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ON EQUAL SUMS OF NINTH POWERS

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ABSTRACT. In this paper, we develop an elementary method to obtain infinitely many solutions of the Diophantine equation

$$x_1^9 + x_2^9 + x_3^9 + x_4^9 + x_5^9 + x_6^9 = y_1^9 + y_2^9 + y_3^9 + y_4^9 + y_5^9 + y_6^9$$

and we give some numerical results.

1. Introduction

Weisstein [7] gives a comprehensive survey of known results concerning equal sums of ninth powers. Moessner [4] in considering the Prouhet-Tarry-Escott problem gives a parametrization that *inter alia* gives infinitely many solutions to the Diophantine equation $\sum_{i=1}^{i=10} x_i^9 = \sum_{i=1}^{i=10} y_i^9$. Only a few numerical solutions of the equation $\sum_{i=1}^{i=6} x_i^9 = \sum_{i=1}^{i=6} y_i^9$ are known. Lander, Parkin, and Selfridge [2] give the single solution

$$(23,\ 18,\ 14,\ 13,\ 13,\ 1,\ 22,\ 21,\ 15,\ 10,\ 9,\ 5); \tag{1}$$

Ekl [1] lists eight other solutions, and Weisstein references nine more. A few additional solutions are listed in Piezas [5].

The solution in (1) satisfies the set of equalities:

$$\begin{cases} 23 + 18 + 14 + 13 + 13 + 1 = 22 + 21 + 15 + 10 + 9 + 5, \\ 23^3 + 18^3 + 14^3 + 13^3 + 13^3 + 1^3 = 22^3 + 21^3 + 15^3 + 10^3 + 9^3 + 5^3, \\ 23^9 + 18^9 + 14^9 + 13^9 + 13^9 + 1^9 = 22^9 + 21^9 + 15^9 + 10^9 + 9^9 + 5^9. \end{cases}$$

Therefore this solution, as well as two other solutions on Ekl's list, satisfies the system $\{(\mathcal{T}_1), (\mathcal{T}_3), (\mathcal{T}_9)\}$, where $(\mathcal{T}_p)$ is the equality:

$$\sum_{k=1}^{6} x_k^p = \sum_{k=1}^{6} y_k^p.$$

In this paper we actually prove that the system $\{(\mathcal{T}_1), (\mathcal{T}_2), (\mathcal{T}_3), (\mathcal{T}_9)\}$ has infinitely many rational solutions.

2. A simpler system

For any rational numbers $a_1, a_2, a_3,\ a_1', a_2', a_3'$, we put

$$\begin{cases} \sigma_1 = a_1 + a_2 + a_3 \\ \sigma_2 = a_2a_3 + a_3a_1 + a_1a_2 \\ \sigma_3 = a_1a_2a_3, \end{cases} \qquad \begin{cases} \sigma_1' = a_1' + a_2' + a_3' \\ \sigma_2' = a_2'a_3' + a_3'a_1' + a_1'a_2' \\ \sigma_3' = a_1'a_2'a_3'. \end{cases}$$

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Proposition 1. *If the rational numbers* $a_1, a_2, a_3, a'_1, a'_2, a'_3$ *satisfy the following system:*

$$(\mathcal{S}) : \begin{cases} (\mathcal{S}_1) : & \sigma_1 = \sigma'_1 \\ (\mathcal{S}_2) : & \sigma_2 = \sigma'_2 \\ (\mathcal{S}_3) : & 3\ \sigma_1^3 - 3\ \sigma_1\sigma_2 + \sigma_3 + \sigma'_3 = 0, \end{cases} \tag{2}$$

then the rational numbers x_k *and* y_k, $1 \leq k \leq 6$, *defined by the following set of equalities:*

$$\begin{cases} x_1 = a_1 + \sigma_1 & x_2 = a_2 + \sigma_1 & x_3 = a_3 + \sigma_1 \\ y_1 = a_1 - \sigma_1 & y_2 = a_2 - \sigma_1 & y_3 = a_3 - \sigma_1 \\ & & \\ x_4 = a'_1 - \sigma'_1 & x_5 = a'_2 - \sigma'_1 & x_6 = a'_3 - \sigma'_1 \\ y_4 = a'_1 + \sigma'_1 & y_5 = a'_2 + \sigma'_1 & y_6 = a'_3 + \sigma'_1 \end{cases} \tag{3}$$

yield a solution of the following system:

(4)

$$(\mathcal{T}) : \begin{cases} (\mathcal{T}_1) : & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = y_1 + y_2 + y_3 + y_4 + y_5 + y_6, \\ (\mathcal{T}_2) : & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2, \\ (\mathcal{T}_3) : & x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 + y_6^3, \\ (\mathcal{T}_9) : & x_1^9 + x_2^9 + x_3^9 + x_4^9 + x_5^9 + x_6^9 = y_1^9 + y_2^9 + y_3^9 + y_4^9 + y_5^9 + y_6^9. \end{cases}$$

Proof. First, make the assumption $(\mathcal{S}_1) : \sigma_1 = \sigma'_1$. Then the system (3) immediately implies $(\mathcal{T}_1)$ and $(\mathcal{T}_2)$.

For any non-negative integer p, put $d_p = \sum_{k=1}^{3} \left(a_k^p - {a'}_k^p \right)$. Then

$$\sum_{k=1}^{6} \left(x_k^3 - y_k^3\right) = 6\sigma_1 d_2$$

and

$$\sum_{k=1}^{6} \left(x_k^9 - y_k^9\right) = 6\sigma_1\left(12 d_2 \sigma_1^6 + 42 d_4 \sigma_1^4 + 28 d_6 \sigma_1^2 + 3 d_8\right).$$

The further assumption $(\mathcal{S}_2) : \sigma_2 = \sigma'_2$ implies $d_2 = 0$, hence $(\mathcal{T}_3)$.

The assumptions $(\mathcal{S}_1)$, $(\mathcal{S}_2)$ also imply the following:

$$\begin{cases} d_4 = 4\ (\sigma_3 - \sigma'_3)\ \sigma_1, \\ d_6 = 3\ (\sigma_3 - \sigma'_3)\ \left(2\ \sigma_1^3 - 4\ \sigma_1\sigma_2 + \sigma_3 + \sigma'_3\ \right), \\ d_8 = 4\ (\sigma_3 - \sigma'_3)\ \left(2\ \sigma_1\ (\ \sigma_1^2 - \sigma_2\ )\ (\ \sigma_1^2 - 3\ \sigma_2\ ) + (\ \sigma_3 + \sigma'_3\ )\ (\ 3\ \sigma_1^2 - 2\ \sigma_2)\right). \end{cases}$$

From this we deduce:

$$\sum_{k=1}^{6} \left(x_k^9 - y_k^9\right) = 144\ \sigma_1\ \left(\sigma_3 - \sigma'_3\right)\ \left(5\ \sigma_1^2 - \sigma_2\right)\ \left(3\ \sigma_1^3 - 3\ \sigma_1\sigma_2 + \sigma_3 + \sigma'_3\right). \tag{5}$$

Consequently, from the assumption $(\mathcal{S}_3) : 3\ \sigma_1^3 - 3\ \sigma_1\sigma_2 + \sigma_3 + \sigma'_3 = 0$, we obtain $(\mathcal{T}_9)$. ☐

Remark 1. Equating to 0 other factors in (5) leads to trivial solutions.

For if $\sigma_1 = 0$, we obtain a trivial solution of $(\mathcal{T})$ satisfying $x_i = y_i$, $i = 1, \ldots, 6$.

If $\sigma_3 = \sigma_3'$, then, taking into consideration the equalities $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$, we deduce $a_i = a_i'$, $i = 1, 2, 3$, which leads to a trivial solution of $(\mathcal{T})$ satisfying

$$(x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)\ =\ (y_4,\ y_5,\ y_6,\ y_1,\ y_2,\ y_3).$$

If $5\ \sigma_1^2 - \sigma_2 = 0$, then necessarily a_1, a_2, a_3, and a_1', a_2', a_3' are zero, which again gives a trivial solution of $(\mathcal{T})$.

Remark 2. The sextuple $(a_1, a_2, a_3, a_1', a_2', a_3') = (17, -18, 5, -19, 9, 14)$ is a solution of the system $(\mathcal{S})$ at (2), with $\sigma_1 = 4$. The numbers x_k and y_k from (3) give the following solution of the system $(\mathcal{T})$ at (4): $(21, -14, 9, -23, 5, 10, 13, -22, 1, -15, 13, 18)$. This permutation of the solution at (1) now satisfies the equation $(\mathcal{T}_2)$.

Henceforth we shall focus attention on the system $(\mathcal{S})$. Observe that if $\sigma_1 = 0$, then $\sigma_1' = 0$, and the system (3) shows that we obtain only trivial solutions of $(\mathcal{T})$, satisfying $x_i = y_i$, $i = 1, \ldots, 6$. We shall refer to solutions of $(\mathcal{S})$ which satisfy $\sigma_1 = 0$ as **trivial** solutions of the system $(\mathcal{S})$.

Remark 3. The only solution of $(\mathcal{S})$ such that $a_1 = a_2 = a_3$ is the zero solution. For if in $(\mathcal{S})$ we replace a_2 and a_3 by a_1, we obtain:

$$a_1' + a_2' + a_3' = 3\ a_1, \quad a_2'a_3' + a_3'a_1' + a_1'a_2' = 3\ a_1^2, \quad a_1'a_2'a_3' = -55\ a_1^3,$$

so that the three numbers a_1', a_2' and a_3' are the solutions of the following equation: $x^3 - 3\ a_1x^2 + 3\ a_1^2x + 55\ a_1^3 = 0$. This may also be written as $(x - a_1)^3 = -7\ (2a_1)^3$. Since x and a_1 are rational, we necessarily have $x = a_1$ and $a_1 = 0$, which implies $(a_1, a_2, a_3, a_1', a_2', a_3') = (0, 0, 0, 0, 0, 0)$.

3. An elliptic surface

We first parametrize the two equations $(\mathcal{S}_1)$, $(\mathcal{S}_2)$, at (2).

Proposition 2. *Let $a = (a_1, a_2, a_3, a_1', a_2', a_3')$ be a sextuple of rational numbers. The following two statements are equivalent:*

I: a satisfies the system: $\begin{cases} a_1 + a_2 + a_3 = a_1' + a_2' + a_3', \\ a_2\ a_3 + a_3\ a_1 + a_1\ a_2 = a_2'\ a_3' + a_3'\ a_1' + a_1'\ a_2'. \end{cases}$

II: There exist triples of rational numbers $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ such that

$$p_1 + p_2 + p_3 = 0, \qquad q_1 + q_2 + q_3 = 0, \tag{6}$$

and

$$\begin{cases} a_1 - a_1' = p_1q_1 \\ a_2 - a_2' = p_2q_1 \\ a_3 - a_3' = p_3q_1, \end{cases} \qquad \begin{cases} a_2 - a_3' = p_1q_2 \\ a_3 - a_1' = p_2q_2 \\ a_1 - a_2' = p_3q_2, \end{cases} \qquad \begin{cases} a_3 - a_2' = p_1q_3 \\ a_1 - a_3' = p_2q_3 \\ a_2 - a_1' = p_3q_3. \end{cases} \tag{7}$$

Proof. It is easily verified that II implies I. Let us prove that I implies II. For this purpose observe first that if $a_1 + a_2 + a_3 = a_1' + a_2' + a_3'$, then each of the following six numbers:

$$\begin{aligned}
&(a_1 - a_1')\,(a_3 - a_1') - (a_2 - a_2')\,(a_2 - a_3')\\
&(a_2 - a_2')\,(a_1 - a_2') - (a_3 - a_3')\,(a_3 - a_1')\\
&(a_3 - a_3')\,(a_2 - a_3') - (a_1 - a_1')\,(a_1 - a_2')\\
-&(a_1 - a_1')\,(a_1 - a_3') + (a_2 - a_2')\,(a_3 - a_2')\\
-&(a_2 - a_2')\,(a_2 - a_1') + (a_3 - a_3')\,(a_1 - a_3')\\
-&(a_3 - a_3')\,(a_3 - a_2') + (a_1 - a_1')\,(a_2 - a_1')
\end{aligned}$$

is equal to $a_2\ a_3 + a_3\ a_1 + a_1\ a_2 - (a_2'\ a_3' + a_3'\ a_1' + a_1'\ a_2')$.
Consequently, if a satisfies the two equalities in I, we obtain the six equalities:

$$\begin{aligned}
(a_1 - a_1')\,(a_3 - a_1') &= (a_2 - a_2')\,(a_2 - a_3')\\
(a_2 - a_2')\,(a_1 - a_2') &= (a_3 - a_3')\,(a_3 - a_1')\\
(a_3 - a_3')\,(a_2 - a_3') &= (a_1 - a_1')\,(a_1 - a_2')\\
(a_1 - a_1')\,(a_1 - a_3') &= (a_2 - a_2')\,(a_3 - a_2')\\
(a_2 - a_2')\,(a_2 - a_1') &= (a_3 - a_3')\,(a_1 - a_3')\\
(a_3 - a_3')\,(a_3 - a_2') &= (a_1 - a_1')\,(a_2 - a_1').
\end{aligned}$$

Now consider the matrix:

$$M = \begin{bmatrix} a_1 - a_1' & a_2 - a_3' & a_3 - a_2' \\ a_2 - a_2' & a_3 - a_1' & a_1 - a_3' \\ a_3 - a_3' & a_1 - a_2' & a_2 - a_1' \end{bmatrix}.$$

From the preceding six equalities, the rank of M is 0 or 1. As is well known, this implies the existence of two triples of rational numbers $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ such that

$$\begin{aligned}
a_1 - a_1' &= p_1 q_1 & a_2 - a_3' &= p_1 q_2 & a_3 - a_2' &= p_1 q_3\\
a_2 - a_2' &= p_2 q_1 & a_3 - a_1' &= p_2 q_2 & a_1 - a_3' &= p_2 q_3\\
a_3 - a_3' &= p_3 q_1 & a_1 - a_2' &= p_3 q_2 & a_2 - a_1' &= p_3 q_3.
\end{aligned}$$

Since $a_1 + a_2 + a_3 = a_1' + a_2' + a_3'$, we obtain (6), (7), the equations defining II, as required. □

Using the parametrization in (6), (7), of the equations $(\mathcal{S}_1)$, $(\mathcal{S}_2)$, we make the following substitution into the equations $(\mathcal{S})$:

$$\begin{aligned}
&a_1 = a_3' + p_2(-q_1 - q_2), \qquad a_2 = a_3' + p_1 q_2, \qquad a_3 = a_3' + (-p_1 - p_2) q_1,\\
&a_1' = a_3' + p_2(-q_1 - q_2) - p_1 q_1, \qquad a_2' = a_3' + p_1 q_2 - p_2 q_1,
\end{aligned} \tag{8}$$

where for the purpose of homogeneity we set $a_3' = p_2 q_4$, say. The third equation $(\mathcal{S}_3)$ delivers a homogeneous cubic equation in $\{q_1, q_2, q_4\}$, with coefficients homogeneous of degree 3 in p_1, p_2. Writing

$$t = p_1/p_2,$$

this equation defines a curve of genus 1 over the function field $\mathbb{Q}(t)$, that is, an elliptic surface, and it is readily checked that the curve contains a point at

$(q_1, q_2, q_4) = (1, 1, 1)$, so that the curve is actually elliptic over $\mathbb{Q}(t)$. Specifically, its equation has the form

$$\begin{aligned} C_t\colon &-3(2+t)(3+3t+t^2)q_1^3 + (-27-2t+16t^2+6t^3)q_1^2q_2 + 7(11+11t+3t^2)q_1^2q_4 \\ &-(15-16t+2t^2+6t^3)q_1q_2^2 - 7(-11+5t+5t^2)q_1q_2q_4 - 56(2+t)q_1q_4^2 \\ &+3(-1+t)(1-t+t^2)q_2^3 + 7(3-5t+3t^2)q_2^2q_4 + 56(-1+t)q_2q_4^2 + 56q_4^3 = 0. \end{aligned}$$

Points on the curve C_t can be pulled back to solutions of the system $(\mathcal{T})$ by means of (3) and (8), namely:

$$\begin{aligned} &x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 = \\ &-(3+t)q_1 + (-2+t)q_2 + 4q_4 : \quad -(2+t)q_1 + (-1+2t)q_2 + 4q_4 : \\ &-(3+2t)q_1 + (-1+t)q_2 + 4q_4 : \quad q_1 - tq_2 - 2q_4 : \\ &(1+t)q_1 + q_2 - 2q_4 : \quad (2+t)q_1 + (1-t)q_2 - 2q_4 : \\ &(1+t)q_1 - tq_2 - 2q_4 : \quad (2+t)q_1 + q_2 - 2q_4 : \\ &q_1 + (1-t)q_2 - 2q_4 : \quad -(3+2t)q_1 + (-2+t)q_2 + 4q_4 : \\ &-(3+t)q_1 + (-1+2t)q_2 + 4q_4 : \quad -(2+t)q_1 + (-1+t)q_2 + 4q_4. \end{aligned} \tag{9}$$

Symmetries of the solutions of $(\mathcal{S})$ under permutation of the a_i, a_i' induce symmetries of the underlying curve C_t, and the effect is that the curves corresponding to the parameters

$$t,\ -1-\frac{1}{t},\ -\frac{1}{1+t},\ -\frac{t}{1+t},\ -1-t,\ \frac{1}{t} \tag{10}$$

are isomorphic. A Weierstrass form for the elliptic curve C_t is discovered to be the following:

$$\begin{aligned} E_t\colon Y^2 = X^3 &+ 1323(1+t+t^2)(23+69t-54t^2-223t^3-54t^4+69t^5+23t^6)X \\ &-2646(947+5682t+16143t^2+28630t^3+22734t^4-18342t^5-46761t^6 \\ &-18342t^7+22734t^8+28630t^9+16143t^{10}+5682t^{11}+947t^{12}). \end{aligned} \tag{11}$$

This latter curve has torsion group $\mathbb{Z}/3\mathbb{Z}$ over $\mathbb{Q}(t)$ with point of order 3 given by

$$\left(147(1+t+t^2)^2,\ 756(1-t+t^2)(3+3t+t^2)(1+3t+3t^2)\right),$$

corresponding to the torsion point on C_t,

$$T_0 = (1, -2, t) \qquad (\text{and where } -T_0 = (2, -1, 1+t)).$$

The condition that we derive a trivial solution of $(\mathcal{S})$ has become

$$(t+2)q_1 - (t-1)q_2 - 3q_4 = 0,$$

and this line cuts C_t in precisely the three points of finite order. Thus only the torsion points of C_t return trivial solutions of (2).

4. Numerical results

We can use standard computer software to determine values of the parameter t for which the rational rank of the curve (11) is positive. The group of points on the associated curve C_t is then infinite, and we are able to deduce an infinity of solutions to the system $(\mathcal{S})$ at (2), and hence an infinity of solutions to the system $(\mathcal{T})$ at (4).

The effect of the symmetries is that it suffices to search with $1 < t$, and in the range $3 \leq \text{numerator}(t) + \text{denominator}(t) \leq 20$ we obtain Table 1, which lists the rational rank of (11) along with corresponding points (q_1, q_2, q_4) on C_t. All computations were performed with Magma [3].

Example. We compute solutions of the system $(\mathcal{T})$ at (4) corresponding to $t = 3$.

The curve C_3 has rank 1 with point of infinite order $Q = (1, 8, -5)$, so that a point on C_3 is of type $nQ + \epsilon T_0$, $n \in \mathbb{Z}$, $\epsilon = 0, \pm 1$. The pullbacks of the three points corresponding to $\epsilon = 0, \pm 1$ give the same solution of the system (4) up to permutation, and so it is only necessary to consider $\epsilon = 0$. Furthermore, the points $\pm nQ$ deliver the same solution up to permutation, and so we can further restrict ourselves to considering only the pullbacks of the points nQ, $n > 0$. Table 2 lists the pullback points for $n = 1, \ldots, 5$.

A similar chain of solutions can be computed for each of the t corresponding to entries in Table 1, with a double infinity of solutions when the rank is 2, and a triple infinity of solutions at $t = 9/4$. We list some "small" solutions arising in this latter instance. Write $Q_1 = (106, -282, 279)$, $Q_2 = (124, -272, 293)$, $Q_3 = (239, -181, 396)$ for the points from Table 1.

Note that the assertion at (10) is equivalent to the fact that the curves C_{p_i/p_j}, $1 \leq i \neq j \leq 3$, are isomorphic. The symmetry in the parametrization (7) interchanging a_2' and a_3' is equivalent to the symmetry interchanging p_i with q_i $(i = 1, 2, 3)$. Thus when C_{p_1/p_2} has positive rank with point $(q_1, q_2, a_3'/p_2)$, there is a corresponding point $(p_1, p_2, a_2'/q_2)$ on C_{q_1/q_2}, and hence the isomorphic curves C_{q_i/q_j}, $1 \leq i \neq j \leq 3$ also have positive rank.

5. Parametrizations over $\mathbb{Q}(\omega)$

It would be useful to know the $\mathbb{Q}(t)$ rank of the curve E_t at (11). If this were positive, then corresponding points on C_t would pull back to solutions of the system $(\mathcal{T})$ that are polynomials in $\mathbb{Z}[t]$. Unfortunately, we are unable to determine whether or not E_t contains any non-torsion points over $\mathbb{Q}(t)$. The large points in Table 1 for $t = 15/2$, $12/7$, $11/9$, together with several specializations of E_t that have rank 0, would suggest that the parametrization of a point is unlikely, and therefore that the $\mathbb{Q}(t)$-rank is 0. The equation E_t is that of an elliptic $K3$-surface, and as such, methods of Shioda [6] allow us to compute an upper bound for the rank of E_t over $\mathbb{C}(t)$, and this turns out to be 3. We are able to spot *two* independent points P_1, P_2 of infinite order on C_t over $\mathbb{C}(t)$, but have not determined whether or not there is a third:

$$P_1 = (6, -4 - 2\omega, 3 + 3t), \qquad P_2 = (-\omega, 1, \frac{1}{2}(1 - \omega)),$$

where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$ is a complex cube root of unity. Summing each point with its Galois conjugate:

$$P_1 + \bar{P}_1 = (1, -2, t), \qquad P_2 + \bar{P}_2 = (1, 1, 1),$$

we obtain points of finite order. Thus no non-trivial rational polynomial parametrization of $(\mathcal{T})$ arises in this way. The two points P_1, P_2 do generate infinitely many polynomial parametrizations of $(\mathcal{T})$, but of course with coefficients over $\mathbb{Q}(\omega)$. For example, the pullbacks of P_1 and P_2 result in trivial solutions of the system (2);

TABLE 1. Values of t with C_t of positive rank

t	independent point(s) on C_t	rank
3	$(1, 8, -5)$	1
4	$(1446, 14711, -15299)$	1
5/2	$(356, 1882, -647)$	1
4/3	$(25, 278, -12)$	1
7	$(207, -52, 785)$	1
8	$(1, 3, -5)$	1
7/2	$(82, -664, 821)$	1
5/4	$(2648, 539468, -54847)$	1
7/3	$(543, 1656, -125)$, $(9137158, 2514571, 13080130)$	2
10	$(59616, 83123, -33345)$	1
6/5	$(6549648283004459, -22665359828265916, 8749022109376884)$	1
11	$(479, -2916, 13421)$, $(14033, -43851, 222497)$	2
12	$(19, 64, -190)$	1
11/2	$(-27783615238, -228089845712, 349090862393)$	1
10/3	$(94087753, 101830636, 85790044)$	1
9/4	$(106, -282, 279)$, $(124, -272, 293)$, $(239, -181, 396)$	3
8/5	$(36194817, 128001892, 11175862)$	1
7/6	$(3049, -8401, 3738)$, $(35004, -61398, 40157)$,	2
13	$(4724, 1055987, -5026528)$	1
9/5	$(121419260, -1377907645, 613796368)$	1
11/4	$(126758321, -30905647, 231572729)$	1
8/7	$(14818297849097, 12821985982426, 14953926106708)$	1
13/3	$(150351, 100917, 224711)$, $(13051968, -12150687, 37914218)$	2
11/5	$(154090, 443445, -3052)$	1
9/7	$(742, -2709, 1116)$, $(2455971, 3614233, 2299483)$	2
16	$(18, 179, -963)$, $(958, 6731, -34507)$	2
15/2	$(1342076090489653760733881569581419 87,$ $289505573647988961162860881783318 79,$ $422719433838369207424332656027729381)$	1
14/3	$(359601, 464652, 185656)$, $(12153, -8250, 35330)$	2
13/4	$(25702, 74362, -22943)$	1
11/6	$(1532, -16154, 7563)$	1
9/8	$(27787, -37949, 30240)$, $(21756703708, -959147282732, 73870303329)$	2
11/7	$(11754403807181, -239314804762243, 71755024964941)$	1
18	$(7901044, -22041517, 187889479)$	1
17/2	$(153, 599, -1262)$, $(1874, 2748, -1015)$	2
15/4	$(4688, 8548, -97)$, $(880834, -4824818, 6950465)$	2
14/5	$(23, 108, -44)$, $(1287580662737, 1391088002287, 1201351021327)$	2
13/6	$(4093367511, 4633565061, 3798630686)$	1
12/7	$(460788234850250312976951 4,$ $735549705551006930092303 9,$ $368874199949089591210638 9)$	1
11/8	$(2720488, 6913952, 1992365)$	1
10/9	$(343956623832196, -126385184284523, 362777970896793)$	1
17/3	$(127, -596, 1405)$	1
11/9	$(888221912602727881028999424599,$ $-492896695539027689262279545737,$ $987459994593608883597631915735)$	1

TABLE 2. Solutions to the system (4) arising from $t = 3$

n	1	2	3	4	5
x_1	18	13825	-176607027	-295503813128476	16781116061381923831545
x_2	-15	-9157	154786818	163694336414557	7388891819632091442261
x_3	13	1092	134510017	453392718089919	-21580737831236008171454
x_4	13	4495	-22242470	129024868182881	-17757024654750594094103
x_5	-22	-14187	168678401	198009483059676	-4986911769004750233739
x_6	1	6812	-202780835	-487825971930557	21449301398866340776666
y_1	10	10945	-232951931	-456295433816476	15486481036492920280369
y_2	-23	-12037	98441914	2902715726557	6094256794743087891085
y_3	5	-1788	78165113	292601097401919	-22875372856125011722630
y_4	21	7375	34102434	289816488870881	-16462389629861590542927
y_5	-14	-11307	225023305	358801103747676	-3692276744115746682563
y_6	9	9692	-146435931	-327034351242557	22743936423755344327842

TABLE 3. Solutions to the system (4) arising from the rank 3 curve

point on $C_{9/4}$	x_1 / y_1	x_2 / y_2	x_3 / y_3	x_4 / y_4	x_5 / y_5	x_6 / y_6
	453	150	-307	-455	-98	281
Q_2	429	174	-331	-431	-122	305
	978	365	-643	-991	-63	490
Q_1	842	501	-779	-855	-199	626
	-1136	583	261	785	1739	-1800
Q_3	-1568	1015	-171	1217	1307	-1368
	-3147	2120	75	1927	4832	-4927
$Q_1 + Q_2$	-4027	3000	-805	2807	3952	-4047
	6190	-18943	17613	3265	-12363	9958
$Q_1 + Q_3$	470	-13223	11893	8985	-18083	15678
	159999	99268	-117995	-175511	67884	21299
$Q_1 + Q_2 + Q_3$	105055	154212	-172939	-120567	12940	76243
	319333	-455078	333993	-72523	-428926	415401
$Q_2 - Q_3$	207133	-342878	221793	39677	-541126	527601

however, the pullbacks of $2P_1$ and $2P_2$ result in the following cubic parametrizations, respectively:

$$
\begin{aligned}
x_1 : x_2 : x_3 : x_4 : x_5 : x_6 :\ y_1 : y_2 : y_3 : y_4 : y_5 : y_6 = \\
(3\omega + 3)t^3 + (\omega + 4)t^2 + (-\omega + 3)t - 3\omega : \\
(-2\omega - 1)t^3 + (3\omega + 3)t^2 + (4\omega + 4)t + \omega + 2 : \\
3\omega t^3 + (2\omega - 1)t^2 + (-9\omega - 6)t - 2\omega - 1 : \\
(-\omega - 2)t^3 + (-5\omega - 6)t^2 + (5\omega - 1)t + \omega - 1 : \\
(2\omega + 1)t^3 + (9\omega + 3)t^2 + (-2\omega - 3)t - 3\omega - 3 :
\end{aligned}
$$

$$
\begin{aligned}
&(-\omega+1)t^3-4\omega t^2-3\omega t+2\omega+1:\\
&(-\omega+1)t^3+(-5\omega+1)t^2+(5\omega+6)t+\omega+2:\\
&(2\omega+1)t^3+(9\omega+6)t^2+(-2\omega+1)t-3\omega:\\
&(-\omega-2)t^3+(-4\omega-4)t^2+(-3\omega-3)t+2\omega+1:\\
&3\omega t^3+(\omega-3)t^2+(-\omega-4)t-3\omega-3:\\
&(-2\omega-1)t^3+3\omega t^2+4\omega t+\omega-1:\\
&(3\omega+3)t^3+(2\omega+3)t^2+(-9\omega-3)t-2\omega-1;
\end{aligned}
$$

and

$$
\begin{aligned}
x_1:x_2:x_3:x_4:x_5:x_6:\ y_1:y_2:y_3:y_4:y_5:y_6 =\\
(2\omega+4)t^3+(4\omega+6)t^2+(5\omega+8)t+3\omega+3:\\
-2t^3+(3\omega+2)t^2+(3\omega+1)t+\omega-1:\\
(-4\omega-6)t^3+(-4\omega-4)t^2+(\omega+7)t+3:\\
(-2\omega-2)t^3+(-5\omega-8)t^2+(-6\omega-9)t-\omega-2:\\
(-2\omega-6)t^3+(-6\omega-14)t^2-3t+\omega+2:\\
(4\omega+8)t^3+(5\omega+12)t^2+2t-2\omega-1:\\
(4\omega+8)t^3+(7\omega+12)t^2+(2\omega+2)t+\omega-1:\\
(-2\omega-6)t^3-4t^2+(6\omega+7)t+3\omega+3:\\
(-2\omega-2)t^3+(-\omega+2)t^2+(-2\omega+1)t-2\omega-1:\\
(-4\omega-6)t^3+(-8\omega-14)t^2+(-3\omega-3)t+\omega+2:\\
-2t^3+(-3\omega-8)t^2+(-3\omega-9)t-\omega-2:\\
(2\omega+4)t^3+(2\omega+6)t^2+(3\omega+8)t+3.
\end{aligned}
$$

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References

[1] R. Ekl, New results in equal sums of like powers, *Math. Comp.* **67**, no. 223 (1998), 1309–1315. MR1474650 (98m:11023)

[2] L.J. Lander, T.R. Parkin, and J.L. Selfridge, A survey of equal sums of like powers, *Math. Comp.* **21** (1967), 446–459. MR0222008 (36:5060)

[3] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language. J. Symbolic Comput., 24(3-4): 235–265, 1997. MR1484478

[4] A. Moessner, On Equal Sums of Like Powers, *Math. Student* **15** (1947), 83–88. MR0033840 (11:500a)

[5] Titus Piezas III, Timeline of Euler's Extended Conjecture, `http://www.geocities.com/titus_piezas/Timeline1.htm`.

[6] T. Shioda, On elliptic modular surfaces, *J. Math. Soc. Japan*, **24** (1972) 20–59. MR0429918 (55:2927)

[7] E.W. Weisstein, Diophantine Equation–9th Powers, http://mathworld.wolfram.com/DiophantineEquation9thPowers.html

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