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GENERAL ORDER MULTIVARIATE PADÉ APPROXIMANTS FOR PSEUDO-MULTIVARIATE FUNCTIONS. II

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ABSTRACT. Explicit formulas for general order multivariate Padé approximants of pseudo-multivariate functions are constructed on specific index sets. Examples include the multivariate forms of the exponential function

$$E\left(\underline{x}\right)=\sum_{j_1,j_2,\ldots,j_m=0}^{\infty}\frac{x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m}}{(j_1+j_2+\cdots+j_m)!},$$

the logarithm function

$$L(\underline{x})=\sum_{j_1+j_2+\cdots+j_m\geq 1}\frac{x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m}}{j_1+j_2+\cdots+j_m},$$

the Lauricella function

$$F_D^{(m)}\left(a,1,\ldots,1;c;x_1,\ldots,x_m\right)=\sum_{j_1,j_2,\ldots,j_m=0}^{\infty}\frac{(a)_{j_1+\cdots+j_m}}{(c)_{j_1+\cdots+j_m}}x_1^{j_1}\cdots x_m^{j_m},$$

and many more. We prove that the constructed approximants inherit the normality and consistency properties of their univariate relatives. These properties do not hold in general for multivariate Padé approximants. A truncation error upperbound is also given.

1. Introduction

In [6] and [11], we explicitly construct multivariate Padé approximants to so-called pseudo-multivariate functions of two variables by using their one variable projections. Our aim in this paper is to first write a pseudo-multivariate function of at least two variables as a finite sum of one variable projections by means of divided differences, and then explicitly construct multivariate Padé approximants to the given pseudo-multivariate function. We prove that the constructed approximants inherit the normality and consistency properties of their univariate relatives, which do not hold in general for multivariate Padé approximants, and present a truncation error upperbound.

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Definition 1.1. Let

$$F(\underline{x}) := F(x_1, \dots, x_m) := \sum_{j_1,\dots,j_m=0}^{\infty} c_{j_1\cdots j_m} x_1^{j_1} \cdots x_m^{j_m}, \quad c_{j_1\cdots j_m} \in \mathbb{C}$$

be a formal power series, and let M, N, E be finite index sets in $\mathbb{N} \times \cdots \times \mathbb{N} = \mathbb{N}^m$. An (M, N) *general order multivariate Padé approximant* to $F(\underline{x})$ on the set E is a rational function

$$[M/N]_E(\underline{x}) := \frac{P(\underline{x})}{Q(\underline{x})},$$

where the polynomials

$$P(\underline{x}) := \sum_{(j_1,\dots,j_m)\in M} a_{j_1\cdots j_m} x_1^{j_1} \cdots x_m^{j_m}, \quad a_{j_1\cdots j_m} \in \mathbb{C},$$

$$Q(\underline{x}) := \sum_{(j_1,\dots,j_m)\in N} b_{j_1\cdots j_m} x_1^{j_1} \cdots x_m^{j_m}, \quad b_{j_1\cdots j_m} \in \mathbb{C},$$

are such that

$$(FQ - P)(\underline{x}) = \sum_{(j_1,\dots,j_m)\in \mathbb{N}^m \setminus E} d_{j_1\cdots j_m} x_1^{j_1} \cdots x_m^{j_m}, \quad d_{j_1\cdots j_m} \in \mathbb{C} \tag{1.1}$$

with E satisfying the inclusion property

$$(j_1, \dots, j_m) \in E,\ 0 \le i_k \le j_k,\ 1 \le k \le m \Longrightarrow (i_1, \dots, i_m) \in E. \tag{1.2}$$

Equation (1.1) translates to the linear system of equations

$$d_{j_1\cdots j_m} = \sum_{\ell_1=0}^{j_1} \cdots \sum_{\ell_m=0}^{j_m} c_{\ell_1\cdots\ell_m} b_{j_1-\ell_1,\dots,j_m-\ell_m} - a_{j_1\cdots j_m} = 0, \qquad (j_1, \dots, j_m) \in E, \tag{1.3}$$

where $b_{k_1\cdots k_m} = 0$ for $(k_1, \dots, k_m) \notin N$ and $a_{k_1\cdots k_m} = 0$ for $(k_1, \dots, k_m) \notin M$. Condition (1.2) takes care of the Padé approximation property, provided $Q(\underline{0}) \neq 0$, namely

$$(F - \frac{P}{Q})(\underline{x}) = \sum_{(j_1,\dots,j_m)\in \mathbb{N}^m \setminus E} e_{j_1\cdots j_m} x_1^{j_1} \cdots x_m^{j_m}, \qquad e_{j_1\cdots j_m} \in \mathbb{C}.$$

The linear system (1.3) can be split into two parts: some of the equations serve to compute the numerator and denominator coefficients $a_{j_1\cdots j_m}$ and $b_{j_1\cdots j_m}$, while the remaining equations are automatically satisfied by $FQ-P$ for the computed P and Q . We refer to the former set of indices $(j_1, \dots, j_m)$ as C and to the latter one as $E \setminus C$. For the class of pseudo-multivariate functions introduced in Definition 1.2 below, very few equations of (1.3) are actually used for the computation of the coefficients. However, in general this is not the case.

One may find a discussion of Definition 1.1 in [6] and more properties of general order multivariate Padé approximants in [3, 4, 5].

Definition 1.2. A multivariate function $F(\underline{x}) := F(x_1, x_2, \dots, x_m)$ is said to be *pseudo-multivariate* if the coefficients of its formal power series

$$F(\underline{x}) = \sum_{j_1,\dots,j_m=0}^{\infty} c_{j_1\cdots j_m} x_1^{j_1} \cdots x_m^{j_m}$$

satisfy

$$c_{j_1\cdots j_m} = g\left(j_1 + \cdots + j_m\right), \quad j_1, \ldots, j_m = 0, 1, \ldots,$$

where $g\left(k\right)$ is a function of k.

A pseudo-multivariate function $F(\underline{x})$ with

$$\lim_{k\to\infty}\left|\frac{g\left(k\right)}{g\left(k+1\right)}\right| = R < \infty,$$

converges in the polydisc centered at the origin with radius R. We recall (see [9] for more information) some background on divided differences.

Definition 1.3. The *divided difference* $f[x_1, x_2, \ldots, x_n]$ of a function $f(x)$ at distinct $x_1, x_2, \ldots, x_n$ is defined recursively by

$$\begin{aligned} f[x_i] &= f(x_i), \quad 1 \leq i \leq n, \\ f[x_i, x_j] &= \frac{f(x_j) - f(x_i)}{x_j - x_i}, \quad 1 \leq i, j \leq n, \\ f[x_{i_1}, x_{i_2}, \ldots, x_{i_p}] &= \frac{f[x_{i_1}, x_{i_2}, \ldots, x_{i_{p-2}}, x_{i_p}] - f[x_{i_1}, x_{i_2}, \ldots, x_{i_{p-2}}, x_{i_{p-1}}]}{x_{i_p} - x_{i_{p-1}}}, \\ &\qquad 1 \leq i_1, i_2, \ldots, i_p \leq n; \end{aligned}$$

It is independent of the order of the involved x_i. If some of the x_i coincide, say $x_i = x_{i+1} = \cdots = x_{i+k}$ for $1 \leq i, i+k \leq n$, then

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] := \lim_{x_{i+1},\ldots,x_{i+k}\to x_i} f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{1}{k!} f^{(k)}\left(x_i\right).$$

Using this expression the recursive scheme for the computation of the divided differences can be continued for a mixture of distinct and coinciding x_i.

Note that for distinct x_i, the divided difference $f[x_1, x_2, \ldots, x_n]$ can be expressed as a linear combination of the $f(x_i), i = 1, 2, \ldots, n$, namely,

$$f[x_1, x_2, \ldots, x_n] = \sum_{i=1}^{n} \frac{f(x_i)}{\omega_n'(x_i)}, \tag{1.4}$$

where

$$\omega_n'(x_i) = \prod_{1\leq j\leq n, j\neq i} (x_i - x_j) \tag{1.5}$$

is the derivative of

$$\omega_n(x) = \prod_{j=1}^{n} (x - x_j)$$

at $x = x_i$. In addition, the divided difference of the function

$$f_n(x) = x^{m+n-1}, \tag{1.6}$$

at $x_1, x_2, \ldots, x_m$ is a homogeneous function of degree n in the m variables $x_1, x_2, \ldots,$ x_m (see [8] for more details),

$$f_n[x_1, x_2, \ldots, x_m] = \sum_{i_1+i_2+\cdots+i_m=n} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}. \tag{1.7}$$

We now discuss how to represent a pseudo-multivariate function as a finite sum of univariate functions. To this end the representation (1.4) is used symbolically, meaning that the evaluation $f(x_i)$ at different x_i is to be regarded as the use of $f(x)$ for distinct variables x_i. Coincidence of some x_i with x_j is to be interpreted as a restriction to the subset of the multidimensional space where $x_i = x_j$ for some $i \neq j$, $i, j = 1, 2, \ldots, m$.

2. Explicit constructions

In Theorem 2.1 we show how to use divided differences to obtain a finite sum representation (see [10] for more on this method) of a pseudo-multivariate function $F(\underline{x})$. Afterwards we use this result to obtain explicit formulas for Padé approximants of pseudo-multivariate functions in Theorem 2.2.

Theorem 2.1. *Let*

$$F(\underline{x}) := \sum_{n=0}^{\infty} g(n) \sum_{j_1+j_2+\cdots+j_m=n} x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m} \tag{2.1}$$

be a pseudo-multivariate function, and let

$$h(z) = \sum_{k=0}^{\infty} g(k) z^k.$$

If $x_i \neq x_j$ for all $i, j = 1, 2, \ldots, m$, we have

$$F(\underline{x}) = \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} h(x_i),$$

where $\omega_m'(x_i)$ is defined by (1.5). If $x_i = x_j$ for some $i, j = 1, 2, \ldots, m$, and if $x_1, \ldots, x_r$ are the distinct variables among the $x_1, \ldots, x_m$, occurring with multiplicities $s_1, \ldots, s_r$, respectively, then we have

$$F(\underline{x}) = \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} \frac{W_i^{(s_i-j-1)}(x_i)}{(s_i-j-1)!} \frac{h^{(j)}(x_i)}{j!},$$

where

$$W_i(x) := \frac{x^{m-1}}{\prod_{1\le j\le r, j\neq i} (x-x_j)^{s_j}}, \ i = 1, 2, \ldots, r. \tag{2.2}$$

Proof. Let $x_i \neq x_j$ for $i, j = 1, 2, \ldots, m$, and let

$$G(x) := \sum_{n=0}^{\infty} g(n) x^{m+n-1} = \sum_{n=0}^{\infty} g(n) f_n(x), \tag{2.3}$$

where $f_n(x)$ is defined by (1.6). Then by (1.4) and (1.7), and the linearity of the divided difference,

$$\begin{aligned} G[x_1, x_2, \ldots, x_m] &= \sum_{n=0}^{\infty} g(n) f_n[x_1, x_2, \ldots, x_m] \\ &= \sum_{n=0}^{\infty} g(n) \sum_{j_1+j_2+\cdots+j_m=n} x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}. \end{aligned} \tag{2.4}$$

Also, from (2.3),

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} g(n)\, x^{n+m-1} \\ &= x^{m-1} \sum_{n=0}^{\infty} g(n)\, x^{n} \\ &= x^{m-1} h(x). \end{aligned}$$

By using (2.1), (2.4), and (1.4), we have

$$\begin{aligned} F(x_1, x_2, \ldots, x_m) &= \sum_{n=0}^{\infty} g(n) \sum_{j_1+j_2+\cdots+j_m=n} x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m} \\ &= G[x_1, x_2, \ldots, x_m] \\ &= \sum_{i=1}^{m} \frac{G(x_i)}{\omega_m'(x_i)} \\ &= \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} h(x_i), \end{aligned}$$

where $\omega_m'(x_i)$ is defined by (1.5). Now let $x_1, \ldots, x_r$ be the distinct variables among all $x_1, \ldots, x_m$, occurring with multiplicities $s_1, \ldots, s_r$, respectively, and with $s_1 + \cdots + s_r = m$. Because of the symmetry property of $F(x_1, x_2, \ldots, x_m)$, we can assume that $x_1 = x_{r+1} = \cdots = x_{r+s_1-1}$, $x_2 = x_{r+s_1} = \cdots = x_{r+s_1+s_2-2}$, $\ldots, x_r = x_{s_1+\cdots+s_{r-1}+2} = \cdots = x_m$. Then for $W_i(x_i)$ defined by (2.2),

$$\begin{aligned} &F(x_1, x_2, \ldots, x_m) \\ &= F(\overbrace{x_1, \ldots, x_1}^{s_1}, \overbrace{x_2, \ldots, x_2}^{s_2}, \ldots, \overbrace{x_r, \ldots, x_r}^{s_r}) \\ &= \lim_{x_{r+1}, \ldots, x_{r+s_1-1} \to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2}, \ldots, x_m \to x_r} \left(\sum_{i=1}^{m} \frac{x_i^{m-1} h(x_i)}{\prod_{1\le j\le m, j\ne i} (x_i - x_j)} \right) \\ &= \lim_{x_{r+1}, \ldots, x_{r+s_1-1} \to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2}, \ldots, x_m \to x_r} \left\{ \left(\frac{x_1^{m-1} h(x_1)}{\prod_{2\le j\le m} (x_1 - x_j)} \right. \right. \\ &\quad \left. + \sum_{i=1}^{s_1-1} \frac{x_{r+i}^{m-1} h(x_{r+i})}{\prod_{1\le j\le m, j\ne r+i} (x_{r+i} - x_j)} \right) + \left(\frac{x_2^{m-1} h(x_2)}{\prod_{1\le j\le m, j\ne 2} (x_2 - x_j)} \right. \\ &\quad \left. + \sum_{i=1}^{s_2-1} \frac{x_{r+s_1-1+i}^{m-1} h(x_{r+s_1-1+i})}{\prod_{1\le j\le m, j\ne r+s_1-1+i} (x_{r+s_1-1+i} - x_j)} \right) \\ &\quad + \cdots + \left(\frac{x_r^{m-1} h(x_r)}{\prod_{1\le j\le m, j\ne r} (x_r - x_j)} \right. \\ &\quad \left. \left. + \sum_{i=1}^{s_r-1} \frac{x_{s_1+\cdots+s_{r-1}+1+i}^{m-1} h(x_{s_1+\cdots+s_{r-1}+1+i})}{\prod_{1\le j\le m, j\ne s_1+\cdots+s_{r-1}+1+i} (x_{s_1+\cdots+s_{r-1}+1+i} - x_j)} \right) \right\} \\ &= \lim_{x_{r+1}, \ldots, x_{r+s_1-1} \to x_1} \left(\frac{W_1(x_1)\, h(x_1)}{\prod_{1\le j\le s_1-1} (x_1 - x_{r+j})} \right. \end{aligned}$$

$$+\sum_{i=1}^{s_1-1}\frac{W_{r+i}\left(x_{r+i}\right)h\left(x_{r+i}\right)}{\left(x_{r+i}-x_1\right)\prod_{1\le j\le s_1-1,j\ne i}(x_{r+i}-x_{r+j})}\Bigg)$$

$$+\lim_{x_{r+s_1},\ldots,x_{r+s_1+s_2-2}\to x_2}\left(\frac{W_2\left(x_2\right)h\left(x_2\right)}{\prod_{1\le j\le s_2-1}(x_2-x_{r+s_1-1+j})}\right.$$

$$+\sum_{i=1}^{s_2-1}\frac{W_{r+s_1-1+i}\left(x_{r+s_1-1+i}\right)h\left(x_{r+s_1-1+i}\right)}{\left(x_{r+s_1-1+i}-x_2\right)\prod_{1\le j\le s_2-1,j\ne i}(x_{r+s_1-1+i}-x_{r+s_1-1+j})}\Bigg)$$

$$+\cdots+\lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r}\left(\frac{W_r\left(x_r\right)h\left(x_r\right)}{\prod_{1\le j\le s_r-1}(x_r-x_{s_1+\cdots+s_{r-1}+1+j})}\right.$$

$$+\sum_{i=1}^{s_r-1}\frac{W_{s_1+\cdots+s_{r-1}+1+i}\left(x_{s_1+\cdots+s_{r-1}+1+i}\right)h\left(x_{s_1+\cdots+s_{r-1}+1+i}\right)}{\left(x_{s_1+\cdots+s_{r-1}+1+i}-x_r\right)\prod_{1\le j\le s_r-1,j\ne i}(x_{s_1+\cdots+s_{r-1}+1+i}-x_{s_1+\cdots+s_{r-1}+1+j})}\Bigg).$$

Now if we let

$$H_i\left(x\right):=W_i\left(x\right)h\left(x\right),\quad i=1,2,\ldots,r,$$

then putting it into the equation above, and using the fact that

$$\lim_{x_2,x_3,\ldots,x_r\to x_1}H[x_1,x_2,\ldots,x_r]=\frac{1}{(r-1)!}H^{(r-1)}\left(x_1\right),$$

and formula (1.4), we have

$$\begin{aligned}F\left(x_1,x_2,\ldots,x_m\right) &= \lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1}H_1[x_1,x_{r+1},\ldots,x_{r+s_1-1}]\\ &\quad+\lim_{x_{r+s_1},\ldots,x_{r+s_1+s_2-2}\to x_2}H_2[x_2,x_{r+s_1},\ldots,x_{r+s_1+s_2-2}]\\ &\quad+\cdots\\ &\quad+\lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r}H_r[x_r,x_{r+s_1+\cdots+s_{r-1}+2},\ldots,x_m]\\ &= \frac{1}{(s_1-1)!}H_1^{(s_1-1)}\left(x_1\right)+\frac{1}{(s_2-1)!}H_2^{(s_2-1)}\left(x_2\right)\\ &\quad+\cdots+\frac{1}{(s_r-1)!}H_r^{(s_r-1)}\left(x_r\right)\\ &= \sum_{i=1}^{r}\sum_{j=0}^{s_i-1}\frac{W_i^{(s_i-j-1)}\left(x_i\right)}{(s_i-j-1)!}\frac{h^{(j)}\left(x_i\right)}{j!}.\end{aligned}$$

This proves Theorem 2.1. ☐

Theorem 2.2. *Let*

$$F(\underline{x}):=\sum_{k=0}^{\infty}g\left(k\right)\sum_{j_1+j_2+\cdots+j_m=k}x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m}$$

be a pseudo-multivariate function. For $u,v\in\mathbb{N}$, let

$$\frac{p_{u,v}}{q_{u,v}}\left(z\right):=\frac{\sum_{j=0}^{u}\alpha_jz^j}{\sum_{j=0}^{v}\beta_jz^j},\qquad \beta_0=1 \tag{2.5}$$

be the (u,v) Padé approximant of the function

$$h\left(z\right):=\sum_{k=0}^{\infty}g\left(k\right)z^k, \tag{2.6}$$

and for $\ell = \max\{u, v-1\}$, *let*

$$N := \{(j_1, j_2, \ldots, j_m) : 0 \le j_1, j_2, \ldots, j_m \le v\}, \tag{2.7}$$

$$M := \{(j_1, j_2, \ldots, j_m) : 0 \le j_1, j_2, \ldots, j_m \le \ell,\ j_1 + j_2 + \cdots + j_m \le \min\{m\ell, u + (m-1)\,v\}\}, \tag{2.8}$$

$$E := \{(j_1, j_2, \ldots, j_m) : 0 \le j_1 + j_2 + \cdots + j_m \le u + v\} \tag{2.9}$$

be index sets in $\mathbb{N}^m$. *Then if* $x_i \neq x_j$ *for all* $i, j = 1, 2, \ldots, m$, *the* (M, N) *general order multivariate Padé approximant to* $F(\underline{x})$ *on the index set* E *is*

$$[M/N]_E(\underline{x}) = \frac{P(\underline{x})}{Q(\underline{x})},$$

where

$$Q(\underline{x}) = q_{u,v}\,(x_1)\, q_{u,v}\,(x_2) \cdots q_{u,v}\,(x_m)\,, \tag{2.10}$$

$$P(\underline{x}) = \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} \left(p_{u,v}\,(x_i) \prod_{1 \le k \le m, k \neq i} q_{u,v}\,(x_k) \right). \tag{2.11}$$

If $x_i = x_j$ *for some* $i, j = 1, 2, \ldots, m$, *and if* $x_1, \ldots, x_r$ *are the distinct variables among the* $x_1, \ldots, x_m$, *occurring with multiplicities* $s_1, \ldots, s_r$, *respectively, then* $P\,(x)$ *becomes*

$$P(\underline{x}) = \sum_{i=1}^{r} \frac{1}{(s_i - 1)!} \left(\prod_{1 \le k \le r, k \neq i} q_{u,v}^{s_k}\,(x_k) \right) \left(W_i p_{u,v} q_{u,v}^{s_i - 1} \right)^{(s_i - 1)} (x_i)\,, \tag{2.12}$$

where $W_i\,(x)$ *is defined by* (2.2).

Proof. From (2.10), we have that

$$Q(\underline{x}) = \prod_{k=1}^{m} q_{u,v}\,(x_k) = \sum_{(j_1, j_2, \ldots, j_m) \in N} b_{j_1 \cdots j_m} x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}, \qquad b_{j_1 \cdots j_m} \in \mathbb{C},$$

where N is defined by (2.7). Now we prove that $P(\underline{x})$ in (2.11) is a polynomial on the index set M. For $m \ge 2$,

$$\begin{aligned}
P(\underline{x}) &= \sum_{k=1}^{m} \frac{x_k^{m-1}}{\omega_m'(x_k)} \left(p_{u,v}\,(x_k) \prod_{1 \le j \le m, j \neq k} q_{u,v}\,(x_j) \right) \\
&= \sum_{k=1}^{m} \frac{x_k^{m-1} p_{u,v}\,(x_k)}{\prod_{1 \le j \le m, j \neq k} (x_k - x_j)} \prod_{1 \le j \le m, j \neq k} q_{u,v}\,(x_j) \\
&= \frac{1}{\prod_{1 \le i < j \le m} (x_i - x_j)} \sum_{k=1}^{m} (-1)^{k-1} x_k^{m-1} p_{u,v}\,(x_k) \\
&\qquad \times \left(\prod_{1 \le i < j \le m, i,j \neq k} (x_i - x_j) \right) \left(\prod_{1 \le j \le m, j \neq k} q_{u,v}\,(x_j) \right).
\end{aligned}$$

Let

$$P^*(\underline{x}) := \sum_{k=1}^{m} (-1)^{k-1} x_k^{m-1} p_{u,v}(x_k) \times \left(\prod_{1\le i<j\le m, i,j\neq k} (x_i - x_j) \right) \left(\prod_{1\le j\le m, j\neq k} q_{u,v}(x_j) \right).$$

Then, if $x_1 = x_2$,

$$\begin{aligned} P^*(\underline{x}) &= x_1^{m-1} p_{u,v}(x_1) \left(\prod_{2\le i<j\le m} (x_i - x_j) \right) \left(\prod_{2\le j\le m} q_{u,v}(x_j) \right) \\ &\quad - x_2^{m-1} p_{u,v}(x_2) \left(\prod_{3\le j\le m} (x_1 - x_j) \right) \left(\prod_{3\le i<j\le m} (x_i - x_j) \right) \\ &\quad \times q_{u,v}(x_1) \left(\prod_{3\le j\le m} q_{u,v}(x_j) \right) \\ &= 0, \end{aligned}$$

and we have that $(x_1 - x_2)$ divides $P^*(\underline{x})$. As $P(\underline{x})$ is symmetric in its variables, then for $1 \le i, j \le m$, if $x_i = x_j$ for some $i \neq j$,

$$P^*(\underline{x}) = 0.$$

So $\prod_{1\le i<j\le m}(x_i - x_j)$ divides $P^*(\underline{x})$ and therefore $P(\underline{x})$ is a polynomial of $x_k,\ k = 1, \ldots, m$. Now observe that the total degree of $\prod_{1\le i<j\le m}(x_i - x_j)$ is $m(m-1)/2$ and the total degree of $P^*(\underline{x})$ is at most

$$(m-1) + u + \frac{(m-1)(m-2)}{2} + (m-1)v = \frac{m(m-1)}{2} + u + (m-1)v,$$

so the total degree of $P(\underline{x})$ is at most $u + (m-1)v$. Also observe that the degree of $P^*(\underline{x})$ in x_1 is $\max\{(m-1)+u, (m-2)+v\}$ and the degree of x_1 in $\prod_{1\le i<j\le m}(x_i - x_j)$ is $m-1$, so the degree of $P(\underline{x})$ in x_1 is $\max\{u, v-1\}$. As $P(\underline{x})$ is symmetric in its variables, the degree of $P(\underline{x})$ in each $x_k,\ k = 1, \ldots, m$, is $\max\{u, v-1\}$. This proves that

$$P(\underline{x}) := \sum_{(j_1,\ldots,j_m)\in M} a_{j_1\cdots j_m} x_1^{j_1} \cdots x_m^{j_m}, \quad a_{j_1\cdots j_m} \in \mathbb{C}.$$

Now we prove that $[M/N]_E(\underline{x}) = P(\underline{x})/Q(\underline{x})$. If $x_i \neq x_j$ for all $i, j = 1, 2, \ldots, m$, we have from Theorem 2.1 that,

$$F(\underline{x}) = \sum_{k=1}^{m} \frac{x_k^{m-1}}{\omega_m'(x_k)} h(x_k),$$

where h is defined by (2.6) and ω_m' is defined by (1.5). Then

$$\begin{aligned}
(QF-P)\left(\underline{x}\right) &= \left(\prod_{j=1}^{m} q_{u,v}\left(x_j\right)\right)\sum_{k=1}^{m}\frac{x_k^{m-1}}{\omega_m'(x_k)}h\left(x_k\right) \\
&\quad -\sum_{k=1}^{m}\frac{x_k^{m-1}}{\omega_m'(x_k)}\left(p_{u,v}\left(x_k\right)\prod_{1\le j\le m, j\ne k} q_{u,v}\left(x_j\right)\right) \\
&= \sum_{k=1}^{m}\frac{x_k^{m-1}}{\omega_m'(x_k)}\left(\prod_{1\le j\le m, j\ne k} q_{u,v}\left(x_j\right)\right)\left(q_{u,v}\left(x_k\right)h\left(x_k\right)-p_{u,v}\left(x_k\right)\right).
\end{aligned}$$

Recall that $p_{u,v}\left(z\right)/q_{u,v}\left(z\right)$ defined by (2.5) is the (u,v) Padé approximant to $h\left(z\right)$, i.e.

$$h\left(z\right)q_{u,v}\left(z\right)-p_{u,v}\left(z\right)=\sum_{j\ge u+v+1}\gamma_j z^j, \qquad \gamma_j\in\mathbb{C}.$$

Then

$$(QF-P)\left(\underline{x}\right)=\sum_{k=1}^{m}\frac{x_k^{m-1}}{\omega_m'(x_k)}\left(\prod_{1\le j\le m, j\ne k} q_{u,v}\left(x_j\right)\right)\sum_{j\ge u+v+1}\gamma_j x_k^j.$$

Similar to the proof that $P(\underline{x})$ is a polynomial on index set M, we can derive that

$$\sum_{k=1}^{m}\frac{x_k^{m-1}}{\omega_m'(x_k)}\left(\prod_{1\le i\le m, i\ne k} q_{u,v}\left(x_i\right)\right)$$

is a polynomial of total degree at most $(m-1)\,v$. Therefore,

$$\begin{aligned}
(QF-P)\left(\underline{x}\right) &= \sum_{j\ge u+v+1, 0\le k\le (m-1)v}\gamma_j\sum_{j_1+\cdots+j_m=j+k}x_1^{j_1}\cdots x_m^{j_m} \\
&= \sum_{(j_1,\ldots,j_m)\in\mathbb{N}^m\setminus E} d_{j_1\cdots j_m}x_1^{j_1}\cdots x_m^{j_m}, \qquad d_{j_1\cdots j_m}\in\mathbb{C},
\end{aligned}$$

where E is defined by (2.9). This proves Theorem 2.2 for the case $x_i\ne x_j$ for all $i,j=1,2,\ldots,m$. Now since $Q(\underline{0})=1$ and

$$\left(F-\frac{P}{Q}\right)\left(\underline{x}\right)=\sum_{(j_1,\ldots,j_m)\in\mathbb{N}^m\setminus E} d^*_{j_1\cdots j_m}x_1^{j_1}\cdots x_m^{j_m},\quad d^*_{j_1\cdots j_m}\in\mathbb{C},$$

we have

$$\begin{aligned}
&\lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1}\cdots\lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r}\left(F-\frac{P}{Q}\right)\left(\underline{x}\right) \\
&= \sum_{(j_1,\ldots,j_m)\in\mathbb{N}^m\setminus E} d^*_{j_1\cdots j_m}x_1^{j_1+j_{r+1}+\cdots+j_{r+s_1-1}}\cdots x_r^{j_r+\cdots+j_m}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1}\cdots\lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r}\frac{P\left(\underline{x}\right)}{Q\left(\underline{x}\right)} \\
&= \lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1}\cdots\lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r}\frac{1}{Q\left(\underline{x}\right)}
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} \left(p_{u,v}\left(x_i\right) \prod_{1\le k\le m, k\neq i} q_{u,v}\left(x_k\right) \right) \\
= \quad & \frac{1}{q_{u,v}^{s_1}\left(x_1\right) q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)} \\
&\times \lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r} \\
&\qquad\qquad \times \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} \left(p_{u,v}\left(x_i\right) \prod_{1\le k\le m, k\neq i} q_{u,v}\left(x_k\right) \right) \\
= \quad & \frac{q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)}{q_{u,v}^{s_1}\left(x_1\right) q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)} \\
&\times \lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1} \left(\frac{W_1\left(x_1\right) p_{u,v}\left(x_1\right) q_{u,v}^{s_1-1}\left(x_1\right)}{\prod_{1\le j\le s_1-1}(x_1 - x_{r+j})} \right. \\
&\qquad \left. + \sum_{i=1}^{s_1-1} \frac{W_{r+i}\left(x_{r+i}\right) p_{u,v}\left(x_{r+i}\right) q_{u,v}^{s_1-1}\left(x_{r+i}\right)}{(x_{r+i}-x_1)\prod_{1\le j\le s_1-1, j\neq i}(x_{r+i}-x_{r+j})} \right) \\
&+ \cdots + \frac{q_{u,v}^{s_1}\left(x_1\right) \cdots q_{u,v}^{s_{r-1}}\left(x_{r-1}\right)}{q_{u,v}^{s_1}\left(x_1\right) q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)} \\
&\times \lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r} \left(\frac{W_r\left(x_r\right) p_{u,v}\left(x_r\right) q_{u,v}^{s_r-1}\left(x_r\right)}{\prod_{1\le j\le s_r-1}(x_r - x_{s_1+\cdots+s_{r-1}+1+j})} \right. \\
&+ \sum_{i=1}^{s_r-1} \left[\frac{W_{s_1+\cdots+s_{r-1}+1+i}\left(x_{s_1+\cdots+s_{r-1}+1+i}\right)}{x_{s_1+\cdots+s_{r-1}+1+i} - x_r} \right. \\
&\qquad\qquad \left.\left. \times \frac{p_{u,v}\left(x_{s_1+\cdots+s_{r-1}+1+i}\right) q_{u,v}^{s_r-1}\left(x_{s_1+\cdots+s_{r-1}+1+i}\right)}{\prod_{1\le j\le s_r-1, j\neq i}\left(x_{s_1+\cdots+s_{r-1}+1+i} - x_{s_1+\cdots+s_{r-1}+1+j}\right)} \right] \right).
\end{aligned}
$$

If we let

$$H_i^*\left(x\right) := W_i\left(x\right) p_{u,v}\left(x\right) q_{u,v}^{s_i-1}(x), \qquad i = 1, 2, \ldots, r,$$

then

$$
\begin{aligned}
&\lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r} \frac{P\left(\underline{x}\right)}{Q\left(\underline{x}\right)} \\
&\quad = \frac{q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)}{q_{u,v}^{s_1}\left(x_1\right) q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)} \\
&\qquad \times \lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1} H_1^*[x_1, x_{r+1}, \ldots, x_{r+s_1-1}] \\
&\qquad + \cdots + \frac{q_{u,v}^{s_1}\left(x_1\right) \cdots q_{u,v}^{s_{r-1}}\left(x_{r-1}\right)}{q_{u,v}^{s_1}\left(x_1\right) q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)} \\
&\qquad \times \lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r} H_r^*[x_r, x_{s_1+\cdots+s_{r-1}+2}, \ldots, x_m] \\
&\quad = \frac{1}{q_{u,v}^{s_1}\left(x_1\right) q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)} \\
&\qquad \times \left(\frac{q_{u,v}^{s_2}\left(x_2\right) \cdots q_{u,v}^{s_r}\left(x_r\right)}{(s_1-1)!} \left(H_1^*\right)^{(s_1-1)}\left(x_1\right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\cdots+\frac{q_{u,v}^{s_1}(x_1)\cdots q_{u,v}^{s_{r-1}}(x_{r-1})}{(s_r-1)!}\left(H_r^*\right)^{(s_r-1)}(x_r)\Bigg)\\
&=\frac{1}{q_{u,v}^{s_1}(x_1)\,q_{u,v}^{s_2}(x_2)\cdots q_{u,v}^{s_r}(x_r)}\\
&\quad\times\sum_{i=1}^{r}\frac{1}{(s_i-1)!}\left(\prod_{1\le k\le r,k\ne i}q_{u,v}^{s_k}(x_k)\right)\left(W_i p_{u,v}q_{u,v}^{s_i-1}\right)^{(s_i-1)}(x_i).
\end{aligned}
$$

So if $x_1,\ldots,x_r$ are the distinct variables among all $x_1,\ldots,x_m$, occurring with multiplicities $s_1,\ldots,s_r$, respectively, then

$$Q(\underline{x})=q_{u,v}^{s_1}(x_1)\,q_{u,v}^{s_2}(x_2)\cdots q_{u,v}^{s_r}(x_r),$$

which is (2.10), and

$$P(\underline{x})=\sum_{i=1}^{r}\frac{1}{(s_i-1)!}\left(\prod_{1\le k\le r,k\ne i}q_{u,v}^{s_k}(x_k)\right)\left(W_i p_{u,v}q_{u,v}^{s_i-1}\right)^{(s_i-1)}(x_i),$$

which is the limiting case of $P(\underline{x})$ in (2.11). This proves Theorem 2.2. □

Remark. It is easy to see that $P(\underline{x})/Q(\underline{x})$ is irreducible, as $p_{u,v}(z)/q_{u,v}(z)$ is irreducible.

3. Properties of pseudo-multivariate Padé approximants

The univariate Padé approximant satisfies a consistency property, meaning that when the given function h is itself rational, then it is reconstructed by $p_{u,v}/q_{u,v}$ when u and v are chosen large enough. This consistency property holds mainly because of the unicity of the irreducible form of $p_{u,v}/q_{u,v}$. For a general order multivariate Padé approximant this is not necessarily the case, because of the possible nonunicity of the irreducible form of the approximant. However, similar to the properties proved in our earlier paper [6], the general order multivariate Padé approximants constructed in Theorem 2.2 in this paper have many nice properties. We prove the consistency and normality properties of these approximants, the latter meaning that if the univariate (u,v) Padé approximant $p_{u,v}/q_{u,v}$ to the function h appears only once in the Padé table, then so does its general multivariate counterpart constructed here. At the end of this section, we also present a truncation error upperbound. The properties proved in [6] constitute the special bivariate case of the properties in this section.

Theorem 3.1. *Let u,v,M,N,E be defined as in Theorem 2.2. If the pseudo-multivariate function $F(\underline{x})$ is a rational function, i.e. if $F(\underline{x})$ has the irreducible form*

$$F(\underline{x}):=\frac{R(\underline{x})}{S(\underline{x})}=\frac{\sum_{(j_1,j_2,\ldots,j_m)\in M}r_{j_1\cdots j_m}x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m}}{\sum_{(j_1,j_2,\ldots,j_m)\in N}s_{j_1\cdots j_m}x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m}},$$

with $r_{j_1\cdots j_m},s_{j_1\cdots j_m}\in\mathbb{C}$ and $S(\underline{0})\ne 0$, then the (M,N) general order multivariate Padé approximant to $F(\underline{x})$ on the index set E, constructed in Theorem 2.2, satisfies

$$\frac{P(\underline{x})}{Q(\underline{x})}=F(\underline{x})$$

for large enough $u,v\in\mathbb{N}$.

Proof. As F is a pseudo-multivariate function, we have

$$F(\underline{x}) = \sum_{n=0}^{\infty} g\left(n\right) \sum_{j_1+j_2+\cdots+j_m=n} x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}$$

with

$$h\left(z\right) = \sum_{k=0}^{\infty} g\left(k\right) z^k.$$

Then $h\left(z\right) = F\left(z, 0, \ldots, 0\right) = F\left(0, z, 0, \ldots, 0\right) = \cdots = F\left(0, \cdots, 0, z\right)$ and hence

$$h(z) = \frac{R\left(z, 0, \ldots, 0\right)}{S\left(z, 0, \ldots, 0\right)} = \cdots = \frac{R\left(0, \ldots, 0, z\right)}{S\left(0, \ldots, 0, z\right)}.$$

From the consistency property of the univariate Padé aproximant, we have for large enough $u, v \in \mathbb{N}$, $p_{u,v}(z) = R\left(z, 0, \ldots, 0\right) = \ldots = R\left(0, \ldots, 0, z\right)$ and $q_{u,v}(z) = S\left(z, 0, \ldots, 0\right) = \cdots = S\left(0, \ldots, 0, z\right)$ after a suitable normalization of $q_{u,v}(z)$. So if $x_i \neq x_j$ for all $i, j = 1, 2, \ldots, m$, then from Theorem 2.1 and Theorem 2.2, we have

$$\begin{aligned}
\frac{P(\underline{x})}{Q(\underline{x})} &= \frac{1}{q_{u,v}\left(x_1\right) \cdots q_{u,v}\left(x_m\right)} \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} \left(p_{u,v}\left(x_i\right) \prod_{1 \leq k \leq m, k \neq i} q_{u,v}\left(x_k\right) \right) \\
&= \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} \frac{p_{u,v}\left(x_i\right)}{q_{u,v}\left(x_i\right)} \\
&= \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} h\left(x_i\right) \\
&= F(\underline{x}),
\end{aligned}$$

Now if $x_i = x_j$ for some $i, j = 1, 2, \ldots, m$, we denote the distinct variables among the $x_1, \ldots, x_m$ by $x_1, \ldots, x_r$, which occur with multiplicities $s_1, \ldots, s_r$, respectively. By Theorem 2.2, the (M, N) general order multivariate Padé approximant to $F(\underline{x})$ on the index set E is the limiting case,

$$\begin{aligned}
\frac{P(\underline{x})}{Q(\underline{x})} &= \lim_{x_{r+1}, \ldots, x_{r+s_1-1} \to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2}, \ldots, x_m \to x_r} \Bigg\{ \frac{1}{q_{u,v}\left(x_1\right) \cdots q_{u,v}\left(x_m\right)} \\
&\quad \times \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} \left(p_{u,v}\left(x_i\right) \prod_{1 \leq k \leq m, k \neq i} q_{u,v}\left(x_k\right) \right) \Bigg\} \\
&= \lim_{x_{r+1}, \ldots, x_{r+s_1-1} \to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2}, \ldots, x_m \to x_r} \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} \frac{p_{u,v}\left(x_i\right)}{q_{u,v}\left(x_i\right)} \\
&= \lim_{x_{r+1}, \ldots, x_{r+s_1-1} \to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2}, \ldots, x_m \to x_r} \sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} h\left(x_i\right) \\
&= \lim_{x_{r+1}, \ldots, x_{r+s_1-1} \to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2}, \ldots, x_m \to x_r} F(\underline{x}) \\
&= F(\underline{x}),
\end{aligned}$$

This completes the proof of Theorem 3.1. ☐

Theorem 3.2. *For $u, v \in \mathbb{N}$, let $M,\ N,\ E,\ F(\underline{x})$ and $h(z)$ be defined as in Theorem 2.1 and Theorem 2.2. If the (u, v) Padé approximant $p_{u,v}\left(z\right)/q_{u,v}\left(z\right)$ to $h(z)$ is* *normal, then the* (M,N) *Padé approximant* $P(\underline{x})/Q(\underline{x})$ *to* $F(\underline{x})$ *on the index set* E *given in Theorem 2.2, is also normal.*

Proof. If the (u,v) Padé approximant to $h(z)$,

$$\frac{p_{u,v}}{q_{u,v}}(z)=\frac{\sum_{j=0}^{u}\alpha_j z^j}{\sum_{j=0}^{v}\beta_j z^j},\qquad \beta_0=1,$$

is normal for $u,v\in\mathbb{N}$, then

$$\alpha_u\neq 0,\qquad \beta_v\neq 0,$$

and

$$\left(hq_{u,v}-p_{u,v}\right)(z)=\sum_{j\geq u+v+1}\gamma_j z^j,$$

with

$$\gamma_{u+v+1}\neq 0.$$

To prove the normality of the multivariate Padé approximant it suffices to prove it for the case that if $x_i\neq x_j$ for all $i,j=1,2,\ldots,m$. The (M,N) Padé approximant to $F(\underline{x})$ on the set E is $P(\underline{x})/Q(\underline{x})$, where

$$Q(\underline{x})=\prod_{k=1}^{m}q_{u,v}(x_k)=\sum_{(j_1,j_2,\ldots,j_m)\in N}b_{j_1\cdots j_m}x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m},\qquad b_{j_1\cdots j_m}\in\mathbb{C},$$

and

$$\begin{aligned}P(\underline{x}) &= \sum_{i=1}^{m}\frac{x_i^{m-1}}{\omega_m'(x_i)}\left(p_{u,v}(x_i)\prod_{1\leq k\leq m,k\neq i}q_{u,v}(x_k)\right)\\ &= \sum_{(j_1,j_2,\ldots,j_m)\in M}a_{j_1\cdots j_m}x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m},\quad a_{j_1\cdots j_m}\in\mathbb{C},\end{aligned}$$

with

$$\begin{aligned}b_{0\cdots 0} &= (\beta_0)^m=1,\\ b_{v\cdots v} &= (\beta_v)^m\neq 0,\\ b_{0\cdots 0v} &= \cdots=b_{v0\cdots 0}=\beta_v(\beta_0)^{m-1}\neq 0,\end{aligned}$$

and

$$a_{u0\cdots 0}=\cdots=a_{0\cdots 0u}=\alpha_u(\beta_0)^{m-1}\neq 0.$$

Now

$$\begin{aligned}(FQ-P)(\underline{x}) &= \left(\prod_{k=1}^{m}q_{u,v}(x_k)\right)\sum_{i=1}^{m}\frac{x_i^{m-1}}{\omega_m'(x_i)}h(x_i)\\ &\quad -\sum_{i=1}^{m}\frac{x_i^{m-1}}{\omega_m'(x_i)}\left(p_{u,v}(x_i)\prod_{1\leq k\leq m,k\neq i}q_{u,v}(x_k)\right)\\ &= \sum_{i=1}^{m}\frac{x_i^{m-1}}{\omega_m'(x_i)}\left(\prod_{1\leq k\leq m,k\neq i}q_{u,v}(x_k)\right)\left(h(x_i)q_{u,v}(x_i)-p_{u,v}(x_i)\right)\\ &= \sum_{i=1}^{m}\frac{x_i^{m-1}}{\omega_m'(x_i)}\left(\prod_{1\leq k\leq m,k\neq i}q_{u,v}(x_k)\right)\sum_{j\geq u+v+1}\gamma_j x_i^j.\end{aligned}\tag{3.1}$$

So if

$$(FQ-P)\left(\underline{x}\right)=\sum_{(j_1,j_2,\ldots,j_m)\in E} d_{j_1\cdots j_m}x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m}, \qquad d_{j_1\cdots j_m}\in\mathbb{C},$$

then for $j_1+\cdots+j_m=u+v+1$,

$$d_{j_1\cdots j_m}=\left(\beta_0\right)^{m-1}\gamma_{u+v+1}=\gamma_{u+v+1}\neq 0.$$

Now assume that for either $u'\neq u$ or $v'\neq v$, $\ell'=\max\{u',v'-1\}$ with

$$\begin{aligned} N'&=\{(j_1,j_2,\ldots,j_m):0\le j_1,j_2,\ldots,j_m\le v'\},\\ M'&=\{(j_1,j_2,\ldots,j_m):0\le j_1,j_2,\ldots,j_m\le \ell',\ j_1+j_2\\ &\qquad\qquad +\cdots+j_m\le u'+(m-1)\,v'\},\\ E'&=\{(j_1,j_2,\ldots,j_m):0\le j_1+j_2+\cdots+j_m\le u'+v'\}\end{aligned}$$

the general order multivariate Padé approximant $[M'/N']_{E'}$ for $F(\underline{x})$ on the set E' equals the same rational function P/Q. Since $\alpha_u=a_{u0\cdots0}\neq0$ and $\beta_v=b_{v0\cdots0}\neq0$, this is only possible for $u'\ge u$ and $v'\ge v$. Hence $u'+v'+1\ge u+v+1$. The fact that $\gamma_{u+v+1}\neq0$ reduces the occurrence of nonnormality to $u'+v'+1\le u+v+1$. Hence $u'+v'+1=u+v+1$. In combination with $u'\ge u$ and $v'\ge v$ this leads to $u'=u$ and $v'=v$. Since the latter is a contradiction with our assumption that either $u'\neq u$ or $v'\neq v$, normality must hold. □

Theorem 3.3. *Let $M,N,E,F(\underline{x})$ and $h(z)$ be defined as in Theorem 2.1 and Theorem 2.2. Let $p_{u,v}(z)/q_{u,v}(z)$ be the (u,v) Padé approximant to $h(z)$ and $P(\underline{x})/Q(\underline{x})$ the (M,N) Padé approximant, constructed in Theorem 2.2 to $F(\underline{x})$ on E. Let the function $h(z)$ be analytic in a disk $B(0,\rho)$ centered at the origin with radius $\rho>\max_{1\le i\le m}|x_i|$. Then, if $x_i\neq x_j$ for all $i\neq j$, $i,j=1,2,\ldots,m$,*

$$\left|\left(F-\frac{P}{Q}\right)(\underline{x})\right|\le\frac{\sup_{\xi\in B(0,\rho)}\left|(hq_{u,v})^{(u+v+1)}(\xi)\right|}{(u+v+1)!}\sum_{i=1}^{m}\left|\frac{x_i^{u+v+m}}{\omega_m'(x_i)q_{u,v}\left(x_i\right)}\right|.$$

If $x_i=x_j$ for some $i\neq j$, $i,j=1,2,\ldots,m$, we assume that $x_1,\ldots,x_r$ are the distinct variables among the $x_1,\ldots,x_m$, occurring with multiplicities $s_1,\ldots,s_r$, respectively, then

$$\begin{aligned}&\left|\left(F-\frac{P}{Q}\right)(\underline{x})\right|\\ &\qquad\le\frac{\sup_{\xi\in B(0,\rho)}\left|(hq_{u,v})^{(u+v+1)}(\xi)\right|}{(u+v+1)!}\sum_{i=1}^{r}\frac{1}{(s_i-1)!}\left|\left(\frac{W_i(x_i)x_i^{u+v+1}}{q_{u,v}(x_i)}\right)^{(s_i-1)}\right|.\end{aligned}$$

Proof. For the univariate function $h(z)$ and its (u,v) Padé approximant $p_{m,n}/q_{m,n}$ we know that

$$|(hq_{u,v}-p_{u,v})(z)|\le\sup_{\xi_z\in[0,z]}\left|(hq_{u,v})^{(u+v+1)}(\xi_z)\right|\frac{|z|^{u+v+1}}{(u+v+1)!}, \tag{3.2}$$

If $x_i\neq x_j$ for all $i,j=1,2,\ldots,m$, then from (3.1) and (3.2),

$$
\begin{aligned}
|(FQ-P)(\underline{x})| &= \left|\sum_{i=1}^{m} \frac{x_i^{m-1}}{\omega_m'(x_i)} \left(\prod_{1\le k\le m, k\ne i} q_{u,v}(x_k)\right) \left(h(x_i)\, q_{u,v}(x_i) - p_{u,v}(x_i)\right)\right| \\
&\le \sum_{i=1}^{m} \left|\frac{x_i^{m-1}}{\omega_m'(x_i)}\right| \left|\prod_{1\le k\le m, k\ne i} q_{u,v}(x_k)\right| \left|h(x_i)\, q_{u,v}(x_i) - p_{u,v}(x_i)\right| \\
&\le \sum_{i=1}^{m} \frac{|x_i|^{u+v+m}}{|\omega_m'(x_i)|\,(u+v+1)!} \left|\prod_{1\le k\le m, k\ne i} q_{u,v}(x_k)\right| \sup_{\xi_i\in[0,x_i]} \left|(hq_{u,v})^{(u+v+1)}(\xi_i)\right|.
\end{aligned}
$$

So

$$
\left|\left(F-\frac{P}{Q}\right)(\underline{x})\right| \le \frac{\sup_{\xi\in B(0,\rho)} \left|(hq_{u,v})^{(u+v+1)}(\xi)\right|}{(u+v+1)!} \sum_{i=1}^{m} \left|\frac{x_i^{u+v+m}}{\omega_m'(x_i) q_{u,v}(x_i)}\right|.
$$

Now if $x_i = x_j$ for some $i, j = 1, 2, \ldots, m$ and if $x_1, \ldots, x_r$ are the distinct variables among the $x_1, \ldots, x_m$, occurring with multiplicities $s_1, \ldots, s_r$, respectively, then by Theorem 2.2 the (M, N) general order multivariate Padé approximant to $F(\underline{x})$ on the index set E is the limiting case. So

$$
\begin{aligned}
&\lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r} \left(F-\frac{P}{Q}\right)(\underline{x}) \\
\le\ & \frac{\sup_{\xi\in B(0,\rho)} \left|(hq_{u,v})^{(u+v+1)}(\xi)\right|}{(u+v+1)!} \\
&\quad \times \lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r} \sum_{i=1}^{m} \left|\frac{x_i^{m-1}}{\omega_m'(x_i)} \frac{x_i^{u+v+1}}{q_{u,v}(x_i)}\right|.
\end{aligned}
$$

By using the same strategy as that in the proofs of Theorem 2.1 and Theorem 2.2, we find

$$
\begin{aligned}
&\lim_{x_{r+1},\ldots,x_{r+s_1-1}\to x_1} \cdots \lim_{x_{s_1+\cdots+s_{r-1}+2},\ldots,x_m\to x_r} \sum_{i=1}^{m} \left|\frac{x_i^{m-1}}{\omega_m'(x_i)} \frac{x_i^{u+v+1}}{q_{u,v}(x_i)}\right| \\
&\qquad = \sum_{i=1}^{r} \frac{1}{(s_i-1)!} \left|\left(\frac{W_i(x_i) x_i^{u+v+1}}{q_{u,v}(x_i)}\right)^{(s_i-1)}\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|\left(F-\frac{P}{Q}\right)(\underline{x})\right| \\
&\quad \le \frac{\sup_{\xi\in B(0,\rho)} \left|(hq_{u,v})^{(u+v+1)}(\xi)\right|}{(u+v+1)!} \sum_{i=1}^{r} \frac{1}{(s_i-1)!} \left|\left(\frac{W_i(x_i) x_i^{u+v+1}}{q_{u,v}(x_i)}\right)^{(s_i-1)}\right|.
\end{aligned}
$$

This completes the proof of Theorem 3.3. □

4. Examples

Throughout this section, we let M, N and E be defined as in Theorem 2.2. We first see some examples of finding the explicit Padé approximants for some pseudo-multivariate functions of which the one variable projections have general explicit formulas of Padé approximants.

Example 4.1. A multivariate form of the exponential function is

$$E\left(\underline{x}\right)=\sum_{j_1,j_2,\ldots,j_m=0}^{\infty}\frac{x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m}}{\left(j_1+j_2+\cdots+j_m\right)!}.$$

It is a pseudo-multivariate function with

$$h\left(z\right)=\exp(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}.$$

From [2], we have that the (u,v) Padé approximant to $h\left(z\right)$ is

$$\frac{p_{u,v}}{q_{u,v}}\left(z\right)=\frac{{}_1F_1\left(-u;-u-v;z\right)}{{}_1F_1\left(-v;-u-v;-z\right)},$$

and then, by Theorem 2.2, if $x_i\neq x_j$ for all $i,j=1,2,\ldots,m$, the (M,N) general order multivariate Padé approximant to $E(\underline{x})$ is $P\left(\underline{x}\right)/Q\left(\underline{x}\right)$, where

$$\begin{aligned}Q\left(\underline{x}\right) &= \prod_{j=1}^{m}{}_1F_1\left(-v;-u-v;-x_j\right)\\ &= \sum_{j_1,\ldots,j_m=1}^{m}\frac{(-1)^{j_1+\cdots+j_m}\left(-v\right)_{j_1}\cdots\left(-v\right)_{j_m}}{\left(-u-v\right)_{j_1}\cdots\left(-u-v\right)_{j_m}}x^{j_1}\cdots x_m^{j_m},\end{aligned}$$

and

$$P\left(\underline{x}\right)=\sum_{i=1}^{m}\frac{x_i^{m-1}}{\omega_m'(x_i)}{}_1F_1\left(-u;-u-v;x_i\right)\prod_{1\leq k\leq m,k\neq i}{}_1F_1\left(-v;-u-v;x_k\right).$$

For pseudo-multivariate functions of which the one variable projection doesn't have an explicit formula for its Padé approximants, we can use software to compute the (M,N) general order multivariate Padé approximant for given M,N. Below is an example of a short procedure in Maple to compute the (M,N) general order multivariate Padé approximant for trivariate pseudo-multivariate functions.

`mpa(f,x,u,v)` — f is the one variable projection of the pseudo-multivariate function F, x is the variable of f and u,v are nonnegative integers. The procedure computes the (M,N) general order multivariate Padé approximant to $F(x,y,z)$ on the set E, where M,N, and E are defined in Theorem 2.2.

```
>with(numapprox):
 mpa:=proc(f,x,u,v)
 local g,px,py,pz,qx,qy,qz,PP,QQ,PQ;
 g:=pade(f,x,[u,v]);
 px:=numer(g); qx:=denom(g);
 py:=subs(x=y,px); qy:=subs(x=y,qx);
 pz:=subs(x=z,px); qz:=subs(x=z,qx);
 PP:=simplify((x^2*px*qy*qz/((x-y)*(x-z))
         +y^2*qx*py*qz/((y-x)*(y-z))
         +z^2*qx*qy*pz/((z-x)*(z-y)));
 QQ:=simplify(qx*qy*qz);
 PQ:=simplify(PP/QQ);
 print(P(x,y,z)=numer(PQ));
 print(Q(x,y,z)=denom(PQ));
 end proc:
```

Example 4.2. A multivariate form of the logarithm series is

$$L(\underline{x}) = \sum_{j_1,j_2,\ldots,j_m \geq 0, j_1+j_2+\cdots+j_m \geq 1} \frac{x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m}}{j_1+j_2+\cdots+j_m}.$$

It is a pseudo-multivariate function with

$$h\left(z\right) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = -\ln\left(1-z\right).$$

Running `mpa(ln(1-x),x,2,3)` gives

$$\begin{aligned} P(x,y,z) = {} & 729000x + 136404x^2yz^2 + 729000y + 136404x^2y^2z \\ & + 136404xy^2z^2 - 601128x^2yz \\ & + 1812780xyz - 601128xyz^2 + 729000z - 32841x^2y^2z^2 - 601128xy^2z \\ & - 461700z^2 - 461700y^2 - 1287900yz + 523260y^2z + 523260yz^2 \\ & - 115830y^2z^2 - 1287900xy + 523260xy^2 + 523260x^2y - 115830x^2y^2 \\ & - 461700x^2 - 1287900xz + 523260xz^2 + 523260x^2z - 115830x^2z^2, \end{aligned}$$

and

$$Q(x,y,z) = (90-102x+21x^2+x^3)(90-102y+21y^2+y^3)(90-102z+21z^2+z^3).$$

Then the (M,N) general order multivariate Padé approximant to the function

$$L(x,y,z) = \sum_{j_1,j_2 j_3 \geq 0, j_1+j_2+j_3 \geq 1} \frac{x^{j_1} y^{j_2} z^{j_3}}{j_1+j_2+j_3}$$

on the index set E is $P\left(x,y,z\right)/Q\left(x,y,z\right)$, with

$$\begin{aligned} N &= \{(j_1,j_2) : 0 \leq j_1, j_2, j_3 \leq 3\}, \\ M &= \{(j_1,j_2) : 0 \leq j_1, j_2, j_3 \leq 2,\ j_1+j_2+j_3 \leq 6\}, \\ E &= \{(j_1,j_2) : 0 \leq j_1+j_2 \leq 5\}. \end{aligned}$$

For some multivariate series, we may not have the one variable function $h(z)$ explicitly, only by its series representation. In this case, we again use `mpa(f, x, u, v)` to compute the (u,v) Padé approximant to $h(z)$ from the partial sum of degree $u+v$ and then use Theorem 2.2 to compute the (M,N) general order multivariate Padé approximant to the multivariate series on the index set E for any given positive integers u,v. We now apply this method to the Lauricella function.

Example 4.3. Our last example is the Lauricella function (see also [1, 7, 10])

$$\begin{aligned} & F_D^{(m)}\left(a, b_1, \ldots, b_m; c; x_1, \ldots, x_m\right) \\ & \quad = \sum_{j_1,j_2,\ldots,j_m=0}^{\infty} \frac{(a)_{j_1+\cdots+j_m} (b_1)_{j_1} \cdots (b_m)_{j_m}}{(c)_{j_1+\cdots+j_m}\, j_1! \cdots j_m!} x_1^{j_1} \cdots x_m^{j_m}, \end{aligned}$$

where $(\alpha)_n$ is the Pochammer symbol defined by

$$(\alpha)_n := \begin{cases} \alpha\left(\alpha+1\right)\cdots\left(\alpha+n-1\right), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

When $b_1 = \cdots = b_m = 1$,

$$F_D^{(m)}(a,1,\ldots,1;c;x_1,\ldots,x_m) = \sum_{j_1,j_2,\ldots,j_m=0}^{\infty} \frac{(a)_{j_1+\cdots+j_m}}{(c)_{j_1+\cdots+j_m}} x_1^{j_1}\cdots x_m^{j_m},$$

which is a pseudo-multivariate function with

$$h(z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^k.$$

We can use Theorem 2.2 and procedure `mpa` to compute the (M,N) general order multivariate Padé approximant to $F_D^{(m)}(a,1,\ldots,1;c;x_1,\ldots,x_m)$. For example, if $c=5, a=3$, and $m=3$, then

$$F_D^{(3)}(3,1,1,1;5;x,y,z) = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{12}{(3+j_1+j_2+j_3)(4+j_1+j_2+j_3)} x^{j_1}y^{j_2}z^{j_3},$$

which is a pseudo-multivariate function with

$$h(z) = 12\sum_{k=0}^{\infty} \frac{1}{(3+k)(4+k)} z^k.$$

Let $h_n(z)$ be the partial sum of degree n. Then `mpa`(h_5 z,z,1,3) gives

$$\begin{aligned}
P(x,y,z) = {} & 662872xy^2z + 225302y^2z^2 - 278663x^2y^2z + 225302x^2y^2 + 78407x^2y^2z^2 \\
& - 278663xy^2z^2 - 353780xy^2 - 353780y^2z - 59930332y + 41930378xy \\
& - 353780x^2y - 31712555xyz + 41930378yz + 662872x^2yz + 662872xyz^2 \\
& - 278663x^2yz^2 - 353780yz^2 + 225302x^2z^2 - 353780xz^2 + 94105480 \\
& - 59930332x - 59930332z + 41930378xz - 353780x^2z, \\
Q(x,y,z) = {} & 5(266 - 329x + 91x^2 + x^3)(266 - 329y \\
& + 91y^2 + y^3)(266 - 329z + 91z^2 + z^3).
\end{aligned}$$

The (M,N) general order multivariate Padé approximant to the function $F_D^{(3)}(3,1,1,1;5;x,y,z)$ on the index set E is $P(x,y,z)/Q(x,y,z)$ with

$$\begin{aligned}
N &= \{(j_1,j_2,j_3) : 0 \le j_1,j_2,j_3 \le 3\}, \\
M &= \{(j_1,j_2,j_3) : 0 \le j_1,j_2,j_3 \le 2,\ j_1+j_2+j_3 \le 6\}, \\
E &= \{(j_1,j_2) : 0 \le j_1+j_2+j_3 \le 4\}.
\end{aligned}$$

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