

## RATIONAL SZEGŐ QUADRATURES ASSOCIATED WITH CHEBYSHEV WEIGHT FUNCTIONS

ADHEMAR BULTHEEL, RUYMÁN CRUZ-BARROSO, KARL DECKERS,  
AND PABLO GONZÁLEZ-VERA

ABSTRACT. In this paper we characterize rational Szegő quadrature formulas associated with Chebyshev weight functions, by giving explicit expressions for the corresponding para-orthogonal rational functions and weights in the quadratures. As an application, we give characterizations for Szegő quadrature formulas associated with rational modifications of Chebyshev weight functions. Some numerical experiments are finally presented.

### 1. INTRODUCTION

Szegő quadrature formulas approximate integrals over the complex unit circle  $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$  and are the analogs of Gauss quadrature formulas that approximate integrals over an interval  $I := [-1, 1]$ . That is,

$$\int f(z) d\mu(z) \approx \sum_{j=1}^n \lambda_j f(z_j),$$

where  $\mu$  is a positive measure on  $\mathbb{T}$ , respectively  $I$ .

Gauss quadrature formulas are optimal in the sense that nodes  $z_j \in I$  and weights  $\lambda_j > 0$  are chosen such that the quadrature formula is exact for all functions  $f \in \mathbb{P}_{2n-1}$  that are polynomials of degree up to  $2n - 1$  and it is not possible to construct a quadrature formula of this form that is exact for all polynomials of degree  $2n$ . They have a maximal domain of validity. Therefore, the nodes have to be chosen as the zeros of the  $n$ th polynomial orthogonal with respect to the inner product  $\langle f, g \rangle = \int f(z)g(z)d\mu(z)$ . These zeros are all simple and in  $I$ .

On the unit circle, one needs to define a positive definite Hermitian inner product  $\langle f, g \rangle = \int f(z)\overline{g(z)}d\mu(z)$ . The  $n$ th orthogonal polynomial with respect to this inner product has all its zeros inside the open unit disk, and thus, they are not very useful as nodes of a quadrature formula. Here, the para-orthogonal polynomials are

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producing the nodes that are needed. Para-orthogonal polynomials are orthogonal to all polynomials of lower degree, except the constants. All their zeros are simple and on  $\mathbb{T}$ . These para-orthogonal polynomials have one free parameter  $\tau \in \mathbb{T}$ . When using the zeros of the  $n$ th para-orthogonal polynomials, then there exist positive weights such that the quadrature formula is exact for all  $f \in \mathcal{R}_{n-1}$ , that is, the space of all Laurent polynomials of degree at most  $n - 1$ , again a space of dimension  $2n - 1$ . Like for Gaussian formulas, this space is a maximal domain of validity since it is not possible to have an  $n$ -point quadrature formula with distinct nodes and positive weights that is exact in a space of Laurent polynomials of a larger degree neither for the positive or the negative powers of  $z$ . The Szegő quadrature formulas were first studied by Jones, Njåstad, and Thron [25] in connection with the trigonometric moment problem.

The Joukowski transform is a map between the unit circle and the interval  $[-1, 1]$ . It implies a relationship between the Szegő and the Gauss quadrature formulas. In fact, this was already studied by Szegő [28, Section 11.5] and Geronimus [21, Chapter 9]. For  $\tau = 1$ , a  $2n$ -point Szegő formula results in an  $n$ -point Gauss quadrature formula and for  $\tau = -1$ , a  $(2n + 2)$ -point Szegő formula can be related to an  $n$ -point Gauss-Lobatto formula, having two extra nodes in the endpoints of the interval. Also Gauss-Radau formulas can be obtained taking  $(2n + 1)$ -point Szegő formulas and  $\tau = \pm 1$ , which fixes one of the endpoints of the interval; see [1]. For other values of  $\tau$ , the quadrature on the interval is not optimal; see [2]. Szegő-Lobatto and Szegő-Radau formulas on the unit circle were recently discussed by Jagels and Reichel [24].

This theory has been generalized in a sequence of papers by Bultheel et al. to the case where the (orthogonal) polynomials in the previous theory are replaced by (orthogonal) rational functions having prescribed poles outside the closed unit disk. If all these poles are at infinity, the polynomials reappear as a special case. For a comprehensive survey see [5]. See also [6] for a survey.

So the Szegő quadrature formulas are replaced by rational Szegő quadrature formulas. If  $\mathcal{L}_n$  is the space of rational functions of degree  $n$  at most whose poles are at  $\{1/\bar{\alpha}_k : k = 1, \dots, n\}$  and  $\mathcal{L}_{n^*}$  is the space of rational functions of degree  $n$  at most whose poles are at  $\{\alpha_k : k = 1, \dots, n\}$ , then the rational Szegő quadrature formulas are exact in the space  $\mathcal{R}_{n-1} = \mathcal{L}_{n-1} + \mathcal{L}_{(n-1)^*}$ . The nodes are the zeros of the  $n$ th rational para-orthogonal function that depends, as in the polynomial case, on a parameter  $\tau \in \mathbb{T}$ .

A relation between orthogonal rational functions on the interval and on the unit circle has been discussed in [16] and [32]. Rational Szegő-Lobatto and Szegő-Radau formulas have been recently studied in [7].

Gauss-Chebyshev quadrature formulas are associated with a measure that is generated by one of the Chebyshev weights. These weights are among the rare examples where a translation to the unit circle gives explicit expressions for the orthogonal polynomials. This is also true for the orthogonal rational functions, which have been under investigation with respect to rational Gauss-Chebyshev quadrature; see e.g. [15], [17], [18], [20], [29] and [30]. To get more general cases where explicit expressions are obtained, a technique of rational modifications of these Chebyshev weights are considered in [14].

In this paper we will continue in this line of developments by considering the construction of explicit expressions for the para-orthogonal rational functions and

for the weights in rational Szegő quadrature formulas that result from Chebyshev weights as they are translated into the unit circle. As an application, characterization results for Szegő quadrature formulas associated with rational modifications of these Chebyshev weights are obtained.

The outline of this paper is as follows. After giving the theoretical preliminaries in Section 2, in Section 3 we give explicit expressions for the orthogonal rational functions on the unit circle, associated with Chebyshev weight functions. Next, in Sections 4 and 5 we characterize rational Szegő quadrature formulas associated with Chebyshev weight functions, respectively, Szegő quadrature formulas associated with rational modifications of Chebyshev weight functions. We conclude with some numerical examples.

### 2. PRELIMINARIES

Suppose  $\mu$  is a positive bounded Borel measure on  $[-\pi, \pi]$ , and consider the general framework of the approximation of integrals on the unit circle  $\mathbb{T}$  in the complex plane, i.e., integrals of the form<sup>1</sup>

$$(1) \quad I_\mu(f) := \int_{-\pi}^\pi f(e^{i\theta})d\mu(\theta) = \int_{\mathbb{T}} f(z)d\mu(z).$$

As usual, estimations of  $I_\mu(f)$  are produced when replacing  $f(z)$  in (1) by an appropriate approximating (interpolating) function  $L(z)$ , so that  $I_\mu(L)$  can now be easily computed. Let  $\Lambda := \mathbb{C}[z, z^{-1}]$  denote the complex vector space of Laurent polynomials in the variable  $z$ . We then set  $\Lambda_{p,q} := \text{span}\{z^p, \dots, z^q\}$  for  $p, q \in \mathbb{Z}$ , with  $p \leq q$ . Because of the density of  $\Lambda$  in  $C(\mathbb{T}) = \{f : \mathbb{T} \rightarrow \mathbb{C}, f \text{ is continuous}\}$  with respect to the uniform norm (see e.g. [13, pp. 304–305]), it seems reasonable to approximate  $f(z)$  in (1) by an appropriate Laurent polynomial. This way, the so-called “quadrature formulas on the unit circle”, or “Szegő rules”, introduced in [25] (see also [22], [23, Chapter 4] and [27]) and of the form

$$(2) \quad I_n(f) := \sum_{j=1}^n \lambda_j f(z_j), \quad z_j \in \mathbb{T}, \quad j = 1, \dots, n, \quad z_j \neq z_k \text{ if } j \neq k,$$

appear as the analogue on  $\mathbb{T}$  of the Gaussian formulas when dealing with the estimation of integrals with respect to a measure supported on an interval  $[a, b]$ , with  $-\infty \leq a < b \leq +\infty$  (see e.g. [19]).

Consider now the Hilbert space  $L_2^\mu(\mathbb{T})$  of measurable functions  $\phi$  for which  $\int_{-\pi}^\pi |\phi(e^{i\theta})|^2 d\mu(\theta) < +\infty$ . Then the inner product induced by  $\mu$  is given by

$$(3) \quad \langle \phi, \psi \rangle_\mu = \int_{-\pi}^\pi \phi(e^{i\theta}) \overline{\psi(e^{i\theta})} d\mu(\theta), \quad \phi, \psi \in L_2^\mu(\mathbb{T}).$$

In this paper we will deal with the more general framework of orthogonal rational functions (ORFs). Suppose a sequence of complex numbers  $\mathcal{A} = \{\alpha_1, \alpha_2, \dots\} \subset \mathbb{D}$  is given, and define the *Blaschke factors*

$$\zeta_k(z) = \frac{z - \alpha_k}{1 - \overline{\alpha_k}z}, \quad k = 1, 2, \dots,$$

and *Blaschke products*

$$B_0(z) \equiv 1, \quad B_k(z) = B_{k-1}(z)\zeta_k(z), \quad k = 1, 2, \dots$$

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<sup>1</sup>The measure  $\mu$  on  $[-\pi, \pi]$  induces a measure on  $\mathbb{T}$  for which we shall use the same notation  $\mu$ .

The space of rational functions with poles in  $\{1/\overline{\alpha_1}, \dots, 1/\overline{\alpha_n}\}$  is then defined as

$$\mathcal{L}_n = \text{span} \{B_0(z), \dots, B_n(z)\},$$

where we set  $\mathcal{L}_{-1} = \emptyset$  to be the trivial subspace. This way, the ordinary polynomial situation is recovered by taking  $\alpha_k = 0$  for every  $k = 1, 2, \dots$ , so that  $\zeta_k(z) = z$  and  $B_k(z) = z^k$ .

We define the substar conjugate of a function  $f$  as  $f_*(z) = \overline{f(1/\overline{z})}$ , and the super-star conjugation of a function  $f_n \in \mathcal{L}_n$  as

$$f_n^*(z) = B_n(z)f_{n*}(z).$$

Note that  $f_*(z) = \overline{f(z)}$  whenever  $z \in \mathbb{T}$ . Furthermore, for a given polynomial  $P_n(z) = \sum_{k=0}^n c_k z^k$  of exact degree  $n$ , the super-star conjugate<sup>2</sup> is given by  $P_n^*(z) = \sum_{k=0}^n \overline{c_{n-k}} z^k$ . Consequently, define

$$\pi_0(z) \equiv 1, \quad \pi_k(z) = \prod_{j=1}^k (1 - \overline{\alpha_j} z), \quad k = 1, 2, \dots,$$

and let  $\mathbb{P}_n$  represent the space of polynomials of degree less than or equal to  $n$ . Then, we equivalently have for every  $k = 1, 2, \dots$  that

$$B_k(z) = \frac{\pi_k^*(z)}{\pi_k(z)}$$

and

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n \in \mathbb{P}_n \right\}.$$

For a fixed natural number  $n$  we obtain a set of orthonormal rational functions  $\{\chi_k(z)\}_{k=0}^n$  by orthonormalizing the basis  $\{B_k(z)\}_{k=0}^n$  (in this order) with respect to the measure  $\mu$  and inner product given by (3). Note that  $\chi_n(z) = \sum_{k=0}^n a_k B_k(z)$  is uniquely determined if we assume the *leading coefficient*  $a_n$  to be strictly positive. Repeating the process for every  $n$ , an orthonormal system  $\{\chi_k(z)\}_{k=0}^\infty$  is obtained, so that  $\chi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ ,  $\chi_n \perp \mathcal{L}_{n-1}$  and  $\langle \chi_n, \chi_n \rangle_\mu = 1$  for every  $n \geq 0$ . We now have the following lemma.

**Lemma 2.1.** *Let  $\chi_n(z) = \frac{P_n(z)}{\pi_n(z)} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ . Then  $P_n$  has exactly  $n$  zeros in  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ .*

*Proof.* Since all the zeros of  $\chi_n$  are in  $\mathbb{D}$  because of [5, Corollary 3.2.2(3)], a zero of  $P_n$  that is not a zero of  $\chi_n$  is only possible if it cancels a zero of  $\pi_n$ . Suppose there exists an  $\alpha \in \mathbb{D}$  such that  $1/\overline{\alpha}$  is a zero of  $P_n$  with multiplicity  $m \geq 1$ . For the special case in which  $\alpha = 0$ , we say that  $P_n$  has a zero at infinity with multiplicity  $m$  iff  $P_n \in \mathbb{P}_{n-m}$ . Hence it follows that there are at least  $m$  indices  $j \in \{1, \dots, n\}$  for which  $\alpha_j = \alpha$ .

Whatever the choice of the sequence of  $\alpha_k$  is, it should always hold that  $\chi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ . Now take all  $\alpha_k$  that are different from  $\alpha$  fixed, and consider  $\alpha$  variable. Note that  $\chi_n$  depends continuously on  $\alpha$ , as do  $P_n$  and  $\pi_n$ . So let us make this explicit by writing  $\chi_n(z, \alpha) = P_n(z, \alpha)/\pi_n(z, \alpha)$ . Since  $P_n(z, \alpha)$  is of the form

$$P_n(z, \alpha) = (1 - \overline{\alpha}z)^m \hat{P}_{n-m}(z, \alpha) \quad \text{with} \quad \hat{P}_{n-m} \in \mathbb{P}_{n-m},$$

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<sup>2</sup>In literature, this is also referred to as the reversed or reciprocal polynomial.

we obtain for  $\alpha \rightarrow \alpha_n$ ,

$$\begin{aligned} \chi_n(z, \alpha_n) &= \lim_{\alpha \rightarrow \alpha_n} \frac{(1 - \bar{\alpha}z)^m \hat{P}_{n-m}(z, \alpha)}{\pi_n(z, \alpha)} \\ &= \frac{(1 - \bar{\alpha}_n z)^{m-1} \hat{P}_{n-m}(z, \alpha_n)}{\pi_{n-1}(z, \alpha_n)} \in \mathcal{L}_{n-1}, \end{aligned}$$

contradicting our assumption that  $\chi_n(z, \alpha) \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$  for all possible choices of  $\alpha \in \mathbb{D}$ . □

Next, consider for every  $n, p, q \geq 0$  the sets

$$\begin{aligned} \mathcal{L}_{n*} &= \{f : f_* \in \mathcal{L}_n\}, \\ \mathcal{L}_{p,q} &= \mathcal{L}_q \cdot \mathcal{L}_{p*} = \{fg : f \in \mathcal{L}_q, g \in \mathcal{L}_{p*}\} = \left\{ \frac{P(z)}{\pi_q(z)\pi_p^*(z)} : P \in \mathbb{P}_{p+q} \right\} \end{aligned}$$

and  $\mathcal{R}_n = \mathcal{L}_{n,n}$ . Following the ordinary polynomial situation (see [25]), we say that a sequence of functions  $\Psi_n(z) \in \mathcal{L}_n$  is *para-orthogonal* whenever  $\Psi_n(z) \perp \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$ , where  $\mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n : f(\alpha_n) = 0\}$ , and  $\langle \Psi_n(z), 1 \rangle_\mu \cdot \langle \Psi_n(z), B_n(z) \rangle_\mu \neq 0$ . Furthermore,  $\Psi_n(z)$  is called “ $\kappa_n$ -invariant” iff there exists a  $\kappa_n \in \mathbb{T}$  so that  $\Psi_n^*(z) = \kappa_n \Psi_n(z)$  for every  $z \in \mathbb{C}$ . Note that the concept of  $\kappa_n$ -invariance is usually defined in literature for  $\kappa_n \in \mathbb{C} \setminus \{0\}$ . However, there can only exist  $\kappa_n$ -invariant rational functions whenever  $\kappa_n \in \mathbb{T}$ . Indeed, if  $\Psi_n(z) = \frac{P_n(z)}{\pi_n(z)} \in \mathcal{L}_n$  is  $\kappa_n$ -invariant, then  $\Psi_n^*(z) = \frac{P_n^*(z)}{\pi_n^*(z)} = \frac{\kappa_n P_n(z)}{\pi_n(z)}$ . Consequently, it remains to prove the statement for ordinary polynomials. If  $P_n(z) = \sum_{k=0}^n c_k z^k$ , with  $c_n \neq 0$ , is  $\kappa_n$ -invariant, then it holds that  $c_0 = P(0) \neq 0$ ,  $\bar{c}_n = \kappa_n c_0$  and  $\bar{c}_0 = \kappa_n c_n$ . Consequently,  $\kappa_n = \frac{\bar{c}_n}{c_0} = \frac{c_n}{\bar{c}_n}$ , which implies that  $|c_n| = |c_0|$ , and hence, that  $\kappa_n \in \mathbb{T}$ .

$\kappa_n$ -invariant para-orthogonal rational functions for  $\mu$  are characterized in [3] as

$$(4) \quad \Psi_n(z) = \Psi_n(z, \tau_n) = C_n [\phi_n(z) + \tau_n \phi_n^*(z)] \in \mathcal{L}_n; \quad n \geq 1, \quad C_n \in \mathbb{C} \setminus \{0\},$$

where  $\tau_n = \frac{\bar{C}_n \kappa_n}{C_n} \in \mathbb{T}$ , and  $\phi_n(z)$  is an  $n$ th orthogonal rational function for  $\mu$ . Furthermore, it is proved in [7, Theorem 2.4] that it suffices to compute  $\phi_{n-1}(z)$  for the computation of  $\Psi_n(z)$ .

The following result, proved by Bultheel et al. (see [5, Chapter 5]), is an extension of a well-known characterization for Szegő quadrature formulas (see [25]) to the rational case.

**Theorem 2.2** (Rational Szegő quadrature). *Let  $n \geq 1$  and  $\tau_n \in \mathbb{T}$ . Then,*

- (1)  $\Psi_n(z, \tau_n)$ , given by (4), has exactly  $n$  distinct zeros on  $\mathbb{T}$ ,
- (2) there exist positive numbers  $\lambda_1, \dots, \lambda_n$  so that

$$(5) \quad I_n(f) = \sum_{j=1}^n \lambda_j f(z_j) = I_\mu(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) d\mu(\theta), \quad \forall f \in \mathcal{R}_{n-1},$$

where  $z_1, \dots, z_n$  are the zeros of  $\Psi_n(z, \tau_n)$ ,

- (3)  $\mathcal{R}_{n-1}$  is a maximal domain of validity, i.e., there cannot be exactness in  $\mathcal{L}_{n-1,n}$ , nor in  $\mathcal{L}_{n,n-1}$ . □

A connection between quadrature formulas on the unit circle and the interval  $[-1, 1]$  is given in [1]. If  $\sigma(x)$  is a weight function on  $[-1, 1]$ , we obtain a weight function on  $\mathbb{T}$  by setting  $\mu'(\theta) = \omega(\theta) = \sigma(\cos \theta) |\sin \theta|$  (see [28]), where

$\mu'$  denotes the Radon-Nikodym derivative of the measure  $\mu$  with respect to the Lebesgue measure. In the special case in which  $\sigma(x)$  is a Jacobi weight function, i.e.,  $\sigma(x) = (1-x)^\alpha(1+x)^\beta$ , with  $\alpha, \beta > -1$ , the corresponding weight function on  $\mathbb{T}$  is given by

$$\begin{aligned} \omega(\theta) &= (1 - \cos \theta)^\alpha (1 + \cos \theta)^\beta |1 - \cos^2 \theta|^{\frac{1}{2}} \\ &= (1 - \cos \theta)^a (1 + \cos \theta)^b, \quad a = \alpha + \frac{1}{2} > -\frac{1}{2}, \quad b = \beta + \frac{1}{2} > -\frac{1}{2}. \end{aligned}$$

Finally, if  $a, b \in \{0, 1\}$ , the so-called Chebyshev weight functions appear. Therefore, we set

$$(6) \quad \omega_1(\theta) \equiv 1, \quad \omega_2(\theta) = \sin^2 \theta, \quad \omega_3(\theta) = 1 - \cos \theta, \quad \omega_4(\theta) = 1 + \cos \theta.$$

In the remainder of this paper we shall be concerned with rational Szegő quadratures associated with the Chebyshev weight functions  $\omega(\theta) = \omega_i(\theta)$ ,  $i = 1, \dots, 4$  given by (6). We start in the next section with giving explicit expressions for the corresponding ORFs.

### 3. ORFS ASSOCIATED WITH CHEBYSHEV WEIGHT FUNCTIONS

As it is known, few measures give rise to explicit expressions for orthogonal polynomials and even less for ORFs; generally, the computation of such a family proceeds by using a recursive process (see e.g. [5, Theorem 4.1.1]).

It is well known that the so-called *Malmquist basis*, given by

$$(7) \quad \phi_0^{(1)}(z) \equiv 1 \quad \text{and} \quad \phi_n^{(1)}(z) = \frac{z B_{n-1}(z)}{1 - \overline{\alpha_n} z}, \quad n > 0,$$

is an orthogonal basis for the Lebesgue measure  $d\mu(\theta) = \omega_1(\theta)d\theta \equiv d\theta$  (see e.g. [5, p. 51]). Recently, explicit expressions are derived in [16] for ORFs associated with the weight functions  $\omega_i(\theta)$ ,  $i \in \{3, 4\}$ , given by (6). Let  $Q_n(z)$  be defined as

$$(8) \quad Q_n(z) = 1 + z \frac{B'_{n-1}(z)}{B_{n-1}(z)} = 1 + \sum_{k=1}^{n-1} \frac{1 - |\alpha_k|^2}{|z - \alpha_k|^2} \quad \text{for } z \in \mathbb{T}.$$

We then have the following theorem ([16]).

**Theorem 3.1.** *Let  $i \in \{3, 4\}$  be fixed and set  $\nu_i = (-1)^{i-1}$ . Next, define*

$$(9) \quad X_n^{(i)}(z) = a_n^{(i)} + z^2 \left( z - b_n^{(i)} \right) \frac{B_{n-1}(z)}{1 - \overline{\alpha_n} z},$$

where

$$(10) \quad a_n^{(i)} = \frac{\nu_i B_{n-1}(\nu_i)}{(1 - \nu_i \overline{\alpha_n}) Q_n^{(i)} + 1}, \quad b_n^{(i)} = \nu_i + \frac{1 - \nu_i \overline{\alpha_n}}{B_{n-1}(\nu_i)} a_n^{(i)}$$

and  $Q_n^{(i)} = Q_n(\nu_i)$  (note that this way,  $X_n^{(i)}(\nu_i) = X_n^{(i)'}(\nu_i) = 0$ ). Then the sequence of rational functions  $\left\{ \phi_n^{(i)}(z) \right\}_{n=0}^\infty$ , where

$$(11) \quad \phi_0^{(i)}(z) \equiv 1 \quad \text{and} \quad \phi_n^{(i)}(z) = \frac{X_n^{(i)}(z)}{(z - \nu_i)^2}, \quad n > 0,$$

forms a set of ORFs with respect to  $\omega_i(\theta) = 1 - \nu_i \cos \theta$ . Furthermore, the sequence  $\{\chi_n^{(i)}(z)\}_{n=0}^\infty$ , with  $\chi_n^{(i)}(z) = c_n^{(i)} \phi_n^{(i)}(z)$ , where

$$(12) \quad \left| c_0^{(i)} \right|^2 = \frac{1}{2\pi} \quad \text{and} \quad \left| c_n^{(i)} \right|^2 = \nu_i \frac{(1 - |\alpha_n|^2)(1 - \nu_i \alpha_n)^2}{\pi \left( b_n^{(i)} - \alpha_n \right) \left( 1 - \alpha_n \overline{b_n^{(i)}} \right)}, \quad n > 0,$$

forms a set of orthonormal rational functions with respect to  $\omega_i(\theta)$ . □

So far, explicit expressions for ORFs associated with the weight function  $\omega_2(\theta)$  in (6) are still not known (as indicated in [16]). So, in the remainder of this section we will deal with this open problem.

Note that

$$\begin{aligned} \omega_2(\theta) &= \sin^2 \theta = (1 + \cos \theta)(1 - \cos \theta) = \left( 1 + \frac{z^2 + 1}{2z} \right) (1 - \cos \theta) \\ &= \frac{1}{2} |z + 1|^2 (1 - \cos \theta) = \frac{1}{2} |z + 1|^2 \omega_3(\theta), \quad z = e^{i\theta}. \end{aligned}$$

Hence, suppose  $\phi_{n+1}^{(3)}$  is a rational function with poles in  $\{0, \alpha_1, \dots, \alpha_n\}$  that is orthogonal on the unit circle with respect to the weight function  $\omega_3(\theta)$ . Further, let  $\phi_n^{(2)}$  be a rational function with poles in  $\{\alpha_1, \dots, \alpha_n\}$  that is orthogonal on the unit circle with respect to the weight function  $\omega_2(\theta)$ . Then for  $n > 0$ , it follows from [14, Theorem 6] that there exist constants  $u_n, t_n$  and  $v_n$  so that

$$(13) \quad (z + 1)^2 \phi_n^{(2)}(z) = (u_n z + t_n) \phi_{n+1}^{(3)}(z) + v_n (1 - \overline{\alpha_n} z) \phi_{n+1}^{(3)*}(z).$$

We now have the following two theorems.

**Theorem 3.2.** *Suppose*

$$(14) \quad \begin{aligned} a^+ &= \frac{B_{n-1}(1)}{1 - \overline{\alpha_n}}, & a^- &= \frac{B_{n-1}(-1)}{1 + \overline{\alpha_n}}, \\ b^+ &= 1 + Q_n^{(3)} + \frac{1}{1 - \overline{\alpha_n}}, & b^- &= 1 + Q_n^{(4)} + \frac{1}{1 + \overline{\alpha_n}}, \end{aligned}$$

where  $Q_n^{(3)}$  and  $Q_n^{(4)}$  are defined as before in Theorem 3.1. Next, let

$$\underline{x} = (d_n, e_n, f_n, g_n)^T, \quad \underline{y} = (-a^+, a^-, -a^+(b^+ + 2), a^-(b^- + 2))^T$$

and

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & a^+ & a^+ \\ 1 & -1 & -a^- & a^- \\ 0 & 1 & a^+ b^+ & a^+(b^+ + 1) \\ 0 & -1 & -a^- b^- & a^-(b^- + 1) \end{pmatrix},$$

and assume

$$(15) \quad \underline{x} = \mathbf{A}^{-1} \underline{y} = \frac{1}{\varrho_n} \begin{pmatrix} 4a^+ a^- (b^+ a^- - b^- a^+) \\ -4a^+ a^- [a^+(b^- + 1) + a^-(b^+ + 1)] \\ (a^+ + a^-)^2 + 4a^+ a^- (b^+ + b^- + b^+ b^-) \\ -2[2a^+ a^- (b^+ - b^-) + (a^+ + a^-)(a^+ - a^-)] \end{pmatrix},$$

where  $\varrho_n$  is given by

$$(16) \quad \varrho_n = (a^+ + a^-)^2 - 4a^+ a^- b^+ b^-.$$

Define  $X_n^{(2)}$  by

$$(17) \quad X_n^{(2)}(z) = d_n + e_n z + z^3 (f_n + g_n z + z^2) \frac{B_{n-1}(z)}{1 - \overline{\alpha_n} z},$$

and set

$$\phi_0^{(2)}(z) \equiv 1 \quad \text{and} \quad \phi_n^{(2)}(z) = \frac{X_n^{(2)}(z)}{(z^2 - 1)^2}.$$

Then the sequence of rational functions  $\{\phi_n^{(2)}(z)\}_{n=0}^\infty$  forms a set of ORFs with respect to  $\omega_2(\theta) = \sin^2 \theta$ .

*Proof.* (The computations are cumbersome; therefore, we will only give the outline of the proof.) From Theorem 3.1 and (13) it follows that for  $n > 0$ ,  $\phi_n^{(2)}(z)$  should be of the form

$$\phi_n^{(2)}(z) = C_n \frac{X_n^{(2)}(z)}{(z^2 - 1)^2}, \quad C_n \neq 0,$$

where  $X_n^{(2)}$  is given by (17). For the sake of simplicity, we may as well assume that  $C_n = 1$ . Furthermore, we should have that  $\phi_n^{(2)} \in \mathcal{L}_n$ . Consequently, it must hold that  $X_n^{(2)}(\pm 1) = X_n^{(2)' }(\pm 1) = 0$ . This leaves us with a system of four equations in the four unknowns  $d_n, e_n, f_n$  and  $g_n$ . Solving this system for the coefficients  $d_n, e_n, f_n$  and  $g_n$  then gives (15). The analytic solution of (15), given by the second equality in (15) and (16), has been computed with the aid of Maple 10<sup>3</sup>.  $\square$

**Theorem 3.3.** *The sequence  $\{\chi_n^{(2)}(z)\}_{n=0}^\infty$ , with  $\chi_n^{(2)}(z) = h_n \phi_n^{(2)}(z)$ , where*

$$(18) \quad |h_0|^2 = \frac{1}{\pi}, \quad |h_n|^2 = \frac{-2(1 - |\alpha_n|^2)(1 - \alpha_n^2)^2}{\pi(f_n + g_n \alpha_n + \alpha_n^2)(\overline{f_n} \alpha_n^2 + \overline{g_n} \alpha_n + 1)}, \quad n > 0,$$

and  $\phi_n^{(2)}(z)$  is defined as before in Theorem 3.2, forms a set of orthonormal rational functions with respect to  $\omega_2(\theta) = \sin^2 \theta$ .

*Proof.* The expression for  $n = 0$  is easily verified; so, we continue for  $n > 0$ . First, note that

$$\sin^2(\theta)d\theta = -(z^2 - 1)^2 \frac{dz}{4iz^3}, \quad z = e^{i\theta}.$$

Hence, we have that

$$\begin{aligned} \left\langle \chi_n^{(2)}, B_n \right\rangle_{\omega_2} &= -\frac{1}{4i} \int_{\mathbb{T}} \chi_n^{(2)}(z) B_{n*}(z) (z^2 - 1)^2 \frac{dz}{z^3} \\ &= -\frac{h_n}{4i} \left[ \int_{\mathbb{T}} \frac{(d_n + e_n z) B_{n*}(z)}{z^3} dz + \int_{\mathbb{T}} \frac{(f_n + g_n z + z^2) B_{n-1}(z)}{(1 - \overline{\alpha_n} z) B_n(z)} dz \right] \\ &= -\frac{h_n}{4i} \left[ -\int_{\mathbb{T}} \frac{(d_n + e_n \overline{z}) B_{n*}(\overline{z})}{\overline{z}^3} d\overline{z} + \int_{\mathbb{T}} \frac{f_n + g_n z + z^2}{z - \alpha_n} dz \right] \\ &= -\frac{h_n}{4i} \left[ \int_{\mathbb{T}} (d_n z + e_n) \overline{B_n(\overline{z})} dz + \int_{\mathbb{T}} G_n(z) dz \right] \\ &= -\frac{h_n}{4i} \left[ \int_{\mathbb{T}} F_n(z) dz + \int_{\mathbb{T}} G_n(z) dz \right]. \end{aligned}$$

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Clearly,  $F_n$  is analytic in  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ , so that

$$\begin{aligned} \left\langle \chi_n^{(2)}, B_n \right\rangle_{\omega_2} &= -\frac{h_n}{4i} \int_{\mathbb{T}} G_n(z) dz = -\frac{h_n}{4i} \int_{\mathbb{T}} \frac{f_n + g_n z + z^2}{z - \alpha_n} dz \\ &= -\frac{h_n \pi}{2} \operatorname{Res} \left\{ \frac{f_n + g_n z + z^2}{z - \alpha_n}, \alpha_n \right\} = -\frac{h_n \pi (f_n + g_n \alpha_n + \alpha_n^2)}{2}. \end{aligned}$$

Finally, suppose  $\chi_n^{(2)}$  is of the form

$$\chi_n^{(2)}(z) = \sum_{k=0}^n a_k B_k(z).$$

Then it is easily verified that  $\left\langle \chi_n^{(2)}, B_n \right\rangle_{\omega_2} = \frac{1}{\bar{a}_n}$ , with

$$\bar{a}_n = \chi_n^{(2)*}(\alpha_n) = \frac{\bar{h}_n (\bar{f}_n \alpha_n^2 + \bar{g}_n \alpha_n + 1)}{(1 - |\alpha_n|^2)(1 - \alpha_n^2)^2}.$$

Consequently,

$$|h_n|^2 = -\frac{2(1 - |\alpha_n|^2)(1 - \alpha_n^2)^2}{\pi(f_n + g_n \alpha_n + \alpha_n^2)(\bar{f}_n \alpha_n^2 + \bar{g}_n \alpha_n + 1)},$$

which ends the proof. □

Finally, we will also need the following lemma.

**Lemma 3.4.** *It holds that*

$$d_n + e_n \epsilon \neq 0,$$

where  $\epsilon \in \{\pm 1\}$ , and  $d_n$  and  $e_n$  are defined as before in Theorem 3.2.

*Proof.* Suppose  $d_n + e_n \epsilon = 0$  for a fixed  $\epsilon \in \{\pm 1\}$ . We then have that

$$d_n + e_n z = e_n(z - \epsilon).$$

Since  $a^+$  and  $a^-$ , given by (14), are different from zero, and  $X_n^{(2)}(\epsilon) = 0$ , we also have that

$$(f_n + g_n z + z^2) = (z - \epsilon)(z - f_n \epsilon).$$

Consequently,

$$X_n^{(2)}(z) = (z - \epsilon) \left[ e_n + z^3(z - f_n \epsilon) \frac{B_{n-1}(z)}{1 - \bar{\alpha}_n z} \right] = (z - \epsilon) \hat{X}_n(z).$$

Note that for  $z \in \mathbb{T}$  it holds that

$$\hat{X}'_n(z) = z^2 \frac{B_{n-1}(z)}{1 - \bar{\alpha}_n z} \left[ z \left( 2 + Q_n(z) + \frac{1}{1 - \bar{\alpha}_n} \right) - f_n \epsilon \left( 1 + Q_n(z) + \frac{1}{1 - \bar{\alpha}_n} \right) \right],$$

where  $Q_n(z)$  is defined as before in (8), so that

$$\begin{aligned} \begin{cases} X_n^{(2)'}(1) &= 0 \\ X_n^{(2)'}(-1) &= 0 \end{cases} &\Leftrightarrow \begin{cases} \hat{X}'_n(1) &= 0 \\ \hat{X}'_n(-1) &= 0 \end{cases} \Leftrightarrow \begin{cases} a^+ [(1 + b^+) - f_n \epsilon b^+] &= 0 \\ a^- [(1 + b^-) + f_n \epsilon b^-] &= 0 \end{cases} \\ &\Leftrightarrow \begin{cases} f_n \epsilon &= 1 + \frac{1}{b^+} \\ f_n \epsilon &= -(1 + \frac{1}{b^-}) \end{cases} \Leftrightarrow \begin{cases} f_n \epsilon &= 1 + \frac{1}{b^+} \\ \frac{1}{b^+} + \frac{1}{b^-} &= -2 \end{cases}. \end{aligned}$$

Furthermore, since  $\alpha_n \in \mathbb{D}$ , we have that  $0 < \Re \left\{ \frac{1}{1 \pm \alpha_n} \right\} < \infty$ . From (8) and (14) we then deduce that  $2 < \Re\{b^+\}, \Re\{b^-\} < \infty$ , so that

$$-2 = \Re \left\{ \frac{1}{b^+} + \frac{1}{b^-} \right\} = \Re \left\{ \frac{1}{b^+} \right\} + \Re \left\{ \frac{1}{b^-} \right\} > 0.$$

This is a contradiction, and hence,  $d_n + e_n \epsilon \neq 0$ . □

#### 4. RATIONAL SZEGŐ QUADRATURES ASSOCIATED WITH CHEBYSHEV WEIGHT FUNCTIONS

The aim of this section is to characterize rational Szegő formulas  $I_n(f)$ , given by (5), associated with the Chebyshev weight functions  $\omega_i(\theta)$ ,  $i = 1, \dots, 4$ , given by (6). Therefore, we will derive explicit expressions for the associated para-orthogonal rational functions  $\Psi_n(z, \tau_n)$  by means of (4) along with the results provided in the previous section. The zeros  $\{z_j\}_{j=1}^n$  of  $\Psi_n(z, \tau_n)$  are the nodes we need for  $I_n(f)$ . First, let us consider the reproducing kernel function for  $\mathcal{L}_n$  associated with a general measure  $\mu$ , namely

$$\mathcal{K}_n(z, \xi) = \sum_{k=0}^n \chi_k(z) \overline{\chi_k(\xi)}.$$

The following Christoffel-Darboux formula has been proved in [5, Theorem 3.1.3]:

$$(19) \quad \mathcal{K}_n(z, \xi) = \frac{\chi_{n+1}^*(z) \overline{\chi_{n+1}^*(\xi)} - \chi_{n+1}(z) \overline{\chi_{n+1}(\xi)}}{1 - \zeta_{n+1}(z) \overline{\zeta_{n+1}(\xi)}}, \quad n \geq 1.$$

Moreover, the following well-known expression for the weights in a Szegő quadrature formula has been proved for the rational case in [5, Theorem 5.4.2]:

$$(20) \quad \lambda_j = \frac{1}{\sum_{k=0}^{n-1} |\chi_k(z_j)|^2}; \quad j = 1, \dots, n.$$

We are now able to prove the following proposition.

**Proposition 4.1.** *Let  $\{z_j\}_{j=1}^n$  and  $\{\lambda_j\}_{j=1}^n$  represent the set of nodes and weights of a rational Szegő quadrature formula (5) for  $\mu$ , and suppose  $\chi_n(z)$  is the corresponding  $n$ th orthonormal rational function. It then holds that*

$$(21) \quad \lambda_j^{-1} = \frac{z_j |z_j - \alpha_n|^2}{1 - |\alpha_n|^2} \left| \begin{array}{cc} \overline{\chi_n(z_j)} & \overline{\chi_n^*(z_j)} \\ (\chi_n^*)'(z_j) & \chi_n'(z_j) \end{array} \right|, \quad j = 1, \dots, n.$$

*Proof.* From (19) and (20) it follows that

$$(22) \quad \lambda_j^{-1} = \lim_{\substack{z \rightarrow z_j \\ z \in \mathbb{T}}} \mathcal{K}_{n-1}(z, z_j) = \frac{(\chi_n^*)'(z_j) \overline{\chi_n^*(z_j)} - \chi_n'(z_j) \overline{\chi_n(z_j)}}{-\zeta_n'(z_j) \overline{\zeta_n(z_j)}}.$$

The statement is now easily verified by taking into account that  $\overline{\zeta_n(z)} = 1/\zeta_n(z)$  and  $\zeta_n'(z) = \frac{1 - |\alpha_n|^2}{z|z - \alpha_n|^2} \zeta_n(z)$  for  $z \in \mathbb{T}$ . □

We start with the Lebesgue measure  $d\mu(\theta) = \omega_1(\theta)d\theta \equiv d\theta$ . From (4) and (7) it follows that, up to a multiplicative factor,

$$\Psi_n^{(1)}(z, \tau_n) = \frac{V_n^{(1)}(z, \tau_n)}{\pi_n(z)}, \quad z, \tau_n \in \mathbb{T},$$

where

$$(23) \quad V_n^{(1)}(z, \tau_n) = p_0^{(1)}(z)\pi_{n-1}(z) + \tau_n z p_0^{(1)*}(z)\pi_{n-1}^*(z)$$

and

$$(24) \quad p_0^{(1)}(z) \equiv \tau_n.$$

Moreover, for this special case it is well known that (see [4])

$$(25) \quad \lambda_j^{(1)} = \frac{2\pi}{Q_n(z_j^{(1)})}, \quad j = 1, \dots, n,$$

with  $Q_n(z)$  given by (8). Indeed, let  $\chi_n^{(1)}(z) = k_n \phi_n^{(1)}(z)$  with  $\phi_n^{(1)}(z)$  given by (7) and  $|k_n|^2 = \frac{1-|\alpha_n|^2}{2\pi}$  for  $n \geq 1$ . Then it follows from Proposition 4.1 that

$$\begin{aligned} \lambda_j^{(1)} &= \frac{1 - |\alpha_n|^2}{z_j^{(1)} |z_j^{(1)} - \alpha_n|^2} \\ &\times \left| \frac{\frac{\overline{k_n}}{(z_j^{(1)} - \alpha_n) B_{n-1}(z_j^{(1)})}}{-\frac{k_n \overline{\alpha_n}}{(1 - \overline{\alpha_n} z_j^{(1)})^2}} \quad \frac{\frac{\overline{k_n z_j^{(1)}}}{(z_j^{(1)} - \alpha_n)}}{\frac{k_n B_{n-1}(z_j^{(1)}) Q_n(z_j^{(1)})}{(1 - \overline{\alpha_n} z_j^{(1)})} - \frac{k_n \overline{\alpha_n} z_j^{(1)} B_{n-1}(z_j^{(1)})}{(1 - \overline{\alpha_n} z_j^{(1)})^2}} \right|^{-1} \\ &= \frac{1 - |\alpha_n|^2}{z_j^{(1)} |z_j^{(1)} - \alpha_n|^2} \times \left[ \frac{|k_n|^2 Q_n(z_j^{(1)})}{(z_j^{(1)} - \alpha_n)(1 - \overline{\alpha_n} z_j^{(1)})} \right]^{-1} = \frac{1 - |\alpha_n|^2}{|k_n|^2 Q_n(z_j^{(1)})}. \end{aligned}$$

Thus, we have the following theorem.

**Theorem 4.2.** *Let  $n \geq 1$  and  $\tau_n \in \mathbb{T}$ . The nodes of an  $n$ -point rational Szegő quadrature  $I_n(f)$  for  $I_{\omega_1}(f)$ , given by (5) and (6), are then the zeros of  $V_n^{(1)}(z, \tau_n)$  given by (23) and (24), while the weights are given by (25).  $\square$*

Next, consider the weight functions  $\omega_i(\theta) = 1 - \nu_i \cos \theta$ , with  $i \in \{3, 4\}$  and  $\nu_i = (-1)^{i-1}$ . From (4) and Theorem 3.1 it follows that, up to a multiplicative constant,

$$\Psi_n^{(i)}(z, \tau_n) = \frac{V_n^{(i)}(z, \tau_n)}{(z - \nu_i)^2 \pi_n(z)}, \quad i = 3, 4, \quad z, \tau_n \in \mathbb{T},$$

where

$$(26) \quad V_n^{(i)}(z, \tau_n) = p_1^{(i)}(z)\pi_{n-1}(z) + \tau_n z^2 p_1^{(i)*}(z)\pi_{n-1}^*(z)$$

and

$$(27) \quad p_1^{(i)}(z) = a_n^{(i)}(1 - \overline{\alpha_n} z) + \tau_n(1 - \overline{b_n^{(i)}} z).$$

Explicit expressions for the weights can be deduced as well from Theorem 3.1 and (21). Suppose  $z_j = z_j^{(i)} \neq \nu_i$  for every  $j \in \{1, \dots, n\}$  and fixed  $i \in \{3, 4\}$ . Next, let

$$T_n^{(i)} = \overline{X_n^{(i)}(z_j)} [(Q_{n+1}(z_j) + 1)(1 - \nu_i z_j) + 2\nu_i z_j] + z_j \left( \overline{X_n^{(i)}(z_j)} \right)' (1 - \nu_i z_j).$$

Then, with  $\lambda_j = \lambda_j^{(i)}$  and  $|c_n^{(i)}|^2$  given by (12), it follows that

$$\begin{aligned}
 \lambda_j^{-1} &= \left| c_n^{(i)} \right|^2 \frac{z_j |z_j - \alpha_n|^2}{1 - |\alpha_n|^2} \left| \begin{array}{cc} \overline{\phi_n^{(i)}(z_j)} & \overline{\phi_n^{(i)*}(z_j)} \\ \left( \phi_n^{(i)*} \right)'(z_j) & \phi_n^{(i)'}(z_j) \end{array} \right| \\
 &= \left| c_n^{(i)} \right|^2 \frac{z_j |z_j - \alpha_n|^2}{1 - |\alpha_n|^2} \left| \begin{array}{cc} \frac{z_j^2}{(1 - \nu_i z_j)^2} \overline{X_n^{(i)}(z_j)} & \frac{X_n^{(i)}(z_j)}{B_n(z_j)(z_j - \nu_i)^2} \\ \frac{z_j B_n(z_j)}{(1 - \nu_i z_j)^3} T_n^{(i)} & \frac{X_n^{(i)'}(z_j)}{(z_j - \nu_i)^2} - \frac{2X_n^{(i)}(z_j)}{(z_j - \nu_i)^3} \end{array} \right| \\
 &= \left| c_n^{(i)} \right|^2 \frac{z_j |z_j - \alpha_n|^2}{(1 - |\alpha_n|^2)(z_j - \nu_i)^4} \left| \begin{array}{cc} z_j^2 \overline{X_n^{(i)}(z_j)} & \frac{X_n^{(i)}(z_j)}{B_n(z_j)} \\ \frac{z_j B_n(z_j)}{1 - \nu_i z_j} T_n^{(i)} & X_n^{(i)'}(z_j) - \frac{2X_n^{(i)}(z_j)}{z_j - \nu_i} \end{array} \right| \\
 &= \left| c_n^{(i)} \right|^2 \frac{|z_j - \alpha_n|^2}{(1 - |\alpha_n|^2) |z_j - \nu_i|^4} \\
 &\quad \times \left\{ z_j \left[ \overline{X_n^{(i)}(z_j)} X_n^{(i)'}(z_j) - X_n^{(i)}(z_j) \overline{\left( X_n^{(i)}(z_j) \right)'} \right] \right. \\
 &\quad \quad \quad \left. - \left| X_n^{(i)}(z_j) \right|^2 (Q_{n+1}(z_j) + 1) \right\} \\
 &= \left| c_n^{(i)} \right|^2 \frac{|z_j - \alpha_n|^2}{(1 - |\alpha_n|^2) |z_j - \nu_i|^4} \\
 &\quad \times \left\{ 2\Re \left[ z_j \overline{X_n^{(i)}(z_j)} X_n^{(i)'}(z_j) \right] - \left| X_n^{(i)}(z_j) \right|^2 (Q_{n+1}(z_j) + 1) \right\}.
 \end{aligned}$$

Here, the last equality is due to the fact that for  $z \in \mathbb{T}$ , and for every  $f(z)$  that is analytic in a small annulus containing the complex unit circle, it holds that (see e.g. [31, Lem. 3.3])

$$(28) \quad \left( \overline{f(z)} \right)' = \frac{d\overline{f}}{dz} = \frac{d\overline{f}}{d\overline{z}} \cdot \frac{d\overline{z}}{dz} = -\frac{\overline{f'(z)}}{z^2}.$$

From (9) we deduce that

$$X_n^{(i)'}(z) = \frac{z^2(z - b_n^{(i)})B_{n-1}(z)}{1 - \overline{\alpha_n}z} \left[ \frac{2}{z} + \frac{B'_{n-1}(z)}{B_{n-1}(z)} + \frac{1}{z - b_n^{(i)}} + \frac{\overline{\alpha_n}}{1 - \overline{\alpha_n}z} \right],$$

so that for  $z \in \mathbb{T}$  it holds that

$$\begin{aligned}
 (29) \quad zX_n^{(i)'}(z) &= \left( X_n^{(i)}(z) - a_n^{(i)} \right) \left( 1 + Q_n(z) + \frac{z}{z - b_n^{(i)}} + \frac{\overline{\alpha_n}z}{1 - \overline{\alpha_n}z} \right) \\
 &= \left( X_n^{(i)}(z) - a_n^{(i)} \right) \left( 1 + Q_{n+1}(z) + \frac{z(b_n^{(i)} - \alpha_n)}{(z - b_n^{(i)})(z - \alpha_n)} \right).
 \end{aligned}$$

Consequently,

$$\lambda_j^{-1} = \left| c_n^{(i)} \right|^2 \frac{|z_j - \alpha_n|^2}{(1 - |\alpha_n|^2) |z_j - \nu_i|^4} \times \left[ \left( |X_n^{(i)}(z_j)|^2 - 2\Re\{a_n^{(i)} \overline{X_n^{(i)}(z_j)}\} \right) [1 + Q_{n+1}(z_j)] + 2\Re \left\{ \frac{z_j(b_n^{(i)} - \alpha_n) \left( |X_n^{(i)}(z_j)|^2 - a_n^{(i)} \overline{X_n^{(i)}(z_j)} \right)}{(z_j - b_n^{(i)})(z_j - \alpha_n)} \right\} \right].$$

Finally, we have that

$$\begin{aligned} |X_n^{(i)}(z)|^2 - 2\Re\{a_n^{(i)} \overline{X_n^{(i)}(z)}\} &= \left| \frac{z - b_n^{(i)}}{z - \alpha_n} \right|^2 - |a_n^{(i)}|^2, \\ |X_n^{(i)}(z)|^2 - a_n^{(i)} \overline{X_n^{(i)}(z)} &= \left| \frac{z - b_n^{(i)}}{z - \alpha_n} \right|^2 + \frac{\overline{a_n^{(i)}} z^2 (z - b_n^{(i)}) B_{n-1}(z)}{1 - \overline{\alpha_n} z} \end{aligned}$$

and<sup>4</sup>

$$(30) \quad z_j^2 B_{n-1}(z_j) = -\frac{p_1^{(i)}(z_j)}{\tau_n p_1^{(i)*}(z_j)},$$

where  $p_1^{(i)}(z)$  is given by (27), so that

$$(31) \quad \lambda_j^{-1} = \frac{\left| c_n^{(i)} \right|^2}{(1 - |\alpha_n|^2) |z_j - \nu_i|^4} \left[ \left( |z_j - b_n^{(i)}|^2 - |a_n^{(i)}|^2 |z_j - \alpha_n|^2 \right) [1 + Q_{n+1}(z_j)] + 2\Re \left\{ (b_n^{(i)} - \alpha_n) \left( \frac{1 - \overline{b_n^{(i)}} z_j}{z_j - \alpha_n} - \frac{\overline{a_n^{(i)}}}{\tau_n} \frac{p_1^{(i)}(z_j)}{p_1^{(i)*}(z_j)} \right) \right\} \right].$$

Whenever  $z_j = \nu_i$  for a certain  $j \in \{1, \dots, n\}$ , computing  $\lambda_j$  by the aid of (21) requires two times the application of l'Hôpital's rule to compute  $\overline{\chi_n^{(i)}(\nu_i)}$ ,  $\chi_n^{(i)*}(\nu_i)$ ,  $\left(\chi_n^{(i)*}\right)'(\nu_i)$  and  $\chi_n^{(i)' }(\nu_i)$ . This gives rise to tedious calculations along with an inappropriate expression for computational purposes. Since  $z_j \neq z_k$  for  $j \neq k$ , there can be at most one index  $j$  for which  $z_j = \nu_i$ . Therefore,  $\lambda_j$  can then be computed as follows:

$$(32) \quad \lambda_j = \int_{-\pi}^{\pi} \omega_i(\theta) d\theta - \sum_{\substack{k=1 \\ k \neq j}}^n \lambda_k = 2\pi - \sum_{\substack{k=1 \\ k \neq j}}^n \lambda_k.$$

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<sup>4</sup>There are no  $\tau_n \in \mathbb{T}$  and  $\gamma \in \mathbb{T} \setminus \{\nu_i\}$  so that  $p_1^{(i)*}(\gamma) = 0$ ; hence, the right-hand side of (30) is well defined. A proof for this statement is given in the Appendix.

Note that there exists an index  $j$  so that  $z_j = \nu_i$  iff  $\tau_n = -\frac{\phi_n^{(i)}(\nu_i)}{\phi_n^{(i)*}(\nu_i)}$ . From (11) and (28) it follows that

$$(33) \quad -\frac{\phi_n^{(i)}(\nu_i)}{\phi_n^{(i)*}(\nu_i)} = \frac{-1}{B_n(\nu_i)} \lim_{\substack{z \rightarrow \nu_i \\ z \in \mathbb{T}}} \frac{X_n^{(i)}(z)}{X_n^{(i)}(z)} = \frac{1}{B_n(\nu_i)} \lim_{\substack{z \rightarrow \nu_i \\ z \in \mathbb{T}}} \frac{X_n^{(i)'}(z)}{X_n^{(i)'}(z)} = \frac{-1}{B_n(\nu_i)} \frac{X_n^{(i)''}(\nu_i)}{X_n^{(i)''}(\nu_i)}.$$

Let

$$Y_n^{(i)}(z) = 1 + Q_{n+1}(z) + \frac{z(b_n^{(i)} - \alpha_n)}{(z - b_n^{(i)})(z - \alpha_n)}.$$

Then it follows from (29) that

$$zX_n^{(i)''}(z) = X_n^{(i)'}(z) \left( Y_n^{(i)}(z) - 1 \right) + \left( X_n^{(i)}(z) - a_n^{(i)} \right) Y_n^{(i)'}(z),$$

and hence,

$$\begin{aligned} X_n^{(i)''}(\nu_i) &= -a_n^{(i)} \nu_i Y_n^{(i)'}(\nu_i) = a_n^{(i)} \left[ \frac{\nu_i(b_n^{(i)} - \alpha_n)(1 - \alpha_n b_n^{(i)})}{(\nu_i - b_n^{(i)})^2(\nu_i - \alpha_n)^2} - \nu_i Q'_{n+1}(\nu_i) \right] \\ &= R_n^{(i)} e^{i\gamma_n^{(i)}}, \quad R_n^{(i)} > 0 \text{ and } \gamma_n^{(i)} \in [0, 2\pi), \end{aligned}$$

where

$$(34) \quad zQ'_{n+1}(z) = 2i \sum_{k=1}^n \frac{\Im\{\overline{\alpha_k} z\} (1 - |\alpha_k|^2)}{|z - \alpha_k|^4} \text{ for } z \in \mathbb{T}.$$

Consequently, there exists an index  $j$  so that  $z_j = \nu_i$  iff  $\tau_n = \frac{e^{i(2\gamma_n^{(i)} - \pi)}}{B_n(\nu_i)}$ .

We now have proved the following theorem.

**Theorem 4.3.** *Let  $i \in \{3, 4\}$  be fixed and set  $\nu_i = (-1)^{i-1}$ . Further, assume  $\tau_n \in \mathbb{T}$ , for  $n \geq 1$ . An  $n$ -point rational Szegő quadrature  $I_n(f)$  for  $I_{\omega_i}(f)$ , given by (5) and (6), is then characterized as follows:*

- (1) *Denote by  $\{\nu_i, \nu_i, z_1, \dots, z_n\}$  the set of zeros of  $V_n^{(i)}(z, \tau_n)$ , given by (26) and (27). Then  $\{z_1, \dots, z_n\}$  is the set of nodes for  $I_n(f)$ .*
- (2) *For every  $j \in \{1, \dots, n\}$  for which  $z_j \neq \nu_i$ , the  $j$ th weight is given by (31). Whenever  $z_j = \nu_i$  for a certain  $j \in \{1, \dots, n\}$ , the associated weight is given by (32). □*

To conclude this section, we consider the weight function  $\omega_2(\theta)$ . From (4) and Theorem 3.2 it now follows that, up to a multiplicative factor,

$$\Psi_n^{(2)}(z, \tau_n) = \frac{V_n^{(2)}(z, \tau_n)}{(z^2 - 1)^2 \pi_n(z)}, \quad z, \tau_n \in \mathbb{T},$$

where

$$(35) \quad V_n^{(2)}(z, \tau_n) = p_2^{(2)}(z) \pi_{n-1}(z) + \tau_n z^3 p_2^{(2)*}(z) \pi_{n-1}^*(z)$$

and

$$(36) \quad p_2^{(2)}(z) = (d_n + e_n z)(1 - \overline{\alpha_n} z) + \tau_n (\overline{f_n} z^2 + \overline{g_n} z + 1).$$

Explicit expressions for the weights can be deduced as well from (21) and Theorems 3.2 and 3.3. Assume  $\epsilon \in \{\pm 1\}$  and suppose  $z_j = z_j^{(2)} \neq \epsilon$  for every  $j \in \{1, \dots, n\}$ . Next, let

$$T_n^{(2)} = \left[ Q_{n+1}(z_j) + 3 - \frac{4z_j^2}{z_j^2 - 1} \right] \overline{z_j X_n^{(2)}(z_j)} - \overline{X_n^{(2)'}(z_j)}.$$

Then, with  $\lambda_j = \lambda_j^{(2)}$  and  $|h_n|^2$  given by (18), it now follows that

$$\begin{aligned} \lambda_j^{-1} &= |h_n|^2 \frac{z_j |z_j - \alpha_n|^2}{1 - |\alpha_n|^2} \left| \begin{array}{cc} \overline{\phi_n^{(2)}(z_j)} & \overline{\phi_n^{(2)*}(z_j)} \\ \left( \phi_n^{(2)*} \right)'(z_j) & \phi_n^{(2)'}(z_j) \end{array} \right| \\ &= |h_n|^2 \frac{z_j |z_j - \alpha_n|^2}{1 - |\alpha_n|^2} \left| \begin{array}{cc} \frac{z_j^4}{(z_j^2 - 1)^2} \overline{X_n^{(2)}(z_j)} & \frac{X_n^{(2)}(z_j)}{B_n(z_j)(z_j^2 - 1)^2} \\ \frac{z_j^2 B_n(z_j)}{(z_j^2 - 1)^2} T_n^{(2)} & \frac{X_n^{(2)'}(z_j)}{(z_j^2 - 1)^2} - \frac{4z_j X_n^{(2)}(z_j)}{(z_j^2 - 1)^3} \end{array} \right| \\ &= |h_n|^2 \frac{z_j |z_j - \alpha_n|^2}{(1 - |\alpha_n|^2)(z_j^2 - 1)^4} \left| \begin{array}{cc} \overline{z_j^4 X_n^{(2)}(z_j)} & \frac{X_n^{(2)}(z_j)}{B_n(z_j)} \\ z_j^2 B_n(z_j) T_n^{(2)} & X_n^{(2)'}(z_j) - \frac{4z_j X_n^{(2)}(z_j)}{z_j^2 - 1} \end{array} \right| \\ &= |h_n|^2 \frac{|z_j - \alpha_n|^2}{(1 - |\alpha_n|^2)|z_j^2 - 1|^4} \\ &\quad \times \left\{ 2\Re \left[ z_j \overline{X_n^{(2)}(z_j)} X_n^{(2)'}(z_j) \right] - \left| X_n^{(2)}(z_j) \right|^2 (Q_{n+1}(z_j) + 3) \right\}. \end{aligned}$$

From (17) we deduce that

$$\begin{aligned} X_n^{(2)'}(z) &= \frac{z^3(f_n + g_n z + z^2)B_{n-1}(z)}{1 - \overline{\alpha_n}z} \\ &\quad \times \left[ \frac{3}{z} + \frac{B'_{n-1}(z)}{B_{n-1}(z)} + \frac{g_n + 2z + e_n \frac{1 - \overline{\alpha_n}z}{z^3 B_{n-1}(z)}}{f_n + g_n z + z^2} + \frac{\overline{\alpha_n}}{1 - \overline{\alpha_n}z} \right], \end{aligned}$$

so that for  $z \in \mathbb{T}$  it holds that

$$\begin{aligned} zX_n^{(2)'}(z) &= \left[ X_n^{(2)}(z) - (d_n + e_n z) \right] \\ &\quad \times \left\{ 2 + Q_n(z) + \frac{z \left( g_n + 2z + e_n \frac{1 - \overline{\alpha_n}z}{z^3 B_{n-1}(z)} \right)}{f_n + g_n z + z^2} + \frac{\overline{\alpha_n}z}{1 - \overline{\alpha_n}z} \right\} \\ &= \left[ X_n^{(2)}(z) - (d_n + e_n z) \right] \\ (37) \quad &\quad \times \left\{ 3 + Q_{n+1}(z) + \frac{z^2 G_n(z)}{(f_n + g_n z + z^2)(z - \alpha_n)} \right\}, \end{aligned}$$

where

$$G_n(z) = \alpha_n f_n \bar{z}^2 - 2f_n \bar{z} - (\alpha_n + g_n) + e_n |z - \alpha_n|^2 \overline{z^3 B_{n-1}(z)}.$$

Consequently,

$$\lambda_j^{-1} = |h_n|^2 \frac{|z_j - \alpha_n|^2}{(1 - |\alpha_n|^2) |z_j^2 - 1|^4} \times \left[ \left( |X_n^{(2)}(z_j)|^2 - 2\Re\{(d_n + e_n z_j) \overline{X_n^{(2)}(z_j)}\} \right) [3 + Q_{n+1}(z_j)] + 2\Re \left\{ \frac{z_j^2 G_n(z_j) \left( |X_n^{(2)}(z_j)|^2 - (d_n + e_n z_j) \overline{X_n^{(2)}(z_j)} \right)}{(f_n + g_n z_j + z_j^2)(z_j - \alpha_n)} \right\} \right].$$

Finally, we have that

$$\begin{aligned} |X_n^{(2)}(z)|^2 - 2\Re\{(d_n + e_n z) \overline{X_n^{(2)}(z)}\} &= \left| \frac{f_n + g_n z + z^2}{z - \alpha_n} \right|^2 - |d_n + e_n z|^2, \\ |X_n^{(2)}(z)|^2 - (d_n + e_n z) \overline{X_n^{(2)}(z)} &= \left| \frac{f_n + g_n z + z^2}{z - \alpha_n} \right|^2 + \frac{\overline{d_n + e_n z} z^3 (f_n + g_n z + z^2) B_{n-1}(z)}{1 - \overline{\alpha_n} z} \end{aligned}$$

and<sup>5</sup>

$$(38) \quad z_j^3 B_{n-1}(z_j) = -\overline{\tau_n} \frac{p_2^{(2)}(z_j)}{p_2^{(2)*}(z_j)},$$

where  $p_2^{(2)}(z)$  is given by (36). Consequently,

$$G_n(z_j) = \alpha_n f_n \overline{z_j}^2 - 2f_n \overline{z_j} - (\alpha_n + g_n) - \tau_n e_n |z_j - \alpha_n|^2 \frac{p_2^{(2)*}(z_j)}{p_2^{(2)}(z_j)},$$

so that

$$(39) \quad \lambda_j^{-1} = \frac{|h_n|^2}{(1 - |\alpha_n|^2) |z_j^2 - 1|^4} \times \left[ (|f_n + g_n z_j + z_j^2|^2 - |d_n + e_n z_j|^2 |z_j - \alpha_n|^2) [3 + Q_{n+1}(z_j)] + 2\Re \left\{ G_n(z_j) \left[ \frac{\overline{f_n z_j^2} + \overline{g_n z_j} + 1}{z_j - \alpha_n} - \overline{\tau_n} (\overline{d_n z_j} + \overline{e_n}) \frac{p_2^{(2)}(z_j)}{p_2^{(2)*}(z_j)} \right] \right\} \right].$$

Whenever  $z_j = \epsilon$  for a certain  $j \in \{1, \dots, n\}$ , computing  $\lambda_j$  by the aid of (21) again requires two times the application of l'Hôpital's rule to compute  $\chi_n^{(2)}(\epsilon)$ ,  $\overline{\chi_n^{(2)*}(\epsilon)}$ ,  $(\chi_n^{(2)*})'(\epsilon)$  and  $\chi_n^{(2)'}(\epsilon)$ . This gives rise to tedious calculations along with an inappropriate expression for computational purposes. Since  $z_j \neq z_k$  for  $j \neq k$ ,

<sup>5</sup>If there are no  $\tau_n \in \mathbb{T}$  and  $\gamma \in \mathbb{T} \setminus \{\pm 1\}$  so that  $p_2^{(2)*}(\gamma) = 0$ , then the right-hand side of (38) is well defined. At this moment of writing no proof has been found for this statement in general, but a proof is given in the Appendix for the special case in which  $n = 1$  or  $\alpha_n = 0$ .



there can be at most one index  $j$  for which  $z_j = \epsilon$ . Therefore, if  $z_k \neq -\epsilon$  for every  $k \in \{1, \dots, n\} \setminus \{j\}$ ,  $\lambda_j$  can be computed as follows:

$$(40) \quad \lambda_j = \int_{-\pi}^{\pi} \omega_2(\theta) d\theta - \sum_{\substack{k=1 \\ k \neq j}}^n \lambda_k = \pi - \sum_{\substack{k=1 \\ k \neq j}}^n \lambda_k.$$

If, on the other hand, there exist indices  $j$  and  $k$  in  $\{1, \dots, n\}$  so that  $z_j = -z_k = 1$ , then  $\lambda_j$  and  $\lambda_k$  can be computed by solving the following system of equations:

$$(41) \quad \begin{cases} \lambda_j + \lambda_k & = \pi - \sum_{\substack{s=1 \\ s \notin \{j,k\}}}^n \lambda_s, \\ \lambda_j \left| \chi_1^{(2)}(1) \right|^2 + \lambda_k \left| \chi_1^{(2)}(-1) \right|^2 & = 1 - \sum_{\substack{s=1 \\ s \notin \{j,k\}}}^n \lambda_s \left| \chi_1^{(2)}(z_s) \right|^2, \end{cases}$$

where we computed  $\chi_1^{(2)}(z)$  with the aid of Maple 10 to find that

$$(42) \quad \left| \chi_1^{(2)}(z) \right|^2 = \frac{1}{\pi} \left( \frac{1 - |\alpha_1|^2}{1 - \left| \frac{\alpha_1}{2 - \alpha_1^2} \right|^2} \right) \left| \frac{z + \frac{\alpha_1}{2 - \alpha_1^2}}{1 - \alpha_1 z} \right|^2.$$

Note that there exists an index  $j$  so that  $z_j = \epsilon$  iff  $\tau_n = -\frac{\phi_n^{(2)}(\epsilon)}{\phi_n^{(2)*}(\epsilon)}$ . From Theorem 3.2 and proceeding as in (33) it follows that

$$-\frac{\phi_n^{(2)}(\epsilon)}{\phi_n^{(2)*}(\epsilon)} = \frac{-1}{B_n(\epsilon)} \frac{X_n^{(2)''}(\epsilon)}{X_n^{(2)''}(\epsilon)}.$$

Let

$$Y_n^{(2)}(z) = 3 + Q_{n+1}(z) + \frac{z^2 G_n(z)}{(f_n + g_n z + z^2)(z - \alpha_n)}.$$

Then it follows from (37) that

$$z X_n^{(2)''}(z) = X_n^{(2)'}(z) \left( Y_n^{(2)}(z) - 1 \right) - e_n Y_n^{(2)}(z) + \left( X_n^{(2)}(z) - (d_n + e_n z) \right) Y_n^{(2)'}(z).$$

It also follows from (37) that

$$0 = \epsilon X_n^{(2)'}(\epsilon) = \left( X_n^{(2)}(\epsilon) - (d_n + e_n \epsilon) \right) Y_n^{(2)}(\epsilon) = -(d_n + e_n \epsilon) Y_n^{(2)}(\epsilon),$$

so that  $Y_n^{(2)}(\epsilon) = 0$  due to Lemma 3.4. Thus,

$$\begin{aligned} X_n^{(2)''}(\epsilon) &= -(d_n + e_n \epsilon) \epsilon Y_n^{(2)'}(\epsilon) \\ &= (d_n + e_n \epsilon) \left[ \frac{\epsilon \left( k_n(\epsilon) + e_n \overline{B_{n-1}(\epsilon)} l_n(\epsilon) \right)}{(1 + f_n + g_n \epsilon)^2 (\epsilon - \alpha_n)^2} - \epsilon Q'_{n+1}(\epsilon) \right] \\ &= R_{n,\epsilon}^{(2)} e^{i\gamma_{n,\epsilon}^{(2)}}, \quad \epsilon \in \{\pm 1\}, R_{n,\epsilon}^{(2)} > 0 \text{ and } \gamma_{n,\epsilon}^{(2)} \in [0, 2\pi), \end{aligned}$$

where

$$\begin{aligned} k_n(\epsilon) &= f_n \alpha_n (6 - f_n) - (1 + g_n \alpha_n) (\alpha_n + g_n) - f_n (g_n + 4\epsilon) (1 + \alpha_n^2) \\ l_n(\epsilon) &= |\epsilon - \alpha_n|^2 [2(g_n - \alpha_n) + \epsilon(3 + f_n - g_n \alpha_n)] \\ &\quad + [|\epsilon - \alpha_n|^2 Q_n(\epsilon) - \epsilon 2i\Im\{\alpha_n\}] (1 + f_n + g_n \epsilon) (\epsilon - \alpha_n), \end{aligned}$$

and with  $Q_n(z)$  and  $zQ'_{n+1}(z)$  given, respectively, by (8) and (34). Consequently, there exists an index  $j$  so that  $z_j = \epsilon$  iff  $\tau_n = \frac{e^{i(2\gamma_{n,\epsilon}^{(2)} - \pi)}}{B_n(\epsilon)}$ .

Finally we have proved the following theorem.

**Theorem 4.4.** *Let  $\tau_n \in \mathbb{T}$  for  $n \geq 1$  and suppose  $\epsilon \in \{\pm 1\}$ . An  $n$ -point rational Szegő quadrature  $I_n(f)$  for  $I_{\omega_2}(f)$ , given by (5) and (6), is then characterized as follows:*

- (1) Denote by  $\{1, 1, -1, -1, z_1, \dots, z_n\}$  the set of zeros of  $V_n^{(2)}(z, \tau_n)$ , given by (35) and (36). Then  $\{z_1, \dots, z_n\}$  is the set of nodes for  $I_n(f)$ .
- (2) For every  $j \in \{1, \dots, n\}$  for which  $z_j \neq \epsilon$ , the  $j$ th weight is given by (39). Whenever  $z_j = \epsilon$  for a certain  $j \in \{1, \dots, n\}$  and  $z_k \neq -\epsilon$  for every  $k \in \{1, \dots, n\} \setminus \{j\}$ , the  $j$ th weight is given by (40). Finally, if there exist two distinct indices  $j$  and  $k$  in  $\{1, \dots, n\}$  so that  $z_j = -z_k = 1$ , the associated weights are given by (41). □

### 5. SZEGŐ QUADRATURES ASSOCIATED WITH RATIONAL MODIFICATIONS OF CHEBYSHEV WEIGHT FUNCTIONS

In this section we will consider a rational modification of a measure  $\mu$  that is a measure of the form

$$(43) \quad d\tilde{\mu}(\theta) = \frac{d\mu(\theta)}{|h(z)|^2}; \quad z = e^{i\theta},$$

with  $h(z)$  a given polynomial of degree  $m$  whose zeros cannot be on  $\mathbb{T}$ . Without loss of generality, we can assume

$$(44) \quad h(z) = \prod_{l=1}^m (z - \alpha_l), \quad |\alpha_l| < 1, \quad \alpha_l \neq 0, \quad l = 1, \dots, m.$$

We will then consider Szegő quadrature formulas with respect to the measure  $\tilde{\mu}$ . In this respect, we recall that an  $n$ -point Szegő rule (2) for a measure  $\mu$  has maximal domain of validity  $\Lambda_{-(n-1),n-1} = \text{span}\{z^{-(n-1)}, \dots, z^{n-1}\}$ , with dimension  $2n - 1$ , and that the nodes are the zeros of a para-orthogonal polynomial associated with  $\mu$ ; see e.g. [1, 8, 9, 10, 11, 22, 24, 25]. We also recall that for  $\kappa_n \in \mathbb{T}$ ,  $\kappa_n$ -invariant para-orthogonal polynomials associated with  $\mu$  are characterized in [25] as

$$(45) \quad \Phi_n(z) = C_n [\rho_n(z) + \tau_n \rho_n^*(z)] \in \mathbb{P}_n, \quad n \geq 1, \quad C_n \in \mathbb{C} \setminus \{0\},$$

where  $\tau_n = \frac{C_n \kappa_n}{C_n} \in \mathbb{T}$  and  $\{\rho_k(z)\}_{k=0}^\infty$  is a sequence of Szegő polynomials associated with  $\mu$ ; see e.g. [26], [28].

When  $d\mu(\theta) = d\theta$ , rational modifications of the Lebesgue measure appear such that Szegő polynomials for  $\tilde{\mu}$  were earlier considered by Szegő in [28]. Observe that when  $m = 1$ ,  $\tilde{\mu}$  gives rise to the so-called Poisson kernel. In relation to this, Waadeland considered in [33] for the first time Szegő quadrature formulas for the measure  $d\tilde{\mu}(\theta) = \frac{d\theta}{|z - \alpha|^2}$ , providing one of the first examples of this type of quadratures in the literature.

Rational modifications of the Lebesgue measure are very suitable in order to approximate other measures. For instance, the following important result is given in [26, Theorem 1.7.8]: Let  $\eta$  be a probability measure supported on  $\mathbb{T}$ , and suppose

$\{\varphi_n(z)\}_{n=0}^\infty$  is a sequence of orthonormal polynomials associated with  $\eta$ . For every  $n$  it then holds that

$$\int_{-\pi}^\pi \frac{d\theta}{2\pi |\varphi_n(e^{i\theta})|^2} = 1$$

and

$$d\eta_n(\theta) = \frac{d\theta}{2\pi |\varphi_n(e^{i\theta})|^2} \longrightarrow d\eta \text{ weakly when } n \rightarrow \infty.$$

Consider now the polynomial  $h(z)$  given by (44) and set  $\alpha_k = 0$  for every  $k > m$ . For every  $n \geq m$  we then have that

$$\pi_n^*(z) = \prod_{j=1}^n (z - \alpha_j) = z^{n-m} h(z), \quad \pi_n(z) = \prod_{j=1}^n (1 - \overline{\alpha_j} z) = \prod_{j=1}^m (1 - \overline{\alpha_j} z) = h^*(z).$$

So, let  $\chi_n(z)$  denote the  $n$ th orthonormal rational function associated with  $\mu$ . Then for every  $n \geq m$ ,  $\chi_n(z) = \frac{P_n(z)}{h^*(z)}$  with  $P_n(z) \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$ . Moreover, if  $n > m$ , we have that  $\langle \chi_n(z), u_k(z) \rangle_\mu = 0$ , where  $u_k(z) = \frac{z^k}{h^*(z)}$  with  $k = 0, 1, \dots, n-1$ , implying that

$$0 = \int_{-\pi}^\pi \frac{P_n(z)}{h^*(z)} \overline{\left(\frac{z^k}{h^*(z)}\right)} d\mu(\theta) = \int_{-\pi}^\pi P_n(z) \overline{z^k} \frac{d\mu(\theta)}{|h^*(z)|^2} = \int_{-\pi}^\pi P_n(z) \overline{z^k} \frac{d\mu(\theta)}{|h(z)|^2}$$

with  $z = e^{i\theta}$  and  $k = 0, 1, \dots, n-1$ . Consequently, the numerator of the  $n$ th orthonormal rational function associated with the measure  $\mu$  coincides for  $n > m$  with an  $n$ th orthonormal polynomial associated with the measure  $\tilde{\mu}$  given by (43).

**Example 5.1.** Take the Lebesgue measure  $d\mu(\theta) = \omega_1(\theta)d\theta \equiv d\theta$ ; see (6). Then for every  $n > m$  it follows from (7) that

$$\begin{aligned} \chi_n(z) &= c_n z \prod_{j=1}^{n-1} \left(\frac{z - \alpha_j}{1 - \overline{\alpha_j} z}\right) = c_n z \prod_{j=1}^m \left(\frac{z - \alpha_j}{1 - \overline{\alpha_j} z}\right) \prod_{j=m+1}^{n-1} \left(\frac{z - \alpha_j}{1 - \overline{\alpha_j} z}\right) \\ &= c_n \frac{z^{n-m} h(z)}{h^*(z)}, \quad c_n \neq 0. \end{aligned}$$

Hence,  $P_n(z) = z^{n-m} h(z)$  represents the  $n$ th monic Szegő polynomial for the measure  $d\tilde{\mu}(\theta) = \frac{d\theta}{|h(z)|^2}$  with  $z = e^{i\theta}$  (compare with the first approach given in [28, pp. 289–290]; actually, this expression for  $P_n(z)$  also holds for  $n = m$ ).

Let us see next what happens with the quadrature formulas. Assume as above  $\alpha_k = 0$  for  $k > m$  and  $\alpha_l \in \mathbb{D} \setminus \{0\}$  for  $l = 1, \dots, m$ . Let  $h(z) = \prod_{l=1}^m (z - \alpha_l) \in \mathbb{P}_m \setminus \mathbb{P}_{m-1}$ , and suppose  $n > m$ . From Theorem 2.2 we then have that

$$I_n(f) = \sum_{j=1}^n \lambda_j f(z_j) = I_\mu(f) = \int_{-\pi}^\pi f(e^{i\theta}) d\mu(\theta), \quad \forall f \in \mathcal{R}_{n-1}.$$

Consider now the case in which  $f(z) = \frac{z^k}{\pi_{n-1}^*(z)\pi_{n-1}(z)} \in \mathcal{R}_{n-1}$  for  $k = 0, 1, \dots, 2n-2$ , and observe that  $\pi_{n-1}(z) = z^{n-1} \overline{\pi_{n-1}^*(z)}$  for  $z \in \mathbb{T}$ . Taking into account that  $k = 0, 1, \dots, 2n-2$ , it then follows that

$$f(z) = \frac{z^{k-(n-1)}}{|\pi_{n-1}^*(z)|^2} = \frac{z^{k+1-n}}{|h(z)|^2} = \frac{z^s}{|h(z)|^2}, \quad -(n-1) \leq s \leq n-1, \quad z \in \mathbb{T}.$$

Hence,

$$\begin{aligned} \int_{-\pi}^{\pi} f(z) d\mu(\theta) &= \int_{-\pi}^{\pi} \frac{z^s}{|h(z)|^2} d\mu(\theta) = \sum_{j=1}^n \lambda_j \frac{z_j^s}{|h(z_j)|^2} = \sum_{j=1}^n \tilde{\lambda}_j z_j^s \\ &= \int_{-\pi}^{\pi} z^s d\tilde{\mu}(\theta), \quad z = e^{i\theta}, \quad -(n-1) \leq s \leq n-1, \end{aligned}$$

where  $\tilde{\lambda}_j = \frac{\lambda_j}{|h(z_j)|^2}$  for  $j = 1, \dots, n$  and  $d\tilde{\mu}(\theta)$  is given by (43). It follows from Theorem 2.2 that  $\{z_j\}_{j=1}^n$  are the zeros of  $\Psi_n(z, \tau_n) = \chi_n(z) + \tau_n \chi_n^*(z)$ . Furthermore, setting  $\chi_n(z) = \frac{P_n(z)}{\pi_n(z)}$  and

$$\chi_n^*(z) = B_n(z) \chi_{n^*}(z) = B_n(z) \frac{\overline{P_n(1/\bar{z})}}{\pi_n(1/\bar{z})} = \frac{B_n(z) P_n^*(z)}{z^n \pi_n(1/\bar{z})} = \frac{P_n^*(z)}{\pi_n(z)},$$

we obtain that

$$\Psi_n(z, \tau_n) = \frac{P_n(z) + \tau_n P_n^*(z)}{\pi_n(z)} = \frac{P_n(z) + \tau_n P_n^*(z)}{h^*(z)}.$$

Thus, we have proved the following theorem.

**Theorem 5.2.** *Let  $h(z) = \prod_{l=1}^m (z - \alpha_l)$  with  $\alpha_l \in \mathbb{D} \setminus \{0\}$  for every  $l = 1, \dots, m$ , and set  $\alpha_k = 0$  for every  $k > m$ . Further, let  $n > m$  and suppose  $\chi_n(z) = \frac{P_n(z)}{h^*(z)}$  with  $P_n(z) \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$  is the  $n$ th orthonormal rational function associated with the measure  $\mu$ . Then for a given  $\tau_n \in \mathbb{T}$ ,  $I_n(f) = \sum_{j=1}^n \tilde{\lambda}_j f(z_j)$  is an  $n$ -point Szegő quadrature formula for  $d\tilde{\mu}(\theta) = \frac{d\mu(\theta)}{|h(e^{i\theta})|^2}$ , if and only if*

- (1) *the nodes  $\{z_j\}_{j=1}^n$  are the zeros of  $P_n(z) + \tau_n P_n^*(z)$ , and*
- (2)  *$\tilde{\lambda}_j = \frac{\lambda_j}{|h(z_j)|^2}$ , where  $\{\lambda_j\}_{j=1}^n$  is the set of weights corresponding to the  $n$ -point rational Szegő quadrature for  $\mu$ . □*

As an application of the previous theorem, we will now deduce characterization theorems for Szegő quadrature formulas associated with the measures  $d\tilde{\mu}(\theta) = \frac{\omega_i(\theta) d\theta}{|h(z)|^2}$ , where  $z = e^{i\theta}$  and  $\omega_i(\theta)$  is given by (6) for  $i \in \{1, \dots, 4\}$ .

We start with the Lebesgue measure (compare with [12, Theorem 4.2]).

**Theorem 5.3.** *Assume  $h(z) = \prod_{l=1}^m (z - \alpha_l)$  with  $\alpha_l \in \mathbb{D} \setminus \{0\}$  for every  $l = 1, \dots, m$ . Let  $\tau_n \in \mathbb{T}$  and set  $d\tilde{\mu}(\theta) = \frac{d\theta}{|h(e^{i\theta})|^2}$ . An  $n$ -point Szegő formula for  $\tilde{\mu}$  and  $n > m$  is then given by*

$$I_n(f) = \sum_{j=1}^n \tilde{\lambda}_j f(z_j),$$

where the nodes  $\{z_j\}_{j=1}^n$  are the zeros of  $z^{n-m} h(z) + \tau_n h^*(z)$  and the weights  $\{\tilde{\lambda}_j\}_{j=1}^n$  are given by

$$\tilde{\lambda}_j = \frac{2\pi}{|h(z_j)|^2 \left[ n - m + \sum_{l=1}^m \frac{1 - |\alpha_l|^2}{|z_j - \alpha_l|^2} \right]}; \quad j = 1, \dots, n. \quad \square$$

In the following theorem we characterize a Szegő quadrature formula associated with a rational modification of the Chebyshev weight functions  $1 \pm \cos \theta$ .

**Theorem 5.4.** *Suppose  $h(z) = \prod_{l=1}^m (z - \alpha_l)$  with  $\{\alpha_l\}_{l=1}^m \subset \mathbb{D} \setminus \{0\}$ ,  $n > m$ ,  $\tau_n \in \mathbb{T}$ ,  $\nu_i = (-1)^{i-1}$  for fixed  $i \in \{3, 4\}$ , and let*

$$d\tilde{\mu}(\theta) = \frac{1 - \nu_i \cos \theta}{|h(e^{i\theta})|^2} d\theta.$$

Next, let  $Q_n(z) = n - m + \sum_{l=1}^m \frac{1-|\alpha_l|^2}{|z-\alpha_l|^2}$  and define  $\tilde{Q}_n^{(i)} = [1 + Q_n(\nu_i)]^{-1}$ . Further, assume  $a_n^{(i)} = \nu_i^n \frac{h(\nu_i)}{h(\nu_i)} \tilde{Q}_n^{(i)}$  and  $b_n^{(i)} = \nu_i \left(1 + \tilde{Q}_n^{(i)}\right)$ . An  $n$ -point Szegő formula for  $\tilde{\mu}$  is then given by  $I_n(f) = \sum_{j=1}^n \tilde{\lambda}_j f(z_j)$ , where the set of nodes and weights are determined as follows:

(1) Denote by  $\{\nu_i, \nu_i, z_1, \dots, z_n\}$  the set of zeros of

$$V_n^{(i)}(z, \tau_n) = z^{n+1-m} \left[ -b_n^{(i)} + \left(1 + \tau_n \overline{a_n^{(i)}}\right) z \right] h(z) + z^m \left[ a_n^{(i)} + \tau_n \left(1 - b_n^{(i)} z\right) \right] \overline{h(z)}.$$

Then  $\{z_1, \dots, z_n\}$  is the set of nodes for  $I_n(f)$ .

(2) For every  $j \in \{1, \dots, n\}$  for which  $z_j \neq \nu_i$ , the  $j$ th weight is given by

$$\tilde{\lambda}_j^{-1} = \nu_i \frac{|h(z_j)|^2 \left( |z_j - b_n^{(i)}|^2 - |a_n^{(i)}|^2 \right)}{\pi b_n^{(i)} |z_j - \nu_i|^4} \times \left[ 1 + Q_{n+1}(z_j) + 2b_n^{(i)} \Re \left\{ \left[ -b_n^{(i)} + \left(1 + \tau_n \overline{a_n^{(i)}}\right) z_j \right]^{-1} \right\} \right].$$

Whenever  $z_j = \nu_i$  for a certain  $j \in \{1, \dots, n\}$ , the associated weight is given by  $\tilde{\lambda}_j = \lambda_j / |h(z_j)|^2$ , where  $\lambda_j$  is given by (32).  $\square$

Finally, for the remainder Chebyshev weight function we have the following theorem.

**Theorem 5.5.** Suppose  $h(z) = \prod_{l=1}^m (z - \alpha_l)$  with  $\{\alpha_l\}_{l=1}^m \subset \mathbb{D} \setminus \{0\}$ ,  $n > m$ ,  $\tau_n \in \mathbb{T}$ ,  $\epsilon \in \{\pm 1\}$ , and let

$$d\tilde{\mu}(\theta) = \frac{\sin^2 \theta}{|h(e^{i\theta})|^2} d\theta.$$

Next, let  $Q_n(z) = n - m + \sum_{l=1}^m \frac{1-|\alpha_l|^2}{|z-\alpha_l|^2}$  and define

$$a^+ = \frac{h(1)}{h(1)}, \quad a^- = (-1)^{n-1} \frac{h(-1)}{h(-1)}, \quad b^+ = 2 + Q_n(1), \quad b^- = 2 + Q_n(-1).$$

Further, assume  $\underline{x} = (d_n, e_n, f_n, g_n)^T$  is the solution of the system given by (15) and (16). An  $n$ -point Szegő formula for  $\tilde{\mu}$  is then given by  $I_n(f) = \sum_{j=1}^n \tilde{\lambda}_j f(z_j)$ , where the set of nodes and weights are determined as follows:

(1) Let  $\{+1 + 1, -1, -1, z_1, \dots, z_n\}$  be the set of zeros of

$$V_n^{(2)}(z, \tau_n) = z^{n+2-m} \left[ \left(1 + \tau_n \overline{d_n}\right) z^2 + (g_n + \tau_n \overline{e_n}) z + f_n \right] h(z) + z^m \left[ \tau_n \overline{f_n} z^2 + (e_n + \tau_n \overline{g_n}) z + \tau_n + d_n \right] \overline{h(z)}.$$

Then  $\{z_1, \dots, z_n\}$  is the set of nodes of  $I_n(f)$ .

(2) For every  $j \in \{1, \dots, n\}$  for which  $z_j \neq \epsilon$ , the  $j$ th weight is given by

$$\tilde{\lambda}_j^{-1} = \frac{2|h(z_j)|^2 (|d_n + e_n z_j|^2 - |f_n + g_n z_j + z_j^2|^2)}{\pi f_n |z_j^2 - 1|^4} \times \left[ 3 + Q_{n+1}(z_j) - \frac{2\Re \left\{ \overline{\tau_n} (2f_n \overline{z_j} + g_n) \overline{z_j} p_2^{(2)}(z_j) + e_n z_j p_2^{(2)}(z_j) \right\}}{|p_2^{(2)}(z_j)|^2} \right],$$

where  $p_2^{(2)}(z_j) = (d_n + e_n z_j) + \tau_n (\overline{f_n z_j^2} + \overline{g_n z_j} + 1)$ . Whenever  $z_j = \epsilon$  for a certain  $j \in \{1, \dots, n\}$  and  $z_k \neq -\epsilon$  for every  $k \in \{1, \dots, n\} \setminus \{j\}$ , the  $j$ th weight is given by  $\tilde{\lambda}_j = \lambda_j / |h(z_j)|^2$ , where  $\lambda_j$  is given by (40). Finally, if there exist two distinct indices  $j$  and  $k$  in  $\{1, \dots, n\}$  so that  $z_j = -z_k = 1$ , the associated weights are given by  $\tilde{\lambda}_j = \lambda_j / |h(z_j)|^2$  and  $\tilde{\lambda}_k = \lambda_k / |h(z_k)|^2$ , where  $\lambda_j$  and  $\lambda_k$  are given by (41) and (42).  $\square$

### 6. NUMERICAL EXAMPLES

The aim of this final section is to present some numerical illustrations of the results given in Section 5 in the construction of Szegő-type quadrature formulas with respect to rational modifications of Chebyshev weight functions.

We start by recalling that the parameter  $\tau_n \in \mathbb{T}$  in (45) can be chosen freely to fix a complex number  $\lambda \in \mathbb{T}$  as a node of the Szegő quadrature formula. By setting  $\tau_n = -\frac{\rho_n(\lambda)}{\lambda^n \rho_n(\lambda)} \in \mathbb{T}$ , the so-called ‘‘Szegő-Radau rules’’ arise (see e.g. [10, Proposition 2.8]). Recently, Jagels and Reichel have characterized Szegő-Lobatto quadrature formulas in [24], i.e. Szegő rules with two prescribed nodes on  $\mathbb{T}$ . Suppose  $z_\alpha$  and  $z_\beta$  are two distinct points on  $\mathbb{T}$ . Let  $N \geq 2$  be fixed, and assume there exist  $N - 2$  distinct nodes  $z_1, \dots, z_{N-2}$  on  $\mathbb{T}$  such that  $z_j \neq z_\alpha$  and  $z_j \neq z_\beta$  for  $1 \leq j \leq N - 2$ . Furthermore, suppose there exist positive weights  $A_1, A_2, \lambda_1, \dots, \lambda_{N-2}$  so that

$$\forall f \in \Lambda_{-(N-2), N-2} :$$

$$I_N(f) := A_1 f(z_\alpha) + A_2 f(z_\beta) + \sum_{j=1}^{N-2} \lambda_j f(z_j) = \int_{\mathbb{T}} f(z) d\mu(z) =: I_\mu(f) .$$

Then,  $I_N(f)$  is called an  $N$ -point Szegő-Lobatto quadrature formula for  $\mu$  with prescribed nodes  $z_\alpha$  and  $z_\beta$ . We now have the following theorem (for the proof, see [24]).

**Theorem 6.1** (Szegő-Lobatto quadrature). *Let  $z_\alpha$  and  $z_\beta$  be two distinct fixed points on  $\mathbb{T}$  and set  $a = z_\alpha^{n-1} \frac{\rho_n(z_\alpha)}{\rho_n(z_\alpha)} \in \mathbb{T}$  and  $b = z_\beta^{n-1} \frac{\rho_n(z_\beta)}{\rho_n(z_\beta)} \in \mathbb{T}$ . Then one of the following statements holds:*

- (1) *If  $az_\alpha = bz_\beta$ , we have that an  $n$ -point Szegő formula  $I_n(f)$  has  $z_\alpha$  and  $z_\beta$  as nodes by setting  $\tau_n = -\frac{\rho_n(z_\alpha)}{\rho_n^*(z_\alpha)} = -\frac{\rho_n(z_\beta)}{\rho_n^*(z_\beta)} \in \mathbb{T}$ .*
- (2) *If  $a = b$ , an  $(n + 1)$ -point Szegő formula  $I_{n+1}(f)$  has  $z_\alpha$  and  $z_\beta$  as nodes by setting  $\tau_{n+1} = -z_\alpha \frac{\rho_n(z_\alpha)}{\rho_n^*(z_\alpha)} = -z_\beta \frac{\rho_n(z_\beta)}{\rho_n^*(z_\beta)} \in \mathbb{T}$ .*
- (3) *Suppose  $az_\alpha \neq bz_\beta$  and  $a \neq b$ , and denote by  $\Gamma$  the circle with center  $c = -\frac{z_\alpha - z_\beta}{az_\alpha - bz_\beta}$  and radius  $r = \left| \frac{a-b}{az_\alpha - bz_\beta} \right|$ . Let  $\tilde{\delta}_{n+1} \in \Gamma \cap \mathbb{D}$ , a circular arc that is proved to be a non-empty set, and set  $\tau_{n+2} = -\frac{az_\alpha - bz_\beta}{a-b} \tilde{\delta}_{n+1} - \frac{z_\alpha - z_\beta}{a-b} \in \mathbb{T}$ . Then, there exists an  $(n + 2)$ -point Szegő-Lobatto quadrature formula whose nodes are the zeros of  $\tilde{\Phi}_{n+2}(z, \tau_{n+2}) = z\tilde{\rho}_{n+1}(z) + \tau_{n+2}\tilde{\rho}_{n+1}^*(z)$  with  $\tilde{\rho}_{n+1}(z) = z\rho_n(z) + \tilde{\delta}_{n+1}\rho_n^*(z)$ .  $\square$*

*Remark 6.2.* In the special case in which  $\tilde{\delta}_{n+1} = \rho_{n+1}(0)$  in Theorem 6.1(3), the  $(n + 2)$ -point Szegő-Lobatto formula is actually an  $(n + 2)$ -point Szegő formula and consequently, exact in  $\Lambda_{-(n+1), n+1}$ .

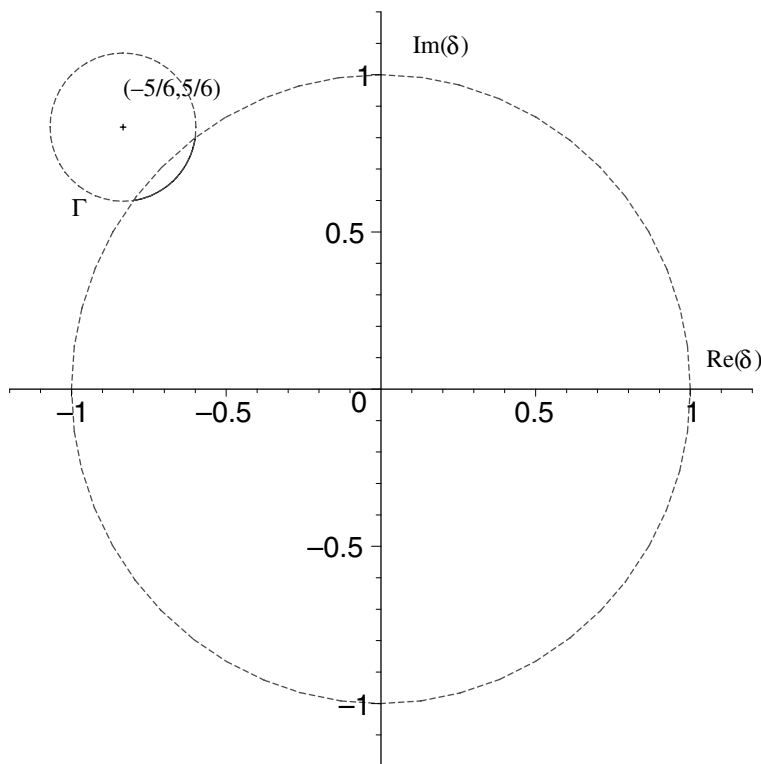


FIGURE 1. The circular arc  $\Gamma \cap \mathbb{D}$ .

In analogy with the real line situation (see e.g. [19]), the set of Szegő, Szegő-Radau and Szegő-Lobatto rules we call Szegő-type quadratures.

**Example 6.3.** Consider the rational modification of the Lebesgue measure given by

$$d\tilde{\mu}(\theta) = \frac{d\theta}{5/4 - \sin(2\theta)} = \frac{d\theta}{|z^2 - \mathbf{i}/2|^2}, \quad z = e^{i\theta}.$$

From Example 5.1 it then follows that the corresponding monic Szegő polynomials are explicitly given by  $\rho_n(z) = z^{n-2}(z^2 - \mathbf{i}/2)$  for every  $n \geq 2$ . Next, let  $n = 10$ , and suppose  $z_\alpha = 1$  and  $z_\beta = \mathbf{i}$ . From Theorem 6.1 we then deduce that

$$a = \frac{3 + 4\mathbf{i}}{5} \quad \text{and} \quad b = \frac{4 + 3\mathbf{i}}{5},$$

where clearly  $a - b = \frac{-1 + \mathbf{i}}{5} \neq 0$  and  $az_\alpha - bz_\beta = \frac{6}{5} \neq 0$ . Consequently, we are dealing with the third situation of Theorem 6.1. Elementary calculation yields that  $\Gamma$  is the circle with center  $c = \frac{5}{6}(-1 + \mathbf{i})$  and radius  $r = \frac{\sqrt{2}}{6}$ , and  $\tilde{\delta}_{11} \in \Gamma \cap \mathbb{D}$  iff (see also Figure 1)

$$(46) \quad \tilde{\delta}_{11} = c + re^{i\theta}, \quad \text{with} \quad -\theta_0 < \theta < -\theta_1, \quad \theta_0 = \arctan(7), \quad \theta_1 = \frac{\pi}{2} - \theta_0.$$

Since  $\rho_{11}(0) = 0 \notin \Gamma \cap \mathbb{D}$ , it follows that the 12-point Szegő-Lobatto formula associated with  $\tilde{\delta}_{11}$  is not a 12-point Szegő formula (see Remark 6.2); thus, the

domain of exactness is  $\Lambda_{-10,10}$ . Therefore, the nodes in the 12-point Szegő-Lobatto formula associated with  $\tilde{\delta}_{11}$  are the zeros of  $\tilde{\Phi}_{12}(z, \tau_{12})$ , given by

$$\tilde{\Phi}_{12}(z, \tau_{12}) = z\tilde{\rho}_{11}(z) + \tau_{12}\tilde{\rho}_{11}^*(z),$$

where

$$\tilde{\rho}_{11}(z) = z\rho_{10}(z) + \tilde{\delta}_{11}\rho_{10}^*(z) = z^9(z^2 - \mathbf{i}/2) + \tilde{\delta}_{11}(1 + z^2\mathbf{i}/2),$$

and  $\tau_{12} = 3(1 + \mathbf{i})\tilde{\delta}_{11} + 5$ . Let  $z_j, j = 1, \dots, 12$ , denote the zeros of  $\tilde{\Phi}_{12}(z, \tau_{12})$ . We have used Maple 10 with 40 digits to compute these zeros for the case in which  $\tilde{\delta}_{11} = -\frac{3}{4} + \frac{10-\sqrt{7}}{12}\mathbf{i}$  (that is, (46) with  $\theta = -\arctan(\sqrt{7})$ ) and  $\tau_{12} = \frac{\sqrt{7}+1}{4} - \frac{\sqrt{7}-1}{4}\mathbf{i}$ , respectively, for the case in which  $\tilde{\delta}_{11} = \frac{2}{3}(-1 + \mathbf{i})$  (that is, (46) with  $\theta = -\pi/4$ ) and  $\tau_{12} = 1$ . The results are given in Table 1, respectively, Table 2.

Finally, since  $\tilde{\delta}_{11} \neq \rho_{11}(0)$ , we are dealing with a modified measure so that we can not use Theorem 5.3 to compute the corresponding weights. Let  $\tilde{\varphi}_{11}(z)$  denote the polynomial of degree 11 that is orthonormal with respect to this modified measure. We then have the following Christoffel-Darboux formula in the ordinary polynomial situation (see also [5, Theorem 3.1.3]):

$$\mathcal{K}_{11}(z, \xi) = \frac{\tilde{\varphi}_{11}^*(z)\overline{\tilde{\varphi}_{11}^*(\xi)} - z\bar{\xi}\tilde{\varphi}_{11}(z)\overline{\tilde{\varphi}_{11}(\xi)}}{1 - z\bar{\xi}}.$$

Setting  $\lambda_j^{-1} = \mathcal{K}_{11}(z_j, z_j)$  and proceeding as in Section 4, it is easily verified that

$$\lambda_j^{-1} = 2\Re \left\{ z_j \overline{\tilde{\varphi}_{11}(z_j)} \tilde{\varphi}_{11}'(z_j) \right\} - 10 |\tilde{\varphi}_{11}(z_j)|^2.$$

Further, with  $\tilde{\varphi}_{11}(z) = c\tilde{\rho}_{11}(z)$  we obtain that

$$\lambda_j^{-1} = |c|^2 \mu_j^{-1}, \quad \mu_j^{-1} := 2\Re \left\{ z_j \overline{\tilde{\rho}_{11}(z_j)} \tilde{\rho}_{11}'(z_j) \right\} - 10 |\tilde{\rho}_{11}(z_j)|^2,$$

so that it remains to determine  $|c|^2$ . It should now hold that

$$\sum_{j=1}^{12} \lambda_j = \int_{-\pi}^{\pi} \frac{d\theta}{|e^{2i\theta} - \mathbf{i}/2|^2} = \frac{8\pi}{3} = \frac{\sum_{j=1}^{12} \mu_j}{|c|^2},$$

and hence,

$$|c|^2 = \frac{3}{8\pi} \sum_{j=1}^{12} \mu_j.$$

The resulting weights are given in Tables 1 and 2 for the case in which  $\tilde{\delta}_{11} = -\frac{3}{4} + \frac{10-\sqrt{7}}{12}\mathbf{i}$  and  $\tau_{12} = \frac{\sqrt{7}+1}{4} - \frac{\sqrt{7}-1}{4}\mathbf{i}$ , respectively, for the case in which  $\tilde{\delta}_{11} = \frac{2}{3}(-1 + \mathbf{i})$  and  $\tau_{12} = 1$ .

With these nodes and weights, the integral is computed to within the numerical precision for

$$\int_{-\pi}^{\pi} \left( \frac{5}{4} - \sin(2\theta) \right) d\tilde{\mu}(\theta), \quad \text{and for} \quad \int_{-\pi}^{\pi} e^{ik\theta} d\tilde{\mu}(\theta), \quad k = -10, \dots, 10.$$

**Example 6.4.** Consider the rational modifications

$$d\tilde{\mu}(\theta) = \frac{\omega_i(\theta)d\theta}{|z - 1/4|^2}, \quad i = 2, 3, 4, \quad z = e^{i\theta},$$



TABLE 1. The nodes  $z_j$  and weights  $\lambda_j$  in the 12-point Szegő-Lobatto formula for the case in which  $\tilde{\delta}_{11} = -\frac{3}{4} + \frac{10-\sqrt{7}}{12}\mathbf{i}$  and  $\tau_{12} = \frac{\sqrt{7}+1}{4} - \frac{\sqrt{7}-1}{4}\mathbf{i}$ .

$j$	$z_j$	$\lambda_j$
1	1	0.4516560678
2	0.9638145989 + 0.2665734776i	0.1232452240
3	0.8316896358 + 0.5552408033i	1.3564329045
4	0.5255107494 + 0.8507869605i	1.3244867113
5	i	0.4907276839
6	-0.5936593393 + 0.8047164649i	0.2935974363
7	-0.9584388804 + 0.2852979365i	0.3549235398
8	-0.9499460287 - 0.3124140564i	0.8469595298
9	-0.7027746634 - 0.7114125192i	1.6733022550
10	-0.2969879126 - 0.9548812386i	0.8202977524
11	0.3024297035 - 0.9531716920i	0.3490662051
12	0.8140918496 - 0.5807361367i	0.2928850997

TABLE 2. The nodes  $z_j$  and weights  $\lambda_j$  in the 12-point Szegő-Lobatto formula for the case in which  $\tilde{\delta}_{11} = \frac{2}{3}(-1 + \mathbf{i})$  and  $\tau_{12} = 1$ .

$j$	$z_j$	$\lambda_j$
1	1	0.4833219467
2	0.8615961242 + 0.5075944433i	1.1348377814
3	$(1 + \mathbf{i})/\sqrt{2}$	0.5132214093
4	0.5075944433 + 0.8615961242i	1.1348377814
5	i	0.4833219467
6	-0.5892211604 + 0.8079717966i	0.2926150925
7	-0.9562662716 + 0.2924975517i	0.3518577037
8	-0.9522800017 - 0.3052258155i	0.8334298010
9	$-(1 + \mathbf{i})/\sqrt{2}$	1.6722343497
10	-0.3052258155 - 0.9522800017i	0.8334298010
11	0.2924975517 - 0.9562662716i	0.3518577037
12	0.8079717966 - 0.5892211604i	0.2926150925

where  $\omega_i(\theta)$  is given by (6), and let

$$f_1(z) = \frac{(z - 1/4) \sin(z)}{8\pi z}, \quad f_2(z) = \frac{(4 - z) \sin(1/z)}{8\pi z} \quad \text{and} \quad f_3(z) = \frac{\sin(z)}{8\pi z}.$$

For these functions, we have used Maple 10 with 40 digits to compute the absolute error of the  $n$ -point Szegő quadrature formula, i.e.,

$$e_{\tilde{\mu}}(f_j) = |I_{\tilde{\mu}}(f_j) - I_n(f_j)|, \quad j = 1, 2, 3,$$

for  $n = 8, 16, 24$  with  $\tau_n = 1$ . The results are given in Tables 3–5.

TABLE 3. The absolute error  $e_{\bar{\mu}}(f_1)$  for the  $n$ -point Szegő quadrature formula with respect to the rational modification of the weight functions  $\omega_i(\theta)$ .

	$i = 2$	$i = 3$	$i = 4$
$n = 8$	$4.381892940 \times 10^{-8}$	$6.032565503 \times 10^{-7}$	$7.738268455 \times 10^{-7}$
$n = 16$	$4.417815525 \times 10^{-17}$	$6.151507562 \times 10^{-16}$	$7.903210143 \times 10^{-16}$
$n = 24$	$1.010117754 \times 10^{-27}$	$1.410431794 \times 10^{-26}$	$1.812756380 \times 10^{-26}$

TABLE 4. The absolute error  $e_{\bar{\mu}}(f_2)$  for the  $n$ -point Szegő quadrature formula with respect to the rational modification of the weight functions  $\omega_i(\theta)$ .

	$i = 2$	$i = 3$	$i = 4$
$n = 8$	$5.094348466 \times 10^{-5}$	$9.843173970 \times 10^{-5}$	$9.705465650 \times 10^{-5}$
$n = 16$	$1.925426989 \times 10^{-13}$	$3.815676087 \times 10^{-13}$	$3.801621369 \times 10^{-13}$
$n = 24$	$9.701673019 \times 10^{-24}$	$1.932272042 \times 10^{-23}$	$1.92904885 \times 10^{-23}$

TABLE 5. The absolute error  $e_{\bar{\mu}}(f_3)$  for the  $n$ -point Szegő quadrature formula with respect to the rational modification of the weight functions  $\omega_i(\theta)$ .

	$i = 2$	$i = 3$	$i = 4$
$n = 8$	$1.750776412 \times 10^{-7}$	$3.438809772 \times 10^{-7}$	$3.376229665 \times 10^{-7}$
$n = 16$	$1.766481103 \times 10^{-16}$	$3.512396863 \times 10^{-16}$	$3.491851390 \times 10^{-16}$
$n = 24$	$4.039751881 \times 10^{-27}$	$8.056536588 \times 10^{-27}$	$8.033580897 \times 10^{-27}$

7. APPENDIX

**Theorem 7.1.** *There are no  $\tau_n \in \mathbb{T}$  and  $\gamma \in \mathbb{T} \setminus \{\nu_i\}$  so that  $p_1^{(i)*}(\gamma) = 0$ .*

*Proof.* Suppose there are  $\tau_n \in \mathbb{T}$  and  $\gamma \in \mathbb{T} \setminus \{\nu_i\}$  so that  $\tau_n p_1^{(i)*}(\gamma) = 0$ . Clearly, it then holds that  $p_1^{(i)}(\gamma) = 0$  as well. Hence, from (27) we then deduce that

$$\tau_n = -\frac{a_n^{(i)}(1 - \overline{\alpha_n}\gamma)}{1 - \overline{b_n^{(i)}}\gamma} = -\frac{\gamma - b_n^{(i)}}{a_n^{(i)}(\gamma - \alpha_n)}, \quad \gamma \in \mathbb{T} \setminus \{\nu_i\}.$$

Therefore,

$$\begin{aligned} (47) \quad 0 &= \gamma \left( |\gamma - b_n^{(i)}|^2 - |a_n^{(i)}|^2 |\gamma - \alpha_n|^2 \right) \\ &= (|a_n^{(i)}|^2 \overline{\alpha_n} - \overline{b_n^{(i)}})\gamma^2 + [1 + |b_n^{(i)}|^2 - |a_n^{(i)}|^2(1 + |\alpha_n|^2)]\gamma + (|a_n^{(i)}|^2 \alpha_n - b_n^{(i)}) \\ &= (\gamma - \nu_i) \left[ (|a_n^{(i)}|^2 \overline{\alpha_n} - \overline{b_n^{(i)}})\gamma - (|a_n^{(i)}|^2 \alpha_n - b_n^{(i)}) \right], \quad \gamma \in \mathbb{T} \setminus \{\nu_i\}, \end{aligned}$$

where the last equality is due to the fact that the equality in (47) clearly holds for  $\gamma = \nu_i$ . Since we assumed  $\gamma \neq \nu_i$ , it follows that

$$\gamma = \nu_i \frac{|a_n^{(i)}|^2 \alpha_n - b_n^{(i)}}{|a_n^{(i)}|^2 \overline{\alpha_n} - \overline{b_n^{(i)}}}.$$

But from (10) we deduce that

$$|a_n^{(i)}|^2 \alpha_n - b_n^{(i)} = \frac{2\Re\{\alpha_n\} - \nu_i[1 - \nu_i \overline{\alpha_n}]^2 Q_n^{(i)} + 1}{|(1 - \nu_i \overline{\alpha_n})Q_n^{(i)} + 1|^2} - \nu_i \in \mathbb{R},$$

so we again find that  $\gamma$  should equal  $\nu_i$ . □

**Theorem 7.2.** *If  $n = 1$  or  $\alpha_n = 0$ , there are no  $\tau_n \in \mathbb{T}$  and  $\gamma \in \mathbb{T} \setminus \{\pm 1\}$  so that  $\tau_n p_2^{(2)*}(\gamma) = 0$ .*

*Proof.* Similarly, as in the proof of Theorem 7.1, we find that there are  $\tau_n \in \mathbb{T}$  and  $\gamma \in \mathbb{T} \setminus \{\pm 1\}$  so that  $\tau_n p_2^{(2)*}(\gamma) = 0$  iff

$$(48) \quad 0 = \gamma^2 (|f_n + g_n \gamma + \gamma^2|^2 - |d_n + e_n \gamma|^2 |\gamma - \alpha_n|^2) \\ = (\gamma^2 - 1) [(e_n \overline{d_n} \overline{\alpha_n} + \overline{f_n}) \gamma^2 + \overline{\varpi_n} \gamma - (\overline{e_n} d_n \alpha_n + f_n)], \quad \gamma \in \mathbb{T} \setminus \{\pm 1\},$$

where the last equality is due to the fact that the equality in (48) clearly holds for  $\gamma = \pm 1$ , and  $\overline{\varpi_n} \in \mathbb{R}$  is given by

$$\begin{aligned} \overline{\varpi_n} &= g_n + f_n \overline{g_n} + \alpha_n (|d_n|^2 + |e_n|^2) - d_n \overline{e_n} (1 + |\alpha_n|^2) \\ &= \overline{g_n} + g_n \overline{f_n} + \overline{\alpha_n} (|d_n|^2 + |e_n|^2) - e_n \overline{d_n} (1 + |\alpha_n|^2). \end{aligned}$$

Since we assumed  $\gamma \notin \{\pm 1\}$ , it should hold that

$$(e_n \overline{d_n} \overline{\alpha_n} + \overline{f_n}) \gamma^2 + \overline{\varpi_n} \gamma - (\overline{e_n} d_n \alpha_n + f_n) = 0.$$

Note that for the parameters  $a^+$ ,  $a^-$ ,  $b^+$  and  $b^-$ , given by (14), we have that

$$\begin{aligned} \overline{a^+} &= \frac{1}{a^+ |1 - \alpha_n|^2}, & \overline{a^-} &= \frac{1}{a^- |1 + \alpha_n|^2}, \\ \overline{b^+} &= b^+ + \frac{2i\Im\{\alpha_n\}}{|1 - \alpha_n|^2}, & \overline{b^-} &= b^- - \frac{2i\Im\{\alpha_n\}}{|1 + \alpha_n|^2}. \end{aligned}$$

With this observation in mind, we computed  $\overline{\varpi_n}$  with the aid of Maple 10 to find that  $\overline{\varpi_n} = 0$ , so it should hold that

$$(49) \quad \gamma^2 = \frac{\overline{e_n} d_n \alpha_n + f_n}{e_n \overline{d_n} \overline{\alpha_n} + \overline{f_n}}.$$

For the special case in which  $\alpha_n = 0$ , it follows from (18) that  $f_n = \frac{-2}{\pi |h_n|^2} \in \mathbb{R}$ , so we again find that  $\gamma$  should equal  $\pm 1$ . For  $n = 1$ , on the other hand, we have that

$$a^+ = \frac{1}{1 - \overline{\alpha_1}}, \quad a^- = \frac{1}{1 + \overline{\alpha_1}}, \quad b^+ = 2 + a^+ \quad \text{and} \quad b^- = 2 + a^-.$$

We then computed  $\gamma^2$ , given by (49) with  $n = 1$ , to find that  $\gamma^2 = 1$ . At this moment of writing, however, we could not verify whether  $\gamma^2$ , given by (49), equals one for the more general case of  $\alpha_n \neq 0$  and  $n > 1$ . □

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DEPARTMENT OF COMPUTER SCIENCE, K.U.LEUVEN, CELESTIJNENLAAN 200 A, B-3001 LEUVEN, BELGIUM.

*E-mail address:* `Adhemar.Bultheel@cs.kuleuven.be`

DEPARTMENT OF COMPUTER SCIENCE, K.U.LEUVEN, CELESTIJNENLAAN 200 A, B-3001 LEUVEN, BELGIUM.

*Current address:* Department of Mathematical Analysis, La Laguna University, 38271 La Laguna, Tenerife, Canary Islands, Spain

*E-mail address:* `szegoquadrature@hotmail.com`

DEPARTMENT OF COMPUTER SCIENCE, K.U.LEUVEN, CELESTIJNENLAAN 200 A, B-3001 LEUVEN, BELGIUM.

*E-mail address:* `Karl.Deckers@cs.kuleuven.be`

DEPARTMENT OF MATHEMATICAL ANALYSIS, LA LAGUNA UNIVERSITY, 38271 LA LAGUNA, TENERIFE, CANARY ISLANDS, SPAIN

*E-mail address:* `pglez@ull.es`