Generators of Function Fields of the Modular Curves $X_1(5)$ and $X_1(6)$

Chang Heon Kim and Ja Kyung Koo

Abstract. We show that the modular functions $j_{1,5}$ and $j_{1,6}$ generate function fields of the modular curves $X_1(N)$ ($N = 5, 6$, respectively) and find some number-theoretic properties of these modular functions.

1. Introduction

Let $\mathcal{H}$ be the complex upper half-plane and let $\Gamma_1(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to $(1 \ 1)$ mod $N$ ($N = 1, 2, 3, \ldots$). Since the group $\Gamma_1(N)$ acts on $\mathcal{H}$ by linear fractional transformations, we get the modular curve $X_1(N) = \Gamma_1(N)\backslash \mathcal{H}$, as the projective closure of the smooth affine curve $\Gamma_1(N)\backslash \mathcal{H}$, with genus $g_1,N$.

Let $r \in \mathbb{Z}$ and $r \neq 0$ mod $N$. For $z \in \mathcal{H}$, Ishii [7] found a family of modular functions $X_r(z)$ defined by

$$X_r(z) = \exp \left( 2\pi i \frac{(r-1)(N-1)}{4N} \right) \prod_{s=0}^{N-1} \frac{K_{r,s}(z)}{K_{1,s}(z)},$$

where $K_{r,v}(z)$ are Klein forms of level $N$. For the Klein forms we refer to Kubert and Lang [14]. For $\zeta_N = e^{2\pi i/N}$, let $\mathfrak{g}_N$ be the field of modular functions for the principal congruence group $\Gamma(N)$ with $\mathbb{Q}(\zeta_N)$-rational Fourier coefficients at the cusp $i\infty$. Then $X_r(z) \in \mathfrak{g}_N$ (resp. $X_r(z)^{\mathbb{Z}_N} \in \mathfrak{g}_N$) if $r$ is odd (resp. if $r$ is even), where $\varepsilon_N$ is $1$ or $2$ according as $N$ is odd or even. When $N \geq 7$, by utilizing such modular functions, Ishida and Ishii showed in [3] that $X_2(z)^{\mathbb{Z}_N}, X_3(z)^N$ are generators of function fields of the modular curves $X_1(N)$. As for the cases $N = 1, 2, 3$, we know that the elliptic modular function $j(z)$ ($N = 1$), and the Thompson series of type $2B$ ($N = 2$, Table 3 in [2]) and the Thompson series of type $3B$ ($N = 3$, Table 3 in [2]) are generators, respectively, because $\Gamma_1(2) = \Gamma_0(2)$ and $\Gamma_1(3) = \Gamma_0(3)$. In the case $N = 4$, we refer to [10]. Thus, in order to find the remaining two cases $N = 5, 6$ we use the following general fact. Since $g_{1,N} = 0$ only for the eleven cases $1 \leq N \leq 10$ and $N = 12$ (10), the function field $\mathcal{C}(X_1(N))$ of the curve $X_1(N)$ is a rational function field over $\mathbb{C}$ for such $N$.

In this article we shall find the field generators $j_{1,5}$ and $j_{1,6}$ as uniformizers of the modular curves $X_1(N)$ when $N = 5$ and $6$, respectively. In §3, $j_{1,5}$ is constructed by making use of the Dedekind eta functions and Eisenstein series of
weight 2, and in §4 we build up $j_{1,6}$ from the Eisenstein series of weight 2. In §5 we estimate the normalized generators (or hauptmodul) $N(j_{1,5})$ and $N(j_{1,6})$, and, when $z \in \mathbb{H} \cap \mathbb{Q}(\sqrt{-d})$ for a square-free positive integer $d$, we show that $N(j_{1,N})$ ($N = 5, 6$) becomes an algebraic integer. In §6 we show that the hauptmodul $N(j_{1,1})$ has integral Fourier coefficients. Lastly, in §7 we find certain connections between the hauptmodul $N(j_{1,N})$ and the parameter $t$ emerging from the moduli problem of elliptic curves.

Throughout the article we adopt the following notation:

$\mathbb{H}$ the extended complex upper half-plane

$\Gamma$ a congruence subgroup of $SL_2(\mathbb{Z})$

$\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) | \gamma \equiv 1 \pmod{N} \}$

$\Gamma_0(N)$ the inhomogeneous group of $\Gamma(= \Gamma/\Gamma(1))$

$X(\Gamma)$ the function field of the curve $X(\Gamma)$

$\mathcal{F}(\Gamma)$ the space of modular forms of weight $k$ with respect to the group $\Gamma$

$M_k(\Gamma, \chi) = \{ f \in M_k(\Gamma_0(N)) | f(\gamma z) = \chi(d)(cz+d)^k f(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \}$

$a \sim b$ means that $a$ is equivalent to $b$

$z \to \infty$ denotes that $z$ goes to $\infty$

$\nu_0(F)$ the sum of orders of zeros of a modular form (or function) $F$

$\nu_\infty(F)$ the sum of orders of poles of a modular form (or function) $F$

$\sigma_\infty(\Gamma)$ the number of $\Gamma$-inequivalent cusps of $\Gamma$.

We shall always take the branch of the square root having argument in $(-\pi, \pi]$.

Thus, $\sqrt{z}$ is a holomorphic function on the complex plane with the negative real axis $(-\infty, 0]$ removed. For any integer $k$, we define $z^{\frac{k}{2}}$ to mean $(\sqrt{z})^k$.

2. Fundamental region of $X_1(N)$

Let $\Gamma$ be a congruence subgroup of $SL_2(\mathbb{Z})$.

**Definition.** An (open) fundamental region $R$ for $\Gamma$ is an open subset of $\mathbb{H}^*$ with the properties:

1. there do not exist $\gamma \in \Gamma$ and $w, z \in R$ for which $w \neq z$ and $w = \gamma z$;
2. for any $z \in \mathbb{H}^*$, there is $\gamma \in \Gamma$ such that $\gamma z \in \overline{R}$, the closure of $R$.

We will examine some necessary results about fundamental regions, which will give us useful geometric information for the modular curve $X_1(N)$. Let $\Gamma^1(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to $\left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ mod $N$ ($N = 1, 2, 3, \ldots$). We note that the two groups $\Gamma_1(N)$ and $\Gamma^1(N)$ are conjugate:

$$
\Gamma^1(N) = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(N) \begin{pmatrix} 1/N & 0 \\ 0 & 1 \end{pmatrix}.
$$
It turns out that the $\Gamma^1$ groups are more convenient than their $\Gamma_1$ counterparts for drawing pictures and making geometric computations. Now we will draw fundamental regions by using Ferenbaugh’s idea ([4], §3). Suppose $c, r \in \mathbb{R}$ with $r > 0$. Then we define the sets

\[
\begin{align*}
\text{arc}(c, r) &= \{z \in \mathbb{H}^* | |z - c| = r\}, \\
\text{inside}(c, r) &= \{z \in \mathbb{H}^* | |z - c| < r\}, \\
\text{outside}(c, r) &= \{z \in \mathbb{H}^* | |z - c| > r\}.
\end{align*}
\]

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\Gamma$, and assume $c \neq 0$. Then we define

\[
\begin{align*}
\text{arc}(\gamma) &= \text{arc}\left(\frac{a}{c}, \frac{1}{|c|}\right), \\
\text{inside}(\gamma) &= \text{inside}\left(\frac{a}{c}, \frac{1}{|c|}\right), \\
\text{outside}(\gamma) &= \text{outside}\left(\frac{a}{c}, \frac{1}{|c|}\right).
\end{align*}
\]

If $c = 0$, $\gamma$ is of the form $z \mapsto z + n$ for some integer $n$. We shall assume $\gamma$ is not the identity, so $n \neq 0$. We then adopt the following conventions: for $n > 0$, we define

\[
\begin{align*}
\text{arc}(\gamma) &= \{z \in \mathbb{H}^* | \Re(z) = \frac{n}{2}\}, \\
\text{inside}(\gamma) &= \{z \in \mathbb{H}^* | \Re(z) > \frac{n}{2}\}, \\
\text{outside}(\gamma) &= \{z \in \mathbb{H}^* | \Re(z) < \frac{n}{2}\}.
\end{align*}
\]

As for the case $n < 0$, we define “arc” in the same way and reverse the inequalities in the definitions of “inside” and “outside”. Then we have

**Proposition 1.** The element $\gamma \in \Gamma \setminus \{I\}$ sends $\text{arc}(\gamma^{-1})$ to $\text{arc}(\gamma)$, $\text{inside}(\gamma^{-1})$ to $\text{outside}(\gamma)$ and $\text{outside}(\gamma^{-1})$ to $\text{inside}(\gamma)$.

*Proof.* See [4], Proposition 3.1. \hfill \Box

**Theorem 2.** With notation as in the above, a fundamental region $R$ for $\Gamma$ is given by

\[
R = \bigcap_{\gamma \in \Gamma \setminus \{I\}} \text{outside}(\gamma).
\]

*Proof.* See [4], Theorem 3.3. \hfill \Box

Now the following theorem enables us to get the generators of the group $\hat{\Gamma}$.

**Theorem 3.** Let $\bar{\Gamma}$ be a congruence subgroup of $\bar{\Gamma}(1)$ of finite index and $R$ be a fundamental region for $\bar{\Gamma}$. Then the sides of $R$ can be grouped into pairs $\lambda_i, \lambda'_i$ ($i = 1, 2, \ldots, s$) in such a way that $\lambda_i \subseteq R$ and $\lambda'_i = \gamma_i \lambda_i$, where $\gamma_i \in \bar{\Gamma}$ ($i = 1, 2, \ldots, s$). The $\gamma_i$’s are called boundary substitutions of $R$. Furthermore, $\bar{\Gamma}$ is generated by the boundary substitutions $\gamma_1, \ldots, \gamma_s$.

*Proof.* See [19], Theorem 2.4.4 (or [10], Theorem 1). \hfill \Box
3. Modular function $j_{1,5}$

Let us take $\Gamma = \Gamma_1(5)$ and put $\gamma_1 = \left( \begin{smallmatrix} 1 & 5 \\ 0 & 1 \end{smallmatrix} \right)$, $\gamma_2 = \left( \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right)$ and $\gamma_3 = \left( \begin{smallmatrix} 9 & 20 \\ 4 & 9 \end{smallmatrix} \right)$. If $R_5$ is a fundamental region of $\Gamma_1(5)$, then by Theorem 2 it is given by

$$R_5 = \bigcap_{i=1}^{3} \text{outside}(\gamma_i^{\pm 1})$$

and is drawn as shown in Figure 1.

![Figure 1. Fundamental domain of $\Gamma_1(5)$.](image)

We denote by $S_{\Gamma}$ the set of inequivalent cusps of $\Gamma$. Then we see from the above figure that $S_{\Gamma_1(5)} = \{\infty, 0, 2, \frac{5}{2}\}$. Furthermore it follows from Theorem 3 that $\overline{\Gamma_1}(5)$ is generated by $\gamma_1$, $\gamma_2$ and $\gamma_3$. Thus we obtain the following theorem by \[11\].

**Theorem 4.** 
(i) $S_{\Gamma_1(5)} = \{\infty, 0, 2, \frac{5}{2}\}$. All cusps of $\Gamma_1(5)$ are regular \[16\], \[22\].
(ii) $\overline{\Gamma_1}(5)$ is generated by $\left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, $\left( \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right)$ and $\left( \begin{smallmatrix} 9 & 20 \\ 4 & 9 \end{smallmatrix} \right)$.

For later use we are in need of calculating the widths of the cusps of $\Gamma_1(5)$.

**Lemma 5.** Let $a/c \in \mathbb{P}^1(\mathbb{Q})$ be a cusp with $(a, c) = 1$. Then the width of $a/c$ in $X_1(N)$ is given by $N/(c, N)$ if $N \neq 4$.

**Proof.** See \[11\], Lemma 3.

Table 1 shows the inequivalent cusps of $\Gamma_1(5)$.

<table>
<thead>
<tr>
<th>Cusp</th>
<th>$\infty$</th>
<th>$0$</th>
<th>$\frac{5}{2}$</th>
<th>$\frac{9}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width</td>
<td>$1$</td>
<td>$5$</td>
<td>$1$</td>
<td>$5$</td>
</tr>
</tbody>
</table>

Table 1. Cusps of $\Gamma_1(5)$

Let $G_2$ be the Eisenstein series of weight 2 defined by

$$G_2(z) = 2\zeta(2) - 8\pi^2 \sum_{n \geq 1} \sigma_1(n)q^n, \quad z \in \mathbb{H}.$$
Then \( G_2 \) has the following transformation formula (\[20\], p.68) for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \) and \( z \in \mathcal{H} \):

\[
G_2 \left( \frac{az + b}{cz + d} \right) = (cz + d)^2 G_2(z) - 2\pi icz + d).
\]

**Lemma 6.** For each prime \( p \), let \( G_2^{(p)}(z) = G_2(z) - pG_2(pz) \). Then \( G_2^{(p)}(z) \in M_2(\Gamma_0(p)) \).

**Proof.** If \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is an element of \( \Gamma_0(p) \), then

\[
G_2^{(p)}(z) |_{\gamma} = (cz + d)^{-2} G_2^{(p)}(\gamma z)
\]

\[
= (cz + d)^{-2} (G_2(\gamma z) - pG_2(p\gamma z))
\]

\[
= (cz + d)^{-2} (G_2(\gamma z) - pG_2(\begin{pmatrix} a & pb \\ c & d \end{pmatrix} \cdot pz))
\]

\[
\text{using } \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & pb \\ c & d \end{pmatrix}
\]

\[
= (cz + d)^{-2} ((cz + d)^2 G_2(z) - 2\pi icz + d)
\]

\[
- p((\frac{c}{p}pz + d)^2 G_2(pz) - 2\pi i \frac{c}{p}(pz + d))) \quad \text{by (3)}
\]

\[
= G_2^{(p)}(z).
\]

Recall that there are 2 cusps \( \infty, 0 \) in \( X_0(p) \). The \( q \)-expansion of \( G_2 \) implies the holomorphicity of \( G_2^{(p)} \) at \( \infty \). At 0,

\[
G_2^{(p)}(z) |_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = z^{-2} G_2^{(p)}(-1/z)
\]

\[
= z^{-2} (G_2(-1/z) - pG_2(-p/z))
\]

\[
= z^{-2} (z^2 G_2(z) - 2\pi iz - p((z/p)^2 G_2(z/p) - 2\pi iz/p)) \quad \text{by (3)}
\]

\[
= G_2(z) - 1/pG_2(z/p);
\]

hence it is holomorphic there.

**Lemma 7.** For \( F \in M_k(\Gamma_0(N), \chi) \), let \( W_N(F) \) be the Fricke involution of \( F \), i.e., \( W_N(F) = F |_{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \). Then for a quadratic character \( \chi \) on \((\mathbb{Z}/N\mathbb{Z})^\times\), \( W_N \) preserves \( M_k(\Gamma_0(N), \chi) \).

**Proof.** See [13], p. 145.

Let \( \eta(z) = e^{\frac{2\pi i z}{c}} \prod_{n=1}^{\infty} (1 - q^n), \ z \in \mathcal{H} \) be the Dedekind eta function. It is well known ([12], p.235) that

\[
\eta(z + 1) = e^{\frac{2\pi i}{c}} \eta(z) \quad \text{and} \quad \eta(-1/z) = (-iz)^{\frac{1}{2}} \eta(z).
\]

**Lemma 8.** (i) \( \eta^p(z)/\eta(pz) \in M_{\frac{p-1}{2}}(\Gamma_0(p), \left( \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix} \right)) \) for a prime \( p > 3 \).

(ii) \( W_p(\eta^p(z)/\eta(pz)) = \text{constant} \times \eta^p(pz)/\eta(z) \in M_{\frac{p-1}{2}}(\Gamma_0(p), \left( \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix} \right)) \).
Proof. For (i) we refer to [18], p. 28.

(ii) We have

\[ W_p(\eta^p(z)/\eta(pz)) = \frac{\eta^p(z)}{\eta(pz)} \mid \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \] \[ = p^{2-1} (pz)^{-2\pi} \eta^p\left( -\frac{1}{pz} \right) / \eta \left( p \cdot \left( -\frac{1}{pz} \right) \right) \]
\[ = p^{2-1} \frac{1}{z} \eta^p(pz)^{-2} \frac{\zeta E(pz)}{\zeta \eta(z)} \text{ by (3)} \]
\[ = \text{constant} \times \eta^p(pz)/\eta(z). \]

Hence, this completes the proof by Lemma 7. □

Now, put \( x(z) = 4 \cdot \eta^5(z)/\eta(5z) + E_2^{(5)}(z) \) and \( y(z) = \eta^5(5z)/\eta(z) \), where \( E_2(z) = G_2(z)/(2\zeta(2)) \) is the normalized Eisenstein series of weight 2 and \( E_2^{(5)}(z) = E_2(z) - 5E_2(5z) \). From the \( q \)-expansions of \( G_2 \) and \( \eta \) it follows that

\[ x(z) = -44q - 52q^2 - 56q^3 - 228q^4 + \cdots, \]
\[ y(z) = q + q^2 + 2q^3 + 3q^4 + 5q^5 + \cdots. \]

We set \( j_{1,5}(z) = x(z)/y(z) \).

**Theorem 9.** (a) \( x, y \in M_2(\Gamma_1(5)) \).
(b) \( \mathbb{C}(X_1(5)) \) is equal to \( \mathbb{C}(j_{1,5}(z)) \).
(c) \( j_{1,5} \) takes the following value at each cusp: \( j_{1,5}(\infty) = -44 \), \( j_{1,5}(0) = -20\sqrt{5} \), \( j_{1,5}(1/2) = 20\sqrt{5} \), and \( j_{1,5}(2/5) = \infty \) (a simple pole).

**Proof.** (a) follows from Lemmas 3 and 8. Next, it is clear by (a) that \( j_{1,5}(z) \in \mathbb{C}(X_1(5)) \). We see from the construction of \( x \) and \( y \) that both \( x \) and \( y \) vanish at \( \infty \). Also, we know from [22], p.39 that \( \nu_0(x) = \nu_0(y) = 2 \). Let \( \infty \) and \( z_0 \) (resp. \( z_0' \)) be the zeros of \( x \) (resp. \( y \)). If \( z_0 \) is equivalent to \( z_0' \) under \( \Gamma_1(5) \), then \( x/y \) has no poles in \( X_1(5) \) so that it would be a constant. However, the \( q \)-expansions of \( x \) and \( y \) show that the quotient \( x/y \) cannot be a constant. Thus \( z_0 \) is not \( \Gamma_1(5) \)-equivalent to \( z_0' \), and \( \nu_0(j_{1,5}) = \nu_{\infty}(j_{1,5}) = 1 \), which implies that \( j_{1,5} \) generates \( \mathbb{C}(X_1(5)) \) over \( \mathbb{C} \). Now we will prove (c). As mentioned in Table 1, we note that there are 4 inequivalent cusps \( \infty, 0, 1/2, 2/5 \) in \( X_1(5) \).

(i) \( s = \infty \):

\[ j_{1,5}(\infty) = \lim_{z \to \infty} \frac{x}{y} = \lim_{q \to 0} \frac{-44q - 52q^2 - 56q^3 - 228q^4 + \cdots}{q + q^2 + 3q^4 + 5q^5 + \cdots} = -44. \]
(ii) \( s = 0 \): Since \( \frac{0}{1} \) sends \( \infty \) to 0,

\[
j_{1,5}(0) = \lim_{z \to +\infty} \frac{4 \cdot \eta^5(z)/\eta(5z) + E_2^{(5)}(z)}{\eta^5(5z)/\eta(z)} \bigg|_{\frac{0}{1}} = \lim_{z \to +\infty} \frac{4 \cdot \eta^5(-1/z)/\eta(-5/z) + E_2^{(5)}(-1/z)}{\eta^5(-5/z)/\eta(-1/z)} = \lim_{z \to +\infty} \frac{4 \cdot (\sqrt{-iz\eta^5(z)})/(\sqrt{-iz/5}\eta(z/5)) + z^2 E_2(z) - (z^2/5)E_2(z/5)}{(\sqrt{-iz/5}\eta^5(z/5))/(\sqrt{-iz}\eta(z))}
\]

by Lemmas \( 3 \) and \( 4 \)

\[
= -20\sqrt{5}.
\]

(iii) \( s = 1/2 \): Now that \( \frac{3}{5} \) \( \frac{0}{1} \) sends \( \infty \) to \( 1/2 \),

\[
j_{1,5}(1/2) = \lim_{z \to +\infty} \frac{4 \cdot \eta^5(z)/\eta(5z) + E_2^{(5)}(z)}{\eta^5(5z)/\eta(z)} \bigg|_{\frac{3}{5}} = \lim_{z \to +\infty} \frac{-4 \cdot \eta^5(z)/\eta(5z) + E_2^{(5)}(z)}{-\eta^5(5z)/\eta(z)} \bigg|_{\frac{0}{1}} \text{ by Lemmas } 6 \text{ and } 8
\]

\[
= 20\sqrt{5} \text{ similarly to (ii).}
\]

(iv) \( s = 2/5 \): \( \frac{2}{5} \) \( \infty \) \( 2/5 \).

\[
j_{1,5}(2/5) = \lim_{z \to +\infty} \frac{4 \cdot \eta^5(z)/\eta(5z) + E_2^{(5)}(z)}{\eta^5(5z)/\eta(z)} \bigg|_{\frac{2}{5}} = \lim_{z \to +\infty} \frac{-4 \cdot \eta^5(z)/\eta(5z) + E_2^{(5)}(z)}{-\eta^5(5z)/\eta(z)} \bigg|_{\frac{0}{1}} \text{ by Lemmas } 6 \text{ and } 8
\]

\[
= \infty \text{ (a simple pole).}
\]

4. Modular function \( j_{1,6} \)

Let us take \( \Gamma = \Gamma^1(6) \) and set \( \gamma_1 = (\frac{1}{0} 6) \), \( \gamma_2 = (\frac{1}{1} 0) \) and \( \gamma_3 = (\frac{5}{2} 12) \). If \( R_6 \) is a fundamental region of \( \Gamma^1(6) \), then \( R_6 \) is described as

\[
R_6 = \bigcap_{i=1}^{3} \text{outside}(\gamma_i^{\pm 1}).
\]

Hence we have a picture for \( R_6 \) as shown in Figure 2.

Then as we see in Figure 2, \( S_{\Gamma(6)} = \{\infty, 0, 2, 3\} \). Furthermore, it follows from Theorem \( 3 \) that \( \Gamma^1(6) \) is generated by \( \gamma_1 \), \( \gamma_2 \) and \( \gamma_3 \). Therefore we obtain the following theorem by \( 11 \).

**Theorem 10.** (i) \( S_{\Gamma(6)} = \{\infty, 0, 1/3, 1/2\} \). All cusps of \( \Gamma_1(6) \) are regular \( (16), (22) \).
(ii) \( \overline{\Gamma_1(6)} \) is generated by \( (0 \ 1) \), \( (1/0 \ 6) \) and \( (5/12 \ 3) \).

Then Table 2 shows the inequivalent cusps of \( \Gamma_1(6) \) by virtue of Lemma \( 8 \).
Thus both $j$ not $\Gamma$

$\textbf{Proof.}$ By Lemma 6, let $G$ be the series as in Lemma 6. Put $\nu$ be the series as in Lemma 6. Put $\nu(z) = 2G(z) - 2G(2z) - G(3z)$ and $Y(z) = 2G(z) - G(z)$.

$\textbf{Theorem 11.}$ (a) $X, Y \in M_2(\Gamma_1(6))$.
(b) $\mathbb{C}(X_1(6))$ is equal to $\mathbb{C}(j_{1,6}(z))$.
(c) $j_{1,6}$ takes the following value at each cusp: $j_{1,6}(\infty) = 1$, $j_{1,6}(0) = 4/3$, $j_{1,6}(1/3) = 0$, and $j_{1,6}(1/2) = 1/3$.

$\textbf{Proof.}$ By Lemma 6 $G_2(p)(z) \in M_2(\Gamma_0(p))$ for a prime $p$. Meanwhile, the identity

$$(\frac{q}{0 \ 1})^{-1} \Gamma_0(p) \left( \frac{q}{0 \ 1} \right) \cap \Gamma_0(p) = \Gamma_0(pq)$$

allows us to have $G_2(p)(z) \in M_2(\Gamma_0(pq))$. Therefore we easily get (a), from which $j_{1,6} = X/Y \in \mathbb{C}(X_1(6))$. By the $q$-expansion of $G_2$ as in [2], we derive that

$$X(z) = -8\pi^2 \cdot (q + q^2 + 3q^3 + q^4 + 6q^5 + \cdots),$$

$$Y(z) = -8\pi^2 \cdot (q - q^2 + 7q^3 - 5q^4 + 6q^5 + \cdots).$$

Thus both $X$ and $Y$ vanish at $\infty$, and the zero formula ([22], p.39) yields $\nu_0(X) = \nu_0(Y) = 2$. If $\infty$ and $w_0$ (resp. $w_0'$) are the zeros of $X$ (resp. $Y$), then $w_0$ is not $\Gamma_1(6)$-equivalent to $w_0'$. Therefore $\nu_0(j_{1,6}) = \nu_\infty(j_{1,6}) = 1$, which means that $j_{1,6}$ generates $\mathbb{C}(X_1(6))$ over $\mathbb{C}$. Next, as for the statement (c), we first recall that there are four $\Gamma_1(6)$-inequivalent cusps $\infty$, 0, 1/3 and 1/2. Put $f_1(z) = G_2(2z)$, $f_2(z) = f_1(3z)$ and $f_3(z) = G_2(3z)$. Then

$$(7) \quad X(z) = f_1(z) - f_2(z) \quad \text{and} \quad Y(z) = 2f_1(z) - f_3(z).$$
We shall then evaluate the values of \( f_i \) (i = 1, 2, 3) at each cusp. First we note that

\[
G_2^{(p)}(\infty) = \lim_{z \to \infty} G_2^{(p)}(z) = 2\zeta(2)(1 - p) \quad \text{by (2)},
\]

\[
G_2^{(p)}(0) = \lim_{z \to \infty} G_2^{(p)}(-1/z) = 2\zeta(2)(1 - 1/p) \quad \text{by (2) and (3)}.
\]

(i) Cusp values of \( f_1 \):

\[
f_1(\infty) = G_2^{(2)}(\infty) = -2\zeta(2) \quad \text{by (3)},
\]

\[
f_1(0) = G_2^{(2)}(0) = \zeta(2) \quad \text{by (4)},
\]

\[
f_1(1/3) = f_1(0) = \zeta(2) \quad \text{since } f_1 \in M_2(\Gamma_0(2)) \text{ and } 1/3 \sim 0 \text{ under } \Gamma_0(2),
\]

\[
f_1(1/2) = f_1(\infty) = -2\zeta(2) \quad \text{since } 1/2 \sim \infty \text{ under } \Gamma_0(2).
\]

(ii) Cusp values of \( f_2 \): Observe that \( f_2(z) = f_1(3z) = \frac{1}{3}f_1\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]_z \):

\[
f_2(\infty) = \lim_{z \to \infty} f_2(z) = \lim_{z \to \infty} f_1(3z) = f_1(\infty) = -2\zeta(2),
\]

\[
f_2(0) = \lim_{z \to \infty} f_2\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]_z = \lim_{z \to \infty} 3f_1\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]_z = \frac{1}{3}f_1(0) \cdot 3 \cdot \frac{1}{9} = \frac{1}{9}\zeta(2),
\]

\[
f_2(1/3) = \lim_{z \to \infty} f_2\left[\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]_z = \lim_{z \to \infty} 3f_1\left[\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]_z = \frac{1}{3}f_1(1) \cdot 3 = f_1(0) = \zeta(2),
\]

\[
f_2(1/2) = \lim_{z \to \infty} f_2\left[\begin{smallmatrix} 0 & 1 \\ 2 & 1 \end{smallmatrix}\right]_z = \lim_{z \to \infty} 3f_1\left[\begin{smallmatrix} 0 & 1 \\ 2 & 1 \end{smallmatrix}\right]_z = \frac{1}{3}f_1(3/2) \cdot 3 \cdot \frac{1}{9} = \frac{1}{9}f_1(1/2) = -\frac{2}{9}\zeta(2).
\]

(iii) Cusp values of \( f_3 \):

\[
f_3(\infty) = G_2^{(3)}(\infty) = -4\zeta(2) \quad \text{by (5)},
\]

\[
f_3(0) = G_2^{(3)}(0) = \frac{4}{3}\zeta(2) \quad \text{by (6)},
\]

\[
f_3(1/3) = f_3(\infty) = -4\zeta(2) \quad \text{since } f_3 \in M_2(\Gamma_0(3)) \text{ and } 1/3 \sim \infty \text{ under } \Gamma_0(3),
\]

\[
f_3(1/2) = f_3(0) = \frac{4}{3}\zeta(2) \quad \text{since } 1/2 \sim 0 \text{ under } \Gamma_0(3).
\]

By (i), (ii), (iii) and (7) we conclude that

\[
X(\infty) = 0, \quad Y(\infty) = 0, \quad j_{1,6}(\infty) = 1 \quad \text{(see (5) and (6))},
\]

\[
X(0) = \frac{6}{3}\zeta(2), \quad Y(0) = \frac{4}{3}\zeta(2), \quad j_{1,6}(0) = 4/3,
\]

\[
X(1/3) = 0, \quad Y(1/3) = 6\zeta(2), \quad j_{1,6}(1/3) = 0,
\]

\[
X(1/2) = -\frac{16}{3}\zeta(2), \quad Y(1/2) = -\frac{16}{3}\zeta(2), \quad j_{1,6}(1/2) = 1/3.
\]

5. Normalized generators

For a modular function \( f \), we call \( f \) normalized if its q-series is

\[
\frac{1}{q} + a_1q + a_2q^2 + \cdots.
\]

Lemma 12. The normalized generator of a genus zero function field is unique.
Proof. See \[10\], Lemma 8. \qed

We will construct the normalized generator (or the hauptmodulus) of the function field \(\mathbb{C}(X_1(N)) (N = 5, 6)\) from the modular function \(j_{1,N}(N = 5, 6)\) described in Theorem \[9\] and Theorem \[11\]. First, we note that

\[
\frac{-8}{j_{1,5}(z) + 44} = \frac{-8y}{x + 44y} = \frac{1}{q} + 5 + 10q + 5q^2 - 15q^3 - 24q^4 + 15q^5 + \cdots ,
\]

which is in \(q^{-1}\mathbb{Z}[[q]]\). This will be justified later in §6. Thus let \(N(j_{1,5}) = \frac{-8}{j_{1,5} + 44} - 5\). As for the modular function \(j_{1,6}\), we observe that

\[
\frac{2}{j_{1,6} - 1} = \frac{2Y}{X - Y} = \frac{2(G_2(z) - 4G_2(2z) + 3G_2(3z))}{2G_2(2z) - 4G_2(3z) + 2G_2(6z)} = \frac{G_2(z) - 4G_2(2z) + 3G_2(3z)}{G_2(z) - 2G_2(2z) + 2G_2(6z)} = \frac{-8\pi^2 \cdot (q - q^2 + 7q^3 - 5q^4 + \cdots)}{-8\pi^2 \cdot (q^2 - 2q^3 + 3q^4 + \cdots)} = \frac{1}{q} + 1 + 6q + 4q^2 - 3q^3 - 12q^4 - 8q^5 + \cdots ,
\]

which is also in \(q^{-1}\mathbb{Z}[[q]]\) because the \(q\)-series of \(\frac{1}{8\pi^2} \cdot (G_2(z) - 4G_2(2z) + 3G_2(3z))\) and \(\frac{1}{8\pi^2} \cdot (G_2(z) - 2G_2(2z) + 2G_2(6z))\) belong to \(\mathbb{Z}[[q]]\), and the leading coefficient of the latter series is 1. Define \(N(j_{1,6}) = \frac{2}{j_{1,6} - 1} - 1\). Then the above computation shows that \(N(j_{1,5})\) and \(N(j_{1,6})\) are the normalized generators of \(\mathbb{C}(X_1(5))\) and \(\mathbb{C}(X_1(6))\), respectively. By Theorem \[9\](c) and \[11\](c) we have Tables 3 and 4.

**Table 3. Cusp values of \(j_{1,5}\) and \(N(j_{1,5})\)**

<table>
<thead>
<tr>
<th>(s)</th>
<th>(\infty)</th>
<th>0</th>
<th>(1/2)</th>
<th>2/5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(j_{1,5}(s))</td>
<td>-44</td>
<td>-20\sqrt{5}</td>
<td>20\sqrt{5}</td>
<td>(\infty)</td>
</tr>
<tr>
<td>(N(j_{1,5})(s))</td>
<td>(\infty)</td>
<td>(\frac{1+5\sqrt{5}}{2})</td>
<td>(\frac{-1-5\sqrt{5}}{2})</td>
<td>-5</td>
</tr>
</tbody>
</table>

**Table 4. Cusp values of \(j_{1,6}\) and \(N(j_{1,6})\)**

<table>
<thead>
<tr>
<th>(s)</th>
<th>(\infty)</th>
<th>0</th>
<th>1/3</th>
<th>1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(j_{1,6}(s))</td>
<td>(1)</td>
<td>(4/3)</td>
<td>0</td>
<td>(1/3)</td>
</tr>
<tr>
<td>(N(j_{1,6})(s))</td>
<td>(\infty)</td>
<td>5</td>
<td>-3</td>
<td>-4</td>
</tr>
</tbody>
</table>

**Lemma 13.** Let \(N\) be a positive integer such that the modular curve \(X_1(N)\) is of genus 0. Let \(t\) be an element of \(\mathbb{C}(X_1(N))\) for which (i) \(\mathbb{C}(X_1(N)) = \mathbb{C}(t)\) and (ii) \(t\) has no poles except for a simple pole at one cusp \(s\). Let \(f \in \mathbb{C}(X_1(N))\). If \(f\) has a pole of order \(n\) only at \(s\), then \(f\) can be written as a polynomial in \(t\) of degree \(n\).
Proof. Take $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma \infty = s$. Let $h$ be the width of $s$. Then we have

$$t|_{\gamma} = \frac{1}{c} \frac{1}{q_h} + \cdots$$

and

$$f|_{\gamma} = b_n \frac{1}{q_h^n} + \cdots$$

for some $c \neq 0$ and $b_n \neq 0$. Thus

$$(f - b_n(ct)^n)|_{\gamma} = \lambda_{n-1} \frac{1}{q_h^{n-1}} + \cdots$$

for some $\lambda_{n-1}$, and

$$(f - b_n(ct)^n - \lambda_{n-1}(ct)^{n-1})|_{\gamma} = \lambda_{n-2} \frac{1}{q_h^{n-2}} + \cdots$$

for some $\lambda_{n-2}$. In this way we can choose $\lambda_i \in \mathbb{C}$ such that

$$(f - b_n(ct)^n - \lambda_{n-1}(ct)^{n-1} - \cdots - \lambda_1(ct))|_{\gamma} \in \mathbb{C}[[q_h]].$$ 

Let $g = f - b_n(ct)^n - \lambda_{n-1}(ct)^{n-1} - \cdots - \lambda_1(ct)$. Then $g$ has no poles in $\mathbb{H}^*$, and so $g$ must be a constant, say $\lambda_0$. Therefore we end up with $f = b_n e^{ct^n + \lambda_{n-1}e^{n-1}t^{n-1}} + \cdots + \lambda_1 ct + \lambda_0$, as desired. \hfill $\square$

**Theorem 14.** Let $d$ be a square-free positive integer and $t$ be the hauptmodulus $N(j_1,N)$ ($N = 5,6$). For $z \in \mathbb{Q}(\sqrt{-d}) \cap \mathbb{H}$, $t(z)$ is an algebraic integer.

**Proof.** Let $j(z) = \frac{1}{q} + 744 + 196884q + \cdots$ be an elliptic modular function. It is well known that $j(z)$ is an algebraic integer for $z \in \mathbb{Q}(\sqrt{-d}) \cap \mathbb{H}$ (15, 22). For algebraic proofs, see 3, 17, 21 and 23. Now, we view $j$ as a function on the modular curve $X_1(N)$. Let $s$ be a cusp of $\Gamma_1(N)$ other than $\infty$, whose width is $h_s$. Then $j$ has a pole of order $h_s$ at the cusp $s$. On the other hand, $t(z) - t(s)$ has a simple zero at $s$. Thus

$$j \times \prod_{s \in S_{\Gamma_1(N)} \setminus \{\infty\}} (t(z) - t(s))^{h_s}$$

has a pole only at $\infty$ whose degree is 12 if $N = 5$ or 6, and so by Lemma 13 it is a monic polynomial in $t$ of degree 12, which we denote by $f(t)$. With the aid of data from Tables 1, 2, 3 and 4, we can compute the product part in the above more explicitly, that is,

$$\prod_{s \in S_{\Gamma_1(N)} \setminus \{\infty\}} (t(z) - t(s))^{h_s} = \left\{ \begin{array}{ll} (t^2 - t - 31)^5(t + 5), & \text{if } N = 5 \\ (t - 5)^6(t + 3)^2(t + 4)^3, & \text{if } N = 6. \end{array} \right.$$ 

Since $j$ and $t$ have integer coefficients in the $q$-expansions, $f(t)$ is a monic polynomial in $\mathbb{Z}[t]$ of degree 12. This claims that $t(z)$ is integral over $\mathbb{Z}[j(z)]$. Therefore $t(z)$ is integral over $\mathbb{Z}$ for $z \in \mathbb{Q}(\sqrt{-d}) \cap \mathbb{H}$. \hfill $\square$
6. INTEGRALITY OF FOURIER COEFFICIENTS OF $N(j_{1,5})$

We recall that $N(j_{1,5}) = \frac{8}{j_{1,5} + 11} - 5 = \frac{8q}{x + 44y} - 5$, where $x(z) = 4 \cdot \eta^5(z)/\eta(5z) + E_2^5(z)$ and $y(z) = \eta^5(5z)/\eta(z)$. Since the $q$-series of $-8y$ and $x + 44y$ start with $-8(q + q^2 + \cdots) \in -8q\mathbb{Z}[q]$ and $-8q^2 + 32q^3 + \cdots \in q^2\mathbb{Z}[q]$, respectively, the $q$-series of $N(j_{1,5})$ is in $q^{-1}\mathbb{Z}[q]$ if all the Fourier coefficients of $x + 44y$ are divisible by 8, in which case we simply write $8\mid x + 44y$. Then

$$8 \mid x + 44y \iff 8 \mid x + 44y \iff 8 \mid 4 \cdot \eta^5(z)/\eta(5z) + 4 \cdot \eta^5(5z)/\eta(z) + E_2^5(z) \iff 2 \mid \eta^5(z)/\eta(5z) + \eta^5(5z)/\eta(z) \text{ except for the constant term}$$

because $24 \mid E_2^5(z)$ except for the constant term. Hence it suffices to show that $2 \mid \eta^5(z)/\eta(5z) + \eta^5(5z)/\eta(z)$ except for the constant term.

Let $\Delta^n$ be the set of $2 \times 2$ integer matrices $(a\ b\ c\ d)$, where $a \in 1 + NZ, c \in NZ$, and $ad - bc = n$. For $f \in M_k(\Gamma_1(N))$ we define the Hecke operator $T_n$ by

$$f |_{T_n} = n^{(k/2) - 1} \sum f|_{\Gamma_1(N)}\sigma_a \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right)_{k},$$

where $\Gamma_1(N)\sigma_j$ runs through the right cosets of $\Gamma_1(N)$ in $\Delta^n$. Then $T_n$ preserves the space $M_k(\Gamma_0(N)), \chi$ for a Dirichlet character $\chi$ (§5). Let $W_N(f) = f|_{\Gamma_0(N)}\chi_k$ be the action of the Fricke involution on $f$.

**Lemma 15.** Let $n$ be a positive integer prime to $N$ and let $f \in M_k(\Gamma_0(N)), \chi$ for a Dirichlet character $\chi$. Then we have $W_N \circ T_n(f) = \chi(n)T_n \circ W_N(f)$.

**Proof.** $\Delta^n$ has the following right coset decomposition (see [13, 16, 22]):

$$\Delta^n = \bigcup_{n}^{n^{-1}} \bigcup_{i=0}^{\frac{n}{n/a}} \Gamma_1(N)\sigma_a \left( \begin{array}{cc} a & i \\ 0 & a \end{array} \right),$$

where $\sigma_a \in SL_2(\mathbb{Z})$ such that $\sigma_a \equiv \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & a \end{array} \right) \mod N$. By (10) and (11),

$$T_n \circ W_N(f) = n^{(k/2) - 1} \sum_{a,b} f|_{\alpha_n\sigma_a \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right)}_{k},$$

where $\alpha_N = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$. Let $\alpha_{a,b} = \sigma_a \sigma_{a,\alpha_N} \sigma_a \left( \begin{array}{cc} a & b \\ 0 & n/a \end{array} \right) \sigma_N^{-1} \in \Delta^n$. Then it is easy to show that $\alpha_{a,b}$ are in distinct cosets of $\Gamma_1(N)$ in $\Delta^n$, and hence form a set of representatives; so by (10),

$$T_n \circ W_N(f) = n^{(k/2) - 1} \sum_{a,b} f|_{\alpha_{a,b}\alpha_N} = n^{(k/2) - 1} \sum_{a,b} f|_{\sigma_{a,\alpha_N} \sigma_a \left( \begin{array}{cc} a & b \\ 0 & n/a \end{array} \right)}_{k} = \chi(n)T_n(W_N(f)) \text{ since } f|_{\sigma_n} = \chi(n)f.$$ 

This completes the proof. \qed

Next, we observe that

$$M_2(\Gamma_1(5)) = \bigoplus_{\chi \in (\mathbb{Z}/5\mathbb{Z})^\times} M_2(\Gamma_0(5), \chi).$$

Since $(\mathbb{Z}/5\mathbb{Z})^\times$ is generated by 2 (= 2 mod 5\mathbb{Z}), any $\chi \in (\mathbb{Z}/5\mathbb{Z})^\times$ is determined by the value at 2. Let $\chi_1$ be the character such that $\chi_1(2) = i$. Then $(\mathbb{Z}/5\mathbb{Z})^\times$
is generated by $\chi_1$ so that $\chi_1^4 = \chi_{\text{triv}}$ and $\chi_1^2 = \left(\frac{5}{2}\right)$. Note that if $\chi$ is an odd character, then $M_2(\Gamma_0(5), \chi) = \{0\}$. Thus

\begin{equation}
M_2(\Gamma_1(5)) = M_2(\Gamma_0(5)) \oplus M_2(\Gamma_0(5), \left(\frac{5}{2}\right)).
\end{equation}

Now that the dimension of the space $M_2(\Gamma)$ is equal to $\sigma_\infty(\Gamma) - 1$, it follows from (12) that $M_2(\Gamma_0(5), \left(\frac{5}{2}\right))$ is two dimensional. In fact it is generated by $\eta^5(z)/\eta(5z)$ and $\eta^5(5z)/\eta(z)$. It then follows from the proof of Lemma 8(ii) that

\begin{equation}
W_5(\eta^5(z)/\eta(5z)) = -5\sqrt{5} \cdot \eta^5(5z)/\eta(z).
\end{equation}

The fact that $W_5$ is an involution and (13) imply that

\begin{equation}
W_5(\eta^5(5z)/\eta(z)) = -5\sqrt{5}^{-1} \cdot \eta^5(z)/\eta(5z).
\end{equation}

Since $T_m$ preserves $M_k(\Gamma_0(N), \chi)$, we may set

\begin{equation}
T_m(\eta^5(z)/\eta(5z)) = p_m \cdot \eta^5(z)/\eta(5z) + q_m \cdot \eta^5(5z)/\eta(z)
\end{equation}

and

\begin{equation}
T_m(\eta^5(5z)/\eta(z)) = r_m \cdot \eta^5(z)/\eta(5z) + s_m \cdot \eta^5(5z)/\eta(z)
\end{equation}

for $p_m, q_m, r_m, s_m \in \mathbb{C}$. Here, we recall from [13], p.163 that if $f(z) = \sum a_n q^n$ and $T_m(f(z)) = \sum b_n q^n$, then

\begin{equation}
b_n = \sum_{d||m, n \atop d > 0} \chi(d) d^{k-1} a_{mn/d^2}.
\end{equation}

If we compare the constant terms in (15), we get $r_m = 0$. In like manner, from (14) we have

\begin{equation}
p_m = \sum_{d||m \atop d > 0} \left(\frac{d}{5}\right) d^{k-1} \cdot 1.
\end{equation}

When $(m, 5) = 1$, by Lemma 13 we obtain

\begin{equation}
T_m \circ W_5 \left(\frac{\eta^5(z)}{\eta(5z)}\right) = \left(\frac{m}{5}\right) W_5 \circ T_m \left(\frac{\eta^5(z)}{\eta(5z)}\right). \tag{16}
\end{equation}

Then, by (13) the LHS of the above is equal to

\begin{equation}
\text{LHS} = -5\sqrt{5} \cdot T_m \left(\frac{\eta^5(5z)}{\eta(z)}\right) = -5\sqrt{5} \left(s_m \cdot \frac{\eta^5(5z)}{\eta(z)}\right).
\end{equation}

On the other hand, the RHS is equal to

\begin{equation}
\text{RHS} = \left(\frac{m}{5}\right) W_5 \left(p_m \cdot \frac{\eta^5(z)}{\eta(5z)} + q_m \cdot \frac{\eta^5(5z)}{\eta(z)}\right) = \left(\frac{m}{5}\right) \left[-5\sqrt{5} \cdot p_m \cdot \frac{\eta^5(z)}{\eta(5z)} + (-5\sqrt{5})^{-1} q_m \cdot \frac{\eta^5(z)}{\eta(5z)}\right].
\end{equation}

Hence, by equating both sides we deduce that $q_m = 0$ and $s_m = \left(\frac{m}{5}\right) p_m = \left(\frac{m}{5}\right) \cdot \sum_{d|m \atop d > 0} \left(\frac{d}{5}\right) d^{k-1}$ by (16). Therefore for each positive integer $m$ prime to 5, it follows
that
\[ T_m \left( \frac{\eta^5(z)}{\eta(5z)} \right) = p_m \cdot \frac{\eta^5(z)}{\eta(5z)} \]
and
\[ T_m \left( \frac{\eta^5(5z)}{\eta(z)} \right) = \left( \frac{m}{5} \right) p_m \cdot \frac{\eta^5(5z)}{\eta(z)}. \]

Let \( \frac{\eta^5(z)}{\eta(5z)} = \sum c_m q^m \) and \( \frac{\eta^5(5z)}{\eta(z)} = \sum d_m q^m \). If we compare the \( q^1 \)-coefficients in (17) and (18), then we get
\[ c_m = -5 \cdot p_m, \quad d_m = \left( \frac{m}{5} \right) p_m \quad \text{for} \quad (m, 5) = 1. \]

Now, let \( m = 5 \). It then follows from (17) that \( p_5 = 1 \). Moreover in (17) and (18), by comparing the \( q^1 \)-coefficients, we have \( q_5 = 0 \) and \( s_5 = 5 \). More generally, we take \( m = 5^l \cdot m_0 \) with \( l \geq 0 \) and \( 5 \nmid m_0 \). Then
\[ T_{5^l \cdot m_0} \left( \frac{\eta^5(z)}{\eta(5z)} \right) = T_{5^l} \circ T_{m_0} \left( \frac{\eta^5(z)}{\eta(5z)} \right) = T_{5^l} \left( p_{m_0} \cdot \frac{\eta^5(z)}{\eta(5z)} \right) \quad \text{by (19)} \]
\[ = (T_5)^l \left( p_{m_0} \cdot \frac{\eta^5(z)}{\eta(5z)} \right) = p_{m_0} \cdot p_5^l \cdot \frac{\eta^5(z)}{\eta(5z)} \]
\[ = p_{m_0} \cdot \frac{\eta^5(z)}{\eta(5z)} \quad \text{since} \quad p_5 = 1. \]

Similarly,
\[ T_{5^l \cdot m_0} \left( \frac{\eta^5(5z)}{\eta(z)} \right) = \left( \frac{m_0}{5} \right) p_{m_0} \cdot 5^l \cdot \frac{\eta^5(5z)}{\eta(z)}. \]

In the equations (20) and (21), if we compare the \( q^1 \)-coefficients, we obtain
\[ c_{5^l \cdot m_0} = -5 \cdot p_{m_0} \quad \text{and} \quad d_{5^l \cdot m_0} = 5^l \cdot \left( \frac{m_0}{5} \right) \cdot p_{m_0} \]
with \( p_{m_0} = \sum_{d|m_0} \left( \frac{d}{5} \right) d^{k-1} \). It is clear that 2 divides \( c_{5^l \cdot m_0} \), \( d_{5^l \cdot m_0} \); hence we conclude that
\[ 2 \cdot \frac{\eta^5(z)}{\eta(5z)} + \frac{\eta^5(5z)}{\eta(z)} \]
extcept for the constant term.

7. Relationship with moduli of elliptic curves

When \( k \) is a field of characteristic prime to \( N \), the \( k \)-rational points on the curve \( X_0(N) \) (\( X_1(N) \), respectively) parametrize pairs \( (E, C) \) (pairs \( (E, P) \), respectively), modulo equivalence over an algebraic closure \( k^{alg} \), of elliptic curves \( E \) with a \( k \)-rational cyclic subgroup \( C \) (\( k \)-rational point \( P \), respectively) of order \( N \). There are “forgetful” maps \( X_1(N) \) to \( X_0(N) \) which send \( (E, P) \to (E, C) \) in terms of the subgroup \( C = \{ P, [2]P, \ldots, [N]P \} \). There are two diagrams of interest coming from these “forgetful” maps:
All of these curves have genus zero, but some of these modular curves are easier to describe than others. For example, there is a canonical bijection $\mathbb{P}^1 \rightarrow X(1)$ of the "j-line" which sends $j \mapsto (E_j, O_j)$ in terms of the normal form

$$E_j : y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$$

with a specified base point $O_j = (0 : 1 : 0)$. Clearly the function field of $X(1)$ is $k(j)$.

Similarly, there are canonical bijections $\mathbb{P}^1 \rightarrow X_1(N)$ which send $t \mapsto (E_t, P_t)$ in terms of the Tate normal forms

$$E_t : \begin{cases} y^2 = x^3 + 2x^2 + tx, & \text{if } N = 2; \\ y^2 + 3xy + ty = x^3, & \text{if } N = 3; \\ y^2 + (1+t)xy + ty = x^3 + tx^2, & \text{if } N = 5; \\ y^2 + (1+t)xy + (t-t^2)y = x^3 + (t-t^2)x^2, & \text{if } N = 6, \end{cases}$$

each with a specified point $P_t = (0 : 0 : 1)$ of order $N$. Such formulas can be found in [B pp.94-95]. Using the "forgetful" maps $X_1(N)$ to $X(1)$, one has the expressions

$$j = \begin{cases} 64(4-3t)^3/(t^2(1-t)), & \text{if } N = 2; \\ 27(9-8t)^3/(t^3(1-t)), & \text{if } N = 3; \\ (1-12t+14t^2+12t^3+t^4)^3/(t^5(1-11t-t^2)), & \text{if } N = 5; \\ ((1-3t)(1-9t+3t^2-3t^3))^3/(t^6(1-t)^3(1-9t)), & \text{if } N = 6. \end{cases}$$

Clearly the function field of $X_1(N)$ is $k(t)$ in these cases; it may be thought of as an algebraic extension of $k(j)$. When the parameter $t$ is interpreted as a modular function $t(z)$, we can find the following identities between our modular function $N(j_{1,N})(z)$ and $t(z)$.

**Theorem 16.** (i) $N(j_{1,5})(z) + 5 = \frac{t^2(z)+1}{-t(z)+\varepsilon^2}$. 
(ii) $N(j_{1,6})(z) + 1 = \frac{1+3t(z)}{1-9t(z)}$.

Here we set $\varepsilon = \zeta_5 + \zeta_5^{-1}$. 

---

[Image of the diagram]
Proof. (i) First we note that \( \varepsilon \) satisfies \( \varepsilon^2 + \varepsilon - 1 = 0 \). Since \( \varepsilon = 2 \cos(2 \pi / 5) > 0 \), we have \( \varepsilon = \frac{-1 + \sqrt{5}}{2} \) and hence \( \varepsilon^5 = \frac{-11 + 5 \sqrt{5}}{2} \). Let \( f(z) = N(j_{1,5})(z) + 5 \). The values of \( f(z) \) at the cusps (obtained from Table 3) are shown in Table 5. Since \( \Delta(E_t) = -t^5(t^2 + 11t - 1) \) from the equation of \( E_t \) in (22), the set of possible values of \( t(z) \) at the cusps are \( \{\infty, 0, \varepsilon^5, -\varepsilon^{-5}\} \). Since \( t(z) \) is a fractional linear transformation of \( f(z) \), we come up with

\[
(f(\infty), f(2/5), f(1/2), f(0)) = [t_1, t_2, t_3, t_4],
\]

\[
[\infty, 0, -\varepsilon^5, f(z)] = [t_1, t_2, t_3, t(z)],
\]

where \( t_1 = t(\infty), t_2 = t(2/5), t_3 = t(1/2), t_4 = t(0) \). Thus we obtain that

\[
\frac{(t(z) - t_1)(t_2 - t_3)}{(t(z) - t_3)(t_2 - t_1)} = \frac{\varepsilon^5}{f(z) + \varepsilon^5}.
\]

Suppose \( t(z) \) has a pole or zero at a cusp \( s \). Let \( h \) be the width of the cusp \( s \). Considering the \( q_h \)-expansion of \( t(z) \) at \( s \) we see from the identity

\[
j = \frac{(1 - 12t + 14t^2 + 12t^3 + t^4)^3}{t^5(1 - 11t - t^2)}
\]

that \( \frac{1}{q} + O(1) = \frac{1}{q_h} + O(1) \). This yields \( h = 5 \). It then follows from Table 1 that \( s = 1/2 \) or \( s = 0 \). This means that \( t_3, t_4 \in \{\infty, 0\} \) and so \( t_1, t_2 \in \{\varepsilon^5, -\varepsilon^{-5}\} \). There are four possibilities for the cusp values \( t(s) \):

Case (i). \( t_1 = \varepsilon^5, t_2 = -\varepsilon^{-5}, t_3 = 0, t_4 = \infty \),

Case (ii). \( t_1 = \varepsilon^5, t_2 = -\varepsilon^{-5}, t_3 = \infty, t_4 = 0 \),

Case (iii). \( t_1 = -\varepsilon^{-5}, t_2 = \varepsilon^5, t_3 = 0, t_4 = \infty \),

Case (iv). \( t_1 = -\varepsilon^{-5}, t_2 = \varepsilon^5, t_3 = \infty, t_4 = 0 \).

We see by a routine check that only the second and third case satisfy the identity (23), from which we conclude that \( t(z) \) should be either

\[
u(z) = \frac{\varepsilon^5 f(z) - 1}{f(z) + \varepsilon^5} \quad \text{or} \quad v(z) = \frac{f(z) + \varepsilon^5}{-\varepsilon^5 f(z) + 1}.
\]

Now we consider the elliptic curve \( E_1 : y^2 + 2xy - y = x^3 + x^2 \). By making an appropriate change of variables we achieve the elliptic curve

\[
E : y^2 = 4x^3 - \frac{4}{3}x + \frac{19}{27},
\]

which is isomorphic to \( E_1 \). We note that under this isomorphism the point \( P_1 = (0, 0) \in E_1 \) is sent to \((2/3, -1) \in E \). The period lattice \( L \) of \( E \) is given by \( L = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} \) with

\[
\omega_1 = 6.346046521397767108443973083772736526087 \cdots,
\]

\[
\omega_2 = 3.173023260698883554221986541886683682630438 \cdots + 1.458816616938495229330889612903675257158 \cdots i
\]

Table 5

<table>
<thead>
<tr>
<th>( s )</th>
<th>( \infty )</th>
<th>( 2/5 )</th>
<th>( 1/2 )</th>
<th>( 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(s) )</td>
<td>( \infty )</td>
<td>( 0 )</td>
<td>( -\varepsilon^5 )</td>
<td>( \varepsilon^{-5} )</td>
</tr>
</tbody>
</table>
from which we can estimate that
\[ g_2(L) = 1.33333 \cdots, \quad g_3(L) = -0.703703703 \cdots, \]
\[ \mathcal{P}(\omega_1/5, L) = 0.66666 \cdots, \quad \mathcal{P}'(\omega_1/5, L) = -1.00000 \cdots. \]
Here \( \mathcal{P}(z, L) \) stands for the Weierstrass \( \mathcal{P} \)-function attached to the lattice \( L \). Thus it turns out that the point of \( X_1(5) \) corresponding to the pair \((E_1, P_1)\) is \( \omega_2/\omega_1 \). Using the Fourier expansion of \( f(z) \) we can find \( u(\omega_2/\omega_1) = 1.00000 \cdots \) and \( v(\omega_2/\omega_1) = -1.00000 \cdots \). Therefore we are forced to have \( t(z) = u(z) \).

(ii) Let \( g(z) = N(j_{1,6})(z) + 1 \). Then it is immediate from Table 4 that the values of \( g(z) \) at the cusps of \( X_1(6) \) are as shown in Table 6. Since \( \Delta(E_1) = (t-1)^6(t^6 + 9t - 1) \) from the equation of \( E_1 \) in (22), the set of possible values of \( t(z) \) at the cusps are \( \{\infty, 1, 0, 1/9\} \). Since \( t(z) \) is a fractional linear transformation of \( g(z) \), we have the equality
\[ (g(\infty), g(0), g(1/3), g(1/2)) = [t_1, t_2, t_3, t_4], \]
\[ \{\infty, 6, -2, g(z)\} = [t_1, t_2, t_3, t_4], \]
where \( t_1 = t(\infty), t_2 = t(0), t_3 = t(1/3), t_4 = t(1/2) \). Thus we establish
\[ (t(z) - t_1)/(t(z) - t_3) = \frac{8}{g(z) + 2}. \]
Suppose \( t(s) = \infty \) for some cusp \( s \). We let \( h \) be the width of the cusp \( s \) and consider the \( q_h \)-expansion of \( t(z) \) at \( s \). We choose an element \( \gamma \in SL_2(\mathbb{Z}) \) such that \( \gamma \infty = s \). It then follows that \( t|_\gamma = \frac{a}{q} + O(1) \) for some \( c \in \mathbb{C} \). Now, from the identity
\[ j = \frac{(1 - 3t)(1 - 9t + 3t^2 - 3t^3)}{t^6(1 - t)^3(1 - 9t)}, \]
we see that \( \frac{1}{q} + O(1) = \frac{1}{q_h} + O(1) \). This yields \( h = 2 \). It then follows from Table 2 that \( s = 1/3 \) and hence \( t_3 = t(1/3) = \infty \). Similarly if \( t(s) = 0 \), then we come up with \( \frac{1}{q} + O(1) = \frac{1}{q_h} + O(1) \). Thus we have \( h = 6 \) and \( s = 0 \), and we deduce that \( t_2 = t(0) = 0 \). Therefore, the identity (24) is simplified to
\[ \frac{t(z) - t_1}{-t_4} = \frac{8}{g(z) + 2}. \]
Here we have two choices for the values \( t_1 \) and \( t_4 \): \( t_1 = 1 \) and \( t_4 = 1/9 \), or \( t_1 = 1/9 \) and \( t_4 = 1 \). Only the latter case fits the identity (25), from which we get the assertion as desired. \( \square \)

According to the referee’s comment we can have canonical bijections \( \mathbb{P}^1 \rightarrow X_0(N) \) which send \( r \mapsto (E_r, C_r) \) in terms of the normal forms
\[
E_r: \begin{cases}
y^2 = x^3 + \frac{2(r+64)}{r^2} x^2 + \frac{r+64}{r^3} x, & \text{if } N = 2; \\
y^2 + \frac{3(r+27)}{r} xy + \frac{r+27}{r^2} y = x^3, & \text{if } N = 3; \\
y^2 + \frac{2(r+25)}{r} xy + \frac{r+25}{r^2} y = x^3 + \frac{r+10}{r^2} y^2, & \text{if } N = 5; \\
y^2 + \frac{5(r+36)}{r} xy + \frac{9(r+8)(r+9)}{r^2} y = x^3 + \frac{2(r+9)}{r^2} x^2, & \text{if } N = 6;
\end{cases}
\]
and cyclic subgroups $C_r = \langle (x : y : 1) \mid \psi_r(x) = 0 \rangle$ of order $N$ which are generated by the roots of certain divisors of the division polynomials:

$$
\psi_r(x) = \begin{cases} 
  x & \text{if } N = 2; \\
  x & \text{if } N = 3; \\
  5x^2 - \frac{4(r^2 + 22 + 125)}{r^2} & \text{if } N = 5; \\
  x & \text{if } N = 6.
\end{cases}
$$

Using the “forgetful” maps $X_1(N) \to X_0(N)$, one has the expressions

$$
r = \begin{cases} 
  64t/(1-t), & \text{if } N = 2; \\
  27t/(1-t), & \text{if } N = 3; \\
  125t/(1 - 11t - t^2), & \text{if } N = 5; \\
  72t/(1 - 9t), & \text{if } N = 6.
\end{cases}
$$

Clearly the function field of $X_0(N)$ is $k(r)$ in these cases; it may be thought of as an algebraic extension of $k(j)$ which is contained in $k(t)$. These curves are chosen on the parameter $r$. For $z \in \mathfrak{h}$, define the hauptmoduli

$$
r(z) = \begin{cases} 
  \left( \frac{\eta(z)}{\eta(2z)} \right)^{24} = \frac{1}{q} - 24 + 276q - 2048q^2 + \cdots & \text{if } N = 2; \\
  \left( \frac{\eta(z)}{\eta(3z)} \right)^{12} = \frac{1}{q} - 12 + 54q - 76q^2 + \cdots & \text{if } N = 3; \\
  \left( \frac{\eta(z)}{\eta(5z)} \right)^{6} = \frac{1}{q} - 6 + 9q + 10q^2 + \cdots & \text{if } N = 5; \\
  \frac{\eta(z)^5 \eta(3z)}{\eta(2z) \eta(6z)^5} = \frac{1}{q} - 5 + 6q + 4q^2 + \cdots & \text{if } N = 6,
\end{cases}
$$

in terms of the Dedekind eta function

$$
\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{for } q = e^{2\pi iz}.
$$

We may summarize all of this discussion in a lattice diagram of function fields. As for $X_1(5)$, the “forgetful” maps correspond to the following for a field of $k$ of characteristic not dividing 5:

$$
\begin{array}{ccc}
X_1(5) & \to & t \\
\downarrow & & \downarrow \\
X_0(5) & \to & k(t) \\
\downarrow & & \downarrow \\
X(1) & \to & j = \frac{(x^2 + 250 + 3125)^3}{r^6} \\
\end{array}
$$
For $X_1(6)$, the “forgetful” maps correspond to the following for a field of $k$ of characteristic not dividing 6:

$$\begin{align*}
X_1(6) & \quad t & \quad k(t) \\
X_1(2) & \quad X_1(3) & \quad k(r) \\
X_0(6) & \quad X_0(2) & \quad X_0(3) \\
X(1) & \quad j = \frac{(r+12)^3(r^3+252r^2+3888r+15552)^3}{r^6(r+8)(r+9)^3} \\
& \quad r = \frac{72t}{1-9t} \\
& \quad k(j) = k\left(\frac{r^3(r+8)}{(r+9)^3}\right) \\
& \quad k\left(\frac{27(r^2-t^2)}{(3t-1)^3}\right)
\end{align*}$$

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References


DEPARTMENT OF MATHEMATICS, HANYANG UNIVERSITY, SEOUL, 133-791 KOREA
E-mail address: chhkim@hanyang.ac.kr

DEPARTMENT OF MATHEMATICAL SCIENCES, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY, DAEGU, 305-701 KOREA
E-mail address: jkkoo@math.kaist.ac.kr