}$ is defined by

$$I_\nu(z) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j}\nu_j}, \tag{1}$$

where z is the Iwasawa form. Examples of the power function in dimensions 2, 3, 4, where we replace the z variables in the argument lists with the y_i on which the functions depend explicitly (but recall the power function is a function of z and the y_i needs to be computed using the Iwasawa form):

$$\begin{aligned} I_\nu(y_1) &= y_1^{\nu_1}, \\ I_\nu(y_1, y_2) &= y_1^{\nu_1+2\nu_2} y_2^{2\nu_1+\nu_2}, \\ I_\nu(y_1, y_2, y_3) &= y_1^{\nu_1+2\nu_2+3\nu_3} y_2^{2\nu_1+4\nu_2+2\nu_3} y_3^{3\nu_1+2\nu_2+\nu_3}. \end{aligned}$$

Note that Friedberg [3] and Stade [13] reverse the order of the y_i's in the Iwasawa form.

Definition 2.1. Let $S = U_n(\mathbb{R})$ be the subgroup of upper triangular unipotent matrices. A function $\psi : S \to \mathbb{C}$ which can be written in the form

$$\psi(u) = \prod_{i=1}^{n-1} e^{2\pi i m_{n-i+1} u_{i,i+1}},$$

for some $n-1$ tuple of integers $m = (m_1, \cdots, m_{n-1})$, is called a **character** or character of $U_n(\mathbb{R})$. We write ψ_m for ψ, and in case each $m_i = 1$ write ψ_1. Note that $\psi(a.b) = \psi(a)\psi(b)$ for $a, b \in U_n(\mathbb{R})$ and that all characters of $U_n(\mathbb{R})$ have this form. Note also that [4] begins with a direct order for the m_i and then reverses the order for the definition of the Jacquet-Whittaker function as given here.

Definition 2.2 ([4, Proposition 2.3.1])**.** The associative algebra D^n is the algebra of operators generated by real linear combinations of the operators $D_{\alpha_1} \circ \cdots \circ D_{\alpha_k}$

where each α_i is an $n \times n$ real matrix, D_α is defined for smooth functions F acting on elements $g \in GL(n,\mathbb{R})$ by

$$D_\alpha F(g) := \frac{\partial}{\partial t} F(g + tg.\alpha))|_{t=0},$$

and $D_\alpha \circ D_\beta$ is the composition of operators. The center of this algebra is denoted $\mathfrak{D}^n$.

Definition 2.3 ([4, Definition 1.3.1])**.** Let $a, b \geq 0$. The **Siegel set** $\Sigma_{a,b} \subset \mathfrak{h}^n$ is the set of all $z = x.y \in \mathfrak{h}^n$ with $|x_{i,j}| \leq b$ for $1 \leq i < j \leq n$ and $y_i > a$ for $1 \leq i \leq n-1$.

From [4, Definition 5.4.1], for $n \geq 2$ and $\nu = (\nu_1, \cdots, \nu_{n-1})$ and ψ a character of $U_n(\mathbb{R})$, a smooth function $W : \mathfrak{h}^n \to \mathbb{C}$ is called an $SL(n,\mathbb{Z})$-Whittaker function of type ν (or for short a **Whittaker function**) if it satisfies the following conditions:

(1) $W(uz) = \psi(u)W(z)$ for all $u \in U_n(\mathbb{R}), z \in \mathfrak{h}^n$,
(2) $DW(z) = \lambda_D W(z)$ for all $D \in \mathfrak{D}^n$, $z \in \mathfrak{h}^n$,
(3) $\displaystyle\int_\Omega |W(z)|^2 d^*z < \infty$ where Ω is $\Sigma_{\frac{\sqrt{3}}{2}, \frac{1}{2}}$ and d^*z is the left invariant quotient measure.

Definition 2.4. The matrix $w_n \in SL(n,\mathbb{Z})$ has a 0 in each row and column except for the reverse leading diagonal entries which are either $(-1)^{[n/2]}$ in the $(1,n)$th position or 1 in every other position. That is to say:

$$w_n = \begin{pmatrix} & & & (-1)^{[n/2]} \\ & & 1 & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix}.$$

Compare the so-called "long-element" permutation matrix w, which is the same as w_n except the $(1,n)$th element has the value 1.

Definition 2.5. The left invariant **quotient measure** on $\mathfrak{h}^n$ [4, Proposition 1.5.3]:

$$\begin{aligned} d^*z &= d^*x \cdot d^*y, \quad \text{where} \\ d^*x &= \prod_{1 \leq i < j \leq n} dx_{i,j} \quad \text{and} \\ d^*y &= \prod_{k=1}^{n-1} y_k^{-k(n-k)-1} dy_k. \end{aligned}$$

The definition of Jacquet's Whittaker function as given in [4, Eqn. 5.5.1] and called "Jacquet's integral" is given by

$$W_J(z; \nu, \psi_m) := \int_{U_n(\mathbb{R})} I_\nu(w_n \cdot u \cdot z)\overline{\psi_m(u)} d^*u,$$

where $z \in \mathfrak{h}^n$, $\nu \in \mathbb{C}^{n-1}$, ψ_m is a character with $m = (m_{n-1}, \cdots, m_1) \in \mathbb{Z}^{n-1}$ and with no $m_i = 0$ and where the measure is inherited from $\mathfrak{h}^n$. The main properties of this function are summarized in:

Theorem 2.1 ([4, Proposition 5.5.2]). *Let $n \geq 2$. Assume $\Re(\nu_i) > 1/n$ for $1 \leq i \leq n-1$ and that non-zero integers m_i with $1 \leq i \leq n-1$ are given. Then Jacquet's integral converges absolutely and uniformly on compact subsets of $\mathfrak{h}^n$ and has meromorphic continuation to all $\nu \in \mathbb{C}^{n-1}$. The function $W_J(z;\nu,\psi_m)$ is an $SL(n,\mathbb{Z})$-Whittaker function of type ν and character ψ_m and satisfies the identity*

$$W_J(z;\nu,\psi_m) = c_{\nu,m} \cdot \psi_m(x) \cdot W_J(My;\nu,\psi_1),$$

where $c_{\nu,m} \neq 0$ depends only on m and ν, and where the diagonal matrix M has ith *entry*

$$|m_1 m_2 \cdots m_{n-i}|$$

for $1 \leq i \leq n-1$ and nth *entry 1 and where the explict value of the constant $c_{\nu,m}$ is given in Theorem 6.1 below.*

Theorem 2.2. *In the definition of Jacquet's Whittaker function the matrix w_n can be replaced by the matrix w.*

Proof. If for $1 \leq j \leq n$, e_j is the standard unit vector, then $e_j.w_n = e_j.w$ for all $j > 1$. The theorem now follows from the exterior product form of the power function given in [4, Lemma 5.7.2]. □

3. Relation to Stade's Whittaker function

The GL(n)pack function `Whittaker` computes a symbolic iterated integral representation of the generalized Jacquet-Whittaker function W_{Jacquet} (also written W_J) of order n, for $n \geq 2$, as defined above. The algorithm uses the recursive representation of the Whittaker function derived by Stade [Stade, 1990, Theorem 2.1], but his Whittaker functions are not the same as those of [4]. Let W_S and W_S^* be Stade's Whittaker and Whittaker starred functions, respectively, and let Γ_ν represent the gamma factors for either form defined below. Now we make the following definitions:

Definition 3.1.

$$\begin{aligned}
H_\nu(y) &:= I_\nu(y_{n-1}, \cdots, y_1),\\
Q = Q_\nu(y) &:= H_\nu(y) \prod_{j=1}^{n-1} y_j^{-\mu_j}, \quad \text{where}\\
\mu_j &:= \sum_{k=1}^{n-j} r_{j,k} \quad \text{for } 1 \leq j \leq n-1, \text{ and where}\\
r_{j,k} &:= \Big(\sum_{i=k}^{k+j-1} \frac{n\nu_i}{2}\Big) - \frac{j}{2} \quad \text{for } 1 \leq j \leq n-1, 1 \leq k \leq n-j.
\end{aligned}$$

From [4, Definition 5.9.2], if $n \geq 2$ and $\nu = (\nu_1, \cdots, \nu_{n-1})$, then

$$\begin{aligned}
\Gamma_\nu &:= \prod_{j=1}^{n-1} \prod_{j \leq k \leq n-1} \pi^{-\frac{1}{2} - v_{j,k}} \Gamma\big(\tfrac{1}{2} + v_{j,k}\big), \quad \text{where}\\
v_{j,k} &:= \sum_{i=0}^{j-1} \frac{n\nu_{n-k+i} - 1}{2}.
\end{aligned}$$

Definition 3.2.

$$\begin{aligned} W_S(y;\nu,\psi_1) = W_{n,\nu}(y) &:= \Gamma_\nu \int_{U_n(\mathbb{R})} H_\nu(w\cdot u\cdot y)\overline{\psi_1(u)}d^*u, \\ W_S^*(y;\nu,\psi_1) = W_{n,\nu}^*(y) &:= W_S(y;\nu,\psi_1)/Q. \end{aligned}$$

The notations $W_{n,\nu}, W_{n,\nu}^*$ are from [13]. Note that the first differs from his $W_{n,a}$ [14].

Theorem 3.1. *Let $n \geq 2$ and for $y = (y_1, \cdots, y_{n-1})$ let $y_r = (y_{n-1}, \cdots, y_1)$ be the vector with coordinates reversed. Then the relationship between the two definitions of Whittaker function for $\mathfrak{h}^n$ may be expressed by the equalities*

$$Q_\nu(y_r) * W_S^*(y_r;\nu,\psi_1) = \Gamma_\nu * W_J(y;\nu,\psi_1) = W_J^*(y;\nu,\psi_1) = W_S(y_r;\nu,\psi_1).$$

Stade's recursive formula for the Whittaker function:

Theorem 3.2 ([13, Theorem 2.1] as amended in [14])**.** *If $n \geq 3$ and $\nu \in \mathbb{C}^{n-1}$, for $2 \leq j \leq n-2$, let $\lambda = (\lambda_1, \cdots, \lambda_{n-3})$ where $\lambda_{j-1} := n\nu_j/(n-2)$, set $u_0 = 0, 1/u_{n-1} = 0$ and $u_{n-1}^0 = 1$.*

$$\begin{aligned}
&W_S^*(y;\nu,\psi_1) = 1 \quad \text{if } n = 0 \text{ or } 1, \\
&W_S^*(y;\nu,\psi_1) = 2K_{\nu-\frac{1}{2}}(2\pi y_1) \quad \text{if } n = 2, \\
&W_S^*(y;\nu,\psi_1) = 8\int_0^\infty u^{\frac{3\nu_1-3\nu_2-2}{2}} K_{\frac{3\nu_1+3\nu_2-2}{2}}(2\pi\sqrt{1+\frac{1}{u_1^2}}y_1) \\
&\qquad\qquad \times K_{\frac{3\nu_1+3\nu_2-2}{2}}(2\pi\sqrt{1+u_1^2}y_2)du_1 \quad \text{for } n = 3, \\
&W_S^*(y;\nu,\psi_1) = 2^{2n-3}\int_{(\mathbb{R}+)^{n-2}} \{\prod_{i=1}^{n-1} u_i^{r_{i,1}-r_{i,n-i}} K_{\mu_1}(2\pi y_i\sqrt{(1+u_{i-1}^2)(1+1/u_i^2)})\} \\
&\qquad\qquad \times W_S^*((\frac{y_2u_1}{u_2}, \cdots, \frac{y_{n-2}u_{n-3}}{u_{n-2}}), (\lambda_1, \cdots, \lambda_{n-3})) \prod_{i=1}^{n-2} \frac{du_i}{u_i}
\end{aligned}$$

for $n \geq 4$, where the quantities $r_{i,j}$ are defined in terms of the ν_i in Definition 3.1 above.

4. Classical Whittaker functions

Properties of classical Whittaker functions are well known. However, we record them here to show the relationship between the classical and GL(n)pack functions.

Whittaker's equation [18, 8, 9] for $W_{k,\mu}(z)$ ([4, p. 57]) is given by

$$w'' + (-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4}-\mu^2}{z^2})w = 0,$$

where $\mu \in \mathbb{C}$, $k \in \mathbb{R}$ and $z \in \mathbb{C}$.

Solutions for this equation have the integral representation [18, 8, 9]:

$$W_{k,\mu}(z) = \frac{z^k e^{-z/2}}{\Gamma(\mu-k+\frac{1}{2})}\int_0^\infty e^{-t}t^{\mu-k-\frac{1}{2}}(1+\frac{t}{z})^{\mu+k-\frac{1}{2}}dt$$

for $\Re\mu - k - \frac{1}{2} > 0$ and $|\arg z| < \pi$.

The solutions also have a series representation [18, 8, 9]: Let $\Psi(\alpha,\gamma;z)$ be the so-called **confluent hypergeometric function of the second kind** satisfying

$$\Psi(\alpha,\gamma;z) = z^{-\alpha}\Big(\sum_{k=0}^{n}\frac{(-1)^k(\alpha)_k(1+\alpha-\gamma)_k}{k!}z^{-k} + O(\frac{1}{|z|^{n+1}})\Big),$$

for $|\arg z| < \pi - \delta$ for all fixed $\delta > 0$. Then we can write

$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-\frac{z}{2}}\Psi(\tfrac{1}{2}-k+\mu, 2\mu+1; z).$$

It follows from the series representations for the K-Bessel and classical Whittaker functions that for all ν and $|\arg z| < \pi - \delta$,

$$\sqrt{\frac{2z}{\pi}}K_\nu(z) = W_{0,\nu}(2z).$$

Comparing this with Stade's W_S in dimension $n = 2$, the reason for the term "Whittaker" for the functions defined on $\mathfrak{h}^n$ is clear.

5. Uniformization of the definitions of Whittaker function in dimensions 2 and 3

Dimension 2: We have [4, Eqn. 5.5.4]:

$$\begin{aligned} W_J(z;\nu,\psi_m) &= \frac{2|m|^{\nu-\frac{1}{2}}\pi^\nu}{\Gamma(\nu)}\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|m|y)\cdot e^{2\pi i m x},\\ \Gamma_\nu &= |m\pi|^{-\nu}\Gamma(\nu). \end{aligned}$$

Stade's form for dimension 2 [13] is

$$W_S(y;\nu,\psi_1) = 2\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi y).$$

Theorem 5.1. *Let $n = 2$ so $z = x + iy$. Assume for $y > 0$ and $\Re\nu > \frac{1}{2}$ that*

$$W_S(y,\nu,\psi_1) = 2\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi y) \text{ and } W_J(z,\nu,\psi_m) = y^{1-\nu}\int_{-\infty}^{\infty}\frac{e^{-2\pi i m u y}}{(1+u^2)^\nu}du\cdot e^{2\pi i m x}.$$

Then

$$W_J(z,\nu,\psi_m) = |m|^{\nu-1}\cdot(\frac{\pi^\nu}{\Gamma(\nu)})\cdot 2\sqrt{y}K_{\nu-\frac{1}{2}}(2\pi|m|y)\cdot e^{2\pi i m x}.$$

Proof. This follows from the assumptions using the Mathematica expression for the infinite integral given above and the relationship $W_J = \frac{\pi^\nu}{\Gamma(\nu)}W_S$. □

Note that the only difference between this form and that of [4, Eqn. 5.5.4] is the constant $c_{\nu,m} = |m|^{\nu-1}$, a form consistent with Theorem 6.1 below.

Dimension 3: By [4, Eqn. 6.1.3] and [13, Page 318] (Note that like Friedberg, Stade swaps the order of the labels on the y_i, but Goldfeld makes the swap back, but no other changes. This is Stade's form:

$$\begin{aligned} W_S((y_1,y_2);(\nu_1,\nu_2),\psi_{1,1}) &= 8y_1^{1-\frac{\nu_1-\nu_2}{2}}y_2^{1+\frac{\nu_1-\nu_2}{2}}\int_0^\infty K_{\frac{3\nu_1+3\nu_2-2}{2}}(2\pi y_1\sqrt{1+u^{-2}})\\ &\times\ K_{\frac{3\nu_1+3\nu_2-2}{2}}(2\pi y_2\sqrt{1+u^2})u^{\frac{3\nu_1-3\nu_2}{2}}\frac{du}{u}\\ \Gamma_\nu &= \pi^{\frac{1}{2}-3\nu_1-3\nu_2}\Gamma(\frac{3\nu_1}{2})\Gamma(\frac{3\nu_2}{2})\Gamma(\frac{3\nu_1+3\nu_2-1}{2}). \end{aligned}$$

GL(n)pack has the same gamma factor. The coefficient 4 (as given by [4, 13], but since corrected [14]) is given its corrected value 8.

6. Evaluation of the constant

Theorem 6.1. *The constant in Theorem 2.1 has value*

$$c_{\nu,m} = \prod_{i=1}^{n-1} |m_i|^{\sum_{j=1}^{n-1} b_{i,j}\nu_j - i(n-i)}.$$

Proof. 1. As in [4, Prop. 5.5.2] let

$$M = \begin{pmatrix} |m_1 m_2 \cdots m_{n-1}| & & & & \\ & |m_1 m_2 \cdots m_{n-2}| & & & \\ & & \ddots & & \\ & & & |m_1| & \\ & & & & 1 \end{pmatrix}.$$

Let $d_n = 1$ and for $1 \le i \le n-1$, $d_i = |m_1| \cdots |m_{n-i}|$, so $u.M = M.\hat{u}$, where for $1 \le i < j \le n-1$,

$$u_{i,j} = \hat{u}_{i,j} \frac{d_i}{d_j} = \hat{u}_{i,j} |m_{n-j+1}| \cdots |m_{n-i}|.$$

If we let $u_i = u_{n-i,n-i+1}$ for $1 \le i \le n-1$, and similarly define $\hat{u}_i$, then under the transformation $u \to \hat{u}$ we have

$$u_i = \hat{u}_i |m_i|, 1 \le i \le n-1.$$

The (absolute value of the determinant of the) Jacobian of the transformation $u \to \hat{u}$ is

$$J(\frac{u}{\hat{u}}) = \prod_{i<j} |m_{n-j+1}| \cdots |m_{n-i}| = \prod_{i=1}^{n-1} |m_i|^{i(n-i)}.$$

2. It follows that

$$\begin{aligned} &W_J(Mz; \nu, \psi_{\epsilon_1,\cdots,\epsilon_{n-1}}) \\ &\quad = \int_{U_n(\mathbb{R})} I_\nu(w.u.Mz) e^{-2\pi i(\epsilon_1 u_1 + \cdots + \epsilon_{n-1} u_{n-1})} d^* u \\ &\quad = J(\frac{u}{\hat{u}}) \int_{U_n(\mathbb{R})} I_\nu(w.M\hat{u}.z) e^{-2\pi i(|m_1|\epsilon_1 \hat{u}_1 + \cdots + |m_{n-1}|\epsilon_{n-1}\hat{u}_{n-1})} d^* \hat{u} \\ &\quad = J(\frac{u}{\hat{u}}) \int_{U_n(\mathbb{R})} I_\nu(w.M.w.w.\hat{u}.z) e^{-2\pi i(m_1 \hat{u}_1 + \cdots + m_{n-1}\hat{u}_{n-1})} d^* \hat{u} \\ &\quad = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} |m_i|^{-b_{i,j}\nu_j} J(\frac{u}{\hat{u}}) \int_{U_n(\mathbb{R})} I_\nu(\omega_n.\hat{u}.z) e^{-2\pi i(m_1 \hat{u}_1 + \cdots + m_{n-1}\hat{u}_{n-1})} d^* \hat{u} \\ &\quad = \gamma_{m,\nu} \cdot W_J(z; \nu, \psi_m) \quad (1), \end{aligned}$$

where we have used $w.M.w$ is a diagonal matrix with elements in the reverse order from those of M and where

$$\gamma_{\nu,m} = \frac{\prod_{i=1}^{n-1} |m_i|^{i(n-i)}}{\prod_{i=1}^{n-1} \prod_{j=1}^{n-1} |m_i|^{b_{i,j}\nu_j}}.$$

3. Next, by taking the Iwasawa form for $z = x.y$, commuting M and x ($M.x = \hat{x}.M$), and then making the transformation $\hat{u} = u.\hat{x}$ (which has Jacobian 1), we obtain the form

$$W_J(Mz;\nu,\psi_{\epsilon_1,\cdots,\epsilon_{n-1}}) = \psi_m(x)\cdot W_J(My;\nu,\psi_{\epsilon_1,\cdots,\epsilon_{n-1}}). \tag{2}$$

4. Now consider the $n-j$th row of the matrix u with $1 \le j \le n-1$:

$$(0,\cdots,0,1,u_j,u_{n-j,n-j+2},\cdots,u_{n-j,n}).$$

If δ_j is the diagonal matrix with 1's in every position except the $n-j$th, which is ϵ_j, and we make the transformation $\hat{u} = \delta_j.u$ with the Jacobian determinant ϵ_j^j, then, since $w.\delta_j = \delta_{n-j+1}.w$ and $I_\nu(\delta_{n-j+1}.z) = I_\nu(z)$, (because δ_{n-j+1} is orthogonal and diagonal matrices commute), we can write

$$\begin{aligned}
&W_J(My;\nu,\psi_{\epsilon_1,\cdots,\epsilon_{n-1}})\\
&\quad= \int_{U_n(\mathbb{R})} I_\nu(w.\delta_j.\delta_j.u.My)e^{-2\pi i(\epsilon_1 u_1+\cdots+\epsilon_{n-1}u_{n-1})}d^*u\\
&\quad= |\epsilon_j^j|\int_{U_n(\mathbb{R})} I_\nu(\delta_{n-j+1}.w.\hat{u}.My)e^{-2\pi i(\epsilon_1\hat{u}_1+\cdots+\hat{u}_j+\cdots+\epsilon_{n-1}\hat{u}_{n-1})}d^*\hat{u}\\
&\quad= W_J(My;\nu,\psi_{\epsilon_1,\cdots,1,\cdots,\epsilon_{n-1}}),
\end{aligned}$$

where the subscript 1 is in the jth position.

One may do the above procedure for each j with $1 \le j \le n-1$ to obtain

$$W_J(My;\nu,\psi_{\epsilon_1,\cdots,\epsilon_{n-1}}) = W_J(My;\nu,\psi_{1,\cdots,1}). \tag{3}$$

Finally, combining the expressions (1), (2), and (3) we derive the equation

$$W_J(z;\nu,\psi_m) = c_{\nu,m}\cdot\psi_m(x)\cdot W_J(My;\nu,\psi_{1,\cdots,1}),$$

where

$$\begin{aligned}
c_{\nu,m} &= \gamma_{\nu,m}^{-1}\\
&= \prod_{i=1}^{n-1}|m_i|^{\sum_{j=1}^{n-1} b_{i,j}\nu_j - i(n-i)}. \qquad \square
\end{aligned}$$

Here is a listing of the first $n=2$ through $n=5$ $c_{m,\nu}$ values:

$$\begin{aligned}
c_{2,\nu} &= |m_1|^{\nu_1-1},\\
c_{3,\nu} &= |m_1|^{\nu_1+2\nu_2-2}|m_2|^{2\nu_1+\nu_2-2},\\
c_{4,\nu} &= |m_1|^{\mathrm{v}_1+2\,\mathrm{v}_2+3\,\mathrm{v}_3-3}\,|m_2|^{2\,v_1+4\,v_2+2\,v_3-4}\,|m_3|^{3\,v_1+2\,v_2+\mathrm{v}_3-3},\\
c_{5,\nu} &= |m_1|^{\mathrm{v}_1+2\,\mathrm{v}_2+3\,\mathrm{v}_3+4\,\mathrm{v}_4-4}\,|m_2|^{2\,v_1+4\,v_2+6\,v_3+3\,v_4-6}\\
&\times\ |m_3|^{3\,v_1+6\,v_2+4\,v_3+2\,v_4-6}\,|m_4|^{4\,v_1+3\,v_2+2\,v_3+\mathrm{v}_1-4}.
\end{aligned}$$

7. Computation of $W_J(z,\nu,\psi)$ and validation

Computation of the Whittaker function was divided into symbolic evaluation and numeric evaluation. The former is more straightforward than the latter and was able to be included in GL(n)pack . The numerical code uses the symbolic form as an initial step. Stade's form W_S^* was computed using his recursive reformulation, Theorem 3.1, which was converted first to a single multiple integral and then, by a change of variables using the inverse hyperbolic tangent in each variable, to an integral over a cube of appropriate dimension. This has a number of decided advantages over any direct use of Jacquet's integral for numerical computation: first, the oscillation implied by the character ψ_1 is removed, and second, the exponential decay of the K-Bessel functions at infinity assists the speed and accuracy of any quadrature application.

Stade's form was then converted into the function W_J using Theorem 3.1.

Examples of the GL(n)pack output are given below: in Figure 1 dimensions 2 and 3, in Figure 2 dimension 4, and in Figure 3 dimension 5 [2]:

In[114]:=
```
Whittaker[{{y1, 0}, {0, 1}}, {v1}, {1}, u][[4]]
```

Out[114]=

$$\frac{2\,\pi^{v1}\,\sqrt{y1}\,K[-\frac{1}{2}+v1,\ 2\,\pi\,y1]}{\mathrm{Gamma}[v1]}$$

In[115]:=
```
Whittaker[{{y1 y2, 0, 0}, {0, y1, 0}, {0, 0, 1}}, {v1, v2}, {1, 1}, u][[4]]
```

Out[115]=

$$\left(8\,\pi^{-\frac{1}{2}+3\,v1+3\,v2}\,y1^{1+\frac{v1}{2}-\frac{v2}{2}}\,y2^{1-\frac{v1}{2}+\frac{v2}{2}}\int_0^\infty u^{-1+\frac{3\,(v1-v2)}{2}}\,K\Big[\frac{1}{2}\,(-2+3\,v1+3\,v2),\ 2\,\pi\,\sqrt{1+u^2}\,y1\Big]\,K\Big[\frac{1}{2}\,(-2+3\,v1+3\,v2),\ 2\,\pi\,\sqrt{1+\frac{1}{u^2}}\,y2\Big]\,du\right)\Big/\left(\mathrm{Gamma}\Big[\frac{3\,v1}{2}\Big]\,\mathrm{Gamma}\Big[\frac{3\,v2}{2}\Big]\,\mathrm{Gamma}\Big[\frac{1}{2}\,(-1+3\,v1+3\,v2)\Big]\right)$$

FIGURE 1. The GL(n)pack Whittaker functions in dimensions 2 and 3.

In[116]:=
```
Whittaker[{{y1 y2 y3, 0, 0, 0}, {0, y1 y2, 0, 0}, {0, 0, y1, 0}, {0, 0, 0, 1}},
  {v1, v2, v3}, {1, 1, 1}, u][[4]]
```

Out[116]=

$$\left(64\,\pi^{-2+6\,v1+8\,v2+6\,v3}\,y1^{\frac{3}{2}+v1-v3}\,y2^2\,y3^{\frac{3}{2}-v1+v3}\int_0^\infty\!\!\int_0^\infty K\Big[-\frac{1}{2}+2\,v2,\ \frac{2\,\pi\,y2\,u[1]}{u[2]}\Big]\,K\Big[-\frac{3}{2}+2\,v1+2\,v2+2\,v3,\ 2\,\pi\,y3\,\sqrt{1+\frac{1}{u[1]^2}}\Big]\,K\Big[-\frac{3}{2}+2\,v1+2\,v2+2\,v3,\ 2\,\pi\,y2\,\sqrt{(1+u[1]^2)\left(1+\frac{1}{u[2]^2}\right)}\Big]\,K\Big[-\frac{3}{2}+2\,v1+2\,v2+2\,v3,\ 2\,\pi\,y1\,\sqrt{1+u[2]^2}\Big]\,(u[1]\,u[2])^{-1+2\,v1-2\,v3}\,du[2]\,du[1]\right)\Big/\left(\mathrm{Gamma}[2\,v1]\,\mathrm{Gamma}[2\,v2]\,\mathrm{Gamma}\Big[-\frac{1}{2}+2\,v1+2\,v2\Big]\,\mathrm{Gamma}[2\,v3]\,\mathrm{Gamma}\Big[-\frac{1}{2}+2\,v2+2\,v3\Big]\,\mathrm{Gamma}[-1+2\,v1+2\,v2+2\,v3]\right)$$

FIGURE 2. The GL(n)pack Whittaker functions in dimension 4.

```
In[1]:= Whittaker[{{y1 y2 y3 y4, 0, 0, 0, 0}, {0, y1 y2 y3, 0, 0, 0}, {0, 0, y1 y2, 0, 0},
          {0, 0, 0, y1, 0}, {0, 0, 0, 0, 1}}, {v1, v2, v3, v4}, {1, 1, 1, 1}, u][[4]]
```

Out[1]=

$$
\Bigg(1024\,\pi^{5\,(-1+2\,v1+3\,v2+3\,v3+2\,v4)}\;
y1^{2+\frac{3\,v1}{2}+\frac{v2}{2}-\frac{v3}{2}-\frac{3\,v4}{2}}\;
y2^{3+\frac{v1}{2}+v2-v3-\frac{v4}{2}}\;
y3^{3-\frac{v1}{2}-v2+v3+\frac{v4}{2}}\;
y4^{2-\frac{3\,v1}{2}-\frac{v2}{2}+\frac{v3}{2}+\frac{3\,v4}{2}}
$$

$$
\int_0^\infty\int_0^\infty\int_0^\infty\left(\int_0^\infty K\left[\frac{1}{2}\,(-2+5\,v2+5\,v3),\ \frac{2\,\pi\,y3\,\sqrt{1+\frac{1}{u[1]^2}}\;u[2]}{u[3]}\right] K\left[\frac{1}{2}\,(-2+5\,v2+5\,v3),\ \frac{2\,\pi\,y2\,\sqrt{1+u[1]^2}\;u[3]}{u[4]}\right] u[1]^{-1+\frac{5\,v2}{2}-\frac{5\,v3}{2}}\,du[1]\right)
$$

$$
K\left[\frac{1}{2}\,(-4+5\,v1+5\,v2+5\,v3+5\,v4),\ 2\,\pi\,y4\,\sqrt{1+\frac{1}{u[2]^2}}\right]
$$

$$
K\left[\frac{1}{2}\,(-4+5\,v1+5\,v2+5\,v3+5\,v4),\ 2\,\pi\,y3\,\sqrt{(1+u[2]^2)\left(1+\frac{1}{u[3]^2}\right)}\right]
$$

$$
K\left[\frac{1}{2}\,(-4+5\,v1+5\,v2+5\,v3+5\,v4),\ 2\,\pi\,y2\,\sqrt{(1+u[3]^2)\left(1+\frac{1}{u[4]^2}\right)}\right]
$$

$$
K\left[\frac{1}{2}\,(-4+5\,v1+5\,v2+5\,v3+5\,v4),\ 2\,\pi\,y1\,\sqrt{1+u[4]^2}\right]
$$

$$
u[3]^{\frac{1}{2}\,(-2+5\,v1+5\,v2-5\,v3-5\,v4)}\,(u[2]\,u[4])^{\frac{1}{2}\,(-2+5\,v1-5\,v4)}\,du[2]\,du[3]\,du[4]\Bigg)\Big/
$$

$$
\Bigg(\mathrm{Gamma}\left[\frac{5\,v1}{2}\right]\mathrm{Gamma}\left[\frac{5\,v2}{2}\right]\mathrm{Gamma}\left[\frac{1}{2}\,(-1+5\,v1+5\,v2)\right]
\mathrm{Gamma}\left[\frac{5\,v3}{2}\right]
\mathrm{Gamma}\left[\frac{1}{2}\,(-1+5\,v2+5\,v3)\right]
\mathrm{Gamma}\left[\frac{1}{2}\,(-2+5\,v1+5\,v2+5\,v3)\right]
$$

$$
\mathrm{Gamma}\left[\frac{5\,v4}{2}\right]
\mathrm{Gamma}\left[\frac{1}{2}\,(-1+5\,v3+5\,v4)\right]
\mathrm{Gamma}\left[\frac{1}{2}\,(-2+5\,v2+5\,v3+5\,v4)\right]
\mathrm{Gamma}\left[\frac{1}{2}\,(-3+5\,v1+5\,v2+5\,v3+5\,v4)\right]\Bigg)
$$

FIGURE 3. The GL(n)pack Whittaker function in dimension 5.

To validate the numerical computations (and thus the symbolic forms computed by GL(n)pack), and give some idea of their accuracy, we used a result highlighted in [14, p. 126], namely that if the power function is defined using some especially chosen new parameters, then the Whittaker functions are invariant under all permutations of those parameters. These permutations give rise to functional equations, which on the face of it differ from those set out in [4, Theorem 5.9.8]. These permutations were used here in a simpler manner: a permutation of an explicit set of values for the new parameters a_i give rise to two corresponding sets of values in the original parameters ν_i. These corresponding sets should be in or close to the domain of absolute convergence of the Whittaker function $\Re\nu_i > 1/n$ to give convergence of the integral forms.

In more detail, set

$$H_{n,a}(y) := \prod_{j=1}^{n-1} y_j \prod_{j=1}^{n-1} y_j^{a_j},$$

where the (a_j) are $n-1$ complex numbers. Then set $a_n = -a_1 - \cdots - a_n$. When defined using this power function the Whittaker function is invariant under all permutations of the (a_i). Then define ν in terms of a by setting $I_\nu(y) = H_{n,a}(y_r)$ and note that the first product term in the definition of H is invariant under reversal of the order of the y_i. These relations in dimensions 3 through 5 are as follows:

$$\begin{aligned}
\text{Dimension } 3: \nu_1 &= (1 + a_1 + 2a_2)/3, \\
\nu_2 &= (1 + a_1 - a_2)/3, \\
\text{Dimension } 4: \nu_1 &= (1 + a_1 + a_2 + 2a_3)/4, \\
\nu_2 &= (1 + a_2 - a_3)/4, \\
\nu_3 &= (1 + a_1 - a_2)/4, \\
\text{Dimension } 5: \nu_1 &= (1 + a_1 + a_2 + a_3 + 2a_4)/5, \\
\nu_2 &= (1 + a_3 - a_4)/5, \\
\nu_3 &= (1 + a_2 - a_3)/5, \\
\nu_4 &= (1 + a_1 - a_2)/5.
\end{aligned}$$

In this study, the Mathematica general adaptive quadrature routine `NIntegrate` was used, with the option method set of `MultiDimensional` and the precision set to `MachinePrecision`. The processor was an Intel Pentium 4. No improvement was found using the function Compile. This is no doubt because most of the work is done by `NIntegrate`, which is already compiled. The values given are for the Whittaker function W_S^*. The timing is from the Mathematica `Timing` function. The results are as follows:

Dimension	v	y	value	timing
3	$\{5/4, 1/4\}$	$\{1, 1\}$	$2.255480212 \times 10^{-8}$	
3	$\{7/6, 5/12\}$	$\{1, 1\}$	$2.255480211 \times 10^{-8}$	0.562s
4	$\{199/520, 23/80, 67/620\}$	$\{1, 1, 1\}$	1.0910×10^{-15}	
4	$\{437/1040, 67/260, 213/1040\}$	$\{1, 1, 1\}$	1.0915×10^{-15}	25703.7s
5	$\{3433/6630, 47/221, 89/510, 3/10\}$	$\{1, 1, 1, 1\}$	5.1976×10^{-28}	
5	$\{558/1105, 3/10, 22/195, 47/221\}$	$\{1, 1, 1, 1\}$	5.1972×10^{-28}	759.8s

Given Theorem 2.1, the uniform nature of the periodicity of the integrand should make the application of modern lattice rule techniques [12] practical for direct numerical evaluation of Jacquet's integral. However, given the unbounded domain and slow convergence of the integrand, this will require considerable adaptation and analysis.

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