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FOURIER EXPANSIONS AND INTEGRAL REPRESENTATIONS FOR THE APOSTOL-BERNOULLI AND APOSTOL-EULER POLYNOMIALS

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ABSTRACT. We investigate Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials using the Lipschitz summation formula and obtain their integral representations. We give some explicit formulas at rational arguments for these polynomials in terms of the Hurwitz zeta function. We also derive the integral representations for the classical Bernoulli and Euler polynomials and related known results.

1. INTRODUCTION

The classical Bernoulli polynomials and Euler polynomials are defined by means of the following generating functions (see [1, pp. 804-806] or [18, pp. 25-32])

$$\frac{ze^{xz}}{e^z-1}=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}\qquad(|z|<2\pi) \tag{1.1}$$

and

$$\frac{2e^{xz}}{e^z+1}=\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}\qquad(|z|<\pi), \tag{1.2}$$

respectively. Obviously, $B_n := B_n(0),\ E_n := 2^nE_n\left(\frac{1}{2}\right)$ are the Bernoulli numbers and Euler numbers respectively.

Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol [2, p. 165, Eq. (3.1)] and (more recently) by Srivastava [20, pp. 83-84]. We begin by recalling here Apostol's definitions as follows:

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Definition 1.1 (Apostol [2]; see also Srivastava [20])**.** The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ in x are defined by means of the generating function

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{z^n}{n!} \tag{1.3}$$

$$(|z| < 2\pi \text{ when } \lambda = 1;\ |z| < |\log \lambda| \text{ when } \lambda \neq 1)$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x;1) \qquad \text{and} \qquad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda),$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers (in fact, it is a function in λ).

Recently, Luo and Srivastava introduced the Apostol-Euler polynomials as follows:

Definition 1.2 (Luo [14]; see also Luo and Srivastava [13])**.** The Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$ in x are defined by means of the generating function

$$\frac{2e^{xz}}{\lambda e^z + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x;\lambda)\frac{z^n}{n!} \qquad (|z| < |\log(-\lambda)|), \tag{1.4}$$

with, of course,

$$E_n(x) = \mathcal{E}_n(x;1) \qquad \text{and} \qquad \mathcal{E}_n(\lambda) := 2^n \mathcal{E}_n\left(\frac{1}{2};\lambda\right),$$

where $\mathcal{E}_n(\lambda)$ denote the so-called Apostol-Euler numbers (in fact, it is a function in λ).

Remark 1.3. In Definition 1.1 and Definition 1.2, the original constraints $|z + \log\lambda| < 2\pi$ and $|z + \log\lambda| < \pi$, respectively, should be replaced by the conditions $|z| < 2\pi$ when $\lambda = 1$; $|z| < |\log\lambda|$ when $\lambda \neq 1$ and $|z| < |\log(-\lambda)|$ for the referee's clear and detailed argumentation. Hence, the corresponding constraints in References [13], [14], [15] and [20] should also be such.

The Apostol-Bernoulli and Apostol-Euler polynomials have been investigated by many people (see, e.g., [2], [4], [5], [9], [13]–[17], [20] and [22]).

D. H. Lehmer [11] gave a new approach to Bernoulli polynomials, starting from a function equation (Rabbe's multiplication theorem). H. Haruki and T. M. Rassias [10] provided the new integral representations for the Bernoulli and Euler polynomials as well as using a similar function equation. Recently, D. Cvijović [7] reproduced the results of H. Haruki and T. M. Rassias in a different way and showed several different integral representations for the Bernoulli and Euler polynomials.

In the present paper, we first investigate Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials based on the Lipschitz summation formula, and then provide their integral representations. We obtain some explicit formulas for the Apostol-Bernoulli and Apostol-Euler polynomials at rational arguments in terms of the Hurwitz zeta function. We also deduce the corresponding uniform integral representations for the classical Bernoulli and Euler polynomials. We will see that the results of Cvijović or H. Haruki and T. M. Rassias are the corresponding direct consequences of our formulas.

The paper is organized as follows. In the first section we rewrite the definitions of Apostol-Bernoulli and Apostol-Euler polynomials. In the second section we derive

Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials. In the third section we show their integral representations. In the fourth section we obtain their explicit formulas at rational arguments in terms of the Hurwitz zeta function. In the fifth section we deduce the corresponding uniform integral representations for the classical Bernoulli and Euler polynomials and related results of Cvijović or H. Haruki and T. M. Rassias. In the sixth section we give some applications and remarks; for example, the classical Euler formula $\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}$ is obtained according to our method.

2. Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials

In this section we investigate Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials by applying the Lipschitz summation formula.

First we recall the Lipschitz summation formula (see [12] or [19]) as follows:

$$\sum_{n+\mu>0} \frac{e^{2\pi i(n+\mu)\tau}}{(n+\mu)^{1-\alpha}} = \frac{\Gamma(\alpha)}{(-2\pi i)^\alpha} \sum_{k\in\mathbb{Z}} \frac{e^{-2\pi ik\mu}}{(\tau+k)^\alpha}, \tag{2.1}$$

where $\mu \in \mathbb{Z}$ and $\Re(\alpha) > 1$ or $\mu \in \mathbb{R}\setminus\mathbb{Z}$ and $\Re(\alpha) > 0$; $\tau \in H$ is the complex upper half plane and Γ denotes the Gamma function.

Theorem 2.1. *For $n = 1$, $0 < x < 1$ and $n > 1$, $0 \le x \le 1$, $\lambda \in \mathbb{C}\setminus\{0\}$, we have*

$$\mathcal{B}_n(x;\lambda) = -\delta_n(x;\lambda) - \frac{n!}{\lambda^x} {\sum}' \frac{e^{2\pi ikx}}{(2\pi ik - \log\lambda)^n} \tag{2.2}$$

$$= -\delta_n(x;\lambda) - \frac{n!i^n}{\lambda^x}\left[\sum_{k=1}^\infty \frac{\exp\left[\left(-2\pi kx + \frac{n\pi}{2}\right)i\right]}{(2\pi ik + \log\lambda)^n} + \sum_{k=1}^\infty \frac{\exp\left[\left(2\pi kx - \frac{n\pi}{2}\right)i\right]}{(2\pi ik - \log\lambda)^n}\right], \tag{2.3}$$

where the symbol ${\sum}'$ denotes the standard convention of a sum over the integers that omits 0; $\delta_n(x;\lambda) = 0$ or $\frac{(-1)^n n!}{\lambda^x \log^n \lambda}$ according as $\lambda = 1$ or $\lambda \ne 1$, respectively.

Proof. For $0 \le x \le 1$, by (1.3) and the generalized binomial theorem, we have

$$\sum_{k=0}^\infty \mathcal{B}_k(x;\lambda)\frac{(2\pi i\tau)^{k-1}}{k!} = \frac{e^{2\pi i\tau x}}{\lambda e^{2\pi i\tau} - 1} = -\sum_{k=0}^\infty \lambda^k e^{2\pi i(k+x)\tau} \tag{2.4}$$

$$\left(|\tau| < 1 \text{ when } \lambda = 1;\ |\tau| < \frac{|\log\lambda|}{2\pi} \text{ when } \lambda \ne 1;\ \Im\tau > \frac{\log|\lambda|}{2\pi}\right).$$

We differentiate both sides of (2.4) with respect to the variable τ, by $n-1$ times and noting that $\mathcal{B}_0(x;\lambda) = \delta_{1,\lambda}$ (see [13, p. 301]). Then we get

$$\sum_{k=n}^\infty \mathcal{B}_k(x;\lambda)\frac{(2\pi i)^{k-1}\tau^{k-n}}{k(k-n)!} + \frac{(-1)^{n-1}(n-1)!}{2\pi i\tau^n}\delta_{1,\lambda} = -(2\pi i)^{n-1}\sum_{k=0}^\infty \lambda^k (k+x)^{n-1} e^{2\pi i(k+x)\tau}, \tag{2.5}$$

where $\delta_{1,\lambda}$ is the Kronecker symbol.

On the other hand, letting $\alpha = n, \mu \mapsto x, \tau \mapsto \tau + \frac{\log \lambda}{2\pi i}$ in (2.1), we find that

$$(-1)^n (n-1)! \sum_{k\in\mathbb{Z}} \frac{e^{-2\pi ikx}}{[2\pi i(\tau+k)+\log\lambda]^n} = \sum_{k=0}^{\infty} \lambda^{k+x}(k+x)^{n-1} e^{2\pi i(k+x)\tau}. \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$\lambda^x \sum_{k=n}^{\infty} \mathcal{B}_k(x;\lambda) \frac{(2\pi i)^{k-1}\tau^{k-n}}{k(k-n)!} + \lambda^x \frac{(-1)^{n-1}(n-1)!}{2\pi i\tau^n}\delta_{1,\lambda} = (-1)^{n-1}(n-1)!(2\pi i)^{n-1} \sum_{k\in\mathbb{Z}} \frac{e^{-2\pi ikx}}{[2\pi i(\tau+k)+\log\lambda]^n}.$$

Separating this $k = 0$ term in the above sum on the right side yields that

$$\lambda^x \sum_{k=n}^{\infty} \mathcal{B}_k(x;\lambda) \frac{(2\pi i)^{k-1}\tau^{k-n}}{k(k-n)!} = (-1)^{n-1}(n-1)!(2\pi i)^{n-1} \times {\sum}' \frac{e^{-2\pi ikx}}{[2\pi i(\tau+k)+\log\lambda]^n} + \frac{(-1)^{n-1}(n-1)!(2\pi i)^{n-1}}{(2\pi i\tau+\log\lambda)^n}(1-\delta_{1,\lambda}). \tag{2.7}$$

Letting $\tau \to 0$ in (2.7) we are led at once to the assertion (2.2) of Theorem 2.1.

Noting that $i^n = e^{\frac{n\pi i}{2}}$, $(-1)^n = e^{-n\pi i}$ and via a simple calculation, then the assertion (2.3) of Theorem 2.1 is a direct consequence of (2.2). This completes our proof. □

In the same manner, we may prove the following.

Theorem 2.2. *For $n = 0$, $0 < x < 1$ and $n > 0$, $0 \le x \le 1$, $\lambda \in \mathbb{C} \setminus \{0, -1\}$, we have*

$$\mathcal{E}_n(x;\lambda) = \frac{2\cdot n!}{\lambda^x} \sum_{k\in\mathbb{Z}} \frac{e^{(2k-1)\pi ix}}{[(2k-1)\pi i - \log\lambda]^{n+1}} \tag{2.8}$$

$$= \frac{2\cdot n! i^{n+1}}{\lambda^x} \left[\sum_{k=0}^{\infty} \frac{\exp\left[\left(\frac{n+1}{2}\pi - (2k+1)\pi x\right) i\right]}{[(2k+1)\pi i + \log\lambda]^{n+1}} + \sum_{k=0}^{\infty} \frac{\exp\left[\left(-\frac{n+1}{2}\pi + (2k+1)\pi x\right) i\right]}{[(2k+1)\pi i - \log\lambda]^{n+1}} \right]. \tag{2.9}$$

By Theorem 2.1 and Theorem 2.2, we can deduce respectively the Fourier expansions for the classical Bernoulli and Euler polynomials as follows:

Corollary 2.3. *For $n = 1$, $0 < x < 1$ and $n > 1$, $0 \le x \le 1$, we have*

$$B_n(x) = -\frac{n!}{(2\pi i)^n} {\sum}' \frac{e^{2\pi ikx}}{k^n} \tag{2.10}$$

$$= -\frac{2\cdot n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos\left(2\pi kx - \frac{n\pi}{2}\right)}{k^n}. \tag{2.11}$$

Corollary 2.4. *For $n=0,\ 0<x<1$ and $n>0,\ 0\leq x\leq 1$, we have*

$$E_n(x)=\frac{2\cdot n!}{(\pi i)^{n+1}}\sum_{k\in\mathbb{Z}}\frac{e^{(2k-1)\pi ix}}{(2k-1)^{n+1}} \tag{2.12}$$

$$=\frac{4\cdot n!}{\pi^{n+1}}\sum_{k=0}^{\infty}\frac{\sin\left[(2k+1)\pi x-\frac{n\pi}{2}\right]}{(2k+1)^{n+1}}. \tag{2.13}$$

Remark 2.5. Replacing τ by $\tau+\frac{\log\lambda}{2\pi i}+\frac{1}{2}$ in (2.1) and applying $\mathcal{E}_0(x;\lambda)=\frac{2}{\lambda+1}$ (see, for details, [13]–[15]) when we prove the assertion (2.8) of Theorem 2.2.

Remark 2.6. We define the n-th Apostol-Bernoulli function as

$$\widehat{\mathcal{B}}_n(x;\lambda):=\mathcal{B}_n(x;\lambda)\ (0\leq x<1),\quad \widehat{\mathcal{B}}_n(x+1;\lambda)=\lambda^{-1}\widehat{\mathcal{B}}_n(x;\lambda), \tag{2.14}$$

which is also called the *quasi-periodicity Apostol-Bernoulli polynomials*. For any $x\in\mathbb{R}, r\in\mathbb{Z}$, we have

$$\widehat{\mathcal{B}}_n(x;\lambda)=\lambda^{-[x]}\mathcal{B}_n(\{x\};\lambda),\quad \widehat{\mathcal{B}}_n(x+r;\lambda)=\lambda^{-r}\widehat{\mathcal{B}}_n(x;\lambda). \tag{2.15}$$

Here the notation $\{x\}$ denotes the fractional part of x, and the notation $[x]$ denotes the greatest integer not exceeding x.

Clearly, the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ $(0\leq x<1)$ are the quasi-periodicity functions in x with period 1. One of the special cases of the quasi-periodicity Apostol-Bernoulli polynomials is just Carlitz's periodic Bernoulli function [3, p. 661] for $\lambda=1$.

Remark 2.7. We define the n-th Apostol-Euler function as

$$\widehat{\mathcal{E}}_n(x;\lambda):=\mathcal{E}_n(x;\lambda)\ (0\leq x<1),\quad \widehat{\mathcal{E}}_n(x+1;\lambda)=-\lambda^{-1}\widehat{\mathcal{E}}_n(x;\lambda), \tag{2.16}$$

which is called the *quasi-periodicity Apostol-Euler polynomials*. For any $x\in\mathbb{R}, r\in\mathbb{Z}$, we have

$$\widehat{\mathcal{E}}_n(x;\lambda)=(-1)^{[x]}\lambda^{-[x]}\mathcal{E}_n(\{x\};\lambda),\quad \widehat{\mathcal{E}}_n(x+r;\lambda)=(-1)^r\lambda^{-r}\widehat{\mathcal{E}}_n(x;\lambda). \tag{2.17}$$

Obviously, the Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$ $(0\leq x<1)$ are the quasi-periodicity functions in x with period 1. One of the special cases of the quasi-periodicity Apostol-Euler polynomials is just Carlitz's periodic Euler function [3, p. 661] for $\lambda=1$.

Remark 2.8. We employ a useful relationship [15, p. 636, Eq. (38)]

$$\mathcal{E}_n(x;\lambda) = \frac{2}{n+1}\left[\mathcal{B}_{n+1}(x;\lambda) - 2^{n+1}\mathcal{B}_{n+1}\left(\frac{x}{2};\lambda^2\right)\right] \tag{2.18}$$

to (2.2) and (2.3), respectively; we can also arrive at the corresponding (2.8) and (2.9).

Remark 2.9. Throughout this paper, we take the principal value of the logarithm $\log\lambda$, i.e., $\log\lambda = \log|\lambda| + i\arg\lambda$ $(-\pi < \arg\lambda \leq \pi)$ when $\lambda \neq 1$; We choose $\log 1 = 0$ when $\lambda = 1$.

3. Integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials

In this section we give the integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials with their Fourier expansions. For convenience, we take $\lambda = e^{2\pi i\xi}$ $(\xi \in \mathbb{R},\ |\xi| < 1)$ in this section.

Theorem 3.1. *For* $n = 1, 2, \ldots,\ 0 \leq \Re(x) \leq 1,\ |\xi| < 1,\ \xi \in \mathbb{R}$, *we have*

$$\begin{aligned}\mathcal{B}_n(x;e^{2\pi i\xi}) &= -\Delta_n(x;\xi) \\ &\quad - ne^{-2\pi ix\xi}\int_0^\infty \frac{U(n;x,t)\cosh(2\pi\xi t) + i\,V(n;x,t)\sinh(2\pi\xi t)}{\cosh 2\pi t - \cos 2\pi x} t^{n-1}\,\mathrm{d}t,\end{aligned} \tag{3.1}$$

where $\Delta_n(x;\xi) = 0$ *or* $\dfrac{(-1)^n n!}{e^{2\pi ix\xi}(2\pi i\xi)^n}$ *according as* $\xi = 0$ *or* $\xi \neq 0$, *respectively, and*

$$\begin{aligned}U(n;x,t) &= \left[\cos\left(2\pi x - \frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)e^{-2\pi t}\right],\\ V(n;x,t) &= \left[\sin\left(2\pi x - \frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)e^{-2\pi t}\right].\end{aligned}$$

Proof. Returning to (2.2) and setting $\lambda = e^{2\pi i\xi},\ k \mapsto -k$ yields

$$\mathcal{B}_n(x;e^{2\pi i\xi}) = -\Delta_n(x;\xi) - \frac{n!e^{-2\pi ix\xi}}{(-2\pi i)^n}\sum{}' \frac{e^{-2\pi ikx}}{(k+\xi)^n}.$$

Using the known integral formula

$$\int_0^\infty t^n e^{-at}\,\mathrm{d}t = \frac{n!}{a^{n+1}} \qquad (n = 0, 1, \ldots;\ \Re(a) > 0), \tag{3.2}$$

and noting that $\left(-\frac{1}{i}\right)^n = e^{\frac{n\pi i}{2}}$ and $(-1)^n = e^{-n\pi i}$, then we have

$$\begin{aligned}\mathcal{B}_n(x;e^{2\pi i\xi}) &= -\Delta_n(x;\xi) - \frac{ne^{-2\pi ix\xi}}{(-2\pi i)^n}\Bigg\{\sum_{k=1}^{\infty} e^{-2\pi ikx}\int_0^{\infty} t^{n-1}e^{-(k+\xi)t}\,\mathrm{d}t \\ &\qquad + (-1)^n\sum_{k=1}^{\infty} e^{2\pi ikx}\int_0^{\infty} t^{n-1}e^{-(k-\xi)t}\,\mathrm{d}t\Bigg\} \\ &= -\Delta_n(x;\xi) - \frac{ne^{-2\pi ix\xi}}{(-2\pi i)^n}\Bigg\{\int_0^{\infty} e^{-\xi t}t^{n-1}\sum_{k=1}^{\infty} e^{-(2\pi ix+t)k}\,\mathrm{d}t \\ &\qquad + (-1)^n\int_0^{\infty} e^{\xi t}t^{n-1}\sum_{k=1}^{\infty} e^{(2\pi ix-t)k}\,\mathrm{d}t\Bigg\} \\ &= -\Delta_n(x;\xi) - \frac{ne^{-2\pi ix\xi}}{(-2\pi i)^n}\Bigg\{\int_0^{\infty} \frac{e^{-2\pi ix}}{e^t - e^{-2\pi ix}}e^{-\xi t}t^{n-1}\,\mathrm{d}t \\ &\qquad + (-1)^n\int_0^{\infty} \frac{e^{2\pi ix}}{e^t - e^{2\pi ix}}e^{\xi t}t^{n-1}\,\mathrm{d}t\Bigg\} \\ &= -\Delta_n(x;\xi) - \frac{ne^{-2\pi ix\xi}}{2(2\pi)^n}\Bigg\{\int_0^{\infty} \frac{e^{\frac{n\pi i}{2}}(e^{-2\pi ix} - e^{-t})}{\cosh t - \cos 2\pi x}e^{-\xi t}t^{n-1}\,\mathrm{d}t \\ &\qquad + \int_0^{\infty} \frac{e^{-\frac{n\pi i}{2}}(e^{2\pi ix} - e^{-t})}{\cosh t - \cos 2\pi x}e^{\xi t}t^{n-1}\,\mathrm{d}t\Bigg\}.\end{aligned}$$

It follows that we make the transformation $t = 2\pi u$, and after simplification we obtain the desired (3.1) immediately. This completes the proof. □

We can obtain the following integral representations for the Apostol-Euler polynomials by a similar method.

Theorem 3.2. *For $n = 1, 2, \dots,\ 0 \le \Re(x) \le 1,\ |\xi| < \frac{1}{2},\ \xi \in \mathbb{R}$, we have*

$$\mathcal{E}_n(x;e^{2\pi i\xi}) = 2e^{-2\pi ix\xi} \times \int_0^{\infty} \frac{X(n;x,t)\cosh(2\pi\xi t) + i\,Y(n;x,t)\sinh(2\pi\xi t)}{\cosh 2\pi t - \cos 2\pi x}t^n\,\mathrm{d}t, \tag{3.3}$$

where

$$X(n;x,t) = \left[e^{-\pi t}\sin\left(\pi x + \frac{n\pi}{2}\right) + e^{\pi t}\sin\left(\pi x - \frac{n\pi}{2}\right)\right],$$
$$Y(n;x,t) = \left[e^{-\pi t}\cos\left(\pi x + \frac{n\pi}{2}\right) - e^{\pi t}\cos\left(\pi x - \frac{n\pi}{2}\right)\right].$$

On the other hand, we can also arrive at the following different integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials.

Theorem 3.3. *For $n = 1, 2, \dots,\ 0 \le \Re(x) \le 1,\ |\xi| < 1,\ \xi \in \mathbb{R}$, we have*

$$\mathcal{B}_n(x;e^{2\pi i\xi}) = -\Delta_n(x;\xi) + \frac{2ne^{-2\pi ix\xi}}{(-2\pi)^n} \times \int_0^{1} \frac{U'(n;x,t)\cosh(\xi\log t) - i\,V'(n;x,t)\sinh(\xi\log t)}{t^2 - 2t\cos 2\pi x + 1}(\log t)^{n-1}\,\mathrm{d}t, \tag{3.4}$$

where $\Delta_n(x;\xi)=0$ *or* $\dfrac{(-1)^n n!}{e^{2\pi ix\xi}(2\pi i\xi)^n}$ *according as* $\xi=0$ *or* $\xi\neq 0$*, respectively, and*

$$U'(n;x,t)=\left[\cos\left(2\pi x-\frac{n\pi}{2}\right)-t\cos\left(\frac{n\pi}{2}\right)\right],$$
$$V'(n;x,t)=\left[\sin\left(2\pi x-\frac{n\pi}{2}\right)+t\sin\left(\frac{n\pi}{2}\right)\right].$$

Proof. First we substitute $\cosh 2\pi t=\frac{e^{2\pi t}+e^{-2\pi t}}{2}$ into (3.1). Then we see that

$$\begin{aligned}\mathcal{B}_n(x;e^{2\pi i\xi})&=-\Delta_n(x;\xi)-2ne^{-2\pi ix\xi}\\&\quad\times\int_0^\infty\frac{U(n;x,t)\cosh(2\pi\xi t)+i\,V(n;x,t)\sinh(2\pi\xi t)}{e^{2\pi t}+e^{-2\pi t}-2\cos 2\pi x}t^{n-1}\,\mathrm{d}t.\end{aligned}\tag{3.5}$$

Then making the transformation $u=e^{-2\pi t}$ in (3.5), we easily obtain formula (3.4). This completes the proof. □

Similarly, we obtain

Theorem 3.4. *For* $n=1,2,\ldots,\ 0\leq\Re(x)\leq 1,\ |\xi|<\frac{1}{2},\ \xi\in\mathbb{R}$*, we have*

$$\begin{aligned}\mathcal{E}_n(x;e^{2\pi i\xi})&=(-1)^n\frac{4e^{-2\pi ix\xi}}{\pi^{n+1}}\\&\quad\times\int_0^1\frac{X'(n;x,t)\cosh(2\xi\log t)-i\,Y'(n;x,t)\sinh(2\xi\log t)}{t^4-2t^2\cos 2\pi x+1}(\log t)^n\,\mathrm{d}t,\end{aligned}\tag{3.6}$$

where

$$X'(n;x,t)=\left[t^2\sin\left(\pi x+\frac{n\pi}{2}\right)+\sin\left(\pi x-\frac{n\pi}{2}\right)\right],$$
$$Y'(n;x,t)=\left[t^2\cos\left(\pi x+\frac{n\pi}{2}\right)-\cos\left(\pi x-\frac{n\pi}{2}\right)\right].$$

Remark 3.5. For any integers ℓ, we see easily that $\mathcal{B}_n(x;e^{2\pi i(\ell+\xi)})=\mathcal{B}_n(x;e^{2\pi i\xi})$, $\mathcal{E}_n(x;e^{2\pi i(\ell+\xi)})=\mathcal{E}_n(x;e^{2\pi i\xi})$. Therefore, the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;e^{2\pi i\xi})$ and the Apostol-Euler polynomials $\mathcal{E}_n(x;e^{2\pi i\xi})$ are the periodicity functions in ξ with period 2π. In view of this observation we say that ξ may take any real numbers in Theorem 3.1–Theorem 3.4.

Remark 3.6. We can also prove Theorem 2.1 and Theorem 2.2 by Theorem 3.1 and Theorem 3.2, respectively, in an inverse process.

4. Explicit formulas for the Apostol-Bernoulli and Apostol-Euler polynomials at rational arguments

In this section we obtain some explicit formulas for the Apostol-Bernoulli and Apostol-Euler polynomials at rational arguments. We can see that some known formulas of Cvijović and Klinowski are the corresponding special cases of our formulas.

The Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (cf., e.g., [21, p. 121, *et seq.*])

$$\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}\tag{4.1}$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C}\text{ when }|z|<1;\ \Re(s)>1\text{ when }|z|=1\right)$$

contains, as its *special* cases, not only the Riemann and Hurwitz zeta functions

$$\zeta(s) := \Phi(1,s,1) = \zeta(s,1) = \frac{1}{2^s-1}\,\zeta\left(s,\frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n^s} \tag{4.2}$$

and

$$\zeta(s,a) := \Phi(1,s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \qquad \left(\Re(s)>1;\ a\notin\mathbb{Z}_0^-\right) \tag{4.3}$$

and the Lerch zeta function (or periodic zeta function)

$$l_s(\xi) := \sum_{n=1}^{\infty}\frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\ \Phi\left(e^{2\pi i\xi},s,1\right) \qquad (\xi\in\mathbb{R};\ \Re(s)>1)\,, \tag{4.4}$$

but also such other functions as the polylogarithmic function

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty}\frac{z^n}{n^s} = z\ \Phi(z,s,1) \qquad \left(s\in\mathbb{C}\quad\text{when}\quad |z|<1;\ \Re(s)>1\quad\text{when}\quad |z|=1\right) \tag{4.5}$$

and the Lipschitz-Lerch zeta function (cf. [21, p. 122, Eq. 2.5 (11)])

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty}\frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi},s,a\right) =: L(\xi,s,a) \qquad \left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \Re(s)>0\quad\text{when}\quad \xi\in\mathbb{R}\setminus\mathbb{Z};\ \Re(s)>1\quad\text{when}\quad \xi\in\mathbb{Z}\right), \tag{4.6}$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions.

Recently, H. M. Srivastava made use of Apostol's formula (see [2, p. 164])

$$\phi(\xi,a,1-n) = \Phi(e^{2\pi i\xi},1-n,a) = -\frac{\mathcal{B}_n\left(a;e^{2\pi i\xi}\right)}{n} \qquad (n\in\mathbb{N}) \tag{4.7}$$

and Lerch's functional equation (see [2, p. 161, (1.4)])

$$\begin{aligned}\phi(\xi,a,1-s) = \frac{\Gamma(s)}{(2\pi)^s}\Bigg\{&\exp\left[\left(\frac{1}{2}s-2a\xi\right)\pi i\right]\phi(-a,\xi,s)\\ &+\exp\left[\left(-\frac{1}{2}s+2a(1-\xi)\right)\pi i\right]\phi(a,1-\xi,s)\Bigg\}\\ &(s\in\mathbb{C};\ 0<\xi<1)\end{aligned} \tag{4.8}$$

to derive the following formula of Apostol-Bernoulli polynomials at rational arguments (see [20, p. 84, Eq. (4.6)]):

(4.9)
$$\mathcal{B}_n\left(\frac{p}{q};e^{2\pi i\xi}\right)=-\frac{n!}{(2q\pi)^n}\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{n}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right.$$
$$\left.+\sum_{j=1}^{q}\zeta\left(n,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}$$

(4.10) $$(n\in\mathbb{N}\setminus\{1\};\ q\in\mathbb{N};\ p\in\mathbb{Z};\ \xi\in\mathbb{R}).$$

Below we obtain similar formulas by using Fourier expansions for the Apostol-Bernoulli polynomials and Apostol-Euler polynomials, respectively.

Theorem 4.1. *For $n\in\mathbb{N}\setminus\{1\}$, $q\in\mathbb{N}$, $p\in\mathbb{Z}$, $\xi\in\mathbb{R}$, $|\xi|<1$, the following formula of Apostol-Bernoulli polynomials at rational arguments*

(4.11)
$$\mathcal{B}_n\left(\frac{p}{q};e^{2\pi i\xi}\right)=-\Delta_n\left(\frac{p}{q};\xi\right)-\frac{n!}{(2\pi q)^n}\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{j+\xi}{q}\right)\exp\left[\left(\frac{n}{2}-\frac{2(j+\xi)p}{q}\right)\pi i\right]\right.$$
$$\left.+\sum_{j=1}^{q}\zeta\left(n,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}$$

holds true in terms of the Hurwitz zeta function, where $\Delta_n(x;\xi)=0$ or $\dfrac{(-1)^n n!}{e^{2\pi i x\xi}(2\pi i\xi)^n}$ according as $\xi=0$ or $\xi\neq 0$, respectively.

Proof. We employ formula (2.3),

$$\mathcal{B}_n(x;\lambda)=-\delta_n(x;\lambda)-\frac{n!i^n}{\lambda^x}\left[\sum_{k=1}^{\infty}\frac{\exp\left[\left(-2\pi kx+\frac{n\pi}{2}\right)i\right]}{(2\pi ik+\log\lambda)^n}+\sum_{k=1}^{\infty}\frac{\exp\left[\left(2\pi kx-\frac{n\pi}{2}\right)i\right]}{(2\pi ik-\log\lambda)^n}\right],$$

so that, in view of the definition (4.1) and the elementary series identity

(4.12) $$\sum_{k=1}^{\infty}f(k)=\sum_{j=1}^{\ell}\sum_{k=0}^{\infty}f(\ell k+j)\qquad(\ell\in\mathbb{N}),$$

we find the formula:

(4.13)
$$\mathcal{B}_n(x;\lambda)=-\delta_n(x;\lambda)-\frac{n!i^n\lambda^{-x}}{(2\pi i\ell)^n}$$
$$\times\left[\sum_{j=1}^{\ell}\Phi\left(e^{2\pi i\ell x},n,\frac{2\pi ij-\log\lambda}{2\pi i\ell}\right)\exp\left[\left(2\pi jx-\frac{n\pi}{2}\right)i\right]\right.$$
$$\left.+\sum_{j=1}^{\ell}\Phi\left(e^{-2\pi i\ell x},n,\frac{2\pi ij+\log\lambda}{2\pi i\ell}\right)\exp\left[-\left(2\pi jx+\frac{n\pi}{2}\right)i\right]\right].$$

Setting $\lambda=\exp(2\pi i\xi)$, $x=\frac{p}{q}$, $\ell=q$ in (4.13), we then obtain the assertion of Theorem 4.1. This completes the proof. □

If we make use of the equivalent of (2.9) as

$$\mathcal{E}_n(x;\lambda) = \frac{2\cdot n!i^{n+1}}{\lambda^x}\left[\sum_{k=1}^{\infty}\frac{\exp\left[\left(\frac{n+1}{2}\pi-(2k-1)\pi x\right)i\right]}{\left[(2k-1)\pi i+\log\lambda\right]^{n+1}} + \sum_{k=1}^{\infty}\frac{\exp\left[\left(-\frac{n+1}{2}\pi+(2k-1)\pi x\right)i\right]}{\left[(2k-1)\pi i-\log\lambda\right]^{n+1}}\right] \tag{4.14}$$

and the elementary series identity (4.12), by an analogous method, we provide that

Theorem 4.2. *For $n,q\in\mathbb{N},\ p\in\mathbb{Z},\ \xi\in\mathbb{R},\ |\xi|<1$, the following formula of Apostol-Euler polynomials at rational arguments*

$$\begin{aligned}\mathcal{E}_n\left(\frac{p}{q};e^{2\pi i\xi}\right) &= \frac{2\cdot n!}{(2q\pi)^{n+1}}\\ &\times\left\{\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j+2\xi-1}{2q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{(2j+2\xi-1)p}{q}\right)\pi i\right]\right.\\ &\left.+\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\right\}\end{aligned} \tag{4.15}$$

holds true in terms of the Hurwitz zeta function.

Upon the special cases of (4.11) and (4.15), for $\xi=0$, are respectively the following known results given earlier by Cvijović and Klinowski.

Corollary 4.3 ([6, p. 1529, Theorem A])**.** *For $n\in\mathbb{N}\setminus\{1\},\ q\in\mathbb{N},\ p\in\mathbb{Z}$, the following formula for the classical Bernoulli polynomials*

$$B_n\left(\frac{p}{q}\right) = -\frac{2\cdot n!}{(2q\pi)^n}\sum_{j=1}^{q}\zeta\left(n,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{n\pi}{2}\right)$$

holds true.

Corollary 4.4 ([6, p. 1529, Theorem B])**.** *For $n,q\in\mathbb{N},\ p\in\mathbb{Z}$, the following formula for the classical Euler polynomials*

$$E_n\left(\frac{p}{q}\right) = \frac{4\cdot n!}{(2q\pi)^{n+1}}\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-1}{2q}\right)\sin\left(\frac{(2j-1)p\pi}{q}-\frac{n\pi}{2}\right)$$

holds true.

Remark 4.5. We may also prove formula (4.15) by applying the relationship (2.18) and Theorem 4.1.

Remark 4.6. In view of Remark 3.5, we say that ξ is any real number in Theorem 4.1 and Theorem 4.2.

Remark 4.7. Obviously, Srivastava's formula (4.9) is an equivalent with our formula (4.11).

5. Integral representations for the Bernoulli and Euler polynomials

In this section we will see that Theorem 5.1 and Theorem 5.2 below involve the results of Cvijović or H. Haruki and T. M. Rassias.

By (3.1) and (3.3) for $\xi = 0$, it follows that we give the uniform integral representations for the classical Bernoulli and Euler polynomials, respectively.

Theorem 5.1. *For* $n = 1, 2, \ldots,\ 0 \le \Re(x) \le 1$, *we have*

$$B_n(x) = -n \int_0^\infty \frac{\cos\left(2\pi x - \frac{n\pi}{2}\right) - e^{-2\pi t}\cos\left(\frac{n\pi}{2}\right)}{\cosh 2\pi t - \cos 2\pi x} t^{n-1}\,\mathrm{d}t. \tag{5.1}$$

Theorem 5.2. *For* $n = 1, 2, \ldots,\ 0 \le \Re(x) \le 1$, *we have*

$$E_n(x) = 2 \int_0^\infty \frac{e^{\pi t}\sin\left(\pi x - \frac{n\pi}{2}\right) + e^{-\pi t}\sin\left(\pi x + \frac{n\pi}{2}\right)}{\cosh 2\pi t - \cos 2\pi x} t^n\,\mathrm{d}t. \tag{5.2}$$

Remark 5.3. Theorem 5.1 and Theorem 5.2 above show the uniform integral representations for the classical Bernoulli and Euler polynomials which were *never* found in the classical literature, for example [1], [8] and [18], etc. So these uniform formulas are interesting in this subject.

By (3.4) and (3.6) for $\xi = 0$, we easily find the following additional integral representations for the classical Bernoulli and Euler polynomials, respectively.

Theorem 5.4. *For* $n = 1, 2, \ldots,\ 0 \le \Re(x) \le 1$, *we have*

$$B_n(x) = (-1)^n \frac{2n}{(2\pi)^n} \int_0^1 \frac{\cos\left(2\pi x - \frac{n\pi}{2}\right) - t\cos\left(\frac{n\pi}{2}\right)}{t^2 - 2t\cos 2\pi x + 1} (\log t)^{n-1}\,\mathrm{d}t. \tag{5.3}$$

Theorem 5.5. *For* $n = 1, 2, \ldots,\ 0 \le \Re(x) \le 1$, *we have*

$$E_n(x) = (-1)^n \frac{4}{\pi^{n+1}} \int_0^1 \frac{\sin\left(\pi x - \frac{n\pi}{2}\right) + t^2\sin\left(\pi x + \frac{n\pi}{2}\right)}{t^4 - 2t^2\cos 2\pi x + 1} (\log t)^n\,\mathrm{d}t. \tag{5.4}$$

We see easily that Theorem 5.4 and Theorem 5.5 imply the *main* results of Cvijović [7, p. 170, Theorem 1] or H. Haruki and T. M. Rassias [10, p. 82, Theorem (ii)(iv)], i.e.,

$$B_{2n}(x) = (-1)^n \frac{2(2n)}{(2\pi)^{2n}} \int_0^1 \frac{\cos 2\pi x - t}{t^2 - 2t\cos 2\pi x + 1} (\log t)^{2n-1}\,\mathrm{d}t, \tag{5.5}$$

$$B_{2n-1}(x) = (-1)^n \frac{2(2n-1)}{(2\pi)^{2n-1}} \int_0^1 \frac{\sin 2\pi x}{t^2 - 2t\cos 2\pi x + 1} (\log t)^{2n-2}\,\mathrm{d}t, \tag{5.6}$$

$$E_{2n}(x) = (-1)^n \frac{4}{\pi^{2n+1}} \int_0^1 \frac{(t^2+1)\sin \pi x}{t^4 - 2t^2\cos 2\pi x + 1} (\log t)^{2n}\,\mathrm{d}t, \tag{5.7}$$

$$E_{2n-1}(x) = (-1)^n \frac{4}{\pi^{2n}} \int_0^1 \frac{(t^2-1)\cos \pi x}{t^4 - 2t^2\cos 2\pi x + 1} (\log t)^{2n-1}\,\mathrm{d}t. \tag{5.8}$$

On the other hand, Theorem 5.1 and Theorem 5.2 also imply the classical integral representations for the Bernoulli polynomials and Euler polynomials, respectively (see [18, pp. 27, 31]):

$$B_{2n}(x) = (-1)^{n+1}(2n)\int_0^\infty \frac{\cos 2\pi x - e^{-2\pi t}}{\cosh 2\pi t - \cos 2\pi x} t^{2n-1}\,\mathrm{d}t, \tag{5.9}$$

$$B_{2n-1}(x) = (-1)^n(2n-1)\int_0^\infty \frac{\sin 2\pi x}{\cosh 2\pi t - \cos 2\pi x} t^{2n-2}\,\mathrm{d}t, \tag{5.10}$$

$$E_{2n}(x) = 4(-1)^n\int_0^\infty \frac{\sin \pi x \cosh \pi t}{\cosh 2\pi t - \cos 2\pi x} t^{2n}\,\mathrm{d}t, \tag{5.11}$$

$$E_{2n-1}(x) = 4(-1)^n\int_0^\infty \frac{\cos \pi x \sinh \pi t}{\cosh 2\pi t - \cos 2\pi x} t^{2n-1}\,\mathrm{d}t. \tag{5.12}$$

Remark 5.6. If we make an appropriate transformation $u = e^{-2\pi t}$ in (5.9) and (5.10), and make a suitable transformation $u = e^{-\pi t}$ in (5.11) and (5.12), respectively, then we can directly obtain (5.5)–(5.8); i.e., the *main* results of Cvijović or H. Haruki and T. M. Rassias are only a very simple *transmogrification* for the corresponding classical cases (5.9)–(5.12). Therefore, in view of this reason, we say that (5.5)–(5.8) are *not* new integral representations for the classical Bernoulli polynomials and Euler polynomials.

6. Further observations and consequences

By formula (2.3) of Theorem 2.1 for $x = 0$, we obtain the relationship between the Apostol-Bernoulli numbers and the Hurwitz zeta function as follows:

$$\mathcal{B}_n(\lambda) = \frac{(-1)^{n-1}n!}{(2\pi i)^n}\left[(-1)^n\zeta\left(n, 1-\frac{\log\lambda}{2\pi i}\right) + \zeta\left(n, \frac{\log\lambda}{2\pi i}\right)\right]. \tag{6.1}$$

Letting $\lambda = 1,\ n \mapsto 2n$ in (6.1), we at once produce the following famous Euler formula (see, e.g, [1, p. 807, 23.2.16]):

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}. \tag{6.2}$$

On the other hand, we define zeta functions as

$$\beta(n;\xi) = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+2\xi+1)^n}, \qquad \beta(n) = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^n} \qquad (\xi\in\mathbb{R}). \tag{6.3}$$

Setting $\lambda = e^{2\pi i\xi}$ in (2.9) of Theorem 2.2, we have

$$\begin{aligned}\mathcal{E}_n(x;e^{2\pi i\xi}) =& \frac{2\cdot n!}{\pi^{n+1}e^{2\pi i\xi x}}\left[\sum_{k=0}^\infty \frac{\exp\left[\left(\frac{n+1}{2}\pi - (2k+1)\pi x\right)i\right]}{(2k+2\xi+1)^{n+1}}\right.\\ &\left.+\sum_{k=0}^\infty \frac{\exp\left[\left(-\frac{n+1}{2}\pi + (2k+1)\pi x\right)i\right]}{(2k-2\xi+1)^{n+1}}\right].\end{aligned} \tag{6.4}$$

Taking $x = \frac{1}{2}$ in (6.4) and noting that $\mathcal{E}_n(\lambda) = 2^n\mathcal{E}_n\left(\frac{1}{2};\lambda\right)$ and (6.3), we readily obtain the following relationship between the Apostol-Euler numbers $\mathcal{E}_n(e^{2\pi i\xi})$ and the zeta function $\beta(n;\xi)$:

$$\mathcal{E}_n(e^{2\pi i\xi}) = \frac{2^{n+1}i^n\cdot n!}{\pi^{n+1}e^{\pi i\xi}}\ \left[\beta(n+1;\xi) + (-1)^n\beta(n+1;-\xi)\right] \tag{6.5}$$

or

$$\beta(2n+1;\xi)+\beta(2n+1;-\xi)=\frac{(-1)^n e^{\pi i\xi}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}\mathcal{E}_{2n}(e^{2\pi i\xi}). \tag{6.6}$$

Further putting $\xi = 0$ in (6.6), we arrive directly at the following well-known formula (see [1, p. 807, 23.2.22]):

$$\beta(2n+1)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}=\frac{(-1)^n}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}E_{2n}. \tag{6.7}$$

We can also obtain the formulas (6.2) and (6.7) by (4.11) and (4.15), respectively.

By Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4, respectively, we easily find the following integral representations for the Apostol-Bernoulli and Apostol-Euler numbers:

(6.8)

$$\begin{aligned}&\mathcal{B}_n(e^{2\pi i\xi})=-\Delta_n(0;\xi)\\ &-n\int_0^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)(1-e^{-2\pi t})\cosh(2\pi\xi t)+i\sin\left(\frac{n\pi}{2}\right)(1+e^{-2\pi t})\sinh(2\pi\xi t)}{\cosh 2\pi t-1}t^{n-1}\,\mathrm{d}t,\end{aligned}$$

(6.9)

$$\begin{aligned}\mathcal{B}_n(e^{2\pi i\xi})&=-\Delta_n(0;\xi)+\frac{2n}{(-2\pi)^n}\\ &\times\int_0^1\frac{\cos\left(\frac{n\pi}{2}\right)(1-t)\cosh(\xi\log t)-i\ \sin\left(\frac{n\pi}{2}\right)(1+t)\sinh(\xi\log t)}{t^2-2t+1}(\log t)^{n-1}\,\mathrm{d}t,\end{aligned}$$

(6.10)

$$\begin{aligned}\mathcal{E}_n(e^{2\pi i\xi})&=2^{n+2}e^{-\pi i\xi}\\ &\times\int_0^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)\cosh\pi t\cosh(2\pi\xi t)+i\sin\left(\frac{n\pi}{2}\right)\cosh\pi t\sinh(2\pi\xi t)}{\cosh 2\pi t+1}t^n\,\mathrm{d}t,\end{aligned}$$

(6.11)

$$\begin{aligned}\mathcal{E}_n(e^{2\pi i\xi})&=(-1)^n\frac{2^{n+2}e^{-\pi i\xi}}{\pi^{n+1}}\\ &\times\int_0^1\frac{\cos\left(\frac{n\pi}{2}\right)\cosh(2\xi\log t)-i\ \sin\left(\frac{n\pi}{2}\right)\sinh(2\xi\log t)}{t^2+1}(\log t)^n\,\mathrm{d}t.\end{aligned}$$

Further setting $\xi = 0$ in (6.8)–(6.11), respectively, we deduce the integral representations for the classical Bernoulli numbers and Euler numbers as follows (see, e.g., [18, pp. 28-32]):

$$\begin{aligned}B_n&=-n\cos\left(\frac{n\pi}{2}\right)\int_0^{\infty}t^{n-1}e^{-\pi t}\mathrm{csch}(\pi t)\,\mathrm{d}t\\ &=\cos\left(\frac{3n\pi}{2}\right)\frac{2n}{(2\pi)^n}\int_0^1\frac{(\log t)^{n-1}}{1-t}\,\mathrm{d}t,\\ E_n&=2^{n+1}\cos\left(\frac{n\pi}{2}\right)\int_0^{\infty}t^n\mathrm{sech}(\pi t)\,\mathrm{d}t\\ &=\cos\left(\frac{3n\pi}{2}\right)\frac{2^{n+2}}{\pi^{n+1}}\int_0^1\frac{(\log t)^n}{1+t^2}\,\mathrm{d}t.\end{aligned}$$

Recently, Garg *et al.* [9] gave an extension of Apostol's formula (4.7) as

$$\mathcal{B}_n\,(a;\lambda) = -n\Phi(\lambda, 1-n, a) \qquad (n \in \mathbb{N},\ \lambda \in \mathbb{C},\ |\lambda| \leq 1,\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-). \tag{6.12}$$

By (1.4) and the binomial theorem, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{E}_n(a;\lambda)\frac{z^n}{n!} &= \frac{2e^{az}}{\lambda e^z + 1} = 2\sum_{k=0}^{\infty}(-\lambda)^k e^{(k+a)z} \\ &= \sum_{n=0}^{\infty}\left[2\sum_{k=0}^{\infty}(-\lambda)^k (k+a)^n\right]\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[2\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(k+a)^{-n}}\right]\frac{z^n}{n!}.\end{aligned} \tag{6.13}$$

Hence, we also obtain the following interesting relationship between the Apostol-Euler polynomials and the Hurwitz-Lerch zeta function:

$$\mathcal{E}_n\,(a;\lambda) = 2\Phi(-\lambda, -n, a) \qquad (n \in \mathbb{N},\ \lambda \in \mathbb{C},\ |\lambda| \leq 1,\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-). \tag{6.14}$$

We can prove Theorem 2.1 and Theorem 2.2, respectively, by applying the relationships (6.12) and (6.14) in conjunction with Lerch's functional equation (4.8). The same as with the elementary series (4.12), we may also prove Theorem 4.1 and Theorem 4.2, respectively.

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