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# THE NORM ESTIMATES FOR THE $q$-BERNSTEIN OPERATOR IN THE CASE $q > 1$

HEPING WANG AND SOFIYA OSTROVSKA

Abstract. The $q$-Bernstein basis with $0 < q < 1$ emerges as an extension of the Bernstein basis corresponding to a stochastic process generalizing Bernoulli trials forming a totally positive system on $[0, 1]$. In the case $q > 1$, the behavior of the $q$-Bernstein basic polynomials on $[0, 1]$ combines the fast increase in magnitude with sign oscillations. This seriously complicates the study of $q$-Bernstein polynomials in the case of $q > 1$.

The aim of this paper is to present norm estimates in $C[0, 1]$ for the $q$-Bernstein basic polynomials and the $q$-Bernstein operator $B_{n,q}$ in the case $q > 1$. While for $0 < q \leq 1$, $\|B_{n,q}\| = 1$ for all $n \in \mathbb{N}$, in the case $q > 1$, the norm $\|B_{n,q}\|$ increases rather rapidly as $n \to \infty$. We prove here that $\|B_{n,q}\| \sim C_q q^{n(n-1)/2}/n$, $n \to \infty$ with $C_q = 2\,(q^{-2}; q^{-2})_\infty/e$. Such a fast growth of norms provides an explanation for the unpredictable behavior of $q$-Bernstein polynomials ($q > 1$) with respect to convergence.

## 1. Introduction

Let $q > 0$. For any nonnegative integer $k$, the *$q$-integer* $[k]_q$ is defined by

$$[k]_q := 1 + q + \cdots + q^{k-1} \ (k = 1, 2, \dots),\ [0]_q := 0,$$

and the *$q$-factorial* $[k]_q!$ is defined by

$$[k]_q! := [1]_q[2]_q \dots [k]_q \ (k = 1, 2, \dots),\ [0]_q! := 1.$$

For integers $k, n$ with $0 \leq k \leq n$, the *$q$-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}\,.$$

We also use the following standard notation (see, e.g., [1], Ch. 10):

$$(a; q)_0 := 1, \quad (a; q)_k := \prod_{s=0}^{k-1}(1 - aq^s), \quad (a; q)_\infty := \prod_{s=0}^{\infty}(1 - aq^s) \ (0 < q < 1).$$

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By $C[0,1]$ we denote the space of continuous functions on $[0,1]$ equipped with the uniform norm. By $\|\cdot\|$ we mean the norm in $C[0,1]$ or the operator norm on $C[0,1]$.

**Definition 1.1.** Let $f:[0,1]\to\mathbb{C}$. The *$q$-Bernstein polynomial* of $f$ is

$$B_{n,q}(f;z) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;z), \quad n=1,2,\dots,$$

where

$$p_{nk}(z)=p_{nk}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (z;q)_{n-k}, \quad k=0,1,\dots,n. \tag{1.1}$$

Note that for $q=1$, $B_{n,q}(f;z)$ is the classical Bernstein polynomial $B_n(f;z)$:

$$B_n(f,z) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} z^k (1-z)^{n-k}.$$

**Definition 1.2.** The *$q$-Bernstein operator* on $C[0,1]$ is given by

$$B_{n,q} : f \mapsto B_{n,q}(f;.).$$

A detailed review of the results on $q$-Bernstein polynomials along with an extensive bibliography is given in [14]. The subject remains under intensive study, and there are new papers constantly coming out (see, for example, papers [15], [21] and [22], which appeared after [14]).

The $q$-Bernstein polynomials inherit some of the properties of the classical Bernstein polynomials, for example, the endpoint interpolation property, the shape-preserving properties in the case $0<q<1$, and the representation via divided differences. Like the classical Bernstein polynomials, the $q$-Bernstein polynomials reproduce linear functions and are degree-reducing on the set of polynomials.

On the other hand, the convergence properties of the $q$-Bernstein polynomials for $q\neq 1$ are essentially *different* from those of the classical ones. What is more, the cases $0<q<1$ and $q>1$ are not similar to each other. This absence of similarity is attributed to the fact that for $0<q<1$, $B_{n,q}$ are *positive* linear operators on $C[0,1]$, while for $q>1$, the positivity does not persist any longer.

The lack of positivity makes the investigation of convergence in the case $q>1$ substantially more difficult than that for $0<q<1$. As a result, the convergence properties of the $q$-Bernstein polynomials in the case $0<q<1$ have been investigated in detail, including the Korovkin type theorem, the properties of the limit operator, the rate of convergence and the saturation phenomenon (see, e.g., [8], [14], [17]–[21]). In contrast, there are only two papers, [13] and [22], dealing systematically with the convergence in the case $q>1$. The results of [13] show nevertheless that for $q>1$ the approximation with $q$-Bernstein polynomials in $C[0,1]$ may be *faster* than that with the classical ones. On the other hand, for some functions analytic on $[0,1]$, their sequences of $q$-Bernstein polynomials ($q>1$) may be divergent, which is totally impossible for $0<q\leq 1$. The problem to describe the class of functions uniformly approximated by their $q$-Bernstein polynomials on $[0,1]$ remains open; there are only a few results available on specific functions (see [15]). The results of [15] reveal the following astonishing fact: if $q$-Bernstein polynomials approximate the logarithmic function $\ln(z+a),\ a>0$, on some interval in $\mathbb{C}$, they have to approximate it on all intervals that are closer to the origin.

One of the reasons for such an unusual behavior of $q$-Bernstein polynomials in the case $q > 1$ is a fast increase of basic polynomials (1.1) near $x = 1$. If we take, for example, $x_0 \in (1/\sqrt{q}, 1)$, then $p_{n0}(q; x_0) = (x_0; q)_n$ and $|p_{n0}(q; x_0)| = q^{n(n-1)/2} x_0^n \left|(1/x_0; 1/q)_n\right| \geq q^{n(n-2)/2} \left|(1/x_0; 1/q)_\infty\right| := C_{q,x_0} q^{n(n-2)/2}$. Similarly, for any fixed $k \in \mathbb{Z}_+$, we have $|p_{nk}(q; x_0)| \geq C_{k,q,x_0} q^{n(n-2)/2}$, while $\sum_{k=0}^{n} p_{nk}(q; x) \equiv 1$. Unlike the situation for $0 < q \leq 1$, when a $q$-Bernstein basis is a totally positive system of functions on [0,1], in the case $q > 1$, polynomials (1.1) combine fast increase in magnitude with sign oscillations (see Theorem 2.1). This creates a serious obstacle for numerical experiments with $q$-Bernstein polynomials in the case $q > 1$.

The polynomials (1.1) form the *$q$-Bernstein basis*, which is shown to be closely related to the $q$-deformed binomial distribution (cf. [5]). The latter plays an important role in the $q$-boson theory giving a $q$-deformation of the quantum harmonic formalism. In particular, it has been used to construct the binomial state for the $q$-boson (cf. [9]). Furthermore, its limit form, called the $q$-deformed Poisson distribution, describes the distribution of energy in a $q$-analogue of the coherent state ([2], [9]).

The $q$-analogue of the boson operator calculus has proved to be a powerful tool in theoretical physics. It provides explicit expressions for the representations of the quantum group $SU_q(2)$. Quantum groups set a common algebraic ground for physical problems seemingly very different from each other, ranging from statistical mechanics to gauge theory and quantum gravity. As a result, these quantum groups have been known to play a significant role in physics (e.g. [4]). Specifically, in condensed-matter physics and quantum field theory, the technique of quantum groups has been used extensively in lattice models, which are ideal to study by the methods of computational physics. Therefore, the properties of the $q$-deformed binomial distribution and related $q$-Bernstein basis (1.1) are very important for applications.

It has to be mentioned here that in the case $0 < q < 1$, various properties of the $q$-Bernstein basis have been studied in [7] and [10]. In the case $0 < q \leq 1$, $q$-Bernstein basic polynomials (1.1) admit a probabilistic interpretation via the stochastic process constructed by A. Il'inskii in [7], in which $p_{nk}(q; x)$ equals the probability of exactly $k$ successes in $n$ trials. For $q = 1$, this reduces to the sequence of Bernoulli trials, where the probability of $k$ successes in $n$ experiments is given by the basic Bernstein polynomials. We notice that for $q < 1$, in distinction from the classical case, trials in this process are not independent and the probability for a success or a failure depends on the history of the process in such a way that each failure diminishes the probability of a success in the next trial. In the case $q > 1$, the role of the probability mass function is played by the sequence $p_{nk}(q; q^{-m})$, $m \in \mathbb{Z}_+$; see [5].

Dealing with the case $q > 1$ in this paper, we present new results related to the $q$-Bernstein basis. Our study involves the norm estimates for the basic polynomials (1.1) in the case $q > 1$. The crux of the matter is Theorem 2.3 related to the asymptotic behavior of the basic polynomials. Moreover, it shows that for $n$ large enough, $|p_{nk}(x)|$ attains its maximum on $[0, 1]$ when $x \in [1/q, 1]$, that is, close to the right endpoint of $[0, 1]$.

The present paper brings out an asymptotic estimate for the norm $\|B_{n,q}\|$ as $n \to \infty$ in the case $q > 1$. The norm of a linear operator characterizes its modulus of continuity. Our computation shows that for $q > 1$ the continuity of $q$-Bernstein operators deteriorates rapidly as $n \to \infty$. This fact reveals one of the ways to look at a recently discovered phenomenon showing that $q$-Bernstein polynomials exhibit a "chaotic" behavior mentioned above. The situations in which norms of finite dimensional operators grow rapidly with the dimension are also studied in the theory of regularizability of inverse linear operators; see [16]. A comprehensive review of results on the approximation of continuous functions defined on compact metric spaces by means of bounded linear operators is presented in [6]. We would like to mention that I. Novikov in [12] has studied asymptotic properties of a particular sequence of Bernstein polynomials from a different point of view.

It is well known that Bernstein polynomials have been used extensively in Computer Aided Geometric Design. In particular, they are crucial for the Bézier method to generate curves and surfaces. Effective computations pertaining to this method are performed with the de Casteljau algorithm and its various modifications. As an example, reference can be made to [3], where a review on the history and the developments concerning this powerful tool has been presented. An extension of the de Casteljau algorithm can be used for evaluating $q$-Bernstein polynomials iteratively and for constructing generalized Bézier curves. The application of this algorithm to the calculations of the $q$-Bernstein polynomials in the case $q > 1$ requires information on either the stability of the computational procedure or the sources of the potential instability. What is decisive in the study of the stability is the rate of growth of the norms $\|B_{n,q}\|$. For this reason, the precise growth estimates of the norms obtained herein are important for the numerical study of $q$-Bernstein polynomials.

Let us restate that for $0 < q < 1$, $\|B_{n,q}\| = 1$ for all $n \in \mathbb{N}$. In contrast to this, our main Theorem 2.6 shows that $\|B_{n,q}\| \to \infty$ as $n \to \infty$ faster than any geometric progression. To be specific, $\|B_{n,q}\| \sim C_q\, q^{n(n-1)/2}/n,\ n \to \infty$ with $C_q = 2(q^{-2}; q^{-2})_\infty / e$.

## 2. Statement of results

From here on we assume that $q > 1$ is fixed. We start with a rather simple statement. However, it is useful to understand the asymptotic behavior of basic polynomials as $n \to \infty$. The formulae below show that the limit involves the function $(1/z; 1/q)_\infty$ analytic in $\mathbb{C} \setminus \{0\}$ and possessing an essential singularity at 0.

**Theorem 2.1.** *(i) For $z \neq 0$, we have*

$$\lim_{n\to\infty} \frac{p_{nk}(z)}{(-1)^n q^{n(n-1)/2} z^n} = \frac{(-1)^k\,(1/z; 1/q)_\infty}{q^{k(k-1)/2}\,(1/q; 1/q)_k}, \quad k \in \mathbb{Z}_+.$$

*The convergence is uniform on any compact set $K \subset \mathbb{C} \setminus \{0\}$.*

*(ii) For $z \neq 0$, we have*

$$\lim_{n\to\infty} \frac{p_{n,n-k}(z)}{q^{nk} z^n} = \frac{(-1)^k (1/z; 1/q)_k}{q^{k(k+1)/2} (1/q; 1/q)_k}, \quad k \in \mathbb{Z}_+.$$

*The convergence is uniform on any compact set $K \subset \mathbb{C} \setminus \{0\}$.*

The following corollary shows that for any $k \in \mathbb{Z}_+$, the values of basic polynomials (1.1) tend to infinity outside of the set $J_q := \{0\} \cup \{q^{-j}\}_{j=0}^{\infty}$. These results will be refined further.

**Corollary 2.2.** *The following equalities hold:*

$$\lim_{n\to\infty} p_{nk}(q;z) = \begin{cases} 0, & if \;\; z \in J_q \setminus \{0\}, \\ 0, & if \;\; z = 0, \; k \neq 0, \\ 1, & if \;\; z = 0, \;\; k = 0, \\ \infty, & otherwise. \end{cases}$$

Theorem 2.1 suggests that it is interesting to investigate the behavior of polynomials (1.1) in Banach spaces of analytic functions. As we know, sometimes complexifications of natural problems lead to unexpected results; see [11].

Our next assertions are concerned with the norm estimates for $p_{nk}(x)$ as $n \to \infty$.

In the sequel, we denote $A(n) \ll B(n)$ if there exists a positive constant $C$ independent of $n$ so that $A(n) \leq CB(n)$, while $A(n) \asymp B(n)$ means that both $A(n) \ll B(n)$ and $B(n) \ll A(n)$.

Clearly, $(1/q;1/q)_k \asymp 1$ and therefore

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{q^{n(n+1)/2}}{q^{k(k+1)/2} q^{(n-k)(n-k+1)/2}} \cdot \frac{(1/q;1/q)_n}{(1/q;1/q)_k (1/q;1/q)_{n-k}} \asymp q^{k(n-k)}. \tag{2.1}$$

For $f \in C[0,1]$, we set

$$\|f\|_s = \|f\|_{C[q^{-s-1},q^{-s}]} = \max_{x\in[q^{-s-1},q^{-s}]} |f(x)|, \;\; s \in \mathbb{Z}_+.$$

The following theorem provides a clue to understanding the asymptotic behavior of basic polynomials (1.1).

**Theorem 2.3.** *Let* $g_{nk} := x^k \prod_{j=0}^{n-k-1} \left(1 - q^j x\right)$. *Then*

$$\|g_{nn}\| = \|g_{nn}\|_0 = 1$$

*and*

$$\|g_{nk}\| \asymp \|g_{nk}\|_0 \asymp q^{(n-k)(n-k-1)/2}/n, \;\; k = 0, 1, \ldots, n-1.$$

Applying (2.1), we come up with this assertion:

**Corollary 2.4.** *The following asymptotic formula holds:*

$$\|p_{nk}(x)\| \asymp \|p_{nk}(x)\|_0 \asymp q^{(n+k-1)(n-k)}/n, \;\; k = 0, 1, \ldots, n-1.$$

The next statement is obtained as a by-product of these results. It gives a weak asymptotic order for the norm of the $q$-Bernstein operator and it will be strengthened further (see Theorem 2.6).

**Corollary 2.5.** *The estimate below is true:*

$$\|B_{n,q}\| \asymp q^{n(n-1)/2}/n, \;\; n \to \infty.$$

The precise rate of growth for $\|B_{n,q}\|$ as $n \to \infty$ is given by Theorem 2.6, which constitutes the main result of the present paper. It estimates the strong asymptotic order of the norm $\|B_{n,q}\|$ as $n \to \infty$.

**Theorem 2.6.** *The following asymptotic equality holds:*

$$\|B_{n,q}\| \sim \frac{2(q^{-2};q^{-2})_\infty \, q^{n(n-1)/2}}{ne} \;\; as \;\; n \to \infty.$$

## 3. Proofs of the theorems

*Proof of Theorem* 2.1. $(i)$ For $z \neq 0$, we have

$$p_{nk}(z) = \begin{bmatrix} n \\ k \end{bmatrix}_q z^n (-1)^{n-k} q^{(n-k)(n-k-1)/2} \left(\frac{1}{z}; \frac{1}{q}\right)_{n-k}.$$

Application of (2.1) yields

$$\begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} = \frac{q^{n(n-1)/2}}{q^{k(k-1)/2}} \cdot \frac{(1/q;1/q)_n}{(1/q;1/q)_k (1/q;1/q)_{n-k}}.$$

Therefore, we obtain

$$\frac{p_{nk}(z)}{(-1)^n q^{n(n-1)/2} z^n} = \frac{(-1)^k (1/q;1/q)_n (1/z;1/q)_{n-k}}{q^{k(k-1)/2} (1/q;1/q)_k (1/q;1/q)_{n-k}}. \tag{3.1}$$

Clearly,

$$\lim_{n\to\infty} (1/q;1/q)_n = \lim_{n\to\infty} (1/q;1/q)_{n-k} = (1/q;1/q)_\infty, \tag{3.2}$$

while $(1/z;1/q)_{n-k} \to (1/z;1/q)_\infty$ as $n \to \infty$ uniformly on any compact set $K \subset \mathbb{C} \setminus \{0\}$. The statement now follows from (3.1).

$(ii)$ Using $\left[ {n \atop n-k} \right]_q = \left[ {n \atop k} \right]_q$, we write

$$p_{n,n-k}(z) = \begin{bmatrix} n \\ k \end{bmatrix}_q z^n (-1)^k q^{k(k-1)/2} \left(\frac{1}{z}; \frac{1}{q}\right)_k.$$

Taking into account (2.1), we derive

$$\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} = q^{nk-k(k+1)/2} \frac{(1/q;1/q)_n}{(1/q;1/q)_k (1/q;1/q)_{n-k}},$$

and hence

$$\frac{p_{n,n-k}(z)}{q^{nk} z^n} = \frac{(-1)^k (1/z;1/q)_k (1/q;1/q)_n}{q^{k(k+1)/2} (1/q;1/q)_k (1/q;1/q)_{n-k}}.$$

With the help of (3.2), we obtain the required statement. □

The following lemma is needed for the sequel.

**Lemma 3.1.** *Let* $h_k(x) := x^k(1-x)(qx-1),\ k \in \mathbb{N}$. *Then*

$$\|h_k\| \asymp \|h_k\|_0 \asymp 1/k.$$

*Proof.* It is easy to see that

$$\max_{x\in[0,1]} x^k(1-x) = \left(\frac{k}{1+k}\right)^k \left(1 - \frac{k}{k+1}\right) = \frac{1}{k(1+1/k)^{k+1}} \asymp 1/k. \tag{3.3}$$

Hence

$$\|h_k\|_0 \le \|h_k\| \ll 1/k.$$

On the other hand, we may suppose that $k \ge \frac{2}{q-1}$. Then $k/(k+1) \in (1/q, 1)$ and

$$\|h_k\| \ge \|h_k\|_0 \ge \left| h_k\left(\frac{k}{k+1}\right) \right| = \frac{1}{k(1+1/k)^{k+1}} \frac{k(q-1)-1}{k+1} \gg \frac{1}{ke} \cdot \frac{q-1}{q+1} \gg 1/k.$$

Thus

$$\|h_k\| \asymp \|h_k\|_0 \asymp 1/k.$$ □

*Proof of Theorem* 2.3. The equality $\|g_{nn}\| = \|g_{nn}\|_0 = 1$ is obvious because $g_{nn} = x^n$. The required asymptotic relations

$$\|g_{nn-1}\| \asymp 1/n, \qquad \|g_{nn-2}\| \asymp 1/n$$

can be easily derived from (3.3) and Lemma 3.1.

We now consider the norm of $g_{nk}$ for $k = 0, 1, \dots, n-3$. Clearly

$$\|g_{nk}\| = \max\{\|g_{nk}\|_{C[0,q^{-(n-k-1)}]}, \|g_{nk}\|_s, s = 0, 1, \dots, n-k-2\}. \tag{3.4}$$

If $x \in [0, q^{-(n-k-1)}]$, then $|g_{nk}(x)| \le 1$, whence $\|g_{nk}\|_{C[0,q^{-(n-k-1)}]} \le 1$.

Now, we take $s_0 = n-k-2$. Then for $x \in (q^{-s_0-1}, q^{-s_0})$, we have

$$|g_{nk}(x)| = x^k \prod_{j=0}^{s_0} (1 - q^j x)(q^{s_0+1}x - 1) \le q^{s_0+1}x - 1 \le q - 1 \ll 1,$$

that is, $\|g_{nk}(x)\|_{n-k-2} \ll 1$.

To estimate $\|g_{nk}\|_0$, that is, to consider the norm on the "small" interval on the right, we take $x \in (1/q, 1)$ and write

$$\begin{aligned}|g_{nk}(x)| &= x^k(1-x)(qx-1)\prod_{j=2}^{n-k-1}(q^j x - 1)\\ &= x^{n-2}(1-x)(qx-1)q^{(n-k)(n-k-1)/2-1}\prod_{j=2}^{n-k-1}\left(1 - \frac{1}{q^j x}\right).\end{aligned}$$

Since

$$1 \ge \prod_{j=2}^{n-k-1}\left(1 - \frac{1}{q^j x}\right) = \prod_{j=1}^{n-k-2}\left(1 - q^{-j}\frac{1}{qx}\right) \ge \prod_{j=1}^{n-k-2}\left(1 - q^{-j}\right) \ge (1/q; 1/q)_\infty,$$

we get, applying Lemma 3.1,

$$\|g_{nk}\|_0 \asymp q^{(n-k)(n-k-1)/2}\|h_{n-2}\|_0 \asymp \frac{q^{(n-k)(n-k-1)/2}}{n}.$$

For $s_0 = 1, \dots, n-k-3$, we take $x \in (q^{-s_0-1}, q^{-s_0})$ so that $y := q^{s_0}x \in (1/q, 1)$. Direct calculations yield

$$\begin{aligned}|g_{nk}(x)| &= x^k(1 - q^{s_0}x)(q^{s_0+1}x - 1)\prod_{s=0}^{s_0-1}(1 - q^s x)\prod_{s=s_0+2}^{n-k-1}(q^s x - 1)\\ &= q^{(s_0+2)+\dots+(n-k-1)}x^{n-k-s_0-2}x^k(1-q^{s_0}x)(q^{s_0+1}x-1)\prod_{s=0}^{s_0-1}(1-q^s x)\prod_{s=s_0+2}^{n-k-1}(1-\frac{1}{q^s x})\\ &= q^{(n-k-1)(n-k)/2-s_0(2n-1-s_0)/2-1}y^{n-s_0-2}(1-y)(qy-1)\prod_{s=0}^{s_0-1}(1-q^s x)\prod_{s=s_0+2}^{n-k-1}(1-\frac{1}{q^s x})\\ &= q^{(n-k-1)(n-k)/2-s_0(2n-1-s_0)/2-1}h_{n-s_0-2}(y)\prod_{s=0}^{s_0-1}(1 - q^s x)\prod_{s=s_0+2}^{n-k-1}(1 - \frac{1}{q^s x}).\end{aligned}$$

Since

$$1 \ge \prod_{s=0}^{s_0-1}(1 - q^s x) = \prod_{t=1}^{s_0}(1 - q^{-t}(q^{s_0}x)) \ge \prod_{s=1}^{s_0}(1 - q^{-s}) \ge (1/q; 1/q)_\infty$$

and

$$1 \geq \prod_{s=s_0+2}^{n-k-1}(1-\frac{1}{q^s x}) = \prod_{t=1}^{n-k-s_0-2}(1-q^{-t}\frac{1}{q^{s_0+1}x})$$
$$\geq \prod_{t=1}^{n-k-s_0-2}(1-q^{-t}) \geq (1/q;1/q)_\infty,$$

we get that

$$\|g_{nk}\|_{s_0} \asymp q^{(n-k-1)(n-k)/2-s_0(2n-1-s_0)/2}/(n-s_0-2).$$

Finally, taking into account (3.4), we arrive at

$$\|g_{nk}\| \asymp \|g_{nk}\|_0 \asymp q^{(n-k-1)(n-k)/2}/n,$$

as required. □

*Proof of Corollary* 2.5. Obviously,

$$\|B_{n,q}\| \geq \|p_{n0}(q;\cdot)\|.$$

Applying Corollary 2.4, we conclude that

$$\|B_{n,q}\| \gg q^{n(n-1)/2}/n.$$

On the other hand,

$$\|B_{n,q}\| \leq \sum_{k=0}^{n}\|p_{nk}(q;\cdot)\| \ll 1+\sum_{k=0}^{n-1}q^{(n+k-1)(n-k)/2}/n$$
$$\ll 1+q^{n(n-1)/2}/n\sum_{k=0}^{n-1}q^{-k(k-1)/2}.$$

Due to the fact that $\sum_{k=0}^{\infty}q^{-k(k-1)}<\infty$, we infer that $\sum_{k=0}^{n-1}q^{-k(k-1)} \ll 1$ and therefore $\|B_{n,q}\| \ll q^{n(n-1)/2}/n$.

Thus

$$\|B_{n,q}\| \asymp q^{n(n-1)/2}/n.$$ □

At the final stage, we estimate the strong asymptotic order of the norm of $\|B_{n,q}\|$.

*Proof of Theorem* 2.6. First, for $x \in (1/q;1)$ and $k \leq n-2$, we have with the help of (3.1):

$$|p_{nk}(x)| = q^{n(n-1)/2-k(k-1)/2}x^{n-1}(1-x)\cdot\frac{(1/q;1/q)_n(1/qx;1/q)_{n-k-1}}{(1/q;1/q)_k(1/q;1/q)_{n-k}}. \tag{3.5}$$

Therefore, in the case of $n$ being large enough to satisfy $1-1/n \in (1/q,1)$, we obtain for $k \leq n-2$ :

$$\left|p_{nk}\left(1-\frac{1}{n}\right)\right| = q^{n(n-1)/2-k(k-1)/2}\cdot\left(1-\frac{1}{n}\right)^{n-1}\cdot\frac{1}{n}\begin{bmatrix}n\\k\end{bmatrix}_{1/q}\left(\frac{n}{q(n-1)};\frac{1}{q}\right)_{n-k-1}$$
$$\geq q^{n(n-1)/2-k(k-1)/2}\cdot\frac{1}{en}\begin{bmatrix}n\\k\end{bmatrix}_{1/q}\left(\frac{n}{q(n-1)};\frac{1}{q}\right)_\infty, \tag{3.6}$$

since $(1-1/n)^{n-1}$ is a decreasing sequence tending to $1/e$. We note that the latter estimate remains true for $k=n-1$ and $k=n$ as well. Indeed, for $k=n-1$,

$$\begin{aligned} p_{n,n-1}(1-1/n) &= [n]_q(1-1/n)^{n-1}\frac{1}{n} \\ \geq q^{n-1}[n]_{1/q}\frac{1}{ne} &\geq q^{n-1}[n]_{1/q}\frac{1}{ne}\left(\frac{n}{q(n-1)};\frac{1}{q}\right)_\infty, \end{aligned}$$

which agrees with (3.6). For $k=n$, and $n\geq 2$,

$$p_{nn}(1-1/n) = (1-1/n)^n \geq 1/e,$$

and (3.6) is also true.

Since

$$\|B_{n,q}\| = \max_{x\in[0,1]}\left(\sum_{k=0}^{n}|p_{nk}(x)|\right) \geq \sum_{k=0}^{n}|p_{nk}(1-1/n)|,$$

by virtue of (3.6) it follows that

$$q^{-n(n-1)/2}ne\|B_{n,q}\| \geq \left(\frac{n}{q(n-1)};\frac{1}{q}\right)_\infty \sum_{k=0}^{n} q^{-k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}_{1/q}.$$

Applying the Rothe identity (cf. [1], Ch. 10, p. 490, Corollary 10.2.2 (c)), we derive

$$q^{-n(n-1)/2}ne\|B_{n,q}\| \geq \left(\frac{n}{q(n-1)};\frac{1}{q}\right)_\infty (-1;q)_n.$$

Since $f(x)=(1/qx;1/q)_\infty$ is continuous at $x=1$, the limit of the right-hand side as $n\to\infty$ exists and equals $(1/q;1/q)_\infty(-1;1/q)_\infty = 2(q^{-2};q^{-2})_\infty$.

As a result, we obtain

$$\liminf_{n\to\infty} q^{-n(n-1)/2}ne\|B_{n,q}\| \geq 2(q^{-2};q^{-2})_\infty. \tag{3.7}$$

Next, we are going to estimate $\|B_{n,q}\|$ from above. Representation (3.5) along with Theorem 2.3 implies that for $k\leq n-2$ and $n$ large enough we have

$$\begin{aligned} \|p_{nk}\| &\leq q^{n(n-1)/2-k(k-1)/2}\cdot\frac{(1/q;1/q)_n(1/q;1/q)_{n-k-1}}{(1/q;1/q)_k(1/q;1/q)_{n-k}}\max_{x\in[1/q,1]} x^{n-1}(1-x) \\ &= q^{n(n-1)/2-k(k-1)/2}\cdot\frac{(1/q;1/q)_n}{(1/q;1/q)_k(1-(1/q)^{n-k})}\left(1-\frac{1}{n}\right)^{n-1}\cdot\frac{1}{n}. \end{aligned} \tag{3.8}$$

In addition, we have $\|p_{nn}\|=1$ and $\|p_{n,n-1}\| \ll q^{n-1}/n$, which means

$$\lim_{n\to\infty} q^{-n(n-1)/2}ne\|p_{nn}\| = \lim_{n\to\infty} q^{-n(n-1)/2}ne\|p_{n,n-1}\| = 0.$$

Therefore, it suffices to estimate $q^{-n(n-1)/2}ne\sum_{k=0}^{n-2}\|p_{nk}\|$. Using (3.8), we obtain

$$q^{-n(n-1)/2}ne\sum_{k=0}^{n-2}\|p_{nk}\| \leq e\left(1-\frac{1}{n}\right)^{n-1}(1/q;1/q)_n\sum_{k=0}^{n-2}\frac{q^{-k(k-1)/2}}{(1/q;1/q)_k\,(1-(1/q)^{n-k})}.$$

Now, for $k,n\in\mathbb{Z}_+$, we set

$$c_{kn} = \begin{cases} \frac{q^{-k(k-1)/2}}{(1/q;1/q)_k(1-(1/q)^{n-k})} & \text{if } k\leq n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\sum_{k=0}^{n-2} \frac{q^{-k(k-1)/2}}{(1/q;1/q)\,(1-(1/q)^{n-k})} = \sum_{k=0}^{\infty} c_{kn}$$

and

$$|c_{kn}| \le \frac{q^{-k(k-1)/2}}{(1/q;1/q)_k\,(1-(1/q)^2)} =: d_k \quad \text{for all} \quad k,n \in \mathbb{Z}_+.$$

Since $\sum_{k=0}^{\infty} d_k < \infty$, we may apply the Lebesgue Dominated Convergence Theorem:

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} c_{kn} = \sum_{k=0}^{\infty} \left( \lim_{n\to\infty} c_{kn} \right) = \sum_{k=0}^{\infty} \frac{q^{-k(k-1)/2}}{(1/q;1/q)_k} = (-1;1/q)_\infty$$

by virtue of the Euler Identity (cf. [1], Ch. 10, p. 490, Corollary 10.2.2 (b)).

Thus, we obtain

$$\limsup q^{-n(n-1)/2} ne\|B_{n,q}\| \le (1/q;1/q)_\infty(-1;1/q)_\infty = 2(q^{-2};q^{-2})_\infty. \tag{3.9}$$

Juxtaposing (3.7) and (3.9), we obtain the needed statement. □

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School of Mathematical Sciences, Capital Normal University, Beijing 100048, People's Republic of China
*E-mail address*: `wanghp@yahoo.cn`

Atilim University, Department of Mathematics, Incek 06836, Ankara, Turkey
*E-mail address*: `ostrovskasofiya@yahoo.com`