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UNCONDITIONAL STABILITY OF EXPLICIT EXPONENTIAL RUNGE-KUTTA METHODS FOR SEMI-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we define unconditional stability properties of exponential Runge-Kutta methods when they are applied to semi-linear systems of ordinary differential equations characterized by a stiff linear part and a non-stiff non-linear part. These properties are related to a class of systems and to a specific norm. We give sufficient conditions in order that an explicit method satisfies such properties. On the basis of such conditions we analyze some of the popular methods.

1. Introduction

Let us consider semi-linear systems of ordinary differential equations (ODEs)

$$\begin{cases} y'(t) = Ly(t) + f(t, y(t)), \ t \geq t_0, \\ y(t_0) = y_0, \end{cases} \tag{1}$$

where $L \in \mathbb{R}^{d\times d}$ has a large norm along with a non-positive or moderately positive logarithmic norm and $f : [t_0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ has a moderate Lipschitz constant with respect to the second argument.

Important examples of such systems of ODEs arise from the spatial discretization by means of finite differences, finite elements or spectral methods of evolutionary partial differential equations.

The right-hand side in (1) is divided into a *linear stiff* part $Ly(t)$ and a *non-linear non-stiff* part $f(t, y(t))$. Since the problem is stiff as a whole, an implicit method should be used in the numerical integration. However, an implicit method requires the solution of a non-linear algebraic system at every step of the integration and the non-linearity of the algebraic system is due to the non-stiff part, which could be explicitly integrated. Therefore, instead of fully implicit methods, one should hopefully use methods which are implicit in the linear stiff part and explicit in the non-linear non-stiff part.

Well-known methods of this type, both Runge-Kutta (RK) and Linear Multistep, are the so-called *IMEX methods* (see e.g. [1], [2] and, for an introduction, [15]) which use an implicit scheme for the stiff linear part and another, explicit, scheme for the non-stiff non-linear part.

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Other such methods are the explicit *exponential integrators*, where the exact flow of the linear part, i.e. a matrix exponential, is used in the construction of the schemes.

Exponential integrators constitute an active field of research in numerical ODEs (see [5], [6], [7], [9], [10], [11], [12], [13], [14], [16], [18], [20], [22], [23], [26] and, for a review, [21]). There is also a MATLAB package for exponential integrators called EXPINT which is described in [4].

In this paper we study unconditional stability properties of explicit exponential RK methods when they are applied to semi-linear systems of ODEs (1). The outline of the paper is the following. In Section 2, we introduce exponential RK methods for semi-linear systems of ODEs. In Section 3, for such methods, we define the properties of unconditional contractivity and unconditional asymptotic stability. In Section 4, some sufficient conditions are given and some of the most known methods are analyzed. Finally, some conclusions are drafted in Section 5.

2. Exponential RK methods

An *exponential RK method* as applied to a semi-linear systems of ODEs (1) takes the form

$$\text{(2a)}\qquad y_{n+1} = e^{h_{n+1}L}y_n + h_{n+1}\sum_{i=1}^{\nu} b_i(h_{n+1}L)f(t_{n+1}^i, Y_{n+1}^i),$$

$$\text{(2b)}\qquad Y_{n+1}^i = e^{c_i h_{n+1}L}y_n + h_{n+1}\sum_{j=1}^{\nu-1} a_{ij}(h_{n+1}L)f(t_{n+1}^j, Y_n^j),\ i=1,...,\nu,$$

where the abscissae c_i are non-negative, $t_{n+1}^i = t_n + c_i h_{n+1}$ and the weights $b_i(h_{n+1}L)$ and the coefficients $a_{ij}(h_{n+1}L)$ are $(d\times d)$-matrices which are analytic functions of the matrix $h_{n+1}L$. As explained in the introduction, we are interested in *explicit* methods, i.e. methods where $a_{ij}(h_{n+1}L) = 0$ for $i \leq j$.

Note that, when the method is applied to an equation with $f = 0$, it yields the exact solution and, when applied to an equation with $L = 0$ it reduces to a one-step method for ODEs called the *underlying method.*

We assume

$$b_i(0) = b_i I_d \text{ and } a_{ij}(0) = a_{ij} I_d,\ i,j = 1,...,\nu,$$

where b_i and a_{ij} are scalar, and so the underlying method turns out to be the classical explicit RK method of weights b_i and coefficients a_{ij}. To avoid confusion between the scalars b_i and a_{ij} and the functions $h_{n+1}L \mapsto b_i(h_{n+1}L)$ and $h_{n+1}L \mapsto a_{ij}(h_{n+1}L)$ we denote, when necessary, such functions by $b_i(\cdot)$ and $a_{ij}(\cdot)$, respectively.

Two particular and important subclasses of exponential RK methods are included in the general class (2): the *Integrating Factor* (IF) methods and the *Exponential Time Differencing* (ETD) methods.

The IF methods, also known as *Lawson methods* after they appeared for the first time in [19], are such that

$$\text{(3a)}\qquad b_i(h_{n+1}L) = b_i e^{(1-c_i)h_{n+1}L},\ i=1,...,\nu,$$

$$\text{(3b)}\qquad a_{ij}(h_{n+1}L) = a_{ij} e^{(c_i-c_j)h_{n+1}L},\ i,j=1,...,\nu,$$

where b_i and a_{ij} are weights and coefficients of a classical RK method for ODEs, which is the underlying method.

The simplest method of this type is the *explicit IF (Lawson) Euler method* given by

$$y_{n+1} = e^{h_{n+1}L} y_n + h_{n+1} e^{h_{n+1}L} f(t_n, y_n),$$

where, of course, the underlying method is the explicit Euler method.

The order of an IF exponential RK method is the same as that of the underlying method.

In order to define the ETD methods, we need to introduce the functions

$$\varphi_l(z) = \int_0^1 e^{(1-s)z} \frac{s^{l-1}}{(l-1)!} ds, \ z \in \mathbb{C}, \ l = 1, 2, ..., \tag{4}$$

which satisfy the recursion

$$\varphi_l(z) = \frac{\varphi_{l-1}(z) - \frac{1}{(l-1)!}}{z}, \ \varphi_0(z) = e^z.$$

Then the weights $b_i(h_{n+1}L)$, $i = 1, ..., \nu$, are linear combinations of the matrices $\varphi_l(h_{n+1}L)$, $l = 1, 2, ...$, and, for any $i = 1, ..., \nu$, the coefficients $a_{ij}(h_{n+1}L)$, $j = 1, ..., \nu$, are linear combinations of $\varphi_{l,i}(h_{n+1}L) := \varphi_l(c_i h_{n+1}L)$, $l = 1, 2, ...$ In other words, weights and coefficients of the scheme (2) are given by

$$b_i(h_{n+1}L) = \int_0^1 e^{(1-s)h_{n+1}L} p_i(s) ds, \ i = 1, ..., \nu, \tag{5a}$$

$$a_{ij}(h_{n+1}L) = \int_0^1 e^{(1-s)c_i h_{n+1}L} p_{ij}(s) ds, \ i, j = 1, ..., \nu, \tag{5b}$$

where p_i and p_{ij} are polynomials. They were presented in this form for the first time in [8].

The simplest method of this type is the *explicit ETD Euler method* given by

$$y_{n+1} = e^{h_{n+1}L} y_n + h_{n+1} \varphi_1(h_{n+1}L) f(t_n, y_n), \tag{6}$$

which equals the exact solution whenever $f(t, y)$ is constant.

We remark that an exponential RK method (2) can be applied also to an abstract semilinear ODE (1) in which $y(t)$ takes values in a Banach space X and L is the infinitesimal generator of an analytic semigroup on X. In this case, the matrix exponentials are replaced by the semigroup operators.

Since exponential methods should be used when the linear part in (1) is stiff, instead of the classical (non-stiff) order it is better to consider the more important concept of *stiff order*.

Definition 1. An exponential RK method (2) has stiff order p if the local error has order $p + 1$ with respect to h_{n+1} when the method is applied to an abstract semi-linear ODE (1) in which $z(t) = f(t, y(t))$ is a sufficiently smooth function of t.

In other words, the stiff order describes the behaviour of the local error independently of the norm of the matrix L in (1).

Stiff order conditions for exponential RK methods were developed in [14, Table 2] up to the order four. Here, we simply recall that the stiff order one condition is

$$\sum_{i=1}^{\nu} b_i(\cdot) = \varphi_1 \tag{7}$$

and the stiff order two condition is

$$\sum_{i=1}^{\nu} b_i(\cdot)\, c_i = \varphi_2,\ \sum_{i=1}^{\nu} a_{ij}(\cdot) = c_i \varphi_{1,i},\ i = 1, ..., \nu. \tag{8}$$

Therefore the explicit ETD Euler method has stiff order one and two-stage explicit exponential RK methods of stiff order two constitute the family with parameter c_2 given by the Butcher tableau

$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2\varphi_{1,2} & \\ \hline & \varphi_1 - \frac{1}{c_2}\varphi_2 & \frac{1}{c_2}\varphi_2 \end{array}. \tag{9}$$

The so-called *ETD2RK method* (see [25, Eq. (3.6)], [22, Section 3] and [7, Eq. (22)])

$$\begin{array}{c|cc} 0 & & \\ 1 & \varphi_1 & \\ \hline & \varphi_1 - \varphi_2 & \varphi_2 \end{array}$$

is a particular method in this family.

It is also useful to consider the concept of *stiff convergence order*.

Definition 2. An exponential RK method has stiff convergence order p if the global error has order p with respect to h, $h = \max_n h_{n+1}$, when the method is applied to an abstract semi-linear ODE (1) in which $z(t) = f(t, y(t))$ is a sufficiently smooth function of t.

In other words, the stiff global order describes the behaviour of the global error independently of the norm of the matrix L in (1).

Of course, an exponential RK method of stiff order p also has stiff convergence order p. However, in order to ensure stiff convergence order p when the method is applied with constant stepsize $h = h_{n+1}$ for all n, it is sufficient to satisfy the stiff order conditions up to the order $p-1$ and the order p condition with $b_i = b_i(0)$ instead of $b_i(\cdot)$, $i = 1, ..., \nu$ (see [14, Theorem 4.7]).

From now on, when we say that a method has stiff convergence order p, we mean that it satisfies the stiff order conditions up to the order $p-1$ and the order p condition in this weak form.

The stiff convergence order one is obtained when

$$\sum_{i=1}^{\nu} b_i = \varphi_1(0) = 1 \tag{10}$$

and the stiff convergence order two is obtained when (7) holds along with

$$\sum_{i=1}^{\nu} b_i c_i = \varphi_2(0) = \frac{1}{2},\ \sum_{i=1}^{\nu} a_{ij}(\cdot) = c_i \varphi_{1,i},\ i = 1, ..., \nu.$$

In particular, a family of two-stage explicit exponential RK methods of stiff convergence order two is given by

$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2\varphi_{1,2} & \\ \hline & \left(1-\frac{1}{2c_2}\right)\varphi_1 & \frac{1}{2c_2}\varphi_1 \end{array} \tag{11}$$

where c_2 is a parameter and, with respect to the methods in (9), the weights involve φ_1 only. The so-called *RKMK2e method* (see [23, Ex. 4]), or Pseudo-Steady-State-Approximation (PSSA) scheme (see [26, Section 2]),

$$\begin{array}{c|cc} 0 & & \\ 1 & \varphi_1 & \\ \hline & \frac{1}{2}\varphi_1 & \frac{1}{2}\varphi_1 \end{array}$$

belongs to this family.

Even if IF methods do not have stiff order one, they achieve stiff convergence order one (and not more than one) whenever condition (10) holds for the underlying RK method. However, they perform to their full non-stiff convergence order on particular problems where the solution satisfies certain smoothness properties such as, for example, the non-linear Schrodinger equation (see [3]).

3. Stability definitions

We begin this section by giving two well-known conditions guaranteeing contractivity and asymptotic stability, respectively, for semi-linear systems of ODEs (1). To this aim, let us introduce a norm $\|\cdot\|$ on $\mathbb{R}^d$ which induces a norm on $\mathbb{R}^{d\times d}$ denoted by the same symbol. Moreover, let $\mu(L)$ and γ denote, respectively, the logarithmic norm of L and the Lipschitz constant (with respect to the second argument) of f in (1) relevant to the given norm on $\mathbb{R}^d$.

We recall the following two properties of the logarithmic norm (see e.g. [24]) which will be widely used later:

- $\left\|e^M\right\| \le e^{\mu(M)}$ for $M \in \mathbb{R}^{d\times d}$,
- $\mu(\alpha M) = \alpha\mu(M)$ for $M \in \mathbb{R}^{d\times d}$ and $\alpha \ge 0$.

Let us consider the semi-linear system (1) with two different initial values:

$$\begin{cases} u'(t) = Lu(t) + f(t, u(t)), \; t \ge t_0, \\ u(t) = u_0, \end{cases} \tag{12}$$

and

$$\begin{cases} v'(t) = Lv(t) + f(t, v(t)), \; t \ge t_0, \\ v(t_0) = v_0, \end{cases} \tag{13}$$

and introduce the difference

$$\delta(t) := u(t) - v(t).$$

We also set $\delta_0 := \delta(0) = u_0 - v_0$.

It is easy to see that the condition

$$\mu(L) + \gamma \le 0 \tag{14}$$

guarantees contractivity for system (1), i.e.

$$\|\delta\,(t)\| \leq \|\delta_0\|\,,\ t \geq t_0,$$

holds for every u_0, v_0. Moreover, when the sign $\leq$ in (14) is replaced by $<$, system (1) is asymptotically stable, i.e.

$$\|\delta\,(t)\| \to 0,\ t \to +\infty,$$

holds for every u_0, v_0.

Now, consider an explicit exponential RK method (2) as applied with stepsize h to (12) and (13). Denoting by u_{n+1} and v_{n+1} the relevant grid approximations and by U^i_{n+1} and V^i_{n+1} the relevant stage-values, the differences

$$\begin{aligned}\delta_{n+1} &:= u_{n+1} - v_{n+1},\\ \Delta^i_{n+1} &:= U^i_{n+1} - V^i_{n+1},\ i = 1, ..., \nu,\end{aligned}$$

satisfy

$$\delta_{n+1} = e^{hL}\delta_n + h\sum_{i=1}^{\nu} b_i(hL)\left[f(t^i_{n+1}, U^i_{n+1}) - f(t^i_{n+1}, V^i_{n+1})\right], \tag{15a}$$

$$\Delta^i_{n+1} = e^{c_i hL}\delta_n + h\sum_{j=1}^{i-1} a_{ij}(hL)\left[f(t^j_{n+1}, U^j_{n+1}) - f(t^j_{n+1}, V^j_{n+1})\right]. \tag{15b}$$

Now, we introduce some definitions concerning stability properties of exponential RK methods. To this purpose, we consider a class $\mathcal{C}$ of semi-linear systems of the type (1).

Definition 3. An exponential RK method (2) is called unconditionally contractive on the class $\mathcal{C}$ with respect to the norm $\|\cdot\|$ if $\|\delta_1\| \leq \|\delta_0\|$ holds for all u_0, v_0 and for all stepsizes h when applied to any system in $\mathcal{C}$ satisfying (14).

Furthermore, it is called unconditionally asymptotically stable if $\delta_n \to 0$ as $n \to \infty$ whenever (14) is satisfied with $\leq$ replaced by $<$.

Remark that, unlike contractivity, asymptotic stability should not be related to a specific norm. Nevertheless, we define the asymptotic stability properties of the methods with respect to a given norm $\|\cdot\|$ because we require the preservation of the asymptotic behaviour of the solutions of systems belonging to subsets of the class $\mathcal{C}$ which depend on that norm via the logarithmic norm of L and the Lipschitz constant γ of the function f (see condition (14)).

4. Sufficient conditions for stability

Let $\mathcal{M}$ be a class of matrices (even of different dimensions) which is closed with respect to the multiplication by positive scalars (so that, $hM \in \mathcal{M}$ for any $h > 0$, if $M \in \mathcal{M}$).

We study stability properties on the class $\mathcal{C}\,(\mathcal{M})$ of equations (1) with $L \in \mathcal{M}$ with respect to a given norm $\|\cdot\|$. To this aim, we introduce, for $\alpha \leq 0$, the $(1 \times \nu)$-vector $\overline{b}\,(\alpha)$ of components

$$\overline{b}_i\,(\alpha) := \sup_{\substack{M\in\mathcal{M}\\ \mu(M)\leq\alpha}} \|b_i(M)\|\,,\ i = 1, ..., \nu, \tag{16}$$

and the $(\nu \times \nu)$-matrix $\overline{A}(\alpha)$ whose elements are

$$\overline{a}_{ij}(\alpha) := \sup_{\substack{M \in \mathcal{M} \\ \mu(M) \leq \alpha}} \|a_{ij}(M)\|, \; i,j = 1, ..., \nu. \tag{17}$$

Note that $\overline{b}(\alpha)$ and $\overline{A}(\alpha)$ depend on the class $\mathcal{M}$ and on the norm $\|\cdot\|$.

Since $\mathcal{M}$ contains all scalars, we have

$$\sup_{x \leq \alpha} |b_i(x)| \leq \overline{b}_i(\alpha) \quad \text{and} \quad \sup_{x \leq \alpha} |a_{ij}(x)| \leq \overline{a}_{ij}(\alpha), \; \alpha \leq 0, i,j = 1, ..., \nu.$$

Moreover, we remark that the functions $\overline{b}_i$ and $\overline{a}_{ij}$, $i,j = 1, ..., \nu$, are non-decreasing.

Now, we give a bound for $\|\delta_{n+1}\|$ in terms of $\|\delta_n\|$ which holds whenever the matrix L in (1) belongs to the class $\mathcal{M}$.

By (15a) and (15b), we obtain

$$\begin{aligned} \|\delta_{n+1}\| &\leq e^{\mu(hL)} \|\delta_n\| + h\gamma \sum_{i=1}^{\nu} \|b_i(hL)\| \left\|\Delta_{n+1}^i\right\|, \\ \left\|\Delta_{n+1}^i\right\| &\leq e^{c_i \mu(hL)} \|\delta_n\| + h\gamma \sum_{j=1}^{i-1} \|a_{ij}(hL)\| \left\|\Delta_{n+1}^j\right\|, \; i = 1, ..., \nu, \end{aligned}$$

where γ is the Lipschitz constant of f, and then

$$\|\delta_{n+1}\| \leq e^{h\mu(L)} \|\delta_n\| + h\gamma \overline{b}(h\mu(L)) \overline{\Delta}_{n+1}, \tag{18}$$

$$\left(I - h\gamma \overline{A}(h\mu(L))\right) \overline{\Delta}_{n+1} \leq e^{ch\mu(L)} 1_\nu \|\delta_n\|, \tag{19}$$

where $\overline{\Delta}_{n+1}$ is the $(\nu \times 1)$-vector of components $\left\|\Delta_{n+1}^i\right\|$, $i = 1, ..., \nu$, c is the $(\nu \times 1)$-vector of components c_i, $i = 1, ..., \nu$, 1_ν is the $(\nu \times 1)$-vector with all components equal to 1, the notation $x \leq y$, $x, y \in \mathbb{R}^\nu$, stands for $x_i \leq y_i$, $i = 1, ..., \nu$, and the notation e^x, $x \in \mathbb{R}^\nu$, stands for the diagonal $(\nu \times \nu)$-matrix $\operatorname{diag}(e^{x_1}, ..., e^{x_\nu})$.

Since the matrix $\overline{A}(h\mu(L))$ is strictly lower triangular, we obtain

$$\|\delta_{n+1}\| \leq \overline{S}(h\mu(L), h\gamma) \|\delta_n\|, \tag{20}$$

where

$$\overline{S}(\alpha, \beta) := e^\alpha + \sum_{k=0}^{\nu-1} \beta^{k+1} \overline{b}(\alpha) \overline{A}(\alpha)^k e^{c\alpha} 1_\nu, \; \alpha \in \mathbb{R} \text{ and } \beta \geq 0. \tag{21}$$

Since the functions $\overline{b}_i$ and $\overline{a}_{ij}$ defined in (16) and (17), respectively, are non-negative and non-decreasing, the function $\overline{S}$ turns out to be increasing in its first argument.

Now we present a sufficient condition for the unconditional stability properties on the class $\mathcal{C}(\mathcal{M})$, relevant to a class $\mathcal{M}$ of matrices, and with respect to a norm $\|\cdot\|$.

Proposition 4. *If an explicit exponential RK method* (2) *satisfies*

$$\overline{S}(-\beta, \beta) \leq 1, \; \beta \geq 0, \tag{22}$$

where $\overline{S}$ is defined in (21)*, then it is unconditionally contractive and asymptotically stable on the class $\mathcal{C}(\mathcal{M})$ with respect to the norm $\|\cdot\|$.*

Proof. Since the function $\overline{S}$ is increasing in its first argument, we have $\overline{S}(\alpha, \beta) \leq \overline{S}(-\beta, \beta)$ for $\alpha + \beta \leq 0$ and $\overline{S}(\alpha, \beta) < \overline{S}(-\beta, \beta)$ for $\alpha + \beta < 0$. The thesis of the proposition now follows. □

Along with the function $\overline{S}$ we consider also the function

$$S(\alpha,\beta) := e^{\alpha} + \sum_{k=0}^{\nu-1} \beta^{k+1} b(\alpha) A(\alpha)^k e^{c\alpha} 1_\nu, \ \alpha,\beta \in \mathbb{R}, \tag{23}$$

where $b(\alpha)$ is the $(1\times\nu)$-vector of components $b_i(\alpha)$, $i=1,...,\nu$ and $A(\alpha)$ is the $(\nu\times\nu)$-matrix of coefficients $a_{ij}(\alpha)$, $i,j=1,...,\nu$.

Thus, as an immediate consequence of the previous proposition, we obtain the following theorem.

Theorem 5. *If an explicit exponential RK method* (2) *satisfies*

$$S(-\beta,\beta) \le 1, \ \beta \ge 0 \tag{24}$$

and

$$b(\alpha) = \overline{b}(\alpha) \ \text{ and } A(\alpha) = \overline{A}(\alpha), \ \alpha \le 0, \tag{25}$$

then it is unconditionally contractive and asymptotically stable on the class $\mathcal{C}(\mathcal{M})$ *with respect to the norm* $\|\cdot\|$.

It is easy to see that condition (24) is satisfied with equality under the "mild" restriction

$$\sum_{i=1}^{\nu} b_i(\cdot) = \varphi_1 \text{ and } \sum_{j=1}^{i-1} a_{ij}(\cdot) = c_i\varphi_{1,i}, \ i=1,...,\nu, \tag{26}$$

i.e., if the method satisfies the stiff order one condition (7) and the second equation in the stiff order two condition (8). Observe that, however, (26) is not satisfied by IF methods.

In this paper, we study the case where $\mathcal{M}$ is the class of all matrices (even of different dimensions) and so $\mathcal{C}(\mathcal{M})$ is the class $\mathcal{C}_{all}$ of all semi-linear equations (1). We start by analyzing IF methods.

Proposition 6. *For an IF method* (3) *with an explicit underlying RK method* (A,b,c), *we have*

$$S(\alpha,\beta) = e^{\alpha}\left(1 + \sum_{k=0}^{\nu-1} \beta^{k+1} b A^k 1_\nu\right), \ \alpha,\beta \in \mathbb{R}. \tag{27}$$

Proof. For such a method we have, for $\alpha,\beta\in\mathbb{R}$,

$$b(\alpha) = b e^{(1_\nu - c)\alpha}, \ A(\alpha) = e^{c\alpha} A e^{-c\alpha}.$$

Thus

$$b(\alpha) A(\alpha)^k e^{c\alpha} = e^{\alpha} b A^k, \ k=0,1,...,\nu-1,$$

and, by (23), we obtain (27). □

As a consequence, condition (24) holds if

$$bA^k 1_\nu \le \frac{1}{(k+1)!}, \ k \ge \overline{k},$$

where

$$\overline{k} = \min\left\{k\in\{0,1,2,...\} \ \middle| \ bA^k 1_\nu \ne \frac{1}{(k+1)!}\right\}. \tag{28}$$

On the other hand, condition (25) holds if

$$0 = c_1 \leq \cdots \leq c_\nu \leq 1 \tag{29}$$

and

$$b_i \geq 0 \text{ and } a_{ij} \geq 0,\ i = 1, ..., \nu \text{ and } j = 1, ..., i-1, \tag{30}$$

since

$$\overline{b}_i(\alpha) \leq |b_i| e^{(1-c_i)\alpha},\ i = 1, ..., \nu, \tag{31a}$$

$$\overline{a}_{ij}(\alpha) \leq |a_{ij}| e^{(c_i-c_j)\alpha},\ i = 1, ..., \nu \text{ and } j = 1, ..., i-1. \tag{31b}$$

Hence an explicit IF method is unconditionally contractive and asymptotically stable on the class $\mathcal{C}_{all}$ with respect to an arbitrary norm if the underlying RK method is a ν-stage explicit RK method of order ν satisfying (29) and (30).

Examples of such methods are well-known up to $\nu = 4$ and the classical RK method of order four is the sole method with four stages (see [17, page 521]). Moreover, it is also known that there do not exist explicit methods of order greater than four satisfying (30) with positive weights (see [17, Corollary 8.7]).

For methods which are not of IF type, condition (24) is implied by (26). For an ETD method (5), condition (25) holds if

$$p_i(s) \geq 0 \text{ and } p_{ij}(s) \geq 0,\ s \in [0,1] \text{ and } i = 1, ..., \nu \text{ and } j = 1, ..., i-1,$$

since

$$\overline{b}_i(\alpha) \leq \int_0^1 e^{(1-s)\alpha} |p_i(s)|\, ds,\ i = 1, ..., \nu,$$

$$\overline{a}_{ij}(\alpha) \leq \int_0^1 e^{(1-s)c_i\alpha} |p_{ij}(s)|\, ds,\ i = 1, ..., \nu \text{ and } j = 1, ..., i-1.$$

As a consequence, we can conclude that explicit ETD Euler method (6) is unconditionally contractive and asymptotically stable on the class $\mathcal{C}_{all}$ with respect to an arbitrary norm.

Moreover, in the family (9) of two-stage exponential RK methods of stiff order two, a method is unconditionally contractive and asymptotically stable on the class $\mathcal{C}_{all}$, with respect to an arbitrary norm, if $c_2 \geq 1$ and, in the family (11), if $c_2 \geq \frac{1}{2}$.

We have not yet found any ETD method of stiff convergence order greater than two that fulfills the above sufficient conditions.

5. Conclusions

In this paper we have studied stability properties of explicit exponential RK methods when they are applied to semi-linear systems of ODEs. The properties of unconditional contractivity and unconditional asymptotic stability have been introduced, and some popular explicit methods have been investigated with respect to such properties.

As for IF methods, we have proved unconditional contractivity and asymptotic stability on the whole class $\mathcal{C}_{all}$ of semi-linear systems with respect to any norm if the underlying explicit RK method has non-negative weights and coefficients and has order equal to the number of stages. Hence, we have proved that there exist several IF methods of order $p \leq 4$ which are unconditionally contractive and asymptotically stable.

On the other hand, it is known that IF methods have stiff convergence order one only. Unfortunately, for methods outside this class, which can reach higher stiff convergence order, there are very few examples which are proved to be unconditionally contractive or asymptotically stable on the class $\mathcal{C}_{all}$ with respect to any norm; they are the explicit ETD Euler method, the methods in the family (9) with $c_2 \geq 1$ and the methods in the family (11) with $c_2 \geq \frac{1}{2}$.

References

[1] U. Ascher, S. Ruuth and R. Spiteri. Implicit-explicit Runge-Kutta methods for time-dependent Partial Differential Equations. Appl. Numer. Math. **25**: 151-167, 1997. MR1485812 (98i:65054)

[2] U. Ascher, S. Ruuth and B. Wetton. Implicit-explicit methods for time-dependent partial differential equations. SIAM J. Numer. Anal. **32** no. 3: 797–823, 1995. MR1335656 (96j:65076)

[3] H. Berland and B. Skaflestad. Solving the non-linear Schrondinger equation using exponential integrators. Technical Report 05/04. The Norwegian Institute of Science and Technology, 2004.

[4] H. Berland and W. Wright. EXPINT—A MATLAB package for exponential integrators. Technical Report 04/05. The Norwegian Institute of Science and Technology, 2005.

[5] G. Beylkin, J. Keiser and L. Vozovoi. A new class of time discretization schemes for the solution of nonlinear PDEs. J. Comput. Phys. **147** no. 2: 362–387, 1998. MR1663563 (99k:65079)

[6] E. Celledoni, A. Marthinsen and B. Owren. Commutator-free Lie group methods. Future Generation Computer Systems, **19**: 341-352. 2003.

[7] S. Cox and P. Matthews. Exponential time differencing for stiff systems. J. Comput. Phys. **176** no. 2: 430–455 (2002). MR1894772 (2003b:65064)

[8] A. Friedli. Verallgemeinerte Runge-Kutta Verfahren zur Losong steifer Differentialgleichungssysteme. Numerical Treatment of Differential Equations, R. Bulirsch, R. Grigorieff and J. Schroder, eds. Lecture Notes in Mathematics, 631, Springer, 1978. MR0494950 (58:13726)

[9] M. Hochbruck and C. Lubich. On Krylov subspace approximations to the matrix exponential operator. SIAM J. Numer. Anal. **34** no. 5: 1911–1925, 1997. MR1472203 (98h:65018)

[10] M. Hochbruck and C. Lubich. Exponential integrators for quantum-classical molecular dynamics. BIT **39** no. 4: 620–645, 1999. MR1735097 (2000k:65231)

[11] M. Hochbruck and C. Lubich. A Gautschi-type method for oscillatory second-order differential equations. Numer. Math. **83** no. 3: 403–426, 1999. MR1715573 (2000i:65098)

[12] M. Hochbruck, C. Lubich and H. Selhofer. Exponential integrators for large systems of differential equations. SIAM J. Sci. Comput. **19** no. 5: 1552–1574, 1998. MR1618808 (99f:65101)

[13] M. Hochbruck and A. Ostermann. Exponential Runge-Kutta methods for parabolic problems. Appl. Numer. Math. **53** no. 2-4: 323–339, 2005. MR2128529 (2005m:65195)

[14] M. Hochbruck and A. Ostermann. Explicit Exponential Runge-Kutta methods for semilinear parabolic problems. SIAM J. Numer. Anal. **43** no. 3: 1069-1090, 2005. MR2177796 (2006h:65137)

[15] W. Hundsdorfer and J. Verwer. Numerical Solution of Time-Dependent Advection-Diffusion Reaction Equations. Springer Series In Computational Mathematics, 33. Springer-Verlag, 2003. MR2002152 (2004g:65001)

[16] A. Kassam and L. Trefethen. Fourth-order time-stepping for stiff PDEs. SIAM J. Sci. Comput. **26** no. 4: 1214–1233, 2005. MR2143482 (2006g:65159)

[17] J. Kraaijevanger. Contractivity of Runge-Kutta methods. BIT **31**: 482-528, 1991. MR1127488 (92i:65120)

[18] S. Krogstad. Generalized integrating factor methods for stiff PDEs. J. Comput. Phys. **203** no. 1: 72–88, 2005. MR2104391 (2005h:65181)

[19] J. Lawson. Generalized Runge-Kutta processes for stable systems with large Lipschitz constants. SIAM J. Numer. Anal. **4**: 372–380, 1967. MR0221759 (36:4811)

[20] Y. Maday, A. Patera and E. Rønquist. An operator-integration-factor splitting method for time-dependent problems: application to incompressible fluid flow. J. Sci. Comput. **5** no. 4: 263–292, 1990. MR1105250 (92a:76068)

[21] B. Minchev and W. Wright. A review of exponential integrators for first order semi-linear problems. Preprint Numerics 2/2005. Norwegian University of Science and Technology, Trondheim Norway.
[22] D. Mott, E. Oran and B. van Leer. A quasi-steady-state solver for stiff ordinary differential equations of reaction kinetics. J. Comput. Phys. **164**: 407-428, 2000. MR1792518
[23] H. Munthe-Kaas. High order Runge-Kutta methods on manifolds. Appl. Numer. Math. **29** no. 1: 115–127, 1999. MR1662814 (99i:65075)
[24] T. Storm. On logarithmic norms. SIAM J. Numer. Anal. **12** no. 5: 741–753, 1975. MR0408227 (53:11992)
[25] K. Strehmel and R. Weiner. B-convergence result for linearly implicit one step methods. BIT **27**: 264-281, 1987. MR894127 (88g:65067)
[26] J. Verwer and M. van Loon. An evaluation of explicit pseudo-steady-state approximation schemes for stiff ODEs systems from chemical kinetics. J. Comp. Phys. **113**: 347-352, 1994. MR1284858 (95g:80013)

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