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CONVERGENCE OF APPROXIMATION SCHEMES FOR NONLOCAL FRONT PROPAGATION EQUATIONS

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ABSTRACT. We provide a convergence result for numerical schemes approximating nonlocal front propagation equations. Our schemes are based on a recently investigated notion of a weak solution for these equations. We also give examples of such schemes, for a dislocation dynamics equation, and for a FitzHugh-Nagumo type system.

1. INTRODUCTION

We are concerned with numerical approximation for nonlocal equations of the form

$$\text{(1.1)} \qquad \begin{cases} u_t(x,t) = H[\mathbf{1}_{\{u\geq 0\}}](x,t,Du,D^2u) & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

which, in the level-set approach for front propagation (see [19, 18, 12] for a complete overview of this method), describe the movement of a family $\{K(t)\}_{t\in[0,T]}$ of compact subsets of $\mathbb{R}^N$ such that

$$K(t) = \{x \in \mathbb{R}^N;\ u(x,t) \geq 0\}$$

for some function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$. Here u_t, Du and D^2u denote, respectively, the time derivative, space gradient and space Hessian matrix of u, while $\mathbf{1}_A$ denotes the indicator function of any set A.

The function H corresponds to the velocity of the front. In our setting, it depends not only on local properties of the front, such as its position, the time, the normal direction and its curvature matrix, but also, at time t, on the family $\{K(s)\}_{s\in[0,t]}$ itself. This nonlocal dependence is carried by the notation $H[\mathbf{1}_{\{u\geq 0\}}]$: for any indicator function χ or more generally for any $\chi \in L^\infty(\mathbb{R}^N \times [0,T])$ with values in $[0,1]$, the Hamiltonian $H[\chi]$ depends on χ in a nonlocal way; typically in our examples, it is obtained by a convolution procedure between χ and a physical kernel (either only in space or in space and time). In particular, $H[\chi]$ is continuous in space but has no particular regularity in time. However, the $H[\chi]$ equation is always well posed.

More precisely, we assume that for any $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$ with bounded support, $H[\chi](x,t,p,A)$ defines a measurable function of $(x,t,p,A) \in \mathbb{R}^N \times [0,T] \times$

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$\mathbb{R}^N \setminus \{0\} \times \mathcal{S}_N$, while for almost every $t \in [0,T]$, $H[\chi](x,t,p,A)$ defines a continuous function of (x,p,A). Here $\mathcal{S}_N$ denotes the set of real square symmetric matrices of size N.

Let us specify the class of equations that we consider: first of all, we are interested in front propagation equations, and therefore we assume that for any $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ with bounded support, the equation $u_t = H[\chi](x,t,Du,D^2u)$ is geometric, and that the upper and lower semicontinuous envelopes of the Hamiltonian $H[\chi]$ with respect to (x,p,A) satisfy, for any $x \in \mathbb{R}^N$ and almost all $t \in [0,T]$,

$$H[\chi]^*(x,t,0,0) = H[\chi]_*(x,t,0,0) = 0. \tag{1.2}$$

We also assume that this equation is degenerate parabolic, which means that for any $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}$, for almost every $t \in [0,T]$ and for all $A, B \in \mathcal{S}_N$, we have

$$H[\chi](x,t,p,A) \leq H[\chi](x,t,p,B) \quad \text{if } A \leq B,$$

where $\leq$ stands for the usual partial ordering for symmetric matrices.

The initial datum $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded and Lipschitz continuous function on $\mathbb{R}^N$ which represents the initial front, i.e. such that

$$\{u_0 \geq 0\} = K_0 \quad \text{and} \quad \{u_0 = 0\} = \partial K_0$$

for some fixed compact set $K_0 \subset \mathbb{R}^N$. Since in the level-set approach, the family $\{K(t)\}_{t\in[0,T]}$ only depends on the 0-level set of u_0 (see [12]), we assume for simplicity that there exists $R_0 > 0$ such that

$$u_0(x) = -1 \quad \text{if } |x| \geq R_0, \tag{1.3}$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^N$. For computational reasons, we ask the equation to preserve this property of compactness of the front. Essentially, this means that there exists a continuous function R on $[0,T]$ such that $R(0) = R_0$ and the solution of $u_t = H[\chi](x,t,Du,D^2u)$ with initial datum u_0 has the following property:

$$\chi(x,t) = 0 \text{ for a.e. } (x,t) \text{ s.t. } |x| \geq R(t) \Rightarrow u(x,t) = -1 \text{ for any } (x,t) \text{ s.t. } |x| \geq R(t).$$

Finally, for the same computational reasons, we point out that even though existence of solutions to (1.1) is known in a more general setting (see [5]), in this article we consider equations depending on the past, which means that $H[\chi](x,t,p,A)$ only depends on $\{\chi(\cdot,s)\}$ for $0 \leq s \leq t$.

The main issue linked with these nonlocal equations is the fact that they do not satisfy a comparison principle (or, geometrically, an inclusion principle on the fronts). Indeed, in general, the fact that $\{u_1 \geq 0\} \subset \{u_2 \geq 0\}$ does not imply that $H[\mathbf{1}_{\{u_1\geq 0\}}] \leq H[\mathbf{1}_{\{u_2 \geq 0\}}]$. A consequence of this absence of monotonicity is that one cannot build viscosity solutions to (1.1) by the classical methods, a comparison principle being crucial for both existence and uniqueness of a solution.

To overcome these difficulties, a notion of weak solution to (1.1) has therefore been introduced in [4, 5]. It uses the notion of L^1-viscosity solution, a notion of a solution adapted to Hamiltonians $H[\chi]$ which are merely measurable in time. We refer to [14, 16, 17, 9, 10] for a complete presentation of the theory of L^1-viscosity solutions. Let us only give their definition:

Definition 1.1. Assume that $H : (x,t,p,A) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}_N \mapsto H(x,t,p,A)$ is measurable and defines a continuous function of (x,p,A) for almost all $t \in [0,T]$.

We say that $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is an L^1-viscosity subsolution (resp. supersolution) of

$$\left\{ \begin{array}{ll} u_t(x,t) = H(x,t,Du,D^2u) & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{array} \right. \tag{1.4}$$

if u is upper semicontinuous (resp. lower semicontinuous), $u(\cdot,0) \le u_0$ (resp $\ge$), and if

(i) for any $\phi \in C^2(\mathbb{R}^N \times (0,T);\mathbb{R})$ and any $b \in L^1((0,T);\mathbb{R})$ such that the function $(x,t) \mapsto u(x,t) - \phi(x,t) - \int_0^t b(s)\,ds$ has a local maximum (resp. minimum) at some $(x_0,t_0) \in \mathbb{R}^N \times (0,T)$,

(ii) and for any continuous function $G : \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \times \mathcal{S}_N \to \mathbb{R}$ such that

$$H^*(x,t,p,A) - b(t) \le G(x,t,p,A) \quad (\text{resp. } H_*(x,t,p,A) - b(t) \ge G(x,t,p,A))$$

for all (x,p,A) in a neighborhood of $(x_0, D\phi(x_0,t_0), D^2\phi(x_0,t_0))$, and almost all t in a neighborhood of t_0,

we have

$$\phi_t(x_0,t_0) \le G(x_0,t_0,D\phi(x_0,t_0),D^2\phi(x_0,t_0)) \quad (\text{resp. } \ge).$$

We say that u is an L^1-viscosity solution of (1.4) if it is both a sub- and supersolution of this equation.

With this notion, we can now recall the definition of a weak solution to (1.1):

Definition 1.2. Let $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ be a continuous function. We say that u is a weak solution of (1.1) if there exists $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ such that:

(1) u is an L^1-viscosity solution of

$$\left\{ \begin{array}{ll} u_t(x,t) = H[\chi](x,t,Du,D^2u) & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N. \end{array} \right. \tag{1.5}$$

(2) For almost all $t \in [0,T]$,

$$\mathbf{1}_{\{u(\cdot,t)>0\}} \le \chi(\cdot,t) \le \mathbf{1}_{\{u(\cdot,t)\ge 0\}} \quad \text{a.e. in } \mathbb{R}^N. \tag{1.6}$$

Moreover, we say that u is a classical viscosity solution of (1.1) if, in addition, for almost all $t \in [0,T]$,

$$\mathbf{1}_{\{u(\cdot,t)>0\}} = \mathbf{1}_{\{u(\cdot,t)\ge 0\}} \quad \text{a.e. in } \mathbb{R}^N.$$

In [5], Barles, Cardaliaguet, Ley and the author proved a general result of existence of weak solutions for these nonlocal equations. In the framework described above, the essential assumptions under which existence is known are the following; they concern the local equation (1.5), where the nonlocal dependence is frozen, that is to say, $\mathbf{1}_{\{u\ge 0\}}$ is replaced by a fixed function $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$:

(A1) If $\chi_n \rightharpoonup \chi$ weak-$*$ in $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$, and if $Supp(\chi_n)$ is uniformly bounded, then for all $(x,t,p,A) \in \mathbb{R}^N \times [0,T] \times \mathbb{R}^N \setminus \{0\} \times \mathcal{S}_N$,

$$\int_0^t H[\chi_n](x,s,p,A)ds \underset{n\to+\infty}{\longrightarrow} \int_0^t H[\chi](x,s,p,A)ds$$

locally uniformly in x,t,p,A.

(A2) A comparison principle holds for (1.5): for any fixed $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ with bounded support, if u is a bounded L^1-viscosity subsolution of (1.5) and v is a bounded L^1-viscosity supersolution of (1.5), then $u \le v$ in $\mathbb{R}^N \times [0,T)$.

These assumptions are the classical ingredients to carry out a stability argument: assumption **(A1)** provides stability for L^1-viscosity solutions under very weak convergence of the Hamiltonians, thanks to a new stability result of Barles [3], while assumption **(A2)** enables us to identify the limit by a comparison principle. This is the idea of the proof of the existence result of [5]. We assume throughout the paper that these assumptions hold, and we refer to [9, 17] for conditions on $H[\chi]$ under which **(A2)** holds.

We also point out that assumption **(A2)** implies that for any fixed $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$ with bounded support, (1.5) has a unique continuous L^1-viscosity solution $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$. Combined with (1.2), which shows that constants are solutions of (1.5), it also implies the existence of uniform bounds on the solutions of (1.5), independent of χ.

Considering this existence result, our motivation is to provide numerical schemes, and a general convergence result, for these nonlocal and nonmonotone front propagation equations with L^1 dependence in time. This work is inspired by [8] where Barles and Souganidis proved a general convergence result for monotone, stable and consistent schemes in the local framework. We also refer to the works of Cardaliaguet and Pasquignon [11] and Slepčev [21] on the approximation of moving fronts in the nonlocal but monotone case.

This paper is organized as follows: in Section 2, we define a class of approximation schemes and prove the general convergence result. In Section 3, we give two explicit examples of such schemes, for a dislocation dynamics equation and FitzHugh-Nagumo type system (see (3.1) and (3.3)).

Notation. In what follows, $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^N$ or $\mathcal{S}_N$, $B(x,R)$ (resp. $\bar{B}(x,R)$) is the open (resp. closed) ball of radius R centered at $x \in \mathbb{R}^N$. We denote the essential supremum of $f \in L^\infty(\mathbb{R}^N)$ with values in $\mathbb{R}$, $\mathbb{R}^N$ or $\mathcal{S}_N$, $f \in L^\infty(\mathbb{R};\mathbb{R})$ or $f \in L^\infty(\mathbb{R}^N \times [0,T];\mathbb{R})$, by $\|f\|_\infty$.

2. Convergence of approximation schemes

Let $h = T/n$ for some $n \in \mathbb{N}^*$, and $\Delta_1, \dots, \Delta_N \in (0,1)$ be our respective time and space steps: a choice of h determines fixed Δ_i's by the relation $\Delta_i = \lambda_i\, h$ for $\lambda_i > 0$ fixed. We define for $(i_1, \dots, i_N) \in \mathbb{Z}^N$, $x_{i_1,\dots,i_N} = (i_1\Delta_1, \dots, i_N\Delta_N)$ and

$$Q_{i_1,\dots,i_N} = \prod_{k=1}^{N} [(i_k - 1/2)\Delta_k, (i_k + 1/2)\Delta_k).$$

Let us also define the space grid

$$\Pi_h = \bigcup_{(i_1,\dots,i_N)\in\mathbb{Z}^N} \{x_{i_1,\dots,i_N}\},$$

and for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, its projection on this grid,

$$x_h := ([x_1/\Delta_1 + 1/2]\Delta_1, \dots, [x_N/\Delta_N + 1/2]\Delta_N) \in \Pi_h,$$

where $[\cdot]$ denotes the integer part, so that if $x \in Q_{i_1,\dots,i_N}$, then $x_h = x_{i_1,\dots,i_N}$.

For $x \in \Pi_h$, $k \in \mathbb{N}$ such that $kh \le T$, $u : \Pi_h \to \mathbb{R}$ and $\chi : \Pi_h \times [0,T] \to [0,1]$ with bounded support, we define an approximate Hamiltonian $H_h[\chi](x, kh, u)$ which depends on

$$\{\chi(x_{i_1,\dots,i_N}, lh)\}_{(i_1,\dots,i_N)\in\mathbb{Z}^N,\, 0\le l\le k} \quad \text{and} \quad \{u(x_{i_1,\dots,i_N})\}_{(i_1,\dots,i_N)\in\mathbb{Z}^N}.$$

We keep in mind that $H_h[\chi](x,kh,u)$ possibly depends on the entire history $\{\chi(\cdot,lh)\}$ for l up to k.

We consider approximation schemes of the following form: for any $k\in\mathbb{N}$ such that $(k+1)h\le T$, and for any $x\in\Pi_h$, we set

$$\begin{cases} u_h(x,(k+1)h)=u_h(x,kh)+h\,H_h[\mathbf{1}_{\{u_h\ge 0\}}](x,kh,u_h(\cdot,kh)),\\ u_h(x,0)=u_0(x). \end{cases} \tag{2.1}$$

We finally extend u_h to a piecewise constant function on $\mathbb{R}^N\times[0,T]$ by setting for any (x,t),

$$u_h(x,t)=u_h(x_h,[t/h]h).$$

In particular we have for any $x\in\mathbb{R}^N$,

$$u_h(x,0)=u_0(x_h).$$

Let us now state our assumptions on H_h; in what follows, $C_b^2(\mathbb{R}^N;\mathbb{R})$ denotes the set of C^2 functions on $\mathbb{R}^N$ such that the norm

$$\|\phi\|=\|\phi\|_\infty+\|D\phi\|_\infty+\|D^2\phi\|_\infty=\sup_{x\in\mathbb{R}^N}|\phi(x)|+\sup_{x\in\mathbb{R}^N}|D\phi(x)|+\sup_{x\in\mathbb{R}^N}|D^2\phi(x)| \tag{2.2}$$

is finite. Let us first state an assumption on the behavior of H_h with respect to its last variable, which represents space derivatives. It is a trivial assumption which is linked to the fact that the equation $u_t=H[\chi](x,t,Du,D^2u)$ is geometric for any fixed χ; it will be satisfied for all reasonable schemes at no cost, so we state it separately:

(H0) *Consistency with respect to derivatives*: (i) For any $x\in\Pi_h$, k,h with $kh\le T$, $u:\Pi_h\to\mathbb{R}$, $\lambda\in\mathbb{R}$, and any function $\chi:\Pi_h\times[0,T]\to[0,1]$ with bounded support,

$$H_h[\chi](x,kh,u+\lambda)=H_h[\chi](x,kh,u)\text{ and }H_h[\chi](x,kh,0)=0.$$

(ii) There exists $r\in\mathbb{N}^*$ such that for any $x\in\Pi_h$, k,h with $kh\le T$, for any $\chi:\Pi_h\times[0,T]\to[0,1]$ with bounded support, and for all $u,v:\Pi_h\to\mathbb{R}$,

$$\text{if } u(y)=v(y)\ \forall\, y\in\Pi_h \text{ s.t. } \forall i,\ |x_i-y_i|\le r\Delta_i,$$
$$\text{then } H_h[\chi](x,kh,u)=H_h[\chi](x,kh,v).$$

We easily deduce from this and (1.3) that there exists $R=R_0+rT\sqrt{N}\max\lambda_i$ such that if u_h is defined by the scheme (2.1), then $u_h(x,t)=-1$ if $x\in\mathbb{R}^N\setminus B(0,R)$, for all $t\in[0,T]$; hence we only need to consider functions χ with uniformly bounded support. This shows in addition that the domain of space computation is uniformly bounded. In particular we set $B_h(\mathbb{R}^N\times[0,T];[0,1])$ to be the set of functions χ defined on $\mathbb{R}^N\times[0,T]$ with values in $[0,1]$ such that $Supp(\chi)\subset\bar{B}(0,R)\times[0,T]$ and χ is constant on each of the sets $Q_{i_1,\dots,i_N}\times[kh,(k+1)h)$.

Our assumptions are the following:

(H1) *H_h is conditionally monotone*: For any $x\in\Pi_h$, k,h with $kh\le T$, for any $\chi\in B_h(\mathbb{R}^N\times[0,T];[0,1])$, and for all $u,v:\Pi_h\to\mathbb{R}$,

$$u\le v\ \Rightarrow\ u(x)+h\,H_h[\chi](x,kh,u)\le v(x)+h\,H_h[\chi](x,kh,v).$$

(H2) *H_h is stable*: There exists $L>0$ such that for any $x\in\Pi_h$, k,h with $kh\le T$, and $\chi\in B_h(\mathbb{R}^N\times[0,T];[0,1])$, the solution u_h of (2.1) satisfies

$$|u_h(x,kh)|\le L.$$

(H3) *H_h is consistent with H*: For any $x \in \mathbb{R}^N$ and $\phi \in C_b^2(\mathbb{R}^N;\mathbb{R})$ such that $D\phi(x) \neq 0$, if $\chi_h \in B_h(\mathbb{R}^N \times [0,T];[0,1])$ is such that $\chi_h \rightharpoonup \chi$ weak-$*$ in $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ as $h \to 0$, then

$$h \sum_{l=0}^{[t/h]-1} H_h[\chi_h](x_h, lh, \phi) \underset{h\to 0}{\longrightarrow} \int_0^t H[\chi](x,s,D\phi(x),D^2\phi(x))\,ds$$

locally uniformly for $t \in [0,T]$ (the sum is set to 0 if $t < h$).

(H4) *Regularity*: For any compact subset K of $\mathbb{R}^N \times C_b^2(\mathbb{R}^N;\mathbb{R})$, there exist uniformly bounded moduli of continuity m_h such that for any $h > 0$, $(x,\phi),(y,\psi) \in K$ with $x, y \in \Pi_h$, for any k, h with $kh \leq T$, and any $\chi \in B_h(\mathbb{R}^N \times [0,T];[0,1])$,

$$\begin{aligned}|H_h[\chi](x,kh,\phi) - &H_h[\chi](y,kh,\psi)| \\ &\leq m_h(|x-y| + |D\phi(x) - D\psi(y)| + |D^2\phi(x) - D^2\psi(y)|),\end{aligned}$$

and such that $m_h(\eta) \to 0$ as $h, \eta \to 0$.

Assumptions **(H1)** to **(H3)** are the classical assumptions introduced by Barles and Souganidis in [8]. Moreover **(H3)** is the discrete equivalent of **(A1)** on the weak convergence of the Hamiltonians. As a matter of fact, the proof of our convergence theorem is based on the proof of the stability result of [3], the key assumption of which is **(A1)**. Finally assumption **(H4)** appears naturally alongside **(H3)**, just as in the continuous case (see [3]).

Remark 2.1. Under assumption **(H0)** (ii), if **(H1)** holds, then it also holds for all functions u and v such that $u(y) \leq v(y)$ for any $y \in \Pi_h$ with $|x_i - y_i| \leq r\Delta_i$ for all $i = 1, \ldots, N$, that is, also for functions that are comparable only locally. Indeed in this case, we can change u and v to 0 out of the set $\{y \in \Pi_h \mid |x_i - y_i| \leq r\Delta_i \ \forall i = 1, \ldots, N\}$. This provides new functions $\tilde{u}$ and $\tilde{v}$ such that $\tilde{u} \leq \tilde{v}$ in Π_h, whence, using **(H1)**,

$$\tilde{u}(x) + h\,H_h[\chi](x,kh,\tilde{u}) \leq \tilde{v}(x) + h\,H_h[\chi](x,kh,\tilde{v}).$$

But $\tilde{u}(x) = u(x)$, $H_h[\chi](x,kh,\tilde{u}) = H_h[\chi](x,kh,u)$ thanks to **(H0)** (ii), and the same holds for v. This proves our assertion.

In the same spirit, we notice that assumption **(H4)** also holds for two functions ϕ and ψ in $C^2(\mathbb{R}^N;\mathbb{R})$, because one can always modify ϕ and ψ to obtain new functions in $C_b^2(\mathbb{R}^N;\mathbb{R})$ without changing the values of $H_h[\chi](x,kh,\phi)$ or $H_h[\chi](y,kh,\psi)$.

Let us now state our main result:

Theorem 2.2. *Assume that assumption* (**A2**) *holds. Let u_0 be a bounded and Lipschitz continuous function which satisfies* (1.3). *Let $(u_h)_h$ be defined by the scheme* (2.1) *satisfying assumptions **(H0)** to **(H4)**.*

Then there exist $h_n \to 0$, $u \in C^0(\mathbb{R}^N \times [0,T];\mathbb{R})$ and $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ such that $u_{h_n} \to u$ locally uniformly in $\mathbb{R}^N \times [0,T]$, $\mathbf{1}_{\{u_{h_n} \geq 0\}} \rightharpoonup \chi$ weak-$*$ in $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ and (u,χ) satisfies (1.5).

Moreover, any such (u,χ) satisfies (1.6)*, so that u is a weak solution of* (1.1). *If in addition* (1.1) *has a unique weak solution u, then the whole sequence* (u_h) *converges locally uniformly to u in $\mathbb{R}^N \times [0,T]$.*

Proof. By compactness of $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ for the weak-$*$ topology, we can find $\chi \in L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ and (h_n) converging to 0 such that

$$\mathbf{1}_{\{u_{h_n} \geq 0\}} \rightharpoonup \chi \quad \text{weak-}* \text{ in } L^\infty(\mathbb{R}^N \times [0,T];[0,1]).$$

By the stability assumption **(H2)**, there exists $L > 0$ such that $\|u_h\|_\infty \le L$ for any h. We can therefore set

$$\begin{aligned}\overline{u}(x,t) &= \limsup{}^*(u_{h_n})(x,t)\\ &= \limsup\{u_{h'_n}(x_n, k_n h'_n) \mid (h'_n) \subset (h_n), x_n \to x \text{ with } x_n \in \Pi_{h_n},\\ &\qquad\qquad k_n h'_n \to t \text{ with } k_n \to +\infty\},\end{aligned}$$

which defines a bounded upper semi-continuous function on $\mathbb{R}^N \times [0,T]$. Let us prove that $\overline{u}$ is an L^1-viscosity subsolution of (1.5). We could prove in the same way that $\underline{u}(x,t) = \liminf\{u_{h'_n}(x_n, k_n h'_n) \mid (h'_n) \subset (h_n), x_n \to x, k_n h'_n \to t\}$ is a bounded L^1-viscosity supersolution of (1.5).

Step 1. We first prove that for any $x \in \mathbb{R}^N$, $\overline{u}(x,0) \le u_0(x)$. To do this we adapt the proof of the same statement in the proof of Theorem 3.1 of [5]. First of all, u_0 is Lipschitz continuous, so that for any fixed $0 < \varepsilon \le 1$, we have, for any $x, y \in \mathbb{R}^N$,

$$u_0(y) \le u_0(x) + \|Du_0\|_\infty |x-y| \le u_0(x) + \frac{|x-y|^2}{2\varepsilon^2} + \frac{\|Du_0\|_\infty \varepsilon^2}{2}\,.$$

We fix x and set $\phi(y) = |x-y|^2/(2\varepsilon^2)$. Using the above inequality, the function defined by

$$\psi_\varepsilon(y, kh_n) = u_0(x) + \phi(y) + \frac{\|Du_0\|_\infty \varepsilon^2}{2} + C_\varepsilon\, kh_n$$

satisfies $u_{h_n}(y,0) = u_0(y) \le \psi_\varepsilon(y,0)$ for all $y \in \Pi_h$. Moreover, using **(H0)** (i), we see that ψ_ε is a supersolution of (2.1) associated to $H_{h_n}[\mathbf{1}_{\{u_{h_n} \ge 0\}}]$ in the ball $B(x, \varepsilon + rT\sqrt{N}\max\lambda_i)$, provided that C_ε is large enough, namely as soon as

$$H_{h_n}[\mathbf{1}_{\{u_{h_n}\ge 0\}}](y, kh_n, \phi) \le C_\varepsilon \quad \text{for all } y \text{ with } |x-y| < \varepsilon + rT\sqrt{N}\max\lambda_i \text{ and } kh_n \le T.$$

This condition can be fulfilled using **(H4)** and the fact that $H_{h_n}[\mathbf{1}_{\{u_{h_n}\ge 0\}}](y, kh_n, 0) = 0$ (assumption **(H0)** (i)). Indeed, for some uniformly bounded moduli of continuity, we have

$$H_{h_n}[\mathbf{1}_{\{u_{h_n}\ge 0\}}](y, kh_n, \phi) \le m_{h_n}(|D\phi(y)| + |D^2\phi(y)|)$$

for any $n \in \mathbb{N}$, $y \in \Pi_{h_n}$ such that $|x-y| < \varepsilon + rT\sqrt{N}\max\lambda_i$ and $kh_n \le T$. The function ϕ does not belong to $C^2_b(\mathbb{R}^N;\mathbb{R})$, but using Remark 2.1, we recall that **(H4)** can also be applied to two functions in $C^2(\mathbb{R}^N;\mathbb{R})$. By the conditional monotonicity assumption **(H1)** (using again Remark 2.1), we obtain that for any $y \in \Pi_{h_n}$ satisfying $|y-x| < \varepsilon + r(T-h_n)\sqrt{N}\max\lambda_i$,

$$u_{h_n}(y, h_n) \le \psi_\varepsilon(y, h_n).$$

Reproducing the argument, we get that for any $y \in \Pi_{h_n}$ with $|y-x| < \varepsilon$ and k, h_n with $kh_n \le T$,

$$u_{h_n}(y, kh_n) \le \psi_\varepsilon(y, kh_n),$$

and in particular

$$\overline{u}(x,0) \le \limsup{}^* \psi_\varepsilon(x,0) = u_0(x) + \frac{\|Du_0\|_\infty \varepsilon^2}{2}.$$

Sending ε to 0 proves the claim.

Step 2. Now let $\phi \in C^2(\mathbb{R}^N \times (0,T);\mathbb{R})$ and $b \in L^1((0,T);\mathbb{R})$ be such that

$$(x,t) \mapsto \overline{u}(x,t) - \phi(x,t) - \int_0^t b(s)\,ds$$

has a local maximum at some $(x_0, t_0) \in \mathbb{R}^N \times (0, T)$. Let G be a continuous function such that for almost all t in a neighborhood of t_0, for all (x, p, A) in a neighborhood of $(x_0, D\phi(x_0, t_0), D^2\phi(x_0, t_0))$,

$$H[\chi]^*(x, t, p, A) - b(t) \leq G(x, t, p, A).$$

To check the L^1-viscosity subsolution property, we have to prove that

$$\phi_t(x_0, t_0) \leq G(x_0, t_0, D\phi(x_0, t_0), D^2\phi(x_0, t_0)).$$

We can assume without loss of generality that the maximum is strict and global and that $\sup_{t \in [0,T]} \|\phi(\cdot, t)\| < +\infty$. Let us set for simplicity $x_h = (x_0)_h$ and introduce the functions

$$f_h : t \mapsto h \sum_{l=0}^{[t/h]-1} H_h[\mathbf{1}_{\{u_h \geq 0\}}](x_h, lh, \phi(\cdot, t_0)) \\ - \int_0^t H[\chi]^*(x_0, s, D\phi(x_0, t_0), D^2\phi(x_0, t_0))\, ds.$$

Two cases arise: if $D\phi(x_0, t_0) \neq 0$, then for almost every $s \in [0, T]$,

$$H[\chi]^*(x_0, s, D\phi(x_0, t_0), D^2\phi(x_0, t_0)) = H[\chi](x_0, s, D\phi(x_0, t_0), D^2\phi(x_0, t_0)),$$

and by the consistency assumption **(H3)**, we know that $f_{h_n}(t) \to 0$ as $n \to +\infty$, locally uniformly for $t \in [0, T]$.

If $D\phi(x_0, t_0) = 0$, then a result by Barles and Georgelin [7] shows that we can also assume that $D^2\phi(x_0, t_0) = 0$. In this case, $H[\chi]^*(x_0, s, D\phi(x_0, t_0), D^2\phi(x_0, t_0)) = 0$ for almost every $s \in [0, T]$, thanks to (1.2). Assumption **(H4)** and the fact that $H_h[\mathbf{1}_{\{u_h \geq 0\}}](x_h, lh, 0) = 0$ (assumption **(H0)** (i)) imply that for some moduli of continuity m_h,

$$\left| h \sum_{l=0}^{[t/h]-1} H_h[\mathbf{1}_{\{u_h \geq 0\}}](x_h, lh, \phi(\cdot, t_0)) \right| \leq T\, m_h(|D\phi(x_h, t_0)| + |D^2\phi(x_h, t_0)|) \underset{h \to 0}{\longrightarrow} 0,$$

because $x_h \to x_0$, $D\phi(x_h, t_0) \to D\phi(x_0, t_0) = 0$ and $D^2\phi(x_h, t_0) \to D^2\phi(x_0, t_0) = 0$. In particular, $f_{h_n}(t) \to 0$ as $n \to +\infty$, locally uniformly for $t \in [0, T]$.

In both cases, the functions

$$v_{h_n} : (x, t) \mapsto u_{h_n}(x, t) - \phi(x, t) - \int_0^t b(s)\, ds - f_{h_n}(t)$$

satisfy

$$\limsup{}^*(v_{h_n})(x, t) = \overline{u}(x, t) - \phi(x, t) - \int_0^t b(s)\, ds.$$

By a standard stability argument, there exists a subsequence of (h_n), still denoted (h_n) for simplicity, and a sequence $(x_n, k_n h_n) \to (x_0, t_0)$ of global maximum points of v_{h_n} with $x_n \in \Pi_{h_n}$. We set

$$\xi_n = v_{h_n}(x_n, k_n h_n),$$

so that

$$u_{h_n}(x, t) \leq \phi(x, t) + \int_0^t b(s)\, ds + f_{h_n}(t) + \xi_n \tag{2.3}$$

for every $(x,t) \in \Pi_{h_n} \times \{0,\dots,[T/h_n]h_n\}$, with equality at $(x_n, k_n h_n)$. Now the definition of the scheme (2.1) shows that if $k_n \geq 1$, then

$$\begin{aligned} u_{h_n}(x_n, k_n h_n) = {} & u_{h_n}(x_n,(k_n-1)h_n) \\ & + h_n\, H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_n,(k_n-1)h_n, u_{h_n}(\cdot,(k_n-1)h_n)). \end{aligned}$$

Replacing u_{h_n} in this expression thanks to (2.3), and using the assumption **(H1)** of conditional monotonicity of the scheme, we therefore have

$$\begin{aligned} & \phi(x_n,k_nh_n) + \int_0^{k_nh_n} b(s)\,ds + f_{h_n}(k_nh_n) + \xi_n \\ & \leq \phi(x_n,(k_n-1)h_n) + \int_0^{(k_n-1)h_n} b(s)\,ds + f_{h_n}((k_n-1)h_n) + \xi_n \\ & + h_n\, H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_n,(k_n-1)h_n, \phi(\cdot,(k_n-1)h_n) + \int_0^{(k_n-1)h_n} b(s)ds \\ & \qquad\qquad + f_{h_n}((k_n-1)h_n) + \xi_n), \end{aligned}$$

which, using assumption **(H0)** (i), reduces to

$$\begin{aligned} & \phi(x_n,k_nh_n) + \int_0^{k_nh_n} b(s)\,ds + f_{h_n}(k_nh_n) \\ & \leq \phi(x_n,(k_n-1)h_n) + \int_0^{(k_n-1)h_n} b(s)\,ds + f_{h_n}((k_n-1)h_n) \\ & + h_n\, H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_n,(k_n-1)h_n, \phi(\cdot,(k_n-1)h_n)). \end{aligned}$$

Replacing f_{h_n} by its value, this transforms into

$$\begin{aligned} & \frac{\phi(x_n,k_nh_n) - \phi(x_n,(k_n-1)h_n)}{h_n} \\ & \leq \frac{1}{h_n}\int_{(k_n-1)h_n}^{k_nh_n} \left\{ H[\chi]^*(x_0,s,D\phi(x_0,t_0),D^2\phi(x_0,t_0)) - b(s) \right\} ds \\ & + H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_n,(k_n-1)h_n,\phi(\cdot,(k_n-1)h_n)) \\ & - H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_{h_n},(k_n-1)h_n,\phi(\cdot,t_0)). \end{aligned}$$

We now use the definition of G to deduce that

$$\begin{aligned} & \frac{\phi(x_n,k_nh_n) - \phi(x_n,(k_n-1)h_n)}{h_n} \\ & \qquad \leq \frac{1}{h_n}\int_{(k_n-1)h_n}^{k_nh_n} G(x_0,s,D\phi(x_0,t_0),D^2\phi(x_0,t_0))\,ds \\ & \qquad\quad + H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_n,(k_n-1)h_n,\phi(\cdot,(k_n-1)h_n)) \\ & \qquad\quad - H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_{h_n},(k_n-1)h_n,\phi(\cdot,t_0)). \end{aligned}$$

Since ϕ and G are sufficiently regular, we have

$$\begin{aligned} & \frac{\phi(x_n,k_nh_n) - \phi(x_n,(k_n-1)h_n)}{h_n} - \frac{1}{h_n}\int_{(k_n-1)h_n}^{k_nh_n} G(x_0,s,D\phi(x_0,t_0),D^2\phi(x_0,t_0))\,ds \\ & \underset{n\to+\infty}{\longrightarrow} \quad \phi_t(x_0,t_0) - G(x_0,t_0,D\phi(x_0,t_0),D^2\phi(x_0,t_0)). \end{aligned}$$

To conclude, it therefore suffices to prove that

$$H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_n,(k_n-1)h_n,\phi(\cdot,(k_n-1)h_n)) \\ - H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_{h_n},(k_n-1)h_n,\phi(\cdot,t_0))$$

has a nonpositive upper limit as $n \to +\infty$. But as n goes to $+\infty$, $x_n \to x_0$, $x_{h_n} \to x_0$, and $\phi(\cdot,(k_n-1)h_n) \to \phi(\cdot,t_0)$, so that thanks to assumption **(H4)**, we have for some moduli of continuity m_{h_n},

$$\begin{aligned}&|H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_n,(k_n-1)h_n,\phi(\cdot,(k_n-1)h_n)) \\ &\qquad - H_{h_n}[\mathbf{1}_{\{u_{h_n}\geq 0\}}](x_{h_n},(k_n-1)h_n,\phi(\cdot,t_0))| \\ &\leq m_{h_n}(|x_n-x_{h_n}| + |D\phi(x_n,(k_n-1)h_n) - D\phi(x_{h_n},t_0)| \\ &\qquad + |D^2\phi(x_n,(k_n-1)h_n) - D^2\phi(x_{h_n},t_0)|),\end{aligned}$$

which converges to 0 as $n \to +\infty$, and the result follows.

Step 3. We just proved that $\overline{u}$ is a bounded L^1-viscosity subsolution of (1.5), while $\underline{u}$ is a bounded L^1-viscosity supersolution of (1.5). The comparison principle **(A2)** for this equation then implies that $\overline{u} \leq \underline{u}$ in $\mathbb{R}^N \times [0,T)$, while the converse inequality is a direct consequence of their definition. This shows that in $\mathbb{R}^N \times [0,T)$, $\overline{u} = \underline{u}$ coincide with the unique continuous L^1-viscosity solution u of (1.5), and that (u_{h_n}) converges locally uniformly in $\mathbb{R}^N \times [0,T)$ to u. Since of course we can extend $H[\chi]$ by 0 after time T, and use the previous argument on the extended time interval, we deduce that the convergence of (u_{h_n}) to u is in fact locally uniform in $\mathbb{R}^N \times [0,T]$. This finally proves the convergence of $(u_{h_n}, \mathbf{1}_{\{u_{h_n}\geq 0\}})$ to a couple (u,χ) which satisfies (1.5).

Moreover, χ being taken as the weak-$*$ limit of $(\mathbf{1}_{\{u_{h_n}\geq 0\}})$, we can prove as in [5] that for almost all $t \in [0,T]$,

$$\mathbf{1}_{\{u(\cdot,t)>0\}} \leq \chi(\cdot,t) \leq \mathbf{1}_{\{u(\cdot,t)\geq 0\}},$$

which means that (u,χ) also satisfies (1.6). In particular u is a weak solution of (1.1).

In fact, this proof shows that any sequence (u_{h_n}) of solutions of the scheme (2.1) admits a subsequence which converges locally uniformly to a weak solution of (1.1). As a consequence, if this equation has a unique weak solution, then the whole sequence (u_h) converges locally uniformly to the weak solution u of (1.1). ☐

3. Applications

3.1. **Dislocation dynamics.** We are interested in particular in the dislocation dynamics equation (see [20, 2, 4] and the references therein), namely

$$\begin{cases} u_t = [c_0(\cdot,t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)]|Du| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.1}$$

where the nonlocal part of the velocity is defined by the space convolution

$$c_0(\cdot,t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) = \int_{\mathbb{R}^N} c_0(x-y,t)\, \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(y)\, dy.$$

We assume that c_0 and c_1 satisfy the following assumptions, under which **(A1)** and **(A2)** are satisfied (see [4, 5]):

(D)(i) $c_0 \in C^0([0,T]; L^1(\mathbb{R}^N))$, $c_1 \in C^0(\mathbb{R}^N \times [0,T]; \mathbb{R})$.

(*ii*) For any $t \in [0,T]$, $c_0(\cdot,t)$ is locally Lipschitz continuous and there exists a constant $C>0$ such that $\|Dc_0\|_{L^\infty([0,T];L^1(\mathbb{R}^N))} \le C$.

(*iii*) There exists a constant $C>0$ such that, for any $x,y \in \mathbb{R}^N$ and $t\in[0,T]$,

$$|c_1(x,t)| \le C \quad \text{and} \quad |c_1(x,t)-c_1(y,t)| \le C|x-y|.$$

Under these assumptions, there exists a weak solution of (3.1), as proved by Barles, Cardaliaguet, Ley and Monneau [4, Theorem 1.2] or Barles, Cardaliaguet, Ley and the author [5, Theorem 3.3]. We are going to study the convergence of the following approximation algorithm proposed by Alvarez, Carlini, Monneau and Rouy [1] for $N=2$, which is a particular case of (2.1). In [1], the authors prove short time existence of a classical viscosity solution to (3.1) and provide a convergence rate for their scheme. We do not obtain such a rate but prove convergence of this scheme to a weak solution of (3.1) for long times. We set, if $x = x_{i_1,\dots,i_N} \in \Pi_h$,

$$\begin{aligned} &H_h[\chi](x,kh,\phi)\\ &= \left\{ \sum_{j_1,\dots,j_N\in\mathbb{Z}} \overline{c_0}(i_1-j_1,\dots,i_N-j_N,k)\,\chi(j_1\Delta_1,\dots,j_N\Delta_N,kh) \right\} |D_h|(\phi)(x)\\ &\quad +\ c_1(x,kh)\,|D_h|(\phi)(x), \end{aligned}$$

where

$$\overline{c_0}(m_1,\dots,m_N,k) = \int_{Q_{m_1,\dots,m_N}} c_0(y,kh)\,dy,$$

and $|D_h|(\phi)(x)$ is a monotone approximation of $|D\phi(x)|$ adapted to the sign of the velocity, such as the one proposed by Osher and Sethian [19] and used in [1]: let $(e_1,\dots,e_N)$ denote the canonical basis of $\mathbb{R}^N$; then for $x\in\Pi_h$,

$$|D_h|(\phi)(x) = \left\{ \sum_{i=1}^N \max\left(\frac{\phi(x+e_i)-\phi(x)}{\Delta_i},0\right)^2 + \min\left(\frac{\phi(x)-\phi(x-e_i)}{\Delta_i},0\right)^2 \right\}^{1/2}$$

if the sum of the nonlocal term and $c_1(x,kh)$ is nonnegative, and

$$|D_h|(\phi)(x) = \left\{ \sum_{i=1}^N \min\left(\frac{\phi(x+e_i)-\phi(x)}{\Delta_i},0\right)^2 + \max\left(\frac{\phi(x)-\phi(x-e_i)}{\Delta_i},0\right)^2 \right\}^{1/2}$$

otherwise. In particular, H_h satisfies **(H0)** with $r=1$. Let $M>0$ be such that

$$\|c_0(\cdot,t)\|_{L^1(\mathbb{R}^N)} + |c_1(x,t)| \le M \quad \text{for any } (x,t)\in\mathbb{R}^N\times[0,T].$$

The CFL condition to ensure the conditional monotonicity **(H1)** of the scheme is

$$\sqrt{2N}\,M\,\frac{h}{\Delta_i} \le 1 \quad \text{for any } i=1,\dots,N. \tag{3.2}$$

The discrete convolution in the definition of H_h is efficiently computed using the Fast Fourier Transform; see [1]. We now state our convergence result:

Theorem 3.1. *Let c_0 and c_1 satisfy **(D)**, and let u_0 be a bounded and Lipschitz continuous function which satisfies* (1.3). *Let us fix space steps $\Delta_i = \lambda_i h$ for any $i=1,\dots,N$, for some constants $\lambda_i>0$ such that* (3.2) *holds.*

Then there exists $h_n\to 0$ such that (u_{h_n}) converges locally uniformly to a weak solution of (3.1) *in $\mathbb{R}^N\times[0,T]$.*

If in addition we have

(D′) *There exist* $\underline{c}, \overline{c} > 0$ *such that, for any* $x \in \mathbb{R}^N$ *and* $t \in [0,T]$,

$$|c_0(x,t)| \le \overline{c},$$
$$0 < \underline{c} \le -\|c_0(\cdot,t)\|_{L^1(\mathbb{R}^N)} + c_1(x,t) \le \|c_0(\cdot,t)\|_{L^1(\mathbb{R}^N)} + c_1(x,t) \le \overline{c},$$

then the whole sequence (u_h) *converges locally uniformly in* $\mathbb{R}^N \times [0,T]$ *to the unique weak solution of* (3.1).

Proof. We check the assumptions of Theorem 2.2, but will assume to avoid repetition that $c_1 = 0$; the treatment of the term c_1 is similar to, but easier than, the treatment of the convolution term involving c_0. To check assumptions (**H2**) to (**H4**), we first notice as in [1] that for $x \in \Pi_h$ and $\chi \in B_h(\mathbb{R}^N \times [0,T]; [0,1])$,

$$H_h[\chi](x, kh, \phi) = \{c_0(\cdot, kh) \star \chi(\cdot, kh)(x)\}\, |D_h|(\phi)(x).$$

Assumption **(H2)** is satisfied with $L = \|u_0\|_\infty$, by a simple comparison with the constant solutions $\pm\|u_0\|_\infty$. It only remains to prove assumptions **(H3)** and **(H4)**. Let us pick $x \in \mathbb{R}^N$, $\phi \in C_b^2(\mathbb{R}^N;\mathbb{R})$, $\chi_h \in B_h(\mathbb{R}^N \times [0,T];[0,1])$ such that $\chi_h \rightharpoonup \chi$ weak-$*$ in $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$, and let us prove that

$$h \sum_{l=0}^{[t/h]-1} \{c_0(\cdot, lh) \star\, \chi_h(\cdot, lh)(x_h)\}\, |D_h|(\phi)(x_h)\, ds \xrightarrow[h\to 0]{} \int_0^t \{c_0(\cdot,s) \star \chi(\cdot,s)(x)\}\, |D\phi(x)|\, ds$$

locally uniformly for $t \in [0,T]$. We decompose the difference of the two above terms as

$$\begin{aligned}
&\int_t^{[t/h]h} \{c_0(\cdot,[s/h]h) \star \chi_h(\cdot,s)(x_h)\}\, |D_h|(\phi)(x_h)\, ds \\
&+ \int_0^t \{c_0(\cdot,[s/h]h) \star \chi_h(\cdot,s)(x_h)\}\, (|D_h|(\phi)(x_h) - |D\phi(x)|)\, ds \\
&+ |D\phi(x)| \int_0^t \{c_0(\cdot,[s/h]h) \star \chi_h(\cdot,s)(x_h) - c_0(\cdot,s) \star \chi_h(\cdot,s)(x_h)\}\, ds \\
&+ |D\phi(x)| \int_0^t \{c_0(\cdot,s) \star \chi_h(\cdot,s)(x_h) - c_0(\cdot,s) \star \chi_h(\cdot,s)(x)\}\, ds \\
&+ |D\phi(x)| \int_0^t \{c_0(\cdot,s) \star \chi_h(\cdot,s)(x) - c_0(\cdot,s) \star \chi(\cdot,s)(x)\}\, ds.
\end{aligned}$$

By definition of $|D_h|$ and regularity of ϕ, the first term of this expression satisfies

$$\begin{aligned}
&\left| \int_t^{[t/h]h} \{c_0(\cdot,[s/h]h) \star \chi_h(\cdot,s)(x_h)\}\, |D_h|(\phi)(x_h)\, ds \right| \\
&\le |t - [t/h]h|\, M\sqrt{2N}\, \|D\phi\|_\infty \le M\sqrt{2N}\, \|D\phi\|_\infty\, h,
\end{aligned}$$

while the second is estimated by

$$\begin{aligned}
&\left| \int_0^t \{c_0(\cdot,[s/h]h) \star \chi_h(\cdot,s)(x_h)\}\, (|D_h|(\phi)(x_h) - |D\phi(x)|)\, ds \right| \\
&\le T\, M\, \big|\, |D_h|(\phi)(x_h) - |D\phi(x)|\, \big| \xrightarrow[h\to 0]{} 0.
\end{aligned}$$

The third term is, in absolute value, less than

$$|D\phi(x)| \int_0^t \|c_0(\cdot,[s/h]h) - c_0(\cdot,s)\|_{L^1(\mathbb{R}^N)}\, ds \le |D\phi(x)|\, T\, m(h),$$

where m is a modulus of continuity for $c_0 \in C^0([0,T];L^1(\mathbb{R}^N))$. We estimate the fourth term by

$$|D\phi(x)|\, T\, C\, |x_h - x| \le \frac{\sqrt{N}}{2}\, T\, C\, |D\phi(x)|\, (\max \lambda_i)\, h$$

using the facts that $\|Dc_0\|_{L^\infty([0,T];L^1(\mathbb{R}^N))} \le C$ and

$$|x_h - x|^2 \le \sum_{i=1}^N \left(\frac{\Delta_i}{2}\right)^2 = \frac{1}{4}\sum_{i=1}^N \lambda_i^2\, h^2 \le \frac{N}{4}\, (\max \lambda_i)^2\, h^2.$$

Finally, the last term is equal to

$$|D\phi(x)| \int_0^t \int_{\mathbb{R}^N} c_0(x-y,s)\, \{\chi_h(y,s) - \chi(y,s)\}\, dyds,$$

which converges to 0 as $h \to 0$ by definition of the weak-$*$ convergence of (χ_h) to χ. This convergence is *a priori* merely pointwise in time but we notice as in [4, Remark 5.2] that the bound

$$\left| \int_{\mathbb{R}^N} c_0(x-y,s)\, \chi_h(y,s)\, dy \right| \le M$$

valid for any $(x,s) \in \mathbb{R}^N \times [0,T]$ and $h > 0$ implies that the convergence is in fact uniform, by Ascoli's theorem.

To check **(H4)**, let K be a compact set of $\mathbb{R}^N$ and let R be a positive constant, and let us fix $x, y \in K \cap \Pi_h$, $k \in \mathbb{N}$ with $kh \le T$, $\phi, \psi \in C_b^2(\mathbb{R}^N;\mathbb{R})$ with $\|\phi - \psi\| \le R$ ($\|\cdot\|$ is defined by (2.2)) and $\chi \in B_h(\mathbb{R}^N \times [0,T];[0,1])$. We want to prove that

$$\begin{aligned}|H_h[\chi](x,kh,\phi) - &H_h[\chi](y,kh,\psi)| \\ &\le m_h(|x-y| + |D\phi(x) - D\psi(y)| + |D^2\phi(x) - D^2\psi(y)|),\end{aligned}$$

for some uniformly bounded moduli of continuity m_h. To do this we write

$$\begin{aligned}
&H_h[\chi](x,kh,\phi) - H_h[\chi](y,kh,\psi)\\
&= \{c_0(\cdot,kh) \star \chi(\cdot,kh)(x)\}\, |D_h|(\phi)(x) - \{c_0(\cdot,kh) \star \chi(\cdot,kh)(y)\}\, |D_h|(\psi)(y)\\
&= \{c_0(\cdot,kh) \star \chi(\cdot,kh)(x)\}\, |D_h|(\phi)(x) - \{c_0(\cdot,kh) \star \chi(\cdot,kh)(x)\}\, |D\phi(x)|\\
&+ \{c_0(\cdot,kh) \star \chi(\cdot,kh)(x)\}\, |D\phi(x)| - \{c_0(\cdot,kh) \star \chi(\cdot,kh)(x)\}\, |D\phi(y)|\\
&+ \{c_0(\cdot,kh) \star \chi(\cdot,kh)(x)\}\, |D\phi(y)| - \{c_0(\cdot,kh) \star \chi(\cdot,kh)(y)\}\, |D\phi(y)|\\
&+ \{c_0(\cdot,kh) \star \chi(\cdot,kh)(y)\}\, |D\phi(y)| - \{c_0(\cdot,kh) \star \chi(\cdot,kh)(y)\}\, |D\psi(y)|\\
&+ \{c_0(\cdot,kh) \star \chi(\cdot,kh)(y)\}\, |D\psi(y)| - \{c_0(\cdot,kh) \star \chi(\cdot,kh)(y)\}\, |D_h|(\psi)(y).
\end{aligned}$$

By definition of $|D_h|$, the first and the last terms of this equality are respectively estimated by

$$M \mid |D_h|(\phi)(x) - |D\phi(x)| \mid \leq M \frac{\sqrt{2N}}{2} \|D^2\phi\|_\infty (\max \lambda_i)\, h$$

$$\text{and} \quad M \mid |D_h|(\psi)(y) - |D\psi(y)| \mid \leq M \frac{\sqrt{2N}}{2} \|D^2\psi\|_\infty (\max \lambda_i)\, h \leq M \frac{\sqrt{2N}}{2} (\|D^2\phi\|_\infty + R) (\max \lambda_i)\, h.$$

The second term is easily dominated by

$$M\, N\, \|D^2\phi\|_\infty\, |x-y|$$

by regularity of ϕ, while the third term is, in absolute value, less than

$$C\, |x-y|\, \|D\phi\|_\infty,$$

because $\|Dc_0\|_{L^\infty([0,T];L^1(\mathbb{R}^N))} \leq C$. Finally, the fourth term is estimated by

$$\begin{aligned} M\,(|D\phi(y)| - |D\psi(y)|) &\leq M\,|D\phi(x) - D\psi(y)| + M\,|D\phi(x) - D\phi(y)| \\ &\leq M\,|D\phi(x) - D\psi(y)| + M\,N\,\|D^2\phi\|_\infty\,|x-y|. \end{aligned}$$

This proves **(H4)** and concludes the proof of the first part of Theorem 3.1.

For the convergence of the entire sequence, we use the result of [6] which states that under assumptions **(D)** and **(D′)**, then (3.1) has a unique weak solution. The convergence of the whole sequence (u_h) to this solution then follows from Theorem 2.2. □

3.2. **A FitzHugh-Nagumo type system.** We are also interested in the following system:

$$\begin{cases} u_t = \alpha(v)|Du| & \text{in } \mathbb{R}^N \times (0,T), \\ v_t - \Delta v = g^+(v)\mathbf{1}_{\{u\geq 0\}} + g^-(v)(1-\mathbf{1}_{\{u\geq 0\}}) & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.3}$$

which is obtained as the asymptotics as $\varepsilon \to 0$ of the following FitzHugh-Nagumo system arising in neural wave propagation or chemical kinetics:

$$\begin{cases} u_t^\varepsilon - \varepsilon\Delta u^\varepsilon = \varepsilon^{-1} f(u^\varepsilon, v^\varepsilon), \\ v_t^\varepsilon - \Delta v^\varepsilon = g(u^\varepsilon, v^\varepsilon) \end{cases} \tag{3.4}$$

in $\mathbb{R}^N \times (0,T)$, where for $(u,v) \in \mathbb{R}^2$,

$$\begin{cases} f(u,v) = u(1-u)(u-a) - v & (0 < a < 1), \\ g(u,v) = u - \gamma v & (\gamma > 0). \end{cases}$$

The functions α, g^+ and $g^- : \mathbb{R} \to \mathbb{R}$ appearing in (3.3) are associated with f and g. This system has been studied in particular by Giga, Goto and Ishii [13] and Soravia and Souganidis [22]. They proved existence of a weak solution to (3.3). Moreover in [22], the convergence of the solution of (3.4) to a solution of (3.3) as $\varepsilon \to 0$ is proved.

If for $\chi \in L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$, v denotes the solution of

$$\begin{cases} v_t - \Delta v = g^+(v)\chi + g^-(v)(1-\chi) & \text{in } \mathbb{R}^N \times (0,T), \\ v(\cdot,0) = v_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.5}$$

and if $c[\chi](x,t) := \alpha(v(x,t))$, then Problem (3.3) reduces to

$$\begin{cases} u_t(x,t) = c[\mathbf{1}_{\{u\geq 0\}}](x,t)|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.6}$$

which is a particular case of (1.1), where $c[\chi]$ depends on χ in a nonlocal way in both space and time. In [5], Barles, Cardaliaguet, Ley and the author were therefore able to recover the existence result of [13, 22], and in [6], they proved uniqueness in the case where $\alpha > \delta$ in $\mathbb{R}$ for some $\delta > 0$.

Let us state the assumptions satisfied by the data; they imply that **(A1)** and **(A2)** hold (see [5]):

(F) (i) α is Lipschitz continuous on $\mathbb{R}$,

(ii) g^+ and g^- are smooth on $\mathbb{R}^N$, and there exist $\underline{g}$ and $\overline{g}$ in $\mathbb{R}$ such that

$$\underline{g} \leq g^-(r) \leq g^+(r) \leq \overline{g} \quad \text{for all } r \text{ in } \mathbb{R}.$$

We set $\gamma = \max\{|\underline{g}|, |\overline{g}|\}$. Moreover we assume that

$$\|(g^+)^{(i)}\|_\infty < +\infty \quad \text{and} \quad \|(g^-)^{(i)}\|_\infty < +\infty \quad \text{for } i = 1,2,3.$$

(iii) v_0 is of class C^5 on $\mathbb{R}^N$ with $\|D^j v_0\|_\infty < +\infty$ for any $j = 0,\ldots,5$.

Here we want to propose a numerical scheme to compute a weak solution, or the weak solution if $\alpha > \delta$, of (3.3)-(3.6). To solve the heat equation part

$$v_t - \Delta v = g^+(v)\chi + g^-(v)(1-\chi),$$

we use an approximation scheme that we write in the following abstract form: we build functions $v_h : \mathbb{R}^N \times [0,T] \to \mathbb{R}$, such that v_h is piecewise constant; i.e., for any $(x,t) \in \mathbb{R}^N \times [0,T]$, $v_h(x,t) = v_h(x_h, [t/h]h)$, and such that for any $k \in \mathbb{N}$ with $(k+1)h \leq T$, for any $x \in \Pi_h$,

$$\begin{cases} v_h(x,(k+1)h) = S_h[\chi](x,kh,v_h), \\ v_h(x,0) = v_{0,h}(x), \end{cases} \tag{3.7}$$

where $S_h[\chi](x,kh,v)$ depends on $\{\chi(x_{i_1,\ldots,i_N}, lh)\}_{(i_1,\ldots,i_N)\in\mathbb{Z}^N}$ for $l \in \mathbb{N}$ up to $k+1$, and on $\{v_h(x_{i_1,\ldots,i_N}, lh)\}_{(i_1,\ldots,i_N)\in\mathbb{Z}^N}$ for $l \in \mathbb{N}$ up to k. Moreover $v_{0,h}$ is an approximation of the initial datum v_0.

The scheme solving the heat equation being fixed, we then use our scheme (2.1) in the following form: for any $k \in \mathbb{N}$ such that $(k+1)h \leq T$, and for any $x \in \Pi_h$, we set

$$\begin{cases} u_h(x,(k+1)h) = u_h(x,kh) + h\,\alpha(v_h(x,kh))|D_h|(u_h(\cdot,kh)), \\ v_h(x,(k+1)h) = S_h[\mathbf{1}_{\{u_h\geq 0\}}](x,kh,v_h), \end{cases} \tag{3.8}$$

with the initial condition

$$\begin{cases} u_h(x,0) = u_0(x), \\ v_h(x,0) = v_{0,h}(x). \end{cases}$$

We recall that $|D_h|(\phi)(x)$ is the monotone approximation of $|D\phi(x)|$ used in the previous section. We easily see that this scheme is of the form (2.1), where $H[\chi](x,kh,u)$ depends on χ through all the values $\chi(\cdot,lh)$ for $0 \leq l \leq k$. We now formulate assumptions on the functions S_h which will guarantee convergence of (3.8) according to Theorem 2.2:

(S) (i) There exists $M > 0$ such that for any fixed $\chi \in B_h(\mathbb{R}^N \times [0,T];[0,1])$, the solution v_h of (3.7) satisfies, for any $x \in \Pi_h$ and $k \in \mathbb{N}$ with $kh \leq T$,

$$|v_h(x,kh)| \leq M \quad \text{independently of } h.$$

(ii) If $\chi_h \in B_h(\mathbb{R}^N \times [0,T];[0,1])$ is such that $\chi_h \rightharpoonup \chi$ in $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ for the weak-$*$ topology as $h \to 0$, then the solution v_h of (3.7) associated to χ_h converges pointwise to the solution v of (3.5) in $\bar{B}(0,R) \times [0,T]$, where we set $R = R_0 + T\sqrt{N} \max\ l_i$ and R_0 is given by (1.3).

(iii) For any compact subset K of $\mathbb{R}^N$, there exist uniformly bounded moduli of continuity m_h such that for any $h > 0$, $x, y \in K \cap \Pi_h$, any $k, h > 0$ with $kh \leq T$ and $\chi \in B_h(\mathbb{R}^N \times [0,T];[0,1])$, the solution v_h of (3.7) satisfies

$$|v_h(x,kh) - v_h(y,kh)| \leq m_h(|x-y|),$$

and such that $m_h(\eta) \to 0$ as $h, \eta \to 0$.

Our convergence result is the following:

Theorem 3.2. *Assume that α, g^+, g^- and v_0 satisfy **(F)**, while u_0 is a bounded and Lipschitz continuous function which satisfies* (1.3). *Let u_h be defined by the scheme* (3.8) *such that **(S)** holds and the Δ_i's satisfy*

$$\sqrt{2N} \max\{|\alpha(r)|, |r| \leq M\} \frac{h}{\Delta_i} \leq 1 \quad \text{for any } i = 1, \dots, N, \tag{3.9}$$

where M is the constant given by assumption **(S)** (i). *Then there exists $h_n \to 0$ such that (u_{h_n}) converges locally uniformly in $\mathbb{R}^N \times [0,T]$ to a weak solution of* (3.6).

If in addition there exists $\delta > 0$ such that $\alpha(r) \geq \delta$ for any $r \in \mathbb{R}$, then the whole sequence (u_h) converges locally uniformly in $\mathbb{R}^N \times [0,T]$ to the weak solution of (3.6).

Proof. Assumption **(S)** (i) guarantees the existence of a constant M such that for any fixed $\chi \in B_h(\mathbb{R}^N \times [0,T];[0,1])$, the solution v_h of (3.7) satisfies, for any $x, y \in \Pi_h$ and $k \in \mathbb{N}$ with $kh \leq T$,

$$|v_h(x,kh)| \leq M \quad \text{independently of } h.$$

The CFL condition to ensure the conditional monotonicity of the first part of the scheme (3.8) is exactly (3.9), while the stability of this scheme follows as in the dislocation case. It only remains to check assumptions **(H3)** and **(H4)** of Theorem 2.2. This verification is very similar to the above proof in the dislocation case: it uses assumption **(S)** and the Lipschitz continuity of α. As a consequence, Theorem 2.2 guarantees the existence of a subsequence (u_{h_n}) converging locally uniformly in $\mathbb{R}^N \times [0,T]$ to a weak solution of (3.6).

If in addition there exists $\delta > 0$ such that $\alpha(r) \geq \delta$ for any $r \in \mathbb{R}$, then (3.6) has a unique weak solution (see [6]). The convergence of the whole sequence (u_h) to this solution follows once more from Theorem 2.2. $\square$

Let us now give an example of scheme (3.7) which satisfies **(S)**. Due to the lack of regularity of the function χ, we will solve an approximate equation in which the term χ is regularized by convolution: for $\varepsilon \in (0,1)$, let (ρ^ε) be a mollifier on $\mathbb{R}^N \times \mathbb{R}$ such that $Supp(\rho^\varepsilon) \subset [-\varepsilon, \varepsilon]^{N+1}$, $\rho^\varepsilon(-x,-t) = \rho^\varepsilon(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

$\|\rho^\varepsilon\|_1 = 1$ and

$$\left\| \frac{\partial^i}{\partial t^i} \circ D^j \rho^\varepsilon \right\|_1 \leq \frac{A}{\varepsilon^{(i+j)(N+1)}} \quad \text{for } (i,j) = (2,0) \text{ or } (i = 0, 1 \text{ and } i + j \leq 3), \tag{3.10}$$

for some constant $A > 0$. To ensure that our scheme is nonanticipative, we shift ρ^ε in time by ε and set

$$\chi^\varepsilon(x,t) = \int_0^T \int_{\mathbb{R}^N} \rho^\varepsilon(x-y, t-s-\varepsilon)\, \chi(y,s)\, dyds.$$

We are going to solve (3.5) by the standard forward Euler scheme, with the regularization χ^ε of χ. This regularization is essential to obtain estimates on the solution v_h^ε, and we can pass to the limit thanks to a good choice of balance between ε and h.

Let us fix the space steps Δ_i by the relation $\Delta_i = \lambda_i\, h$ for some fixed constants $\lambda_i > 0$ to be made precise later. Recall that these conditions are essential to guarantee compactness of the front $\{u_h(\cdot, t) \geq 0\}$ for any time $t \in [0,T]$, when u_h satisfies (2.1). However, for the forward Euler scheme to be stable and monotone, these conditions are not adapted.

For this reason, we need to solve (3.5) on a refined time grid: let h' be another time step such that $h/h' =: p \in \mathbb{N}^*$; the integer p may depend on h. We define the operator $T_{h'}^{kh'}[\chi]$ corresponding to the k-th step of the forward Euler scheme for (3.5) on this refined grid; that is, for any function $v : \Pi_h \to \mathbb{R}$, $\chi \in B_h(\mathbb{R}^N \times [0,T]; [0,1])$, for any $x \in \Pi_h$ and k, h' such that $(k+1)h' \leq T$,

$$\begin{aligned} T_{h'}^{kh'}[\chi](v)(x) = v(x) &+ h' \sum_{i=1}^N \frac{v(x+\Delta_i e_i) - 2v(x) + v(x - \Delta_i e_i)}{\Delta_i^2} \\ &+ h'\, g^+(v(x))\chi^\varepsilon(x, kh') + h\, g^-(v(x))(1 - \chi^\varepsilon(x, kh')), \end{aligned} \tag{3.11}$$

where $(e_1, \ldots, e_N)$ is the canonical basis of $\mathbb{R}^N$.

Then we set for any $v : \Pi_h \to \mathbb{R}$, $\chi \in B_h(\mathbb{R}^N \times [0,T]; [0,1])$, for any $x \in \Pi_h$ and k, h such that $(k+1)h \leq T$,

$$S_h[\chi](x, kh, v) = T_{h'}^{kh+(p-1)h'}[\chi] \circ \cdots \circ T_{h'}^{kh+h'}[\chi] \circ T_{h'}^{kh}[\chi](v)(x), \tag{3.12}$$

and we denote by v_h^ε the solution of (3.7) with initial condition

$$v_{0,h}(x) = v_0^\varepsilon(x) \tag{3.13}$$

for some regularization v_0^ε of v_0 of class C^∞ with $\|D^j v_0^\varepsilon\|_\infty \leq \|D^j v_0\|_\infty$ for any $j = 0, \ldots, 5$, and such that $v_0^\varepsilon \to v_0$ uniformly as $\varepsilon \to 0$.

This means that, to define $v_h^\varepsilon(x, (k+1)h)$ knowing $v_h^\varepsilon(x, kh)$, we split the time interval $[kh, (k+1)h]$ in $p = p(h)$ intervals of length h' and make p iterations of the operator $T_{h'}$, starting from $v_h^\varepsilon(x, kh)$.

To explain the choice of h', we notice that the linear part of (3.11), which is represented by the operator

$$G(h') : v = (v(x))_{x \in \Pi_h} \mapsto \left(v(x) + h' \sum_{i=1}^N \frac{v(x+\Delta_i e_i) - 2v(x) + v(x-\Delta_i e_i)}{\Delta_i^2} \right)_{x \in \Pi_h},$$

is monotone and satisfies

$$\|G(h')v\|_\infty \leq \|v\|_\infty$$

under the CFL condition

$$\max \frac{h'}{\Delta_i^2} \leq \frac{1}{2N}. \tag{3.14}$$

Since in addition we have for any k, h' such that $kh' \leq T$,

$$|h'\, g^+(v_h^\varepsilon(x,kh'))\chi^\varepsilon(x,kh') + h'\, g^-(v_h^\varepsilon(x,kh'))(1-\chi^\varepsilon(x,kh'))| \leq \gamma\, h',$$

it is easy to see that under condition (3.14), for any h and ε we have

$$\|v_h^\varepsilon\|_\infty \leq \|v_0\|_\infty + \gamma\, T = M.$$

We therefore choose our time step h' by the relation $\Delta_i = \mu_i \sqrt{h'}$ for some constant $\mu_i > 0$ such that $h/h' \in \mathbb{N}^*$ and (3.14) holds: more precisely, we fix constants $\mu_i \geq \sqrt{2N}$ independent of h such that $\lambda_i/\mu_i =: \nu$ does not depend on i, and set

$$h' = (\nu h)^2, \quad \text{where} \quad h = \frac{1}{\nu^2 p} \tag{3.15}$$

for some $p \in \mathbb{N}^*$. For this particular scheme, we have the following convergence result:

Proposition 3.3. *Assume that α, g^+, g^- and v_0 satisfy **(F)**, while u_0 is a bounded and Lipschitz continuous function which satisfies* (1.3)*. Let us fix $\Delta_i = \lambda_i\, h$ for some fixed constants $\lambda_i > 0$ such that* (3.9) *holds with*

$$M = \|v_0\|_\infty + \gamma\, T,$$

and let us define h' by (3.15)*. We also assume that ε is linked to h by the relation*

$$\varepsilon^{3(N+1)} = h^{2\beta} \tag{3.16}$$

for some fixed $\beta \in (0,1)$.

Let us define the scheme (3.7) *with S_h and $v_{0,h}$ defined by* (3.11), (3.12), *and* (3.13)*. Then the assumptions of Theorem* 3.2 *are satisfied.*

Proof. First of all, as explained above, **(S)** (i) is satisfied with $M = \|v_0\|_\infty + \gamma\, T$, and the Δ_i's were chosen so as to satisfy (3.9) with this M.

To check **(S)** (ii), let us fix a sequence of functions $\chi_h \in B_h(\mathbb{R}^N \times [0,T];[0,1])$ such that $\chi_h \rightharpoonup \chi$ weak-$*$ in $L^\infty(\mathbb{R}^N \times [0,T];[0,1])$ as $h \to 0$. We want to prove that for the choice of $\varepsilon(h)$ given by (3.16), the solution v_h^ε of (3.7) associated to χ_h with initial condition v_0^ε converges pointwise to the solution v of (3.5) in $\bar{B}(0,R) \times [0,T]$ as $h \to 0$. To do so, we set $\chi_h^\varepsilon := (\chi_h)^\varepsilon$ and write

$$v_h^\varepsilon - v = (v_h^\varepsilon - w_h^\varepsilon) + (w_h^\varepsilon - w_h) + (w_h - v),$$

where w_h (resp. w_h^ε) denotes the solution of (3.5) associated to χ_h (resp. χ_h^ε) with initial condition v_0 (resp. v_0^ε). That is, we split the error into three parts, the first part concerning the approximation error coming from the scheme, but with regular source terms χ_h^ε, the second part taking into account the error on exact solutions of (3.5), but as we relax the regularity of χ^ε by letting $\chi_h^\varepsilon \to \chi_h$, and the third part dealing with the weak convergence of χ_h to χ.

Step 1: the term $v_h^\varepsilon - w_h^\varepsilon$. Let us set

$$E^k = (E^k(x))_{x\in\Pi_h} := (v_h^\varepsilon(x,kh') - w_h^\varepsilon(x,kh'))_{x\in\Pi_h}$$

to be the approximation error at step k. Let us also set $e^k = (e^k(x))_{x\in\Pi_h}$, where

$$e^k(x) := \frac{w_h^\varepsilon(x,(k+1)h') - G(h')w_h^\varepsilon(x,kh')}{h'} - g^+(w_h^\varepsilon(x,kh'))\chi_h^\varepsilon(x,kh') - g^-(w_h^\varepsilon(x,kh'))(1-\chi_h^\varepsilon(x,kh')),$$

which represents the consistency error of the scheme. Classical error estimates on the explicit Euler scheme for the heat equation imply that there exists a constant $C>0$ such that for any $x\in\Pi_h$ and k,h' with $kh'\le T$,

$$|e^k(x)| \le \frac{C}{\varepsilon^{3(N+1)}}\,(h' + \max\Delta_i^2). \tag{3.17}$$

Indeed, the Hölder theory for parabolic equations (see for example [15]) shows that

$$\left\|\frac{\partial^2 w_h^\varepsilon}{\partial t^2}\right\|_\infty \le \frac{A}{\varepsilon^{3(N+1)}} \quad\text{and}\quad \left\|D^4 w_h^\varepsilon\right\|_\infty \le \frac{A}{\varepsilon^{3(N+1)}}$$

for some constant $A>0$, thanks to (3.10) and the bounds on the derivatives of g^+, g^- and the initial datum v_0. Then we remark that

$$\begin{aligned}
&E^{k+1}(x)\\
&= v_h^\varepsilon(x,(k+1)h') - w_h^\varepsilon(x,(k+1)h')\\
&= G(h')v_h^\varepsilon(\cdot,kh')(x) + h'\,g^+(v_h^\varepsilon(x,kh'))\chi_h^\varepsilon(x,kh')\\
&\quad + h'\,g^-(v_h^\varepsilon(x,kh'))(1-\chi_h^\varepsilon(x,kh')) - G(h')w_h^\varepsilon(\cdot,kh')(x)\\
&\quad - h'\,g^+(w_h^\varepsilon(x,kh'))\chi_h^\varepsilon(x,kh') - h'\,g^-(w_h^\varepsilon(x,kh'))(1-\chi_h^\varepsilon(x,kh')) - h'\,e^k(x),
\end{aligned}$$

which we rewrite as

$$\begin{aligned}
E^{k+1}(x) = G(h')&[v_h^\varepsilon(\cdot,kh') - w_h^\varepsilon(\cdot,kh')](x) - h'\,e^k(x)\\
&+ h'\,[g^+(v_h^\varepsilon(x,kh')) - g^+(w_h^\varepsilon(x,kh'))]\chi_h^\varepsilon(x,kh')\\
&+ h'\,[g^-(v_h^\varepsilon(x,kh')) - g^-(w_h^\varepsilon(x,kh'))](1-\chi_h^\varepsilon(x,kh')).
\end{aligned}$$

If D denotes a Lipschitz constant for g^+ and g^-, then we obtain, using the fact that $\|G(h')\|\le 1$,

$$\|E^{k+1}\|_\infty \le \|E^k\|_\infty + h'\|e^k\|_\infty + D\,h'\,\|E^k\|_\infty = (1+Dh')\|E^k\|_\infty + h'\,\|e^k\|_\infty.$$

By induction, and using the fact that $E^0=0$, we easily deduce that for any k with $kh'\le T$,

$$\|E^k\|_\infty \le h'\sum_{i=0}^{k}(1+Dh')^i\,\|e^{k-i}\|_\infty.$$

Using (3.17), we obtain that for any k with $kh'\le T$,

$$\begin{aligned}
\|E^k\|_\infty &\le Te^{DT}\,\frac{C}{\varepsilon^{3(N+1)}}\,(h' + \max\Delta_i^2)\\
&\le Te^{DT}\,\frac{C}{\varepsilon^{3(N+1)}}\,(1+\max\mu_i^2)\,\nu^2\,h^2,
\end{aligned} \tag{3.18}$$

thanks to the choices of $\Delta_i = \mu_i\sqrt{h'}$ and $h' = (\nu h)^2$. We therefore see that if we choose ε as in (3.16), i.e. $\varepsilon^{3(N+1)} = h^{2\beta}$ for some $\beta\in(0,1)$, then $v_h^\varepsilon - w_h^\varepsilon$ converges to 0 uniformly on Π_h as $h\to 0$. Moreover, an easy consequence of the explicit resolution of (3.5) (see Lemma 3.5 in [5]) is that there exists a constant $k_N > 0$ depending only on N such that for any $x, y \in \mathbb{R}^N$,

$$|w_h^\varepsilon(x, kh) - w_h^\varepsilon(y, kh)| \leq \left(\|Dv_0\|_\infty + k_N \, \gamma \sqrt{T}\right) |x - y|.$$

As a consequence, $v_h^\varepsilon - w_h^\varepsilon$ also converges to 0 uniformly on $\mathbb{R}^N$ as $h \to 0$.

Step 2: the term $w_h^\varepsilon - w_h$. Let us first prove that $\chi_h^\varepsilon - \chi_h \rightharpoonup 0$ in $L^\infty(\mathbb{R}^N \times [0, T]; [0, 1])$ weak-$*$ as $h \to 0$. For any $\phi \in L^1(\mathbb{R}^N \times [0, T]; \mathbb{R})$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \chi_h^\varepsilon(x, t) \, \phi(x, t) \, dxdt - \int_0^T \int_{\mathbb{R}^N} \chi_h(x, t) \, \phi(x, t) \, dxdt \\
&= \int_0^T \int_{\mathbb{R}^N} \left(\int_0^T \int_{\mathbb{R}^N} \chi_h(y, s) \, \rho^\varepsilon(x - y, t - s - \varepsilon) \, dyds \right) \phi(x, t) \, dxdt \\
&\qquad\qquad - \int_0^T \int_{\mathbb{R}^N} \chi_h(x, t) \, \phi(x, t) \, dxdt.
\end{aligned}$$

Exchanging the variables (x, t) and (y, s) in the first integral, which is permitted by the facts that χ_h takes values in $[0, 1]$ and that ρ^ε and $\phi \in L^1$, we transform this difference of integrals into

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \chi_h(y, s) \left(\int_0^T \int_{\mathbb{R}^N} \rho^\varepsilon(x - y, t - s - \varepsilon) \, \phi(x, t) \, dxdt \right) dyds \\
&\qquad\qquad - \int_0^T \int_{\mathbb{R}^N} \chi_h(y, s) \, \phi(y, s) \, dyds,
\end{aligned}$$

which, in absolute value, is less than

$$\int_0^T \int_{\mathbb{R}^N} \left| \left(\int_0^T \int_{\mathbb{R}^N} \rho^\varepsilon(x - y, t - s - \varepsilon) \, \phi(x, t) \, dxdt \right) - \phi(y, s) \right| dyds,$$

since $|\chi_h| \leq 1$. Using the fact that ρ^ε is symmetric, this integral is equal to

$$\int_0^T \int_{\mathbb{R}^N} \left| \left(\int_0^T \int_{\mathbb{R}^N} \rho^\varepsilon(y - x, s - t + \varepsilon) \, \phi(x, t) \, dxdt \right) - \phi(y, s) \right| dyds,$$

that is to say,

$$\|\rho^\varepsilon(\cdot, \cdot + \varepsilon) \star \tilde{\phi} - \tilde{\phi}\|_{L^1(\mathbb{R}^N \times [0,T])},$$

where $\tilde{\phi}$ is the extension of ϕ to $\mathbb{R}^N \times \mathbb{R}$ by $\tilde{\phi}(\cdot, t) = 0$ if $t \notin [0, T]$. Reproducing the standard proof on approximation by convolution (using the approximation of $\tilde{\phi}$ by a function of class C^1), we see that this term converges to 0 as $\varepsilon = \varepsilon(h) \to 0$. This proves the claim.

We deduce from this assertion and the fact that $v_0^\varepsilon \to v_0$ uniformly, that $w_h^\varepsilon - w_h$ converges locally uniformly to 0 as $h \to 0$. This verification is similar to the proof of Theorem 3.4 of [5], based on the explicit resolution of (3.5) in terms of the Green function of the heat equation.

Step 3: the term $w_h - v$. We prove in the same manner that this term converges locally uniformly to 0 as $h \to 0$, since $\chi_h \rightharpoonup \chi$ weak-$*$ in $L^\infty(\mathbb{R}^N \times [0, T]; [0, 1])$. This concludes the verification of **(S)** (ii).

Let us finally check **(S)** (*iii*) for the choice of ε given by (3.16): let K be a compact subset of $\mathbb{R}^N$, let $x, y \in K \cap \Pi_h$, $kh \leq T$ and $\chi \in B_h(\mathbb{R}^N \times [0,T]; [0,1])$. To estimate $v_h^\varepsilon(x,kh) - v_h^\varepsilon(y,kh)$, where v_h^ε is the solution of (3.7), we write

$$\begin{aligned} v_h^\varepsilon(x,kh) - v_h^\varepsilon(y,kh) = (v_h^\varepsilon(x,kh) - w_h^\varepsilon(x,kh)) + (w_h^\varepsilon(x,kh) - w_h^\varepsilon(y,kh)) \\ + (w_h^\varepsilon(y,kh) - v_h^\varepsilon(y,kh)). \end{aligned}$$

Using the error estimate (3.18), we know that

$$|v_h^\varepsilon(x,kh) - w_h^\varepsilon(x,kh)| + |w_h^\varepsilon(y,kh) - v_h^\varepsilon(y,kh)| \leq 2T\, e^{DT}\, \frac{C}{\varepsilon^{3(N+1)}}\, (1 + \max \mu_i^2)\, \nu^2\, h^2.$$

Moreover, as recalled above, the solution w_h^ε of (3.5) associated to χ_h^ε satisfies

$$|w_h^\varepsilon(x,kh) - w_h^\varepsilon(y,kh)| \leq \left(\|Dv_0\|_\infty + k_N\, \gamma \sqrt{T} \right) |x-y|.$$

With the previous choice of ε, we therefore obtain that **(S)** (*iii*) is satisfied with

$$m_h(\eta) = 2T\, e^{DT}\, C\, (1 + \max \mu_i^2)\, \nu^2\, h^{2(1-\beta)} + \left(\|Dv_0\|_\infty + k_N\, \gamma \sqrt{T} \right) \eta.$$

This concludes the proof of Proposition 3.3 and implies the convergence of our scheme according to Theorem 3.2. □

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