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A NEW MULTIDIMENSIONAL CONTINUED FRACTION ALGORITHM

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ABSTRACT. It has been believed that the continued fraction expansion of (α,β) $(1,\alpha,\beta$ is a $\mathbb{Q}$-basis of a real cubic field) obtained by the modified Jacobi-Perron algorithm is periodic. We conducted a numerical experiment (cf. Table B, Figure 1 and Figure 2) from which we conjecture the non-periodicity of the expansion of $(\langle\sqrt[3]{3}\rangle,\langle\sqrt[3]{9}\rangle)$ ($\langle x\rangle$ denoting the fractional part of x). We present a new algorithm which is something like the modified Jacobi-Perron algorithm, and give some experimental results with this new algorithm. From our experiments, we can expect that the expansion of (α,β) with our algorithm always becomes periodic for any real cubic field. We also consider real quartic fields.

1. Introduction

The study of continued fractions has a long history dating back to J. Wallis (1616–1703) and Ch. Huygens (1629–1695) [8]. In particular, many kinds of higher-dimensional continued fractions have been studied starting with K.G. Jacobi (1804–1851) [7]. A central problem has been to find a higher-dimensional generalization of Legendre's theorem concerning the periodic continued fractions. In fact, the following conjecture has been believed.

Conjecture. *Let $1,\alpha_1,\dots,\alpha_s$ be a $\mathbb{Q}$-basis of real number field K with $[K:\mathbb{Q}]=s+1$. Then, the expansion of $(\alpha_1,\dots,\alpha_s)$ by the Jacobi-Perron algorithm is eventually periodic; cf.* [11].

Although Bernstein [1] gave some classes of periodic continued fractions obtained by the Jacobi-Perron algorithm, the conjecture is still open. For example, the periodicity of the Jacobi-Perron algorithm for $(\sqrt[3]{4},\sqrt[3]{4^2})$ has not been established (see [2], [12]). We also gave a numerical experiments (cf. Table B, Figure 1 and Figure 2) from which we conjecture the non-periodicity of the expansion of $(\langle\sqrt[3]{3}\rangle,\langle\sqrt[3]{9}\rangle)$ by the modified Jacobi-Perron algorithm.

In this paper, we give some candidates of algorithms of continued fraction expansion of dimensions 2 and 3, which can be easily generalized to any dimension.

By numerical experiments, we checked that, for instance, $(\langle\sqrt[3]{m}\rangle,\langle\sqrt[3]{m^2}\rangle)$ $(2\le m\le 5000, m\in\mathbb{Z}, \sqrt[3]{m}\notin\mathbb{Q})$ and $(\langle\sqrt[4]{m}\rangle,\langle\sqrt[4]{m^2}\rangle,\langle\sqrt[4]{m^3}\rangle)$ $(2\le m\le 5000, m\in\mathbb{Z}, \sqrt{m}\notin\mathbb{Q})$ obtained by our algorithm become periodic. We showed the periodicity

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of the expansion of $(\alpha,\beta)\in K^2$ by our algorithm for small classes of cubic fields K, including totally real cases, and pure cubic cases; cf. Theorems 2.4, 2.5.

The algorithms given in this paper are motivated by the algorithms given in [12], which are quite different from the Jacobi-Perron algorithm, but our algorithms are related to the so-called modified Jacobi-Perron algorithm; cf. [3], [4], [5], [10].

2. The cubic case

In this section we consider real cubic fields K (including totally real cases and not totally real cases). We denote by X_K the set defined by

$$X_K := \{(\alpha,\beta)\in K^2 | 1,\alpha,\beta \text{ are linearly independent over } \mathbb{Q}\}\cap I^2, \text{ where } I=[0,1).$$

We define the transformation T_K on X_K by

$$T_K(\alpha,\beta) := \begin{cases} (\frac{1}{\alpha}-\lfloor\frac{1}{\alpha}\rfloor, \frac{\beta}{\alpha}-\lfloor\frac{\beta}{\alpha}\rfloor) & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} > \frac{\beta}{\sqrt{|N(\beta)|}}, \\ (\frac{\alpha}{\beta}-\lfloor\frac{\alpha}{\beta}\rfloor, \frac{1}{\beta}-\lfloor\frac{1}{\beta}\rfloor) & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} < \frac{\beta}{\sqrt{|N(\beta)|}} \end{cases}$$

for $(\alpha,\beta)\in X_K$, where $\lfloor x\rfloor$ is the floor function of x and $N(x)$ is the norm of $x\in K$ over $\mathbb{Q}$.

Lemma 2.1. *The transformation T_K is well defined.*

Proof. Let $(\alpha,\beta)\in X_K$. It suffices to show that $\frac{\alpha}{\sqrt{|N(\alpha)|}} \neq \frac{\beta}{\sqrt{|N(\beta)|}}$. We suppose $\frac{\alpha}{\sqrt{|N(\alpha)|}} = \frac{\beta}{\sqrt{|N(\beta)|}}$. Then, we have $\alpha = \sqrt{\frac{|N(\alpha)|}{|N(\beta)|}}\beta$. Since α and β are linearly independent over $\mathbb{Q}$, so that $\sqrt{\frac{|N(\alpha)|}{|N(\beta)|}} \notin \mathbb{Q}$. Hence $\sqrt{\frac{|N(\alpha)|}{|N(\beta)|}}$ is a quadratic irrational and $\sqrt{\frac{|N(\alpha)|}{|N(\beta)|}} \in K$, which is a contradiction. It is easy to see that $T_K(\alpha,\beta)\in X_K$. □

We define the integer-valued functions a, b and e on X_K as follows:

$$a(\alpha,\beta) := \begin{cases} \left\lfloor\frac{1}{\alpha}\right\rfloor & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} > \frac{\beta}{\sqrt{|N(\beta)|}}, \\ \left\lfloor\frac{\alpha}{\beta}\right\rfloor & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} < \frac{\beta}{\sqrt{|N(\beta)|}}, \end{cases}$$

$$b(\alpha,\beta) := \begin{cases} \left\lfloor\frac{\beta}{\alpha}\right\rfloor & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} > \frac{\beta}{\sqrt{|N(\beta)|}}, \\ \left\lfloor\frac{1}{\beta}\right\rfloor & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} < \frac{\beta}{\sqrt{|N(\beta)|}}, \end{cases}$$

$$e(\alpha,\beta) := \begin{cases} 0 & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} > \frac{\beta}{\sqrt{|N(\beta)|}}, \\ 1 & \text{if } \frac{\alpha}{\sqrt{|N(\alpha)|}} < \frac{\beta}{\sqrt{|N(\beta)|}} \end{cases}$$

for $(\alpha,\beta)\in X_K$.

We put

$$\begin{aligned}(a_n, b_n, e_n) &= (a_n(\alpha,\beta), b_n(\alpha,\beta), e_n(\alpha,\beta))\\ &:= (a(T_K^{n-1}(\alpha,\beta), b(T_K^{n-1}(\alpha,\beta), e(T_K^{n-1}(\alpha,\beta)) \quad (n \in \mathbb{Z}_{>0}),\\ \mathcal{S}(\alpha,\beta) &:= \{(a_n(\alpha,\beta), b_n(\alpha,\beta), e_n(\alpha,\beta))\}_{n=1}^{\infty}.\end{aligned}$$

The sequence $\mathcal{S}(\alpha,\beta)$ will be referred to as *the expansion of* $(\alpha,\beta) \in X_K$ *by* T_K; T_K gives rise to a 2-dimensional continued fraction expansion, which will be called the Algebraic Jacobi-Perron Algorithm(AJPA). Throughout our paper α_n, β_n are numbers defined by

$$(\alpha_n, \beta_n) := T_K^n(\alpha,\beta), \quad (n \in \mathbb{Z}_{\geq 0}).$$

Notice that

$$(a_n, b_n, e_n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \{0,1\}, \quad (n \in \mathbb{Z}_{\geq 0}).$$

For each $(a', b', e') \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0} \times \{0,1\}$, we put

$$A_{(a',b',e')} := \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & b' \\ 1 & 0 & a' \end{pmatrix} & \text{if } e' = 0, \\ \\ \begin{pmatrix} 1 & 0 & a' \\ 0 & 0 & 1 \\ 0 & 1 & b' \end{pmatrix} & \text{if } e' = 1, \end{cases} \tag{2.1}$$

$$M_n(\alpha,\beta) = \begin{pmatrix} p_n''(\alpha,\beta) & p_n'(\alpha,\beta) & p_n(\alpha,\beta) \\ q_n''(\alpha,\beta) & q_n'(\alpha,\beta) & q_n(\alpha,\beta) \\ r_n''(\alpha,\beta) & r_n'(\alpha,\beta) & r_n(\alpha,\beta) \end{pmatrix} := A_{(a_1,b_1,e_1)} \cdots A_{(a_n,b_n,e_n)}. \tag{2.2}$$

Definition. We denote by $\mathcal{P}_K$ the set

$\{(\alpha,\beta) \in X_K |$there exist $m, n \in \mathbb{Z}_{>0}$ such that $m \neq n$ and $T_K^m(\alpha,\beta) = T_K^n(\alpha,\beta)\}$.

If $(\alpha,\beta) \in \mathcal{P}_K$, the expansion $\mathcal{S}(\alpha,\beta)$ by T_K becomes periodic, and vice versa. In what follows we mean by "the period" the period obtained by choosing the shortest period and preperiod. For the periodic continued fraction obtained by AJPA, we have the following proposition. In a way similar to Perron [9], we can show

Proposition 2.2. *Let* $(\alpha,\beta) \in \mathcal{P}_K$*. Then, there exists a constant* $c(\alpha,\beta) > 0$ *and* $\eta(\alpha,\beta) > 0$ *such that* $\eta(\alpha,\beta) \leq \frac{3}{2}$ *and*

$$|\alpha - \frac{p_n}{r_n}| \leq \frac{c(\alpha,\beta)}{r_n^{\eta(\alpha,\beta)}},$$

$$|\beta - \frac{q_n}{r_n}| \leq \frac{c(\alpha,\beta)}{r_n^{\eta(\alpha,\beta)}},$$

holds. Furthermore, $\eta(\alpha,\beta) = \frac{3}{2}$ *holds if and only if* K *is not a totally real cubic field.*

Remark 2.3.

(1) Based on our many experiments (cf. Tables A, C, D), we can hope that

$$X_K = P_K.$$

(2) In view of Proposition 2.2, we see that

$$(\alpha,\beta)\neq(\alpha',\beta')\Longleftrightarrow \mathcal{S}(\alpha,\beta)\neq\mathcal{S}(\alpha',\beta'),$$

for $(\alpha,\beta),(\alpha',\beta')\in\mathcal{P}_K$.

Bernstein [1] gave some classes of periodic continued fractions obtained by the Jacobi-Perron algorithm. We can also give some examples of periodic expansions obtained by the AJPA, for example:

Theorem 2.4. *Let $K=\mathbb{Q}(\sqrt[3]{m^3+1})$ with $m\in\mathbb{Z}_{>0}$. Let $(\alpha,\beta)=(\sqrt[3]{m^3+1}-m,\sqrt[3]{(m^3+1)^2}-m^2)$. Then, $(\alpha,\beta)\in\mathcal{P}_K$ and the length of the period is 2.*

Proof. It is easy to see that $(\alpha,\beta)\in X_K$. We have $|N(\alpha)|=1$ and $|N(\beta)|=(m^3+1)^2-m^6=2m^3+1$.

First, we consider the case where $m\geq 2$. Since $\sqrt{2m^3+1}>\sqrt[3]{m^3+1}+m$, we have $\frac{\alpha}{\sqrt{|N(\alpha)|}}>\frac{\beta}{\sqrt{|N(\beta)|}}$, so that $e_1=0$. Therefore, we get

$$\begin{aligned}(\alpha_1,\beta_1)=T_K(\alpha,\beta)&=(\frac{1}{\alpha}-a_1,\frac{\beta}{\alpha}-b_1)\\&=(\sqrt[3]{(m^3+1)^2}+m\sqrt[3]{m^3+1}+m^2-3m^2,\sqrt[3]{m^3+1}+m-2m)\\&=(\sqrt[3]{(m^3+1)^2}+m\sqrt[3]{m^3+1}-2m^2,\sqrt[3]{m^3+1}-m).\end{aligned}$$

We see that $|N(\alpha_1)|=9m^3+1$ and $|N(\beta_1)|=1$. One can see that

$$\frac{\sqrt[3]{(m^3+1)^2}+m\sqrt[3]{m^3+1}-2m^2}{\sqrt{9m^3+1}}<\sqrt[3]{m^3+1}-m.$$

Therefore, we have $e_2=1$. Thus, we get

$$\begin{aligned}(\alpha_2,\beta_2)=T_K(\alpha_1,\beta_1)&=(\frac{\alpha_1}{\beta_1}-a_2),\frac{1}{\beta_1}-b_2)\\&=(\sqrt[3]{m^3+1}+2m-3m,\sqrt[3]{(m^3+1)^2}+m\sqrt[3]{m^3+1}+m^2-3m^2)\\&=(\sqrt[3]{m^3+1}-m,\sqrt[3]{(m^3+1)^2}+m\sqrt[3]{m^3+1}-2m^2)\\&=(\beta_1,\alpha_1).\end{aligned}$$

Therefore, we have $(\alpha_2,\beta_2)=(\beta_1,\alpha_1)$. Hence, $\{T_K^n(\alpha,\beta)\}_{n=0}^{\infty}$ is periodic with 2 as the length of its (shortest) period. Thus, we get

n	1	2	3
a_n	$3m^2$	$3m$	$3m^2$
b_n	$2m$	$3m^2$	$3m$
e_n	0	1	0

$$a_n=a_{n-2},\ b_n=b_{n-2}\text{ and }e_n=e_{n-2}\text{ for all }n\geq 4.$$

Second, we consider the case $m=1$, i.e., $(\alpha_0,\beta_0)=(\sqrt[3]{2}-1,\sqrt[3]{4}-1)$. Then, we see that

$$\begin{aligned}&(\alpha_1,\beta_1)=(\frac{1}{3}-\frac{\sqrt[3]{2}}{3}+\frac{\sqrt[3]{4}}{3},-\frac{2}{3}+\frac{2\sqrt[3]{2}}{3}+\frac{\sqrt[3]{4}}{3}),\\&(\alpha_2,\beta_2)=(\alpha_0,\beta_0).\end{aligned}$$

We get

n	1	2
a_n	0	2
b_n	1	1
e_n	1	0

$a_n = a_{n-2}$, $b_n = b_{n-2}$ and $e_n = e_{n-2}$ for all $n \geq 3$. □

We put, for $\frac{p}{q}(p, q \in \mathbb{Z}$ are coprime),

$$\mathrm{dh}(\frac{p}{q}) := \max\{\lfloor \log_{10}|p| + 1\rfloor, \lfloor \log_{10}|q| + 1\rfloor\}, \quad \mathrm{dh}(0) := 0.$$

The function dh can be extended to $\mathbb{Q}[x]$: for $g(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Q}[x]$, put

$$\mathrm{dh}(g) := \max_{0\leq i\leq n}\{\mathrm{dh}(a_i)\}.$$

Furthermore, we put, for $(\alpha, \beta) \in K^2$,

$$\overline{\mathrm{dh}}(\alpha, \beta) := \max\{\mathrm{dh}(\mathrm{mpol}(\alpha)), \mathrm{dh}(\mathrm{mpol}(\beta))\},$$

where $\mathrm{mpol}(\gamma) \in \mathbb{Q}[x]$ ($\gamma \in K$) is the monic minimal polynomial of γ. We put, for $n \in \mathbb{Z}_{\geq 0}$ and $(\alpha, \beta) \in X_K$,

$$\mathrm{dh}_{\mathrm{AJPA}}(n; \alpha, \beta) := \overline{\mathrm{dh}}(\alpha_n, \beta_n),$$
$$\mathrm{rdh}_{\mathrm{AJPA}}(n; \alpha, \beta) := \frac{\overline{\mathrm{dh}}(\alpha_n, \beta_n)}{\overline{\mathrm{dh}}(\alpha_0, \beta_0)},$$

where $(\alpha_n, \beta_n) := T_K^n(\alpha, \beta)$. The function $\mathrm{dh}_{\mathrm{AJPA}}(n; \alpha, \beta)$ (resp., $\mathrm{rdh}_{\mathrm{AJPA}}(n; \alpha, \beta)$) is referred to as *the nth decimal height of* (α, β) (resp., *the nth relative decimal height of* (α, β)) with respect to the AJPA.

Let $K = Q(\sqrt[3]{m})$ with $m \in \mathbb{Z}_{>0}$ and $\sqrt[3]{m} \notin \mathbb{Q}$. We computed the length of the periods of the expansion $\mathcal{S}(\langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m^2}\rangle)$ for all m with $2 \leq m \leq 5000$ ($\sqrt[3]{m} \notin \mathbb{Q}$) and these decimal heights; cf. Table A. For the calculation of the tables, we used a computer equipped with GiNaC [6] on GNU C++. We confirmed that $(\langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m^2}\rangle) \in \mathcal{P}_K$ for all m with $2 \leq m \leq 5000(\sqrt[3]{m} \notin \mathbb{Q})$. There are no reports on the periodicity of any of these pairs of numbers obtained by the Jacobi-Perron algorithm or any modified Jacobi-Perron algorithms except for the algorithms of [12].

In [2] Elsner and Hasse gave numerical results for 36 pairs of cubic numbers with the Jacobi-Perron algorithm. They found 14 cases of periodicity but no sign of periodicity for the other 22 cases.

In [12] the first author computed the $\overline{\mathrm{dh}}(T^n(\langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m}\rangle^2)$ for some n and m, where T is the transformation related to the Jacobi-Perron algorithm. Tamura observed that it becomes gradually bigger and bigger as a function of n for many $2 \leq m \leq 104$. This suggests that for some m,

$$\lim_{n\to\infty} \overline{\mathrm{dh}}(T^n(\langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m}\rangle^2) = \infty,$$

i.e., an explosion of the size of minimal polynomials. We computed $\overline{\mathrm{dh}}(\bar{T}^n(\sqrt[3]{3} - 1, \sqrt[3]{9} - 2)$ for $0 \leq n \leq 2 \times 10^4$, where the transformation $\bar{T}$, which is associated with the modified Jacobi-Perron algorithm, is defined by the following:

For $(x,y)\in[0,1)^2$ ($1,x,y$ are linearly independent over $\mathbb{Q}$),

$$\bar{T}(x,y) := \begin{cases} (\frac{y}{x}, \frac{1}{x} - \lfloor\frac{1}{x}\rfloor) & \text{if } \ x > y, \\ \\ (\frac{1}{y} - \lfloor\frac{1}{y}\rfloor, \frac{x}{y}) & \text{if } \ x < y; \end{cases}$$

cf. Podsypanin [10]. This algorithm has been studied in connection with simultaneous diophantine approximation [3], [4] and [5].

Table B gives $\overline{\mathrm{dh}}(\bar{T}^n(\sqrt[3]{3}-1, \sqrt[3]{9}-2)$ for $0 \le n \le 2\times 10^4$. The table also suggests an explosion phenomenon related to the modified Jacobi-Perron algorithm.

Theorem 2.5. *Let δ_m be the root of $x^3 - mx + 1 = 0$ ($m \in \mathbb{Z}, m \ge 3$) determined by $0 < \delta_m < 1$. Then, $K = \mathbb{Q}(\delta_m)$ is a cubic number field and $(\delta_m, \delta_m^2) \in \mathcal{P}_K$.*

Proof. The irreduciblity of $g := x^3 - mx + 1$ is clear. It is also clear that g has a unique root δ_m in $[0, 1)$. We consider $T_K^n(\alpha, \beta)$ with $(\alpha, \beta) = (\delta_m, \delta_m^2)$.

Since $|N(\delta_m)| = |N(\delta_m^2)| = 1$ and $0 < \delta_m < 1$, we have $\frac{\delta_m}{\sqrt{|N(\delta_m)|}} > \frac{\delta_m^2}{\sqrt{|N(\delta_m^2)|}}$, so that $e_1 = 0$, and

$$\begin{aligned}(\alpha_1, \beta_1) = T_K(\alpha, \beta) &= (\frac{1}{\alpha} - a(\alpha, \beta), \frac{\beta}{\alpha} - b(\alpha, \beta)) \\ &= (m - \delta_m^2 - (m-1), \delta_m) \\ &= (1 - \delta_m^2, \delta_m).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\alpha_1}{\sqrt{|N(\alpha_1)|}} = \frac{1-\delta_m^2}{\sqrt{|N(1-\delta_m^2)|}} = \frac{1-\delta_m^2}{\sqrt{m(m-2)}}, \\ &\frac{\beta_1}{\sqrt{|N(\beta_1)|}} = \delta_m.\end{aligned}$$

One can check that $\dfrac{1-\delta_m^2}{\sqrt{m(m-2)}} > \delta_m$ if and only if $(1-\delta_m^2)^2 > m(m-2)\delta_m^2$. Since $(1-\delta_m^2)^2 - m(m-2)\delta_m^2 = 2(m-2)\delta_m^2 + 2(\delta_m^2 - \delta_m^3) + \delta_m^4 - \delta_m^6 > 0$, we get $\dfrac{\alpha_1}{\sqrt{|N(\alpha_1)|}} > \dfrac{\beta_1}{\sqrt{|N(\beta_1)|}}$. Therefore, $e_2 = 0$, and

$$\begin{aligned}(\alpha_2, \beta_2) = T_K(\alpha_1, \beta_1) &= (\frac{1}{\alpha_1} - a(\alpha_1, \beta_1), \frac{\beta_1}{\alpha_1} - b(\alpha_1, \beta_1)) \\ &= \left(\frac{-(m-1)\delta_m^2 - \delta_m + m^2 - 2m + 1}{m^2 - 2m} - 1, \frac{-\delta_m^2 - (m-1)\delta_m + (m-1)}{m^2-2m}\right) \\ &= \left(\frac{-(m-1)\delta_m^2 - \delta_m + 1}{m^2 - 2m}, \frac{-\delta_m^2 - (m-1)\delta_m + (m-1)}{m^2-2m}\right).\end{aligned}$$

In view of $N(\alpha_2) = \dfrac{1}{m(m-2)}$ and $N(\beta_2) = \dfrac{1}{m(m-2)}$, we get

$$\frac{\alpha_2}{\sqrt{|N(\alpha_2)|}} = \frac{\delta^2\sqrt{m(m-2)}}{1-\delta_m^2} < \frac{\delta\sqrt{m(m-2)}}{1-\delta_m^2} = \frac{\beta_2}{\sqrt{|N(\beta_2)|}}.$$

Thus, we get $e_3 = 1$, and

$$\begin{aligned}(\alpha_3, \beta_3) = T_K(\alpha_2, \beta_2) &= (\frac{\alpha_2}{\beta_2} - a(\alpha_2, \beta_2), \frac{1}{\beta_2} - b(\alpha_2, \beta_2)) \\ &= (\delta_m, -\delta_m^2 - \delta_m + m - (m-1)) \\ &= (\delta_m, -\delta_m^2 - \delta_m + 1).\end{aligned}$$

Since $|N(\beta_3)| = m^2 - 4m + 4$, $\frac{\alpha_3}{\sqrt{|N(\alpha_3)|}} < \frac{\beta_3}{\sqrt{|N(\beta_3)|}}$ holds if and only if

$$\beta_3\sqrt{N(\alpha_3)} - \alpha_3\sqrt{N(\beta_3)} = (-\delta_m^2 - \delta_m + 1) - \delta_m(m-2) > 0,$$

which is equivalent to

$$-\delta_m^4 + \delta_m^2 - 2\delta_m + 1 > 0. \tag{2.3}$$

The inequality (2.3) holds, since $0 < \delta_m < \frac{1}{2}$, which can be easily seen. Therefore, we get $e_4 = 1$, and we have

$$\begin{aligned}(\alpha_4, \beta_4) = T_K(\alpha_3, \beta_3) &= (\frac{\alpha_3}{\beta_3} - a(\alpha_3, \beta_3), \frac{1}{\beta_3} - b(\alpha_3, \beta_3)) \\ &= \left(\frac{-\delta_m + 1}{m-2}, \frac{-\delta_m^2 + m - 1}{m-2} - 1\right) \\ &= \left(\frac{-\delta_m + 1}{m-2}, \frac{-\delta_m^2 + 1}{m-2}\right),\end{aligned}$$

and $|N(\alpha_4)| = \frac{1}{(m-2)^2}$ and $|N(\beta_4)| = \frac{m}{(m-2)^2}$ follows. Thus, we have $\frac{\alpha_4}{\sqrt{|N(\alpha_4)|}} > \frac{\beta_4}{\sqrt{|N(\beta_4)|}}$. Therefore, we see $e_5 = 0$ and

$$\begin{aligned}(\alpha_5, \beta_5) = T_K(\alpha_4, \beta_4) &= (\frac{1}{\alpha_4} - a(\alpha_4, \beta_4), \frac{\beta_4}{\alpha_4} - b(\alpha_4, \beta_4)) \\ &= (-\delta_m^2 - \delta_m + m - 1 - (m-2), \delta_m + 1 - 1) \\ &= (-\delta_m^2 - \delta_m + 1, \delta_m),\end{aligned}$$

which implies $(\alpha_6, \beta_6) = (\frac{-\delta_m^2+1}{m-2}, \frac{-\delta_m+1}{m-2})$ and $(\alpha_7, \beta_7) = (\alpha_3, \beta_3)$. Thus, we obtain $(\alpha_{3+4j}, \beta_{3+4j}) = (\alpha_3, \beta_3)$ for all integer $j \geq 0$.

Thus, we get

n	1	2	3	4	5	6	7
a_n	$m-1$	1	0	0	$m-2$	1	1
b_n	0	0	$m-1$	1	1	0	$m-2$
e_n	0	0	1	1	0	0	1

$$a_n = a_{n-4},\ b_n = b_{n-4} \text{ and } e_n = e_{n-4} \text{ for all } n \geq 8.$$ □

We also confirmed that $(\langle\tau_m\rangle, \langle\tau_m^2\rangle) \in \mathcal{P}_K$ for all integers m with $2 \leq m \leq 5000$, where τ_m is the maximal root of $x^3 - mx + 1$, while the length of the (shortest) period is very long in some cases. Table C gives these results.

3. The quartic case

Let K be a real quartic field over $\mathbb{Q}$. In this section, we mean by X'_K and X_K the sets defined by

$$X'_K := \{(\alpha_1,\alpha_2,\alpha_3) \in K^3 | 1, \alpha_1, \alpha_2, \alpha_3 \text{ are linearly independent over } \mathbb{Q}\} \cap I^3,$$
$$X_K := \{(\alpha_1,\alpha_2,\alpha_3) \in X'_K | \text{there exists } i \in \{1,2,3\} \text{ such that } K = \mathbb{Q}(\alpha_i)\}.$$

For $x \in K \cap I$ we define $\phi(x)$ by

$$\phi(x) := \begin{cases} \frac{x}{\sqrt[3]{|N(x)|}} & \text{if } K = \mathbb{Q}(x), \\ -1 & \text{if } K \neq \mathbb{Q}(x). \end{cases}$$

In a similar manner to the cubic case, one can show the following.

Lemma 3.1. *Let $(\alpha_1,\alpha_2,\alpha_3) \in X_K$. If $\phi(\alpha_i), \phi(\alpha_j) > 0$ and $\phi(\alpha_i) = \phi(\alpha_j)$ for integers i and j with $1 \leq i, j \leq 3$, then $i = j$.*

For $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in X_K$, we define $\rho(\alpha)$ and

$$\rho(\alpha) = \max\{\phi(\alpha_i) | i \in \{1,2,3\}\}.$$

For $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in X_K$, from Lemma 3.1 it follows that

$$\sharp\{i \in \{1,2,3\} | \rho(\alpha) = \phi(\alpha_i)\} = 1,$$

and we denote by $\omega(\alpha)$ the uniquely determined number $i \in \{1,2,3\}$ with $\rho(\alpha) = \phi(\alpha_i)$. We define the transformation T_K on X_K as follows: For $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in X_K$, $T_K(\alpha) = (\beta_1,\beta_2,\beta_3)$ with

$$\beta_i := \begin{cases} \frac{1}{\alpha_i} - \lfloor \frac{1}{\alpha_i} \rfloor & \text{if } i = \omega(\alpha), \\ \frac{\alpha_i}{\alpha_{\omega(\alpha)}} - \lfloor \frac{\alpha_i}{\alpha_{\omega(\alpha)}} \rfloor & \text{if } i \neq \omega(\alpha) \end{cases} \qquad (i = 1,2,3).$$

We define $\mathcal{P}_K$, in a similar fashion in Section 2, that is,

$$\mathcal{P}_K := \{\alpha \in X_K | \text{there exist } m, n \in \mathbb{Z}_{>0} \text{ such that } m \neq n \text{ and } T_K^n(\alpha) = T_K^m(\alpha)\}.$$

We also define $\overline{\mathrm{dh}}$, $\mathrm{dh}_{\mathrm{AJPA}}$ and $\mathrm{rdh}_{\mathrm{AJPA}}$ in a similar manner to Section 2, namely, for $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in X_K$ and $n \in \mathbb{Z}_{\geq 0}$

$$\overline{\mathrm{dh}}(\alpha) := \max_{i \in \{1,2,3\}} \{\mathrm{dh}(p_i^{4/deg(p_i)})\},$$
$$\mathrm{dh}_{\mathrm{AJPA}}(n;\alpha) := \overline{\mathrm{dh}}(T_K^n(\alpha)),$$
$$\mathrm{rdh}_{\mathrm{AJPA}}(n;\alpha) := \frac{\overline{\mathrm{dh}}(T_K^n(\alpha))}{\overline{\mathrm{dh}}(\alpha)},$$

where $p_i = \mathrm{mpol}(\alpha_i) \in \mathbb{Q}[x]$ $(i \in \{1,2,3\})$ is the monic minimal polynomial of α_i.

We computed the length of the period of the expansion of $(\langle \sqrt[4]{m} \rangle, \langle \sqrt[4]{m^2} \rangle, \langle \sqrt[4]{m^3} \rangle)$ obtained by T_K for m all $2 \leq m \leq 5000$ and relative decimal heights given above; cf. Table D. We confirmed that $(\langle \sqrt[4]{m} \rangle, \langle \sqrt[4]{m^2} \rangle, \langle \sqrt[4]{m^3} \rangle) \in \mathcal{P}_K$ for all $2 \leq m \leq 5000 (\sqrt{m} \notin \mathbb{Q})$.

4. A conjecture

Conjecture. *Let K be a real cubic field or a real quartic field. Then:*

(1) $X_K = \mathcal{P}_K$.
(2) *There exists an absolute constant c independent of K and $\alpha \in X_K$ such that* $\mathrm{rdh}_{\mathrm{AJPA}}(\alpha) \le c$ *holds.* (*Probably we can take $c = 6$.*)
(3) *There are infinitely many $\alpha \in X_K$ (including $(\langle\sqrt[3]{3}\rangle, \langle\sqrt[3]{9}\rangle)$) such that the expansion of α obtained by the Jacobi-Perron Algorithm modified by Podsypanin is* **not** *periodic.*

In Table A, $L(m_1, m_2)$, $H(m_1, m_2)$ and $R(m_1, m_2)$ are numbers defined by $L(m_1, m_2) :=$ the maximum value of the length of the shortest period of the expansion of $(\langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m^2}\rangle)$ by our algorithm for $m_1 \le m \le m_2$ with $\sqrt[3]{m} \notin \mathbb{Q}$,

$$H(m_1, m_2) := \max_{m_1 \le m \le m_2,\, \sqrt[3]{m} \notin \mathbb{Q}, 0 \le n < \infty} \mathrm{dh}_{\mathrm{AJPA}}(n; \langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m^2}\rangle),$$

$$R(m_1, m_2) := \max_{m_1 \le m \le m_2,\, \sqrt[3]{m} \notin \mathbb{Q}, 0 \le n < \infty} \mathrm{rdh}_{\mathrm{AJPA}}(n; \langle\sqrt[3]{m}\rangle, \langle\sqrt[3]{m^2}\rangle),$$

which are well defined by the periodicity.

Table A

range of m $m_1 \le m \le m_2$	$L(m_1, m_2)$	$H(m_1, m_2)$	$R(m_1, m_2)$
$2 \le m \le 200$	494	5	3/2
$201 \le m \le 400$	898	6	3/2
$401 \le m \le 600$	1702	6	3/2
$601 \le m \le 800$	1938	6	6/5
$801 \le m \le 1000$	2802	7	7/5
$1001 \le m \le 1200$	4062	7	7/5
$1201 \le m \le 1400$	5586	7	7/5
$1401 \le m \le 1600$	5090	8	8/5
$1601 \le m \le 1800$	8022	7	7/5
$1801 \le m \le 2000$	7854	8	8/5
$2001 \le m \le 2200$	5486	7	7/5
$2201 \le m \le 2400$	6422	7	7/5
$2401 \le m \le 2600$	7758	8	7/5
$2601 \le m \le 2800$	6026	8	4/3
$2801 \le m \le 3000$	9970	8	4/3
$3001 \le m \le 3200$	11562	8	4/3
$3201 \le m \le 3400$	6734	9	3/2
$3401 \le m \le 3600$	6650	8	4/3
$3601 \le m \le 3800$	12350	8	4/3
$3801 \le m \le 4000$	19230	8	4/3
$4001 \le m \le 4200$	11454	8	4/3
$4201 \le m \le 4400$	16410	8	4/3
$4401 \le m \le 4600$	14618	8	4/3
$4601 \le m \le 4800$	18158	8	4/3
$4801 \le m \le 5000$	14918	8	4/3

In Table B, $\underline{U}(m_1, m_2)$, $\underline{V}(m_1, m_2)$ are numbers defined by

$$\underline{U}(m_1, m_2) := \max_{m_1 \le n \le m_2} \overline{\mathrm{dh}}(\bar{T}^n(\sqrt[3]{3}-1, \sqrt[3]{9}-2),$$
$$\underline{V}(m_1, m_2) := \min_{m_1 \le n \le m_2} \overline{\mathrm{dh}}(\bar{T}^n(\sqrt[3]{3}-1, \sqrt[3]{9}-2).$$

We define integral-valued functions U and V by

$$U(x) := \underline{U}(1000n_1+1, 1000(n_1+1)),\ \text{if } 1000n_1 < x \le 1000(n_1+1),$$
$$V(x) := \underline{V}(1000n_1+1, 1000(n_1+1)),\ \text{if } 1000n_1 < x \le 1000(n_1+1),$$

for $x, n_1 \in \mathbb{Z}_{\ge 0}$.

TABLE B

range of n	$\underline{U}(m_1, m_2)$	$\underline{V}(m_1, m_2)$
$0 \le n \le 1000$	93	1
$1001 \le n \le 2000$	175	92
$2001 \le n \le 3000$	253	174
$3001 \le n \le 4000$	321	252
$4001 \le n \le 5000$	397	319
$5001 \le n \le 6000$	484	396
$6001 \le n \le 7000$	555	483
$7001 \le n \le 8000$	624	554
$8001 \le n \le 9000$	716	624
$9001 \le n \le 10000$	773	716
$10001 \le n \le 11000$	850	772
$11001 \le n \le 12000$	923	847
$12001 \le n \le 13000$	1000	921
$13001 \le n \le 14000$	1074	1000
$14001 \le n \le 15000$	1139	1073
$15001 \le n \le 16000$	1212	1137
$16001 \le n \le 17000$	1284	1211
$17001 \le n \le 18000$	1370	1284
$18001 \le n \le 19000$	1444	1370
$19001 \le n \le 20000$	1514	1444

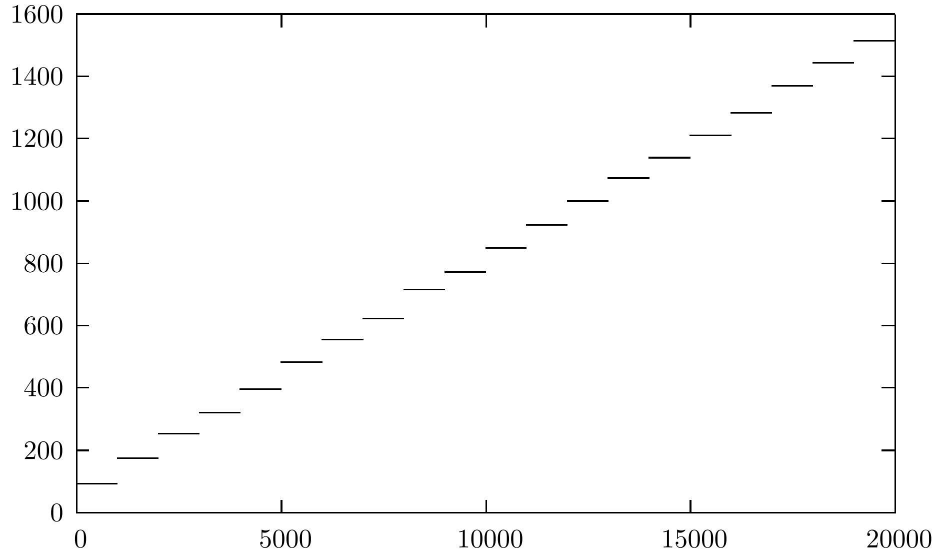

FIGURE 1. The graph of U

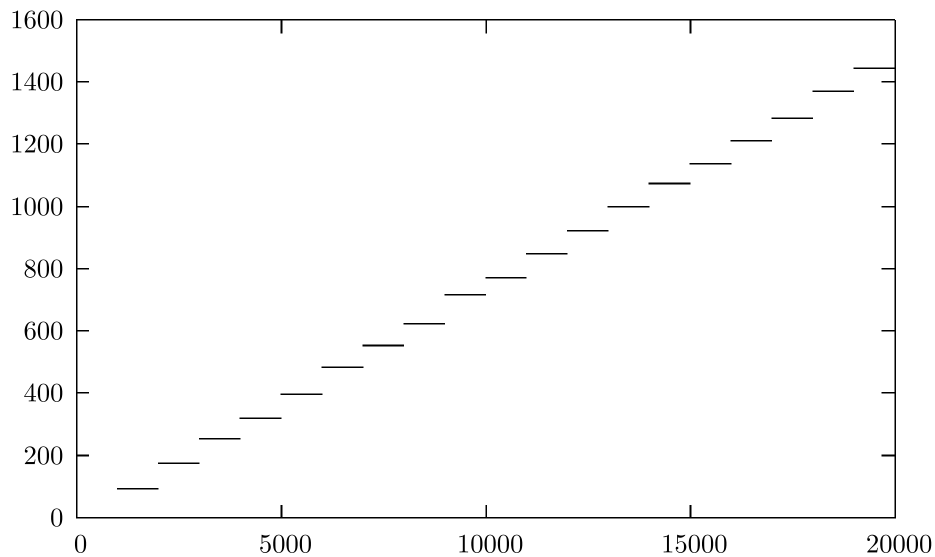

FIGURE 2. The graph of V

In Table C, $L(m_1, m_2)$, $H(m_1, m_2)$ and $R(m_1, m_2)$ are numbers defined by $L(m_1, m_2) :=$ the maximum value of the length of the shortest period of the expansion of $(\langle \tau_m \rangle, \langle \tau_m^2 \rangle)$ by our algorithm for $m_1 \le m \le m_2$, where τ_m is a maximal root of $x^3 - mx + 1$,

$$H(m_1, m_2) := \max_{m_1 \le m \le m_2, 0 \le n < \infty} \mathrm{dh}_{\mathrm{AJPA}}(n; \langle \tau_m \rangle, \langle \tau_m^2 \rangle),$$

$$R(m_1, m_2) := \max_{m_1 \le m \le m_2, 0 \le n < \infty} \mathrm{rdh}_{\mathrm{AJPA}}(n; \langle \tau_m \rangle, \langle \tau_m^2 \rangle),$$

which are well defined by the periodicity.

In Table D, $L(m_1, m_2)$, $H(m_1, m_2)$ and $R(m_1, m_2)$ are numbers defined by $L(m_1, m_2) :=$ the maximum value of the length of the shortest period of the expansion of $(\langle \sqrt[4]{m} \rangle, \langle \sqrt[4]{m^2} \rangle, \langle \sqrt[4]{m^3} \rangle)$ by our algorithm for $m_1 \le m \le m_2$ with $\sqrt{m} \notin \mathbb{Q}$,

$$H(m_1, m_2) := \max_{m_1 \le m \le m_2, \sqrt{m} \notin \mathbb{Q}, 0 \le n < \infty} \mathrm{dh}_{\mathrm{AJPA}}(n; \langle \sqrt[3]{m} \rangle, \langle \sqrt[3]{m^2} \rangle, \langle \sqrt[3]{m^3} \rangle),$$

$$R(m_1, m_2) := \max_{m_1 \le m \le m_2, \sqrt{m} \notin \mathbb{Q}, 0 \le n < \infty} \mathrm{rdh}_{\mathrm{AJPA}}(n; \langle \sqrt[3]{m} \rangle, \langle \sqrt[3]{m^2} \rangle, \langle \sqrt[3]{m^3} \rangle),$$

which are well defined by the periodicity.

Table C

range of m $m_1 \le m \le m_2$	$L(m_1, m_2)$	$H(m_1, m_2)$	$R(m_1, m_2)$
$3 \le m \le 200$	866	6	2
$201 \le m \le 400$	3312	7	7/3
$401 \le m \le 600$	5378	7	7/3
$601 \le m \le 800$	10578	9	9/4
$801 \le m \le 1000$	11808	8	2
$1001 \le m \le 1200$	19264	8	2
$1201 \le m \le 1400$	17254	8	2
$1401 \le m \le 1600$	25792	9	9/4
$1601 \le m \le 1800$	33562	10	5/2
$1801 \le m \le 2000$	36476	9	9/4
$2001 \le m \le 2200$	23274	9	9/4
$2201 \le m \le 2400$	38938	9	9/4
$2401 \le m \le 2600$	54046	10	5/2
$2601 \le m \le 2800$	57246	9	9/4
$2801 \le m \le 3000$	51964	9	9/4
$3001 \le m \le 3200$	57036	9	9/4
$3201 \le m \le 3400$	92332	9	9/4
$3401 \le m \le 3600$	55698	10	5/2
$3601 \le m \le 3800$	96972	9	9/4
$3801 \le m \le 4000$	86784	10	5/2
$4001 \le m \le 4200$	94188	10	5/2
$4201 \le m \le 4400$	116912	10	5/2
$4401 \le m \le 4600$	109288	10	5/2
$4601 \le m \le 4800$	113792	10	5/2
$4801 \le m \le 5000$	109426	10	5/2

TABLE D

range of m $m_1 \le m \le m_2$	$L(m_1, m_2)$	$H(m_1, m_2)$	$R(m_1, m_2)$
$2 \le m \le 200$	4194	13	13/5
$201 \le m \le 400$	8994	12	2
$401 \le m \le 600$	9730	11	11/6
$601 \le m \le 800$	18894	13	13/7
$801 \le m \le 1000$	14172	13	13/7
$1001 \le m \le 1200$	27876	14	2
$1201 \le m \le 1400$	34308	13	12/7
$1401 \le m \le 1600$	16452	13	13/8
$1601 \le m \le 1800$	32870	14	7/4
$1801 \le m \le 2000$	44244	13	13/8
$2001 \le m \le 2200$	48732	13	13/8
$2201 \le m \le 2400$	58974	14	7/4
$2401 \le m \le 2600$	62706	13	13/8
$2601 \le m \le 2800$	41678	13	13/8
$2801 \le m \le 3000$	45066	14	7/4
$3001 \le m \le 3200$	41382	13	13/8
$3201 \le m \le 3400$	112670	15	15/8
$3401 \le m \le 3600$	121296	15	15/8
$3601 \le m \le 3800$	58782	15	5/3
$3801 \le m \le 4000$	60890	13	13/9
$4001 \le m \le 4200$	92190	14	14/9
$4201 \le m \le 4400$	43882	14	14/9
$4401 \le m \le 4600$	94542	16	16/9
$4601 \le m \le 4800$	79714	13	13/9
$4801 \le m \le 5000$	51786	15	5/3

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