COLLOCATION METHODS FOR INDEX 1 DAES WITH A SINGULARITY OF THE FIRST KIND

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ABSTRACT. We study the convergence behavior of collocation schemes applied to approximate solutions of BVPs in linear index 1 DAEs which exhibit a critical point at the left boundary. Such a critical point of the DAE causes a singularity within the inherent ODE system. We focus our attention on the case when the inherent ODE system is singular with a *singularity of the first kind*, apply polynomial collocation to the original DAE system and consider different choices of the collocation points such as equidistant, Gaussian or Radau points. We show that for a well-posed boundary value problem for DAEs having a sufficiently smooth solution, the global error of the collocation scheme converges with the order $O(h^s)$, where s is the number of collocation points. Superconvergence cannot be expected in general due to the singularity, not even for the differential components of the solution. The theoretical results are illustrated by numerical experiments.

1. INTRODUCTION

In recent years, a lot of effort has been put into the analysis and numerical treatment of BVPs in ODEs which can exhibit singularities. Such problems are typically given in the following form:

(1a) $$t^\alpha z'(t) = M(t)z(t) + f(t, z(t)), \quad t \in (0, 1],$$

(1b) $$g(z(0), z(1)) = 0, \quad z \in C[0, 1],$$

where $\alpha \geq 1$, z is an n-dimensional real function, M is a smooth $n \times n$ matrix, and $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^p$ are smooth functions. For $\alpha = 1$ the problem is called singular with a singularity of the first kind; for $\alpha > 1$ it is essentially singular (singularity of the second kind). The search for efficient numerical methods to solve (1a) is strongly motivated by numerous applications from physics [9, 10, 23, 42], chemistry [13, 39, 41], mechanics [12], ecology [30, 37], or economics [14, 15, 24]. Also, research activities in related fields, such as the computation of connecting orbits in dynamical systems [38], or singular Sturm-Liouville problems [6], benefit from techniques developed for problems of the form (1a). In this paper we will extend the techniques developed in the context of ODEs to DAEs.

Our first objective was to provide a sound theoretical basis and the implementation of an open domain MATLAB code for the numerical solution of BVPs with a singularity of the first kind, $\alpha = 1$. To compute the numerical solution of (1a), we use collocation at points in the interior of a collocation interval. Our decision to use collocation was motivated by its advantageous convergence properties for (1a).

The convergence order is at least equal to the *stage order* of the method, while in the presence of a singularity, other high-order methods show order reductions and become inefficient (see for example [22]); cf. [5, 21, 25]. The above convergence results mean that a collocation scheme with s inner collocation points constitutes a high-order basic solver ($O(h^s)$ uniformly in t), robust with respect to the singularity of the first kind. Here, we denote by h the maximal stepsize in a (nonequidistant) grid.

In order to solve the ODE systems efficiently, the meshes have to be adapted to the solution behavior. For singular problems, we aim at meshes which are not affected by the steep direction field, staying coarse also close to the singularity when the solution is smooth in that region. To design a mesh adaptation procedure, we need an efficient asymptotically correct a posteriori estimate for the error of the numerical solution. Such a global error estimate was introduced in [4] and is based on the defect correction principle. We could show that for a collocation method of order $O(h^s)$, the error of the estimate (the difference between the exact global error and its estimate) is of order $O(h^{s+1})$; see [5, 25]. This asymptotically correct error estimate yields a reliable basis for an efficient mesh selection procedure. Our grid adaptation procedure results in grids which adequately reflect the solution behavior.

The final step was to implement the above algorithm and to provide an open domain MATLAB code `sbvp1.0` for nonlinear problems with an error estimation routine and a grid selection strategy; see [2, 3]. Due to its advantageous convergence properties, collocation is the most robust and dependable solver for singular ODEs; cf. [3, 21]. Consequently, we expected collocation to have similar properties in the context of singular DAEs and we will be able to confirm this hypothesis in this article.

In the past and also more recently, several authors successfully applied collocation to well-posed BVPs in index 1 DAEs with no singularities. In [11, 31], nonlinear systems of DAEs with constant kernel of $G_{x'}$ have been studied and superconvergence results for Gaussian and Lobatto points were given. The first attempts to provide respective software go back to 1994; see [1]. In the scope of the collocation code COLDAE are semi-explicit problems. Collocation methods applied to solve linear and nonlinear BVPs in index 1 DAEs (without singularities) have been recently analyzed in [32, 33], respectively. Here, the system is assumed to be given in a form of separated sets of equations involving derivatives and derivative free equations. Collocation at Gaussian points and Lobatto points is used to treat those separated equation subsets, respectively. According to our experience, in case of singularities, we need to restrict ourselves to Gaussian points, because the use of Radau or Lobatto points often results in an ill-posed discretized system, due to the unboundedness of the associated canonical projector; cf. [28].

Much progress has been made concerning DAE theory and applications, but there are still many questions left open. In particular, the numerical treatment of critical points and singularities is just emerging. With this paper we are giving the first insight into the behavior of polynomial collocation in the context of singular linear index 1 DAE systems.

Only a few years ago, the concept of DAEs *with properly stated leading term* was introduced and studied; cf. [7, 16, 34]. In contrast to the widely used DAE formulation $E(t)x'(t) + F(t)x(t) = g(t)$, this concept aims at a proper and natural

description of the involved solution derivatives. In particular, one considers linear DAEs written in the form

$$(2) \qquad A(t)(D(t)x(t))' + B(t)x(t) = g(t), \quad t \in [a, b],$$

with continuous coefficient matrices $A(t) \in \mathbb{R}^{m \times n}$, $D(t) \in \mathbb{R}^{n \times m}$, $B(t) \in \mathbb{R}^{m \times m}$, where $A(t)$ and $D(t)$ are assumed to be well-matched in the sense that $\ker A(t) \oplus \mathcal{R}(D(t)) = \mathbb{R}^n$. One of the advantages of this precise description of the problem structure is that there exists an *inherent explicit regular ODE* (IERODE) uniquely determined by the problem data; see [16, 17, 18]. Under mild assumptions, DAEs in standard form can be reformulated to have properly stated leading terms. For DAEs with properly stated leading terms arising in applications, see [16]. Linear BVPs for regular DAEs with tractability index 1 are investigated in detail in [8].

In [36], linear index 1 DAEs with properly stated leading term and type $1A$-critical points have been analyzed. This means that after decoupling the system using the matrix chain technique developed in [7] into the differential and algebraic components, the related inherent ODE exhibits a singularity of the first or second kind.

In [28], our experimental results showing the convergence of collocation schemes applied to solve (2) have been collected, and in [27], a convergence analysis for the above problem subject to initial conditions has been given. It turns out that for appropriately smooth problem data in (2) the stage order is retained. This means that the global error of the collocation scheme is at least $O(h^s)$ uniformly in t. In this article, we will formulate the respective convergence results for BVPs in index 1 DAEs with a singularity of the first kind.

2. Convergence of collocation methods

2.1. **Problem specification.** We investigate the convergence of collocation for index 1 DAEs, where the so-called *inherent ODE* may have a singularity of the first kind. In this section we analyze the error of collocation methods applied to a linear system of DAEs given in the following form:

$$(3) \qquad A(t)(D(t)x(t))' + B(t)x(t) = g(t), \quad t \in (0, 1],$$

where $A(t) \in \mathbb{R}^{m \times n}$, $D(t) \in \mathbb{R}^{n \times m}$, $B(t) \in \mathbb{R}^{m \times m}$, $g(t)$, $x(t) \in \mathbb{R}^m$, with $n \leq m$. All data in (3), that is, the matrix functions A, D, and B, and the function g, are assumed to be at least continuous on $[0, 1]$. Moreover, we require that

$$(4) \qquad\qquad \ker A(t) = \{0\}, \quad t \in (0, 1],$$
$$(5) \qquad\qquad \mathcal{R}(D(t)) = \mathbb{R}^n, \quad t \in [0, 1].$$

Condition (5) means that the matrix $D(t)$ has full constant rank on the closed interval. The structure (4) and (5) means that the system (3) has a properly stated leading term on $(0, 1]$; cf. [35]. In order to describe the boundary conditions which are necessary and sufficient for (3) to be well-posed, we decouple this system using techniques from [7]. To this end we define

$$(6) \qquad\qquad N_0(t) := \ker(A(t)D(t)), \quad t \in (0, 1].$$

Note that $N_0(t) = \ker D(t)$, $t \in [0, 1]$.

Let us denote by Q_0 a continuous pointwise projector function onto $\ker D$, i.e. $Q_0(t)^2 = Q_0(t)$, $\mathcal{R}(Q_0(t)) = \ker D(t)$, $t \in [0, 1]$, and let $P_0 := I - Q_0$. Next define

$$(7) \qquad\qquad G_0(t) := A(t)D(t), \quad t \in [0, 1],$$

$$(8) \qquad\qquad G_1(t) := G_0(t) + B(t)Q_0(t), \quad t \in [0, 1].$$

In the following, we discuss systems (3) which are regular with tractability index 1 on the interval $(0, 1]$, which means that the matrix $G_1(t)$ is nonsingular for $t \in (0, 1]$. We recall that the concept of a tractability index is related to properly defined projector functions and a special matrix function sequence characterizing the structure of the DAE; cf. [40].

Finally, we introduce the pointwise generalized inverse D^- of D uniquely defined by the following requirements:

$$(9) \qquad D^- D D^- = D^-, \ \ DD^- D = D, \ \ DD^- = I, \ \ D^- D = P_0,$$

which need to hold pointwise on $[0, 1]$. Note that D^- is also continuous on $[0, 1]$.

We want to incorporate the case where the inherent ODE associated with (3) exhibits a singularity of the first kind; see [20].

It was demonstrated in [7] that with the above assumptions the solutions of the DAE (3) can be decoupled on $(0, 1]$ into the *differential components* Dx and the *algebraic components* $Q_0 x$. While $u = Dx$ satisfies the explicit inherent ODE,

$$(10) \qquad u'(t) + D(t)G_1^{-1}(t)B(t)D(t)^- u(t) = D(t)G_1^{-1}(t)g(t), \quad t \in (0, 1],$$

the algebraic components are given by

$$(11) \qquad Q_0(t)x(t) = -Q_0(t)G_1^{-1}(t)B(t)D(t)^- u(t) + Q_0(t)G_1^{-1}(t)g(t)$$

and the solutions of (3) can be expressed as

$$x(t) = D(t)^- u(t) + Q_0(t)x(t), \quad t \in (0, 1].$$

We are interested in solutions being smooth on the whole interval $[0, 1]$ since we intend to apply a high-order collocation scheme for their approximation. However, $G_1(t)$ becomes singular for $t = 0$, in general. The asymptotic behavior of (10) related to a singularity of the first kind arises when we assume that $G_1(0)$ is singular but $tG_1^{-1}(t)$ has a continuous extension on $[0, 1]$. Consequently, we can rewrite the matrix $D(t)G_1^{-1}(t)B(t)D(t)^-$ and obtain

$$(12) \qquad\qquad D(t)G_1^{-1}(t)B(t)D(t)^- =: -\frac{1}{t}M(t),$$

where $M \in C[0, 1]$. For the subsequent existence and uniqueness analysis we require $M \in C^1[0, 1]$, which means that the problem data needs to be appropriately smooth. Let us denote the right-hand side of (10) by $f(t)$. Then we arrive at the inherent ODE of the form

$$(13) \qquad\qquad u'(t) = \frac{1}{t}M(t)u(t) + f(t), \quad t \in (0, 1].$$

As mentioned before, we are especially interested in smooth solutions x and therefore u needs to be at least in $C[0, 1]$. It turns out that the smoothness of u depends on the smoothness of f and the eigenstructure of $M(0)$. The boundary conditions associated with (13) have to be chosen such that a well-posed singular boundary value problem results. The theoretical background for this problem class, where $f \in C[0, 1]$, is discussed in detail in [20] and [26], for example. In order to use this standard theory, we assume that $G_1^{-1}(t)g(t)$ and thus $f(t)$ are in $C[0, 1]$.

In this paper we focus our attention on boundary value problems for singular ODE systems (13), which can equivalently be expressed as a well-posed initial value problem with initial conditions at $t^* = 0$ or terminal conditions at $t^* = 1$. This means a restriction on the spectrum of the matrix $M(0)$ from (12); see [26, 29] for a detailed explanation of this fact. The reason for the above assumption is that we intend to employ a shooting argument in the course of the analysis.

A singular initial value problem posed at $t^* = 0$ for the differential equation (13) is well-posed if and only if the spectrum of $M(0)$ contains no eigenvalues with positive real parts and the initial value satisfies $u(0) = \gamma \in \ker M(0)$. A singular terminal value problem posed at $t^* = 1$ is well-posed if and only if the spectrum of $M(0)$ contains no eigenvalues with negative real parts and the invariant subspace associated with the eigenvalue zero coincides with the nullspace of $M(0)$.

The inherent ODE (13) is augmented by the boundary conditions

$$(14) \qquad\qquad B_a u(0) + B_b u(1) = \beta,$$

with appropriately chosen $n \times n$ matrices B_a, B_b, and right-hand side β. For the original DAE (3), this yields the boundary conditions

$$(15) \qquad\qquad B_a D(0) x(0) + B_b D(1) x(1) = \beta.$$

For a well-posed boundary value problem (13)–(14), the continuous solution can be represented in the form

$$(16) \qquad\qquad u(t) = Ec + tF(t), \quad t \in [0, 1],$$

where the columns of E are a basis of $\ker M(0)$ and $F(t) \in C[0, 1]$; cf. [20]. Consequently, the solution of the full problem (3) and (15) is continuous on $[0, 1]$, cf. (11), if the limit

$$\Theta_c := \lim_{t \to 0^+} Q_0(t) G_1^{-1}(t) B(t) D(t)^- Ec$$

exists, and hence we can set

$$
(17) \qquad
\begin{aligned}
x_0 := x(0) &= D(0)^- Ec - Q_0(0)\Theta_c \\
&+ Q_0(0)\left(-\lim_{t \to 0^+} tG_1^{-1}(t) B(t) D(t)^- F(t) + \lim_{t \to 0^+} G_1^{-1}(t) g(t)\right).
\end{aligned}
$$

In order to ensure that for any continuous solution u of the inherent ODE (13) there exists a continuous solution x of the DAE (3), we additionally assume that

$$\Theta := \lim_{t \to 0^+} Q_0(t) G_1^{-1}(t) B(t) D(t)^- R$$

exists, where $R \in \mathbb{R}^{n \times n}$ is a projector onto $\ker M(0)$. Consequently, we have $\Theta_c = \Theta Ec$, and

$$
(18) \qquad
\begin{aligned}
x_0 &= D(0)^- Ec - Q_0(0)\Theta Ec \\
&+ Q_0(0)\left(-\lim_{t \to 0^+} tG_1^{-1}(t) B(t) D(t)^- F(t) + \lim_{t \to 0^+} G_1^{-1}(t) g(t)\right).
\end{aligned}
$$

It is important to note that in order to be able to set up a well-posed analytical problem, three (sufficient) conditions have been imposed: We have assumed that

$$(19) \qquad\qquad tG_1^{-1}(t), \quad G_1^{-1}(t) g(t), \quad Q_0(t) G_1^{-1}(t) B(t) D(t)^- E$$

have continuous extensions on $[0,1]$. The last assumption can also be expressed in terms of the so-called canonical projector,

$$(20) \qquad Q_{can}(t) := Q_0(t)G_1^{-1}(t)B(t).$$

In general, $Q_{can}(t)$ has to be continuous on $[0,1]$ to guarantee (19).

Remark. We exemplify the meaning of assumptions (19) by considering a special DAE system given by

$$(21a) \qquad tx_1'(t) + B_{11}(t)x_1(t) + B_{12}(t)x_2(t) = g_1(t),$$
$$(21b) \qquad B_{21}(t)x_1(t) + B_{22}(t)x_2(t) = g_2(t),$$

where $B_{11}(t), B_{12}(t), B_{21}(t), B_{22}(t) \in \mathbb{R}^{m\times m}$, $g_1(t), g_2(t) \in \mathbb{R}^m$, and $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^{2m}$. It follows immediately that we can write the above system in the form (3), where

$$A(t) = \begin{pmatrix} tI \\ 0 \end{pmatrix}, \quad D(t) = \begin{pmatrix} I & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}.$$

Let

$$Q_0(t) := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Then

$$G_1(t) = \begin{pmatrix} tI & B_{12}(t) \\ 0 & B_{22}(t) \end{pmatrix}, \quad G_1^{-1}(t) = \begin{pmatrix} \frac{1}{t}I & -\frac{1}{t}B_{12}(t)B_{22}^{-1}(t) \\ 0 & B_{22}^{-1}(t) \end{pmatrix},$$

and

$$Q_0(t)G_1^{-1}(t)B(t)D(t)^- = \begin{pmatrix} 0 \\ B_{22}^{-1}(t)B_{21}(t) \end{pmatrix}.$$

Let us assume that $B_{22}(t)$ is nonsingular on $[0,1]$. Then we can decouple (21a), and (3) is an index 1 DAE on $(0,1]$. Moreover, $tG_1^{-1}(t)$ and $Q_0(t)G_1^{-1}(t)B(t)D(t)^-R$ are continuous on $[0,1]$. The data of the corresponding inherent ODE (13) read:

$$M(t) = B_{11}(t) - B_{12}(t)B_{22}^{-1}(t)B_{21}(t), \quad f(t) = \frac{1}{t}\big(g_1(t) - B_{12}(t)B_{22}^{-1}(t)g_2(t)\big),$$

where $M(t), f(t) = G_1^{-1}(t)g(t) \in C[0,1]$. Clearly, the differential component is $u(t) = D(t)x(t) = x_1(t)$, and the algebraic component is

$$Q_0(t)x(t) = \begin{pmatrix} 0 \\ x_2(t) \end{pmatrix}, \quad x_2(t) = -B_{22}^{-1}(t)B_{21}(t)u(t) + B_{22}^{-1}(t)g_2(t).$$

If we only assume that $B_{22}(t)$ is nonsingular on $(0,1]$ and $B_{22}(0)$ is singular, then it is clear that we will require additional conditions to guarantee that the inherent ODE exhibits a singularity of the first kind, that it has a continuous solution u, and that this solution yields a smooth solution x. In particular, these requirements are satisfied if the matrices $tB_{22}^{-1}(t)$, $B_{22}^{-1}(t)B_{21}(t)$, and $B_{12}(t)B_{22}^{-1}(t)$ have continuous extensions on the closed interval $[0,1]$. $\qquad\square$

For a high-order method to work efficiently, it is necessary that the analytical solution x is appropriately smooth. This means that the original problem data need to be smooth enough and, additionally, all positive real parts of the eigenvalues of $M(0)$ need to be sufficiently large; cf. [20].

In the next section, we apply polynomial collocation to approximate solutions of (3) by means of an enlarged system,

$$(22) \qquad A(t)u'(t) + B(t)x(t) = g(t),$$

$$(23) \qquad D(t)x(t) - u(t) = 0, \quad t \in (0,1],$$

which can be brought into the standard form

$$(24) \qquad \widehat{A}(t)(\widehat{D}(t)\widehat{x}(t))' + \widehat{B}(t)\widehat{x}(t) = \widehat{g}(t), \quad t \in (0,1],$$

where $\widehat{x}(t) = (x(t), u(t))^T$, $\widehat{g}(t) = (g(t),0)^T$, and

$$\widehat{A}(t) = \begin{pmatrix} A(t) \\ 0 \end{pmatrix}, \quad \widehat{D}(t) = \begin{pmatrix} 0 & I \end{pmatrix} =: \widehat{D}, \quad \widehat{B}(t) = \begin{pmatrix} B(t) & 0 \\ D(t) & -I \end{pmatrix}.$$

Problem (24) is a regular DAE system with properly stated leading term and tractability index 1 on $(0,1]$. To see this, note that $\widehat{D}(t)$ is constant. Therefore, we define the related matrices $\widehat{G}_0(t)$, $\widehat{Q}_0$, and $\widehat{G}_1(t)$ as

$$\widehat{G}_0(t) := \widehat{A}(t)\widehat{D} = \begin{pmatrix} 0 & A(t) \\ 0 & 0 \end{pmatrix}, \quad \widehat{Q}_0 := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\widehat{G}_1(t) := \widehat{G}_0(t) + \widehat{B}(t)\widehat{Q}_0 = \begin{pmatrix} B(t) & A(t) \\ D(t) & 0 \end{pmatrix},$$

respectively. Moreover,

$$\ker \widehat{G}_1 = \{z \in \mathbb{R}^{m+n}, \; z = (z_1, z_2)^T; \; z_1 = Q_0 w, \; z_2 = Dw, \; w \in \ker G_1\}$$

for all $t \in [0,1]$, which means that $\widehat{G}_1$ is nonsingular on $(0,1]$ and $\widehat{G}_1(0)$ is singular, simultaneously with G_1. From the standard decoupling procedure applied to (24), the same inherent ODE (13) results, and for the solution $\widehat{x}$ on $(0,1]$ the following representation holds:

$$(25) \qquad \widehat{x}(t) = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} (I - Q_0(t)G_1^{-1}(t)B(t))D(t)^- u(t) + Q_0(t)G_1^{-1}(t)g(t) \\ u(t) \end{pmatrix}.$$

It is easily seen that the matrices $t\widehat{G}_1^{-1}(t)$ and $\widehat{Q}_0(t)\widehat{G}_1^{-1}(t)\widehat{B}(t)\widehat{D}(t)^-\widehat{R}$, where $\widehat{R} = R$, are continuous on $[0,1]$.

2.2. Collocation methods.
For the theoretical discussion of collocation methods, we define the meshes

$$\Delta := \{\tau_0, \tau_1, \ldots, \tau_N\},$$

and $h_i := \tau_{i+1} - \tau_i$, $i = 0, \ldots, N-1$, $\tau_0 = 0$, $\tau_N = 1$. For reasons of simplicity, we restrict the discussion to equidistant meshes, $h_i = h$, $i = 0, \ldots, N-1$. However, the results also hold for nonuniform meshes, which have a limited variation in the step sizes. For collocation, s distinct points $t_{i,j} := \tau_i + h_i\rho_j$, $j = 1, \ldots, s$, are inserted in each subinterval (τ_i, τ_{i+1}). Since we want to focus on Gaussian points, we restrict ourselves to interior collocation points, where $\rho_1 > 0$ and $\rho_s < 1$. A grid with equidistant interior collocation points is illustrated in Figure 1.

Now, let us denote by $\mathcal{B}_s$ the Banach space of continuous, piecewise polynomial functions $q \in \mathbb{P}_s$ of degree $\leq s$, $s \in \mathbb{N}$, equipped with the maximum norm $\|\cdot\|_\infty$. In the following, we denote by q the vector-valued functions from $\mathcal{B}_s$ independently of the number of their components. By $p \in \mathcal{B}_s$ we denote an approximation to the exact solution x of (3), (15), and by $q \in \mathcal{B}_s$ an approximation to the exact solution
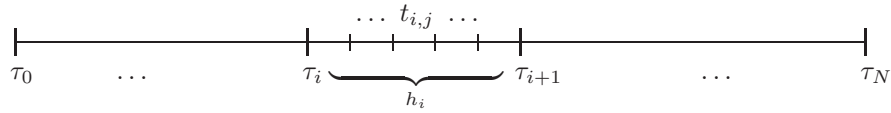
FIGURE 1. The computational grid

u of the inherent ODE (13); cf. (25). The numerical scheme defining p and q has the form

$$A(t_{i,j})q'(t_{i,j}) + B(t_{i,j})p(t_{i,j}) = g(t_{i,j}), \tag{26}$$

$$D(t_{i,j})p(t_{i,j}) - q(t_{i,j}) = 0, \tag{27}$$

$$B_a q(0) + B_b q(1) = \beta, \tag{28}$$

where $j = 1, \ldots, s$ and $i = 0, \ldots, N - 1$. It is clear by inspection of the number of unknowns and equations that further conditions will be necessary to close the system for the numerical treatment. Clearly, these additional conditions have to be consistent with the original DAEs. Various choices are possible, e.g.,

$$B(0)p(0) - g(0) \in \lim_{t \to 0^+} \mathcal{R}(A(t)), \quad q(0) = D(0)p(0), \tag{29}$$

or

$$B(1)p(1) - g(1) \in \mathcal{R}(A(1)), \quad q(1) = D(1)p(1). \tag{30}$$

We first treat the case of boundary value problems which can equivalently be posed as initial value problems at $t^* = 0$. As we will demonstrate at the end of this section, it is sufficient to investigate the convergence for the IVP case. Consequently, we discuss the solution of

$$A(t)(D(t)x(t))' + B(t)x(t) = g(t), \tag{31}$$

$$D(0)x(0) = \gamma, \tag{32}$$

which is at least continuous on $[0, 1]$ and satisfies, see (17),

$$Q_0(0)x(0) = Q_0(0)x_0. \tag{33}$$

Recall that in the initial condition (32) the vector γ can be freely chosen in $\ker M(0)$, and (33) is a consistent initial condition closing the system for the numerical treatment.

When applying collocation to (31)–(33), we are seeking piecewise polynomial functions p and q in $\mathcal{B}_s$, which for $j = 1, \ldots, s$ and $i = 0, \ldots, N - 1$ satisfy

$$A(t_{i,j})q'(t_{i,j}) + B(t_{i,j})p(t_{i,j}) = g(t_{i,j}), \tag{34}$$

$$D(t_{i,j})p(t_{i,j}) - q(t_{i,j}) = 0, \tag{35}$$

$$D(0)p(0) = \gamma, \tag{36}$$

$$Q_0(0)p(0) = Q_0(0)x_0, \quad P_0(0)p(0) = D(0)^- q(0). \tag{37}$$

The following arguments are very similar to those given in the analysis of collocation methods applied to ordinary differential equations with a singularity of the first kind. Thus, to keep the presentation concise we refer to [5, 21, 25] for many of the technical details. First, we study the existence and uniqueness of polynomial functions $p, q \in \mathcal{B}_s$ satisfying (34)–(37). As in the analytical case discussed

in Section 2.1, the system can be decoupled, yielding collocation equations for the inherent ODE, $j = 1, \ldots, s$, $i = 0, \ldots, N-1$,

$$(38) \qquad q'(t_{i,j}) = \frac{1}{t_{i,j}} M(t_{i,j}) q(t_{i,j}) + f(t_{i,j}),$$

$$(39) \qquad q(0) = \gamma \in \ker M(0),$$

and the value assignments, $j = 1, \ldots, s$, $i = 0, \ldots, N-1$,

$$(40) \qquad \begin{aligned} p(t_{i,j}) &= (I - Q_0(t_{i,j}) G_1^{-1}(t_{i,j}) B(t_{i,j})) D(t_{i,j})^- q(t_{i,j}) \\ &\quad + Q_0(t_{i,j}) G_1^{-1}(t_{i,j}) g(t_{i,j}), \end{aligned}$$

$$(41) \qquad p(0) = D(0)^- \gamma + Q_0(0) x_0.$$

The results in [5, 21, 25] show that the collocation equations (38), (39) have a unique solution which satisfies

$$(42) \qquad q(t) = Ec + t\varphi(t) = \gamma + t\varphi(t), \quad \varphi \in \mathcal{B}_s;$$

cf. (16). Conditions (40) and (41) uniquely define a piecewise polynomial $p \in \mathcal{B}_s$.

2.3. **Error analysis: Initial value problems at** $t^* = 0$. We now analyze the error of the approximation p provided by (34)–(37). Let us introduce an error function $\widehat{e} \in \mathcal{B}_s$ defined by

$$(43) \qquad \widehat{e}'(t_{i,j}) = \widehat{x}'(t_{i,j}) - \widehat{p}'(t_{i,j}), \quad j = 1, \ldots, s, \ i = 0, \ldots, N-1,$$

$$(44) \qquad \widehat{e}(0) = 0,$$

cf. [5, 21], where $\widehat{x}(t) = (x(t), u(t))^T$ is the exact solution, and $\widehat{p}(t) = (p(t), q(t))^T$ is its approximation by a collocation polynomial. Moreover, $\widehat{e}(t) := (e(t), e_u(t))^T$. Trivially,

$$(45) \qquad \widehat{e}'(t) = \sum_{\mu=1}^{s} L_\mu \left(\frac{t - \tau_i}{h} \right) \widehat{x}'(t_{i,\mu}) - \widehat{p}'(t), \quad t \in (\tau_i, \tau_{i+1}),$$

where L_μ denotes the μth Lagrange polynomial (of degree $s - 1$) associated with the abscissae $\rho_1, \ldots, \rho_s$ on the interval $[0, 1]$. In our analysis we wish to take into account that possibly the differential and the algebraic solution components have different smoothness. Thus, we assume that

$$u = Dx \in C^{\tilde{k}+1}[0, 1], \quad x \in C^{\tilde{\ell}+1}[0, 1],$$

and set

$$k := \min\{\tilde{k}, s\}, \quad \ell := \min\{\tilde{\ell}, s\}.$$

From standard results for interpolation, see [19], we can conclude that

$$(46) \qquad \widehat{e}'(t) = \widehat{x}'(t) - \widehat{p}'(t) + \begin{pmatrix} O(h^\ell) \\ O(h^k) \end{pmatrix}.$$

Due to (44), integration of (46) yields

$$(47) \qquad \widehat{e}(t) = \widehat{x}(t) - \widehat{p}(t) + t \begin{pmatrix} r(t) \\ s(t) \end{pmatrix},$$

with $r(t) = O(h^\ell)$ and $s(t) = O(h^k)$. Thus, $\widehat{e}$ satisfies the collocation scheme

(48) $$A(t_{i,j})e_u'(t_{i,j}) + B(t_{i,j})e(t_{i,j}) = t_{i,j}B(t_{i,j})r(t_{i,j}),$$

(49) $$D(t_{i,j})e(t_{i,j}) - e_u(t_{i,j}) = t_{i,j}(D(t_{i,j})r(t_{i,j}) - s(t_{i,j})),$$

(50) $$e(0) = 0, \quad e_u(0) = 0.$$

Again, we can reduce the problem to the collocation scheme applied to the inherent ODE related to (48)–(49), cf. (38), and obtain

(51) $$e_u'(t_{i,j}) = \frac{1}{t_{i,j}}M(t_{i,j})e_u(t_{i,j}) - M(t_{i,j})O(h^k),$$

(52) $$e_u(0) = 0.$$

To prove this statement we multiply (48) by $D(t_{i,j})G_1^{-1}(t_{i,j})$ and obtain

(53) $$e_u'(t_{i,j}) + D(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j})D(t_{i,j})^- D(t_{i,j})e(t_{i,j})$$
$$= t_{i,j}D(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j})r(t_{i,j}),$$

taking into account the relations $DG_1^{-1}A = I$, $DG_1^{-1}BQ_0 = 0$, and $DG_1^{-1}B = DG_1^{-1}BD^- D$. We now express $D(t_{i,j})e(t_{i,j})$ using (49), as

$$D(t_{i,j})e(t_{i,j}) = e_u(t_{i,j}) + t_{i,j}D(t_{i,j})r(t_{i,j}) - t_{i,j}\,s(t_{i,j}),$$

and insert this into (53). Rearranging yields

(54) $$e_u'(t_{i,j}) - \frac{1}{t_{i,j}}M(t_{i,j})e_u(t_{i,j})$$
$$= -M(t_{i,j})s(t_{i,j}) + \big(M(t_{i,j})D(t_{i,j}) + t_{i,j}D(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j})\big)r(t_{i,j}).$$

The second term on the right-hand side vanishes, since

$$MD + tDG_1^{-1}B = -tDG_1^{-1}BD^- D + tDG_1^{-1}B$$
$$= tDG_1^{-1}B(I - D^- D) = tDG_1^{-1}BQ_0 = 0,$$

and this completes the argument.

Note that the inhomogeneity in (51) remains uniformly bounded for $t \in [0, 1]$. Therefore, we can use [5, 21], see also (42), to conclude that

(55) $$e_u(t) = tO(h^k).$$

Consequently, the error function $e \in \mathcal{B}_s$ can be uniquely described by assigning its values,

(56) $$e(t_{i,j}) = (I - Q_0(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j}))D(t_{i,j})^- e_u(t_{i,j})$$
$$+ Q_0(t_{i,j})G_1^{-1}(t_{i,j})t_{i,j}B(t_{i,j})r(t_{i,j}),$$

(57) $$e(0) = 0.$$

Finally, we have $e(t_{i,j}) = O(h^{\min\{\ell,k\}})$ and since $e \in \mathcal{B}_s$, it immediately follows that $e(t) = O(h^{\min\{\ell,k\}})$ and

(58) $$x(t) - p(t) = O(h^{\min\{\ell,k\}})$$

hold.

2.4. Error analysis: Terminal value problems at $t^* = 1$. Consider the case where the boundary value problem can equivalently be posed as a terminal value problem at $t^* = 1$. The convergence proof given above requires a few modifications in that case. The collocation equations (34)–(35) are now augmented by the terminal conditions, cf. (11), (25),

$$\begin{aligned}
D(1)p(1) &= \beta, \\
Q_0(1)p(1) &= -Q_0(1)G_1^{-1}(1)B(1)D(1)^-\beta + Q_0(1)G_1^{-1}(1)g(1), \\
P_0(1)p(1) &= D(1)^-q(1).
\end{aligned}$$

Clearly, (44) is replaced by $\widehat{e}(1) = 0$, whence (47) becomes

$$\widehat{e}(t) = \widehat{x}(t) - \widehat{p}(t) + (1-t)\begin{pmatrix} r(t) \\ s(t) \end{pmatrix}. \tag{59}$$

Consequently, the error function satisfies the collocation scheme

$$A(t_{i,j})e_u'(t_{i,j}) + B(t_{i,j})e(t_{i,j}) = (1-t_{i,j})B(t_{i,j})r(t_{i,j}), \tag{60}$$

$$D(t_{i,j})e(t_{i,j}) - e_u(t_{i,j}) = (1-t_{i,j})D(t_{i,j})r(t_{i,j}) - (1-t_{i,j})s(t_{i,j}), \tag{61}$$

$$e(1) = 0, \quad e_u(1) = 0. \tag{62}$$

The collocation scheme for the associated inherent ODE reads

$$e_u'(t_{i,j}) = \frac{1}{t_{i,j}}M(t_{i,j})e_u(t_{i,j}) - \frac{1}{t_{i,j}}M(t_{i,j})(1-t_{i,j})O(h^k), \tag{63}$$

$$e_u(1) = 0. \tag{64}$$

This yields, see [25] for the proof,

$$e_u(t) = O(h^k). \tag{65}$$

Moreover,

$$\begin{aligned}
e(t_{i,j}) &= (I - Q_0(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j}))D(t_{i,j})^-e_u(t_{i,j}) \\
&\quad + (1-t_{i,j})Q_0(t_{i,j})G_1^{-1}(t_{i,j})B(t_{i,j})r(t_{i,j}),
\end{aligned} \tag{66}$$

$$e(1) = 0. \tag{67}$$

According to assumptions (19), this yields $e(t_{i,j}) = t_{i,j}^{-1}O(h^{\min\{\ell,k\}})$ and therefore $e(t) = O(h^{\min\{\ell,k\}-1})$; i.e., we observe an order reduction if compared with (58). The error bound $e(t_{i,j}) = O(h^{\min\{\ell,k\}})$ from (58), however, follows under the additional assumption that the canonical projector

$$Q_{can}(t) := Q_0(t)G_1^{-1}(t)B(t) \tag{68}$$

has a continuous extension on $[0,1]$.

2.5. Error analysis: Boundary value problems. To conclude the error analysis, we give the shooting argument which demonstrates that the analysis of initial value problems given above is sufficient for the analysis of boundary value problems for the DAE (3). This argument is the same as the one given in [5, 21] for boundary value problems of singular ODEs, and carries over to our present analysis. Recall that $tG_1^{-1}(t)$, $G_1^{-1}(t)g(t)$, and $Q_0(t)G_1^{-1}(t)B(t)D(t)^-E$ are assumed to have continuous extensions on $[0,1]$. The columns of E form a basis for $\ker M(0)$.

The solution of the inherent ODE (13) can be written as

$$u(t) = \sum_{\mu=1}^{r} a_\mu u_\mu(t) + \tilde{u}(t),$$

where $U = (u_\mu)_{\mu=1,\ldots,r}$ is the solution of the homogeneous matrix ODE with initial condition $U(0) = E$, and the particular solution $\tilde{u}$ satisfies the inhomogeneous equation, albeit with homogeneous initial condition. It is easy to see from Section 2.1 that this implies a representation of x as

$$(69) \qquad x(t) = \sum_{\mu=1}^{r} a_\mu x_\mu(t) + \tilde{x}(t),$$

where $X = (x_\mu)_{\mu=1,\ldots,r}$ with

$$(70) \qquad x_\mu(t) = (I - Q_0(t)G_1^{-1}(t)B(t))D(t)^- u_\mu(t)$$

solves the matrix DAE

$$(71) \qquad A(t)(D(t)X(t))' + B(t)X(t) = 0,$$

$$(72) \qquad D(0)X(0) = E,$$

and $\tilde{x} = (I - Q_0 G_1^{-1}B)D^- \tilde{u} + Q_0 G_1^{-1}g$ satisfies

$$(73) \qquad A(t)(D(t)\tilde{x}(t))' + B(t)\tilde{x}(t) = g(t),$$

$$(74) \qquad D(0)\tilde{x}(0) = 0.$$

Due to our assumptions, X and $\tilde{x}$ are continuous on $[0,1]$. An analogous representation holds for the collocation solution p,

$$(75) \qquad p(t) = \sum_{\mu=1}^{r} b_\mu p_\mu(t) + \tilde{p}(t),$$

where $P = (p_\mu)_{\mu=1,\ldots,r}$ and $\tilde{p}$ are the collocation solutions of the problems (71)–(72) and (73)–(74), respectively. Similarly as in the ODE case, see [5, 21], substitution of the solution representations (69) and (75) into (14) yields

$$a_\mu - b_\mu = O(h^k)$$

provided that the matrix $B_a E + B_b P(1)$ is nonsingular. Consequently, the convergence result

$$\|u - q\|_\infty = O(h^k), \quad \|x - p\|_\infty = O(h^{\min\{\ell,k\}})$$

holds also for the solution of the boundary value problem (3), (15).

The considerations for problems posed at $t^* = 1$ are analogous.

3. Numerical experiments

In this section, we illustrate the convergence behavior of the quantities

$$gex_{\text{tau}} := \|p - x\|_{\text{tau}}, \qquad gex_{\text{tcol}} := \|p - x\|_{\text{tcol}},$$
$$geu_{\text{tau}} := \|q - u\|_{\text{tau}}, \qquad geu_{\text{tcol}} := \|q - u\|_{\text{tcol}},$$

numerically. The first two expressions are the maximal values of the global errors of the solution x in the mesh points only (tau)[1] and in all grid points, including

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[1] Cf. the definition of Δ in Section 2.2.

τ_i and collocation points $t_{i,j}$, (tcol), respectively. The last two expressions are the respective global errors for the differential solution components u.

Before continuing with the results of our numerical experiments, we stress that in the above theory the important assumptions (19),

(76) $tG_1^{-1}(t) \in C[0,1], \quad G_1^{-1}(t)g(t) \in C[0,1], \quad Q_0(t)G_1^{-1}(t)B(t)D(t)^-E \in C[0,1]$

have been made. They are sufficient for the stage order s of the collocation scheme to hold. Let x and u denote the solutions of the DAE and the inherent ODE, respectively. Provided that u is sufficiently smooth, (76) yields, for equidistant interior collocation points,

(77a) $$geu_{\text{tau}} = O(h^{s+1}), \quad geu_{\text{tcol}} = O(h^{s+1}),$$

(77b) $$geu_{\text{tau}} = O(h^s), \qquad geu_{\text{tcol}} = O(h^s),$$

for s odd or even, respectively; see [28]. Moreover, note that for Gaussian collocation points the superconvergence behavior $O(h^{2s})$ in Δ does not hold in general, a well-known fact in the context of singular ODEs. Rather, the same orders as in (77a) are observed (uniform superconvergence).

If the inherent ODE is an initial value problem and if x is sufficiently smooth, theory predicts the same order of convergence for gex_{tau} and gex_{tcol} as for u; cf. Section 2.3. If the inherent ODE is a terminal value or boundary value problem, our analysis additionally requires

(78) $$Q_{can}(t) := Q_0(t)G_1^{-1}(t)B(t) \in C[0,1]$$

to ensure this optimal convergence behavior; cf. Section 2.4. In this case, theory predicts that

(79) $$gex_{\text{tau}} = O(h^s), \qquad gex_{\text{tcol}} = O(h^s).$$

If the assumptions (76) and (78) are violated, order reductions in the algebraic components might occur. In particular, order reductions can be due to the behavior of the canonical projector $Q_{can}(t)$ for $t \to 0^+$, in the case when Q_{can} becomes unbounded in this limit. In the following, we consider two problem classes and try to illustrate important aspects of the theory by appropriately constructed model problems; cf. Tables 1 and 2 for a summary. Therein, we highlight order reductions in the sense that the stage order is not observed for the numerical experiment, although the exact solutions are sufficiently smooth.

All experiments have been carried out in MATLAB 7 by our new version of sbvp [3, 4].

3.1. **Problem class 1 — setting and analytical properties.** The system of DAEs in this problem class has the form

$$\begin{pmatrix} t^k \\ 1 \end{pmatrix} (x_1(t) + \alpha(t)x_2(t))' + \begin{pmatrix} t^k\beta(t) & -t^k\,\alpha'(t) \\ 0 & \gamma(t) - \alpha'(t) \end{pmatrix} x(t) = \begin{pmatrix} t^k g_1(t) \\ g_2(t) \end{pmatrix} =: g(t),$$

where $k \geq 0$. Model problems can be constructed in such a way that the inherent differential equation is regular or singular, with singularity of the first or of the second kind; see [28] for comprehensive experimental results on the convergence of collocation schemes applied to solve such problems.

For the models considered here, $k = 0$ and the above system has the form

(80) $$\begin{pmatrix} 1 \\ 1 \end{pmatrix} (x_1(t) + \alpha(t)x_2(t))' + \begin{pmatrix} \beta(t) & -\alpha'(t) \\ 0 & \gamma(t) - \alpha'(t) \end{pmatrix} x(t) =: g(t).$$

TABLE 1. Overview of numerical experiments, where the inherent ODEs are classified as initial (IVP), terminal (TVP), or boundary value problem (BVP). We stress that the exact solutions for all but the final Problem 2.4 are smooth. Order reductions are indicated in the fifth column

Experiment	Assumptions (76)	Q_{can}, cf. (78)	ODE	Order	Reference
Prbl. 1.1	satisfied	$O(1/t)$	IVP	stage	[28, Prbl. 1.22]
Prbl. 1.2	not satisfied	$O(1/t^2)$	TVP	reduction	[28, Prbl. 1.21]
Prbl. 2.1	satisfied	continuous	BVP	stage	[28, Prbl. 4.2]
Prbl. 2.2	satisfied	continuous	BVP	stage	[28, Prbl. 4.3]
Prbl. 2.3	satisfied	$O(1/t)$	TVP	stage	[28, Prbl. 4.5]
Prbl. 2.4	satisfied	$O(1/t)$	TVP	reduction	[28, Prbl. 4.6]

TABLE 2. Experimentally observed convergence rates for different collocation schemes with $s = 3, 4$; cf. [28] for details. Order reductions are highlighted in italic.

| | | $s = 3$ | | | | $s = 4$ | | | |
| | | *gex* | | *geu* | | *gex* | | *geu* | |
Experiment	Collocation	tau	tcol	tau	tcol	tau	tcol	tau	tcol
Prbl. 1.1	equidistant	4	4	4	4	4	4	4	4
	Gaussian	4	4	4	4	4	4	5	5
Prbl. 1.2	equidistant	3	3	4	4	*3*	*3*	4	4
	Gaussian	3	3	5	4	*3*	*3*	5	5
Prbl. 2.1	equidistant	4	4	4	4	4	4	4	4
	Gaussian	4	4	4	4	4	4	5	5
	Radau	4	4	4	4	5	5	5	5
Prbl. 2.2	equidistant	4	4	4	4	4	4	4	4
	Gaussian	4	4	4	4	4	4	5	5
	Radau	4	4	4	4	5	5	5	5
Prbl. 2.3	equidistant	3	3	4	4	4	4	4	4
	Gaussian	3	3	4	4	4	4	5	5
Prbl. 2.4	equidistant	*0.3*	*0.3*	1.2	1.2	*0.3*	*0.3*	1.2	1.2
	Gaussian	*0.3*	*0.3*	1.3	1.3	*0.3*	*0.3*	1.2	1.2

In order to specify x we need to calculate the canonical projector (20),

$$(81) \quad Q_{can}(t) = Q_0(t)G_1^{-1}(t)B(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \alpha(t)\beta(t) & -\alpha(t)\gamma(t) \\ -\beta(t) & \gamma(t) \end{pmatrix}.$$

Moreover, with

$$Q_0(t)G_1^{-1}(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \alpha(t) & -\alpha(t) \\ -1 & 1 \end{pmatrix},$$

we obtain

$$P_{can}(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \gamma(t) & \alpha(t)\gamma(t) \\ \beta(t) & \alpha(t)\beta(t) \end{pmatrix},$$

and finally, the explicit formula for the solution x of the DAE (80) follows from

$$x(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \gamma(t) \\ \beta(t) \end{pmatrix} u(t) + \frac{1}{\alpha(t)\beta(t) + \gamma(t)} \begin{pmatrix} \alpha(t)g_1(t) - \alpha(t)g_2(t) \\ -g_1(t) + g_2(t) \end{pmatrix}.$$

It is important to note that the sign of the term

$$-\frac{\beta(t)(\gamma(t) - \alpha'(t))}{\varphi(t)},$$

cf. (10), with $\beta(t)(\gamma(t) - \alpha'(t))$ evaluated at $t = 0$, is crucial and decides the boundary conditions necessary for the solution u (and x) to be at least in $C[0,1]$; cf. [20] for the analysis of singular ODEs.

Problem 3.1. We choose $\alpha(t) = -1$, whence $\varphi(t) = \gamma(t) - \beta(t)$. With $\gamma(t) = t+2$ and $\beta(t) = 2$, we obtain $\varphi(t) = t$ and the equation for u reads:

$$u'(t) = -\frac{2(t+2)}{t} u(t) + \frac{1}{t}\Big((t+2)\, g_1(t) - 2\, g_2(t)\Big).$$

The requirement $\varphi(t) \neq 0$ for $t \in (0,1]$ is satisfied because $\gamma(t) - \beta(t) = t \neq 0$ for $t \in (0,1]$. We additionally choose $u(t) = te^{5t}$. The right-hand side g is specified below, and the solution of the problem has the form $x(t) = (-(6t+1)e^{5t}/2, -(8t+1)e^{5t}/2)^T$. Finally, we arrive at the system of DAEs

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}(x_1(t) - x_2(t))' + \begin{pmatrix} 2 & 0 \\ 0 & t+2 \end{pmatrix} x(t) \;\; = \;\; \begin{pmatrix} -te^{5t} \\ -\frac{8t+7}{2}\, te^{5t} \end{pmatrix},$$

subject to the boundary conditions $x_1(0) - x_2(0) = 0$, $2x_1(1) - 3x_2(1) = 6.5e^5$. Concerning the boundary conditions, we note that here

$$\frac{1}{t}M(t) = -\frac{2(t+2)}{t}, \quad M(0) = -4,$$

and therefore we need to prescribe the value $u(0) = x_1(0) - x_2(0) = 0$. The second boundary condition is equivalent to $\beta(t)x_1(t) - \gamma(t)x_2(t) = g_1(t) - g_2(t)$; cf. (30). For this example, condition (76) is satisfied. From $\alpha(t) = -1$, $\gamma(t) = t+2$, $\beta(t) = 2$, $\alpha(t)\beta(t) + \gamma(t) = t$, we obtain

$$G_1^{-1}(t) = \frac{1}{\alpha(t)\beta(t) + \gamma(t)}\begin{pmatrix} \alpha(t) + \gamma(t) - \alpha'(t) & -\alpha(t) + \alpha(t)\beta(t) + \alpha'(t) \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{t}\begin{pmatrix} t+1 & -1 \\ -1 & 1 \end{pmatrix}.$$

This means that

$$G_1^{-1}(t) = O\left(\frac{1}{t}\right) \;\; \Rightarrow \;\; tG_1^{-1}(t) = O(1).$$

Moreover

$$G_1^{-1}(t)g(t) = \frac{1}{t}\begin{pmatrix} t+1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -te^{5t} \\ -\frac{8t+7}{2}\, te^{5t} \end{pmatrix}$$

$$= \begin{pmatrix} t+1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -e^{5t} \\ -\frac{8t+7}{2}\, e^{5t} \end{pmatrix} \in C[0,1]$$

and $E = 0$. The associated canonical projector (20) is

$$Q_{can}(t) = \begin{pmatrix} -\frac{2}{t} & \frac{2+t}{t} \\ -\frac{2}{t} & \frac{2+t}{t} \end{pmatrix} = O\left(\frac{1}{t}\right).$$

Since the solution x is appropriately smooth, in Table 3 we observe the following convergence behavior for $s = 4$ equidistant collocation points,

$$gex_{\text{tau}} = gex_{\text{tcol}} = geu_{\text{tau}} = geu_{\text{tcol}} = O(h^s) = O(h^4),$$

TABLE 3. Convergence results for Problem 1.1: $s = 4$, equidistant collocation points.

Uniform Mesh		gex_{tau}			gex_{tcol}		
N	h	error	order	const.	error	order	const.
20	$5.00e - 02$	$2.321e - 03$	4.0	$3.548e + 02$	$2.321e - 03$	4.0	$3.548e + 02$
40	$2.50e - 02$	$1.459e - 04$	4.0	$3.627e + 02$	$1.459e - 04$	4.0	$3.627e + 02$
80	$1.25e - 02$	$9.155e - 06$	4.0	$3.652e + 02$	$9.155e - 06$	4.0	$3.652e + 02$
160	$6.25e - 03$	$5.744e - 07$	4.0	$3.659e + 02$	$5.744e - 07$	4.0	$3.659e + 02$
Uniform Mesh		geu_{tau}			geu_{tcol}		
N	h	error	order	const.	error	order	const.
20	$5.00e - 02$	$1.032e - 04$	4.0	$1.742e + 01$	$1.032e - 04$	4.0	$1.742e + 01$
40	$2.50e - 02$	$6.429e - 06$	4.0	$1.674e + 01$	$6.429e - 06$	4.0	$1.674e + 01$
80	$1.25e - 02$	$4.015e - 07$	4.0	$1.653e + 01$	$4.015e - 07$	4.0	$1.653e + 01$
160	$6.25e - 03$	$2.509e - 08$	4.0	$1.646e + 01$	$2.509e - 08$	4.0	$1.646e + 01$

TABLE 4. Convergence results for Problem 1.1: $s = 4$, Gaussian collocation points.

Uniform Mesh		gex_{tau}			gex_{tcol}		
N	h	error	order	const.	error	order	const.
20	$5.00e - 02$	$8.633e - 04$	4.0	$1.319e + 02$	$8.633e - 04$	4.0	$1.319e + 02$
40	$2.50e - 02$	$5.426e - 05$	4.0	$1.349e + 02$	$5.426e - 05$	4.0	$1.349e + 02$
80	$1.25e - 02$	$3.406e - 06$	4.0	$1.358e + 02$	$3.406e - 06$	4.0	$1.358e + 02$
160	$6.25e - 03$	$2.137e - 07$	4.0	$1.361e + 02$	$2.137e - 07$	4.0	$1.361e + 02$
Uniform Mesh		geu_{tau}			geu_{tcol}		
N	h	error	order	const.	error	order	const.
20	$5.00e - 02$	$2.479e - 09$	5.5	$3.804e - 02$	$6.356e - 06$	4.8	$1.243e + 01$
40	$2.50e - 02$	$6.377e - 11$	5.3	$1.839e - 02$	$2.100e - 07$	4.9	$1.597e + 01$
80	$1.25e - 02$	$1.801e - 12$	5.1	$1.120e - 02$	$6.749e - 09$	5.0	$1.856e + 01$
160	$6.25e - 03$	$5.684e - 14$	5.0	$5.533e - 03$	$2.139e - 10$	5.0	$2.020e + 01$

which is predicted by theory. For $s = 4$ Gaussian collocation points, we observe

$$gex_{\text{tau}} = gex_{\text{tcol}} = O(h^s) = O(h^4), \quad geu_{\text{tau}} = geu_{\text{tcol}} = O(h^{s+1}) = O(h^5);$$

cf. Table 4. Note that although the canonical projector $Q_{can}(t) = O(1/t)$ is unbounded for $t \to 0^+$, no order reduction is observed. This is due to the fact that the involved inherent ODE is an *initial value problem*, and therefore an additional factor t helps to balance the unboundedness of $Q_{can}(t) = O(1/t)$; see (55)–(56).

Problem 3.2. Functions $\alpha(t)$ and $\varphi(t)$ are as in Problem 1.1. With $\gamma(t) = \sin t$ and $\beta(t) = t$, we obtain $\varphi(t) = \sin t - t$ and the equation for u reads:

$$u'(t) = -\frac{t \sin t}{\sin t - t} u(t) + \frac{1}{\sin t - t}\Big(\sin t\, g_1(t) - t\, g_2(t) \Big).$$

The requirement $\varphi(t) \neq 0$ for $t \in (0, 1]$ is satisfied. We set $u(t) = t - \sin t$, $g_1(t) = g_2(t)$ are given below, and the solution of the system is $x(t) = (-\sin t, -t)^T$. The related system of DAEs is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} (x_1(t) - x_2(t))' + \begin{pmatrix} t & 0 \\ 0 & \sin t \end{pmatrix} x(t) = \begin{pmatrix} 1 - \cos t - t \sin t \\ 1 - \cos t - t \sin t \end{pmatrix},$$

boundary conditions are given by $x_1(1) - x_2(1) = -\sin(1) + 1$, $x_2(1) = -1$, and it is clear that this problem must be treated as a terminal value problem from right $(t = 1)$ to left $(t = 0)$. Since

$$\lim_{t \to 0^+} \Big(-\frac{t \sin t}{\sin t - t} \Big) = \lim_{t \to 0^+} \Big(\frac{6}{t} \Big) = \lim_{t \to 0^+} \Big(\frac{\lambda}{t} \Big), \ \lambda > 0,$$

TABLE 5. Convergence results for Problem 1.2: $s = 2$, equidistant collocation points. One observes order reductions for both components of $x = (x_1, x_2)$.

Uniform Mesh		$gex1_{\mathbf{tau}}$			$gex1_{\mathbf{tcol}}$		
N	h	error	order	const.	error	order	const.
20	$5.00e-02$	$3.165e-02$	1.0	$5.587e-01$	$3.165e-02$	1.0	$5.587e-01$
40	$2.50e-02$	$1.604e-02$	1.0	$5.975e-01$	$1.604e-02$	1.0	$5.975e-01$
80	$1.25e-02$	$8.072e-03$	1.0	$6.200e-01$	$8.072e-03$	1.0	$6.200e-01$
160	$6.25e-03$	$4.049e-03$	1.0	$6.330e-01$	$4.049e-03$	1.0	$6.330e-01$
Uniform Mesh		$gex2_{\mathbf{tau}}$			$gex2_{\mathbf{tcol}}$		
N	h	error	order	const.	error	order	const.
20	$5.00e-02$	$3.165e-02$	1.0	$5.575e-01$	$3.165e-02$	1.0	$5.575e-01$
40	$2.50e-02$	$1.604e-02$	1.0	$5.971e-01$	$1.604e-02$	1.0	$5.971e-01$
80	$1.25e-02$	$8.072e-03$	1.0	$6.199e-01$	$8.072e-03$	1.0	$6.199e-01$
160	$6.25e-03$	$4.049e-03$	1.0	$6.330e-01$	$4.049e-03$	1.0	$6.330e-01$
Uniform Mesh		$geu_{\mathbf{tau}}$			$geu_{\mathbf{tcol}}$		
N	h	error	order	const.	error	order	const.
20	$5.00e-02$	$6.260e-06$	2.5	$1.247e-02$	$6.691e-06$	2.4	$9.998e-03$
40	$2.50e-02$	$1.538e-06$	2.0	$2.697e-03$	$1.605e-06$	2.1	$3.197e-03$
80	$1.25e-02$	$3.829e-07$	2.0	$2.519e-03$	$3.920e-07$	2.0	$2.911e-03$
160	$6.25e-03$	$9.563e-08$	2.0	$2.466e-03$	$9.681e-08$	2.0	$2.712e-03$

we prescribe the value of u at $t = 1$, $u(1) = x_1(1) - x_2(1)$. The second boundary condition is again the consistency condition (30).

The conditions (76) and (78) are not satisfied. From $\alpha(t) = -1$, $\gamma(t) = \sin t$, $\beta(t) = t$, $\alpha(t)\beta(t) + \gamma(t) = \sin t - t$, it follows that

$$(82) \quad G_1^{-1}(t) = \frac{1}{\sin t - t} \begin{pmatrix} \sin t - 1 & 1 - t \\ -1 & 1 \end{pmatrix} = O\left(\frac{1}{t^3}\right) \quad \Rightarrow \quad tG_1^{-1}(t) = O\left(\frac{1}{t^2}\right).$$

However,

$$G_1^{-1}(t)g(t) = \frac{1}{\sin t - t} \begin{pmatrix} \sin t - 1 & 1 - t \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 - \cos t - t\sin t \\ 1 - \cos t - t\sin t \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \cos t - t\sin t \\ 0 \end{pmatrix}$$

and $E = 0$. The canonical projector (20) has the form

$$(83) \qquad\qquad Q_{can}(t) = \begin{pmatrix} -\frac{t}{\sin t - t} & \frac{\sin t}{\sin t - t} \\ -\frac{t}{\sin t - t} & \frac{\sin t}{\sin t - t} \end{pmatrix} = O\left(\frac{1}{t^2}\right).$$

Due to (82)–(83), we observe clear order reductions; see Tables 5 and 6. Here, we are dealing with a terminal value problem and there are no additional t factors to balance the behavior of $Q_{can}(t)$; see (65)–(66). Note that for singular boundary value problems in ODEs, the uniform superconvergence order $O(h^{s+1})$ for Gaussian collocation nodes holds in general at both the mesh points τ_i and the collocation points t_{ij}.

3.2. Problem class 2.

3.2.1. *Problems with bounded canonical projector, B_{22} nonsingular.* The problems considered in the previous section exhibited scalar inherent ODEs with $M(0) \in \mathbb{R}$. Therefore, the associated BVP could be posed either as an *initial* value problem, $M(0) < 0$, or as a *terminal* value problem, $M(0) > 0$; see Problem 1.1 and Problem

TABLE 6. Convergence results for Problem 1.2: $s = 2$, Gaussian collocation points. One observes order reductions for both components of $x = (x_1, x_2)$.

Uniform Mesh		$gex1_{\textbf{tau}}$			$gex1_{\textbf{tcol}}$		
N	h	error	order	const.	error	order	const.
20	$5.00e - 02$	$3.994e - 02$	1.0	$7.413e - 01$	$3.994e - 02$	1.0	$7.413e - 01$
40	$2.50e - 02$	$2.013e - 02$	1.0	$7.716e - 01$	$2.013e - 02$	1.0	$7.716e - 01$
80	$1.25e - 02$	$1.010e - 02$	1.0	$7.889e - 01$	$1.010e - 02$	1.0	$7.889e - 01$
160	$6.25e - 03$	$5.062e - 03$	1.0	$7.988e - 01$	$5.062e - 03$	1.0	$7.988e - 01$
Uniform Mesh		$gex2_{\textbf{tau}}$			$gex2_{\textbf{tcol}}$		
N	h	error	order	const.	error	order	const.
20	$5.00e - 02$	$3.994e - 02$	1.0	$7.405e - 01$	$3.994e - 02$	1.0	$7.405e - 01$
40	$2.50e - 02$	$2.013e - 02$	1.0	$7.713e - 01$	$2.013e - 02$	1.0	$7.713e - 01$
80	$1.25e - 02$	$1.010e - 02$	1.0	$7.888e - 01$	$1.010e - 02$	1.0	$7.888e - 01$
160	$6.25e - 03$	$5.062e - 03$	1.0	$7.987e - 01$	$5.062e - 03$	1.0	$7.987e - 01$
Uniform Mesh		$geu_{\textbf{tau}}$			$geu_{\textbf{tcol}}$		
N	h	error	order	const.	error	order	const.
20	$5.00e - 02$	$3.070e - 06$	3.0	$2.432e - 02$	$3.070e - 06$	3.0	$2.432e - 02$
40	$2.50e - 02$	$3.840e - 07$	3.0	$2.450e - 02$	$3.840e - 07$	3.0	$2.450e - 02$
80	$1.25e - 02$	$4.801e - 08$	3.0	$2.456e - 02$	$4.801e - 08$	3.0	$2.456e - 02$
160	$6.25e - 03$	$6.001e - 09$	3.0	$2.457e - 02$	$6.001e - 09$	3.0	$2.457e - 02$

1.2, respectively. In order to study the influence of other eigenstructures of the matrix $M(0)$, we now consider the following higher-dimensional semi-explicit problem posed on the interval $(0, 1]$:

$$(84) \qquad \begin{pmatrix} t I \\ 0 \end{pmatrix} \left((I \ 0) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right)' + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix},$$

or explicitly,

$$(85a) \qquad\qquad t x_1'(t) + B_{11} x_1(t) + B_{12} x_2(t) = g_1(t),$$
$$(85b) \qquad\qquad\qquad B_{21} x_1(t) + B_{22} x_2(t) = g_2(t),$$

where $B_{11}, B_{12}, B_{21}, B_{22} \in \mathbb{R}^{2 \times 2}$ are constant matrices, $g_1(t), g_2(t) \in C[0, 1]$ and the matrix B_{22} is nonsingular. Due to the problem structure we can immediately rewrite (85) to decouple the inherent ODE system and the system of algebraic constraints.

We first express $x_2(t)$ from (85b), $x_2(t) = B_{22}^{-1}(g_2(t) - B_{21} x_1(t))$, and rewrite (85a) as

$$x_1'(t) = -\frac{1}{t} \left(B_{11} x_1(t) - B_{12} B_{22}^{-1} B_{21} x_1(t) \right) + \frac{1}{t} \left(g_1(t) - B_{12} B_{22}^{-1} g_2(t) \right).$$

Consequently, the inherent ODE system is singular with a singularity of the first kind and has the form

$$(86) \qquad\qquad x_1'(t) = \frac{1}{t} M x_1(t) + f(t), \quad t \in (0, 1],$$

where $M = B_{12} B_{22}^{-1} B_{21} - B_{11}$ and $f(t) = \left(g_1(t) - B_{12} B_{22}^{-1} g_2(t) \right) / t$. Problems 2.1 and 2.2 given below are specified in such a way that M has a special eigenstructure: Problem 2.1 has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 5$, and for Problem 2.2, $\lambda_1 = \lambda_2 = 0$.

The initial conditions which guarantee that for $f \in C[0, 1]$ the solution $x_1(t)$ of (86) is at least in $C[0, 1]$ read $Q x_1(0) = 0$, where Q is a projection onto the subspace of $\mathbb{R}^2$ spanned either by the eigenvector associated with the negative eigenvalue of M or by the principal vector associated with the eigenvalue zero (Problem 2.2). For uniqueness we need to prescribe the value of $P x_1$, where $P =$

$I - Q$. Clearly, $P = S + R$, where S is a projection onto the subspace of $\mathbb{R}^2$ spanned by the eigenvector associated with the positive eigenvalue of M (Problem 2.1). Recall that R is a projection onto the subspace of $\mathbb{R}^2$ spanned by the eigenvector associated with the eigenvalue zero (Problems 2.1 and 2.2). According to theory, one needs to prescribe Sx_1 at $t = 1$, but Rx_1 can be specified at either $t = 0$ or $t = 1$. We now give the specification of Problems 2.1 and 2.2; for details see [28].

Problem 3.3. The matrices B_{ij} are

$$B_{11} = \begin{pmatrix} 9 & 12 \\ -8 & -11 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

For the system (86) we have $\lambda_1 = 0$, $\lambda_2 = 5$ and

$$M = \begin{pmatrix} -10 & -15 \\ 10 & 15 \end{pmatrix}, \quad ev_0 = \begin{pmatrix} -1.5 \\ 1 \end{pmatrix}, \quad ev_5 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad R = \begin{pmatrix} 3 & 3 \\ -2 & -2 \end{pmatrix}, \quad S = \begin{pmatrix} -2 & -3 \\ 2 & 3 \end{pmatrix}.$$

The functions g and f are specified in such a way that the solution of the Problem 2.1 reads:

$$x_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} e^{5t}\sin(t) - 1.5 + t^5 \\ \cos(5t) - t^5 \end{pmatrix}, \quad x_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} \cos(5t) - 1 \\ e^{5t}\sin(t) \end{pmatrix}.$$

Here, $Q = 0$ and the solution is continuous. We just need to specify boundary conditions which are necessary for uniqueness. We prescribe $(Sx_1(1))_2 = 2x_{11}(1) + 3x_{12}(1) = 2e^5\sin(1) + 3\cos(5) - 4$, and for the second condition we specify $(Rx_1(0))_1$, which means $x_{11}(0) + x_{12}(0) = -0.5$. For x_2, we prescribe the consistency condition $B_{22}x_2(t) + B_{21}x_1(t) = g_2(t)$ at $t = 0$,

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} x_2(0) + \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} x_1(0) = \begin{pmatrix} -0.5 \\ 0 \end{pmatrix}.$$

Problem 3.4. For this model we define the matrices B_{ij} as

$$B_{11} = \begin{pmatrix} -9 & -15 \\ 8 & 13 \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

In (86) we now have $\lambda_1 = \lambda_2 = 0$ and

$$M = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}, \quad ev_0 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \quad pv_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R = \begin{pmatrix} 3 & 3 \\ -2 & -2 \end{pmatrix}, \quad Q = \begin{pmatrix} -2 & -3 \\ -2 & -3 \end{pmatrix}.$$

Moreover,

$$x_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} e^{4t}\sin(t) - 3 \\ \cos(4t) + 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} \cos(4t) - 1 \\ e^{4t}\sin(t) \end{pmatrix}.$$

We require $(Qx_1(0))_1 = 0$, or equivalently, $2x_{11}(0) + 3x_{12}(0) = 0$ for the solution x_1 to be continuous on $[0, 1]$. As a second condition we choose to specify $(Rx_1(1))_1$, which means $x_{11}(1) + x_{12}(1) = e^4\sin(1) + \cos(4) - 2$. The consistency conditions for x_2 read $B_{22}x_2(0) + B_{21}x_1(0) = g_2(0)$, which means

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} x_2(0) + \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} x_1(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

We stress that, for Problems 2.1 and 2.2, condition (76) is satisfied due to the Remark on page 286. The numerical results are summarized in Table 2. In [28] we have also considered other eigenstructures of the matrix M, and it is worth mentioning that collocation at Gaussian points proved robust for all situations. The experiments for this problem class show convergence rates which do not differ from those given in (77a) and (79). Here, we have also used Radau points as

collocation points. (Lobatto points cannot be used because of the singularity.) As for Gaussian points, superconvergence does not always hold; cf. Table 2.

3.2.2. *Problems with unbounded canonical projector, B_{22} singular.* Here, we continue the study of the behavior of the canonical projector $Q_{can}(t)$. The following examples can be written as a modified system (84) and have the form

$$
(87) \qquad \begin{pmatrix} t^\alpha I \\ 0 \end{pmatrix} \left((I \ 0) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \right)' + \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}.
$$

Here again,

$$
D = (I \ 0), \quad D^- = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.
$$

Moreover, $B_{22}(t) := t^\beta B_{22}$, $\beta > 0$, $\alpha + \beta = 1$, and B_{22} is nonsingular. In this setting,

$$
G_1(t) = \begin{pmatrix} t^\alpha I & B_{12}(t) \\ 0 & B_{22}(t) \end{pmatrix}, \quad G_1^{-1}(t) = \begin{pmatrix} \frac{1}{t^\alpha} I & -\frac{1}{t^\alpha} B_{12}(t) B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{pmatrix},
$$

which means that $t G_1^{-1}$ has a continuous extension to $[0,1]$ in the case where all involved matrices are continuous on $[0,1]$. However, the canonical projector

$$
(88) \qquad Q_{can}(t) = Q_0 G_1^{-1}(t) B(t) = \begin{pmatrix} 0 & 0 \\ \frac{1}{t^\beta} B_{22}^{-1} B_{21}(t) & I \end{pmatrix}
$$

is unbounded on $[0,1]$ for $\beta > 0$. Moreover,

$$
DG_1^{-1}(t) B(t) D^- = \frac{1}{t^\alpha} B_{11}(t) - \frac{1}{t} B_{12}(t) B_{22}^{-1} B_{21}(t)
$$

$$
= \frac{1}{t} \big(t^\beta B_{11}(t) - B_{12}(t) B_{22}^{-1} B_{21}(t) \big),
$$

and hence

$$
M(t) = -t^\beta B_{11}(t) + B_{12}(t) B_{22}^{-1} B_{21}(t), \quad M(0) = B_{12}(0) B_{22}^{-1} B_{21}(0).
$$

For the experiments, we choose the matrices $B_{12}(t), B_{21}(t)$ to be constant and the matrices $B_{22}, B_{21}(t)$ to be nonsingular. Moreover, with the choice $B_{12} = B_{21}^{-1}$ and $B_{22}^{-1} = \mathrm{diag}(\lambda_1, \lambda_2)$, $\lambda_i \neq 0$, the eigenvalues of $M(0)$ are λ_1 and λ_2. Consequently, since $M(0)$ is nonsingular, $R = 0$, and the matrix $Q_0 G_1^{-1}(t) B(t) D^- R = 0$ has a continuous extension on $[0,1]$.

Problem 3.5. The matrices B_{ij} are

$$
B_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ B_{12} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, \ B_{21} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \ B_{22}(t) = t^\beta B_{22} = t^\beta \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}.
$$

Also,

$$
B_{22}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \ M(0) = B_{12} B_{22}^{-1} B_{21} = \begin{pmatrix} -7 & -12 \\ 8 & 13 \end{pmatrix}.
$$

System (87) is subject to the boundary conditions

$$
x_1(1) = \sin(1), \quad x_2(1) = e^1,
$$
$$
x_1(1) + x_2(1) + x_3(1) = \sin(1) + e^1 + \cos(1),
$$
$$
2x_1(1) + 3x_2(1) + \frac{1}{5} x_4(1) = 2\sin(1) + 3e^1 + \frac{1}{5} e^{-1}.
$$

Since the eigenvalues of $M(0)$ are both positive, we prescribe the values of the differential components $x_1(t)$ and $x_2(t)$ at $t = 1$. The remaining two conditions are consistent boundary conditions for the algebraic components. Note that we solve a terminal value problem which generically shows order reductions when $Q_{can}(t)$ becomes unbounded for $t \to 0^+$.

The right-hand side g which depends on parameters α, β, and $\ell \geq \alpha$ has been chosen such that the problem has the solution $x(t) = (t\sin(t), te^t, \cos(t), t^\ell e^{-t})^T$; cf. [28] for details. For the experiment, the parameters have been set as follows: $\alpha = 0$, $\beta = 1$, and $\ell = 3$.

The numerical results for this example can be found in Table 2. For the case where the differential solution components $u(t)$ are smooth, no order reduction is observed, although the projection matrix $Q_{can}(t) = O(1/t)$ is unbounded for $t \to 0^+$.

Problem 3.6. We use the same data as in Problem 2.3 except for the right-hand side g which here depends on parameters α, β, γ, δ and ℓ such that $\ell \geq \alpha \geq 0$, $\alpha + \beta = 1$, $\beta > 0$, $\delta \geq 1$, $\gamma \geq 0$, $\gamma + \alpha \geq 0$.

The boundary conditions for the resulting terminal value problem are given by

$$x_1(1) = \sin(1), \quad x_2(1) = e^1,$$
$$x_1(1) + x_2(1) + x_3(1) = \sin(1) + e^1 + \cos(1),$$
$$2x_1(1) + 3x_2(1) + \frac{1}{5}x_4(1) = 2\sin(1) + 3e^1 + \frac{1}{5}e^{-1}.$$

Again, since the eigenvalues of $M(0)$ are both positive, we prescribe the values of the differential components $x_1(t)$ and $x_2(t)$ at $t = 1$. The remaining two conditions are consistent boundary conditions for the algebraic components. Note that (76) is again satisfied. The solution now has the form $x(t) = (t^\gamma \sin(t), t^\delta e^t, \cos(t), t^\ell e^{-t})^T$, and the differential components x_1 and x_2 may become unsmooth. We set $\alpha = 0$ and $\beta = 1$, and the remaining parameters are specified as $\ell = \frac{5}{2}$, $\gamma = \frac{6}{5}$, $\delta = \frac{5}{2}$.

The experiments for this example are reported in Table 2. Now, we observe order reductions due to the fact that the canonical projector (88) is unbounded for $t \to 0^+$ ($\beta = 1$). One would expect to see the convergence order $O(h^{1.2})$ owing to the properties of x, especially the differential components. However, one loses approximately one additional power of h in x which can be attributed to the $O(1/t)$ behavior of $Q_{can}(t)$.

4. Summary

We investigated the convergence behavior of collocation schemes applied to solve BVPs in linear index 1 DAEs with a singularity of the first kind. We have considered a very general analytical problem setting, linear index 1 DAE systems with properly stated leading term, which were required to be well-posed and have sufficiently smooth solutions. For the discussion of the analytical problem and in the convergence analysis of the collocation scheme, we utilized a theoretical decoupling of the system to derive the explicit inherent ODE and the algebraic constraints. We could show that in the case of initial, terminal, and boundary value problems, the global error of the collocation scheme for s equidistant, Gaussian or Radau

collocation points is $O(h^s)$ uniformly in t. More precisely, for equidistant nodes,

$$gex_{\text{tau}} = O(h^s), \ \ gex_{\text{tcol}} = O(h^s), \ \ \ geu_{\text{tau}} = O(h^{s+1}), \ \ geu_{\text{tcol}} = O(h^{s+1}),$$

$$gex_{\text{tau}} = O(h^s), \ \ gex_{\text{tcol}} = O(h^s), \ \ \ geu_{\text{tau}} = O(h^s), \ \ geu_{\text{tcol}} = O(h^s)$$

hold for s odd and even, respectively. By means of experiments we could illustrate the fact that conditions on the problem data which are sufficient for the analytical problem to be well posed turn out to be necessary for the numerical scheme to have the stage order s. The superconvergence order $2s$ for Gaussian points does not hold in general, due to the singularity in the inherent ODE.

Similar convergence results hold also for systems of DAEs with no singularities in the inherent ODE. For initial, terminal, and boundary value problems with smooth solutions, the above convergence orders hold in the case that condition (78) is satisfied. Otherwise, order reductions in the algebraic components may occur.

Clearly, when the solution of the problem is not sufficiently smooth, corresponding order reductions are observed, in line with classical collocation theory.

For Radau collocation superconvergence, $gex_{\text{tau}} = O(h^{2s-1})$, $geu_{\text{tau}} = O(h^{2s-1})$ is observed, and the uniform convergence behavior is $gex_{\text{tcol}} = O(h^{s+1})$, $geu_{\text{tcol}} = O(h^{s+1})$, for all values of s, except for the case $s = 1$, corresponding to the backward Euler rule showing the expected linear convergence.

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