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SOME COMPLETELY MONOTONIC FUNCTIONS OF POSITIVE ORDER

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Abstract. We completely determine the set of $(\alpha, \beta) \in \mathbb{R}^2$ for which the function $\frac{e^{\alpha x} - e^{\beta x}}{e^x - 1}$ is convex on $(0, \infty)$ and use this result to give some special classes of completely monotonic functions of positive order related to gamma and psi functions.

1. Introduction

Euler's gamma function $\Gamma(x)$ is defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ for $x > 0$. Its logarithmic derivative $\psi(x) = \Gamma'(x)/\Gamma(x)$ is called the psi or digamma function, and the derivatives $\psi^{(n)}(x)$ are called polygamma functions. In this paper we shall extend and strengthen some of the results obtained in [4] regarding inequalities for ratios of gamma functions and differences of digamma and polygamma functions. For background information and an extensive bibliography concerning such inequalities, we refer to [4].

Many of the inequalities presented in [4] are obtained by verifying the complete monotonicity of certain functions. A function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic if it has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, \text{ for all } x > 0 \text{ and } n \geq 0. \tag{1.1}$$

J. Dubourdieu [1] proved that if a nonconstant function f is completely monotonic, then strict inequality holds in (1.1). See also [2] for a simpler proof of this result. A necessary and sufficient condition for complete monotonicity is given by Bernstein's theorem (see [11, p. 161]), which states that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

In [6], Koumandos and Pedersen called a function f completely monotonic of order $n = 0, 1, 2, \ldots$ if $x^n f(x)$ is completely monotonic on $(0, \infty)$. Thus, completely monotonic functions of order 0 are the classical completely monotonic functions,

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while completely monotonic functions of order 1 are the strongly completely monotonic functions that have been introduced in [10]. It is easy to see that a function f is completely monotonic if $xf(x)$ is completely monotonic, and therefore a function that is completely monotonic of order n is completely monotonic of order $m = 0, 1, \ldots, n-1$.

In [6, Thm. 1.3], it is furthermore shown that a function f is completely monotonic of order $n \geq 1$ on $(0, \infty)$ if, and only if,

$$f(x) = \int_0^\infty e^{-xt} p(t) dt,$$

where the integral converges for all $x > 0$ and where p is $n-1$ times differentiable on $[0, \infty)$ with $p^{(n-1)}(t) = \mu([0, t])$ for some Radon measure μ and $p^{(k)}(0) = 0$ for $0 \leq k \leq n-2$. This has already been proven for the case $n = 1$ in [10, Thm. 1] and for the case $n = 2$ in [4, Lem. 2] and will be used here in the case $n = 3$ in order to strengthen some of the results obtained in [4].

The applications of [4, Lem. 2] that are presented in [4] lead to the question for which $(\alpha, \beta) \in \mathbb{R}^2$ the function $f_{\alpha,\beta}(x)$, defined by

$$f_{\alpha,\beta}(x) := \frac{e^{\alpha x} - e^{\beta x}}{e^x - 1} \quad \text{for} \quad x \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad f_{\alpha,\beta}(0) = \alpha - \beta,$$

is convex in $(0, \infty)$. We will give a complete solution to this question and thus extend some further results from [4].

In order to state our results, for α, $\beta \in \mathbb{R}$ set

$$\begin{aligned}
\varepsilon_1(\alpha, \beta) &:= 2\alpha\beta + 2\alpha^2 - 3\alpha + 2\beta^2 - 3\beta + 1, \\
\varepsilon_2(\alpha, \beta) &:= 4\alpha^2\beta^2 - 4\alpha^2\beta - 4\alpha\beta^2 + 4\alpha\beta - \alpha^2 + \alpha - \beta^2 + \beta, \\
\varepsilon_3(\alpha, \beta) &:= (\alpha - \tfrac{1}{2})^2 + (\beta - \tfrac{1}{2})^2 - \tfrac{1}{2},
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_1 &:= \{(\alpha, \beta) : 0 \leq \beta \leq 1 < \alpha\} \cap \{(\alpha, \beta) : \varepsilon_1(\alpha, \beta) = 0\}, \\
\Gamma_2 &:= \{(\alpha, \beta) : 0 \leq \beta \leq 1 \leq \alpha\} \cap \{(\alpha, \beta) : \varepsilon_2(\alpha, \beta) = 0\}, \\
\Gamma_3 &:= \{(\alpha, \beta) : \beta \leq \tfrac{1}{2} - |\alpha - \tfrac{1}{2}|\} \cap \{(\alpha, \beta) : \varepsilon_1(\alpha, \beta) = 0\},
\end{aligned}$$

and let C and D be the open bounded sets whose boundaries are given by the Jordan curves $\Gamma_1 \cup \Gamma_2 \cup \{(1, \beta) : \frac{1}{2} \leq \beta \leq 1\}$ and $\Gamma_2 \cup \Gamma_3 \cup \{(\alpha, \alpha) : \frac{1}{6}(3 - \sqrt{3}) \leq \alpha \leq 1\}$, respectively. Let H denote the half-plane $\{(\alpha, \beta) : \beta \leq \alpha\}$ and set

$$A := (H \cup \{(\alpha, 1) : 0 \leq \alpha \leq \tfrac{1}{2}\}) \setminus D$$

and $B := D \setminus (C \cup \{(1, \beta) : 0 \leq \beta \leq \frac{1}{2}\})$. (Cf. Figure 1. The graphs in this paper have been created by using the KETpic package for Maple [3].)

It is perhaps interesting to note that the set $\{(\alpha, \beta) : \varepsilon_1(\alpha, \beta) = 0\}$ describes an ellipse with center $(\frac{1}{2}, \frac{1}{2})$ and semi-axes $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{6}}$ with the major axis forming an angle of $-\frac{\pi}{4}$ with the α-axis. Γ_1 and Γ_3 are therefore elliptical arcs (Γ_3 is even a quarter-ellipse).

Continuing with necessary definitions, for α, $\beta \in \mathbb{R}$ and $t > 0$, define $g_{\alpha,\beta}(t) := g_\alpha(t) - g_\beta(t)$, where

$$g_\alpha(t) := t^{\alpha-1} \left[(1-\alpha)^2 t^2 + (1 + 2\alpha - 2\alpha^2) t + \alpha^2 \right] - t - 1,$$

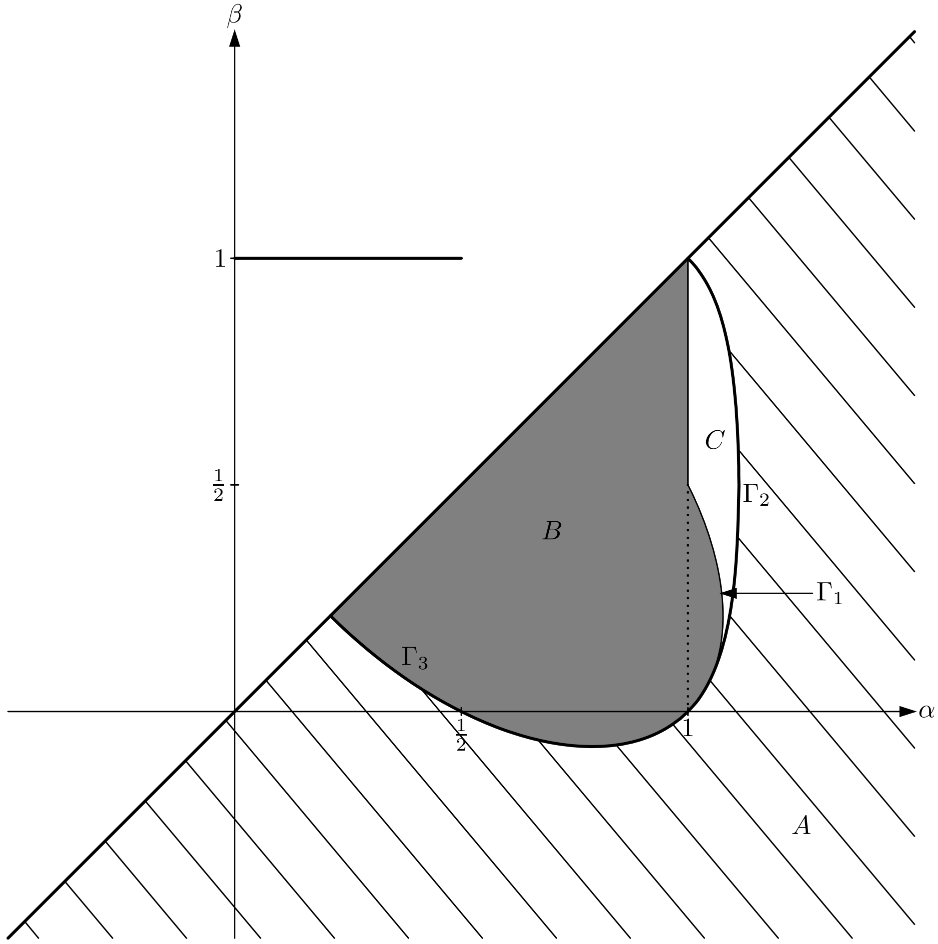

FIGURE 1. The sets A, B and C. The bold curves are ∂A. Note that $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 = (1,0)$ and that the dotted line $\{(1,\beta) : 0 < \beta \le \frac{1}{2}\}$ belongs to neither A, B or C, but to A^*.

and, for α, $\beta \in \mathbb{R} \setminus \{1\}$ with $\varepsilon_2(\alpha,\beta)\varepsilon_3(\alpha,\beta) \le 0$,

$$t^*(\alpha,\beta) = \frac{\varepsilon_2(\alpha,\beta) - 2\alpha\beta(1-\alpha)(1-\beta) + \sqrt{-\varepsilon_2(\alpha,\beta)\varepsilon_3(\alpha,\beta)}}{2(1-\alpha)^2(1-\beta)^2}.$$

Finally, for any set $M \subset \mathbb{R}^2$, let M^* denote its reflection with respect to the straight line $\{(\alpha,\alpha) : \alpha \in \mathbb{R}\}$.

Theorem 1.1. (1) *For $(\alpha,\beta) \in A$ the function $f_{\alpha,\beta}(x)$ is convex in $(0,\infty)$ and for $(\alpha,\beta) \in A^*$ it is concave there.*

(2) *For $(\alpha,\beta) \in B \cup B^*$ the function $f''_{\alpha,\beta}(x)$ changes sign in $(0,\infty)$.*

(3) *In $C \cup C^*$ the sign of $f''_{\alpha,\beta}(x)$ is constant in $(0,\infty)$ if, and only if, $(\alpha,\beta) \in C_{conv} \cup C^*_{conv}$, where*

$$C_{conv} := \{(\alpha,\beta) \in C : g_{\alpha,\beta}(t^*(\alpha,\beta)) \ge 0\}.$$

$f_{\alpha,\beta}(x)$ is convex in $(0,\infty)$ if $(\alpha,\beta) \in C_{conv}$ and concave if $(\alpha,\beta) \in C^*_{conv}$ (cf. Figure 2).

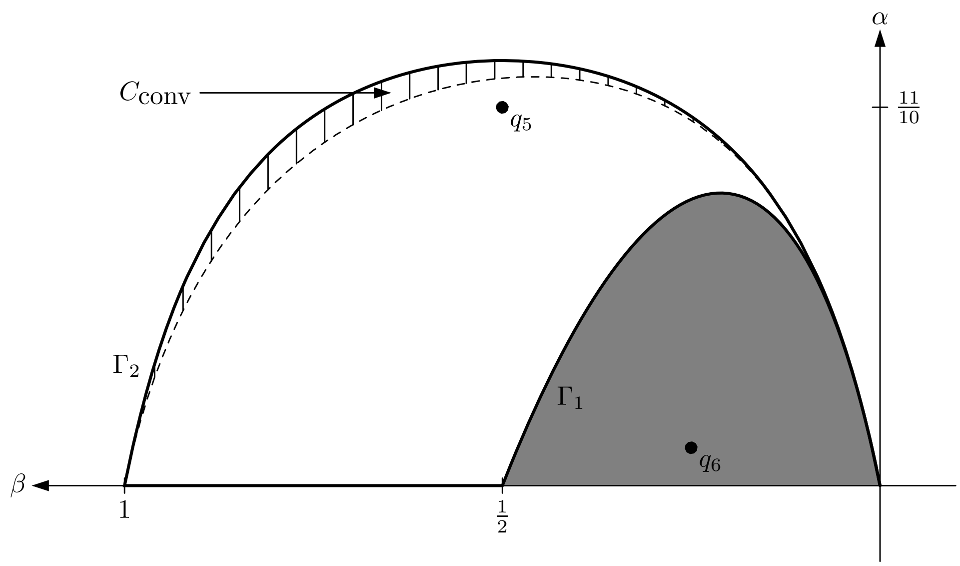

FIGURE 2. The set C. The bold curves are ∂C. The dashed curve describes the set of points $(\alpha, \beta) \in C$ for which $g_{\alpha,\beta}(t^*(\alpha, \beta)) = 0$.

Let S_1 denote the set of $(s,t) \in \mathbb{R}^2$ such that $(1-t, 1-s) \in A \cup C_{\text{conv}}$ and set

$$T := (\{(s,t) : 0 \le s, t\} \cup \{(s,t) : t \le 1-s\}) \setminus \{(s,t) : 0 < t < 1-s < 1\}$$

and $S_2 := S_1 \cap T$.

Theorem 1.1 will allow us to prove the following extensions of results from [4].

Theorem 1.2. (1) *Let*

$$L(x) := x - \frac{\Gamma(x+t)}{\Gamma(x+s)} x^{s-t+1}.$$

Then

$$\Phi(x) := -\frac{\Gamma(x+s)}{\Gamma(x+t)} x^{t-s-1} L''(x)$$

is completely monotonic of order 2 on $(0,\infty)$ for all $(s,t) \in S_1$, and for $(s,t) \in S_2$ the function $L'(x)$ is completely monotonic on $(0,\infty)$. In particular, for $(s,t) \in S_2$, the function $L(x)$ is strictly increasing and concave on $(0,\infty)$ and the inequality

$$0 < x - \frac{\Gamma(x+t)}{\Gamma(x+s)} x^{s-t+1} < \frac{1}{2}(s-t)(s+t-1) \tag{1.2}$$

holds for all $x > 0$ (cf. [4, Thm. 1]*).*

(2) *For $(s,t) \in S_2$ the inequality*

$$\psi(x+t) - \psi(x+s) + \frac{s-t+1}{x} < \frac{\Gamma(x+s)}{\Gamma(x+t)} x^{t-s-1}$$

holds for all $x > 0$ and the function

$$\psi(x+s) - \psi(x+t) - \frac{s-t}{x} + \frac{(s-t)(s+t-1)}{2x^2}$$

is completely monotonic in $(0,\infty)$ for all $(s,t) \in S_1$ (cf. [4, Cor. 1]*).*

(3) *For* $m,\, n \in \mathbb{N}$ *with* $m > n$, *let*

$$U_{n,m}(x) := \sum_{k=n}^{m} \frac{(t)_k}{(s)_k} e^{ikx}, \quad V_{n,m}(x) := \frac{\Gamma(s)}{\Gamma(t)} \sum_{k=n}^{m} \frac{1}{k^{s-t}} e^{ikx},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$. *If* $(s,t) \in S_2 \cap \{(s,t) : s \geq 1\}$, *then for* $\frac{\pi}{n} \leq x < \pi$, $n > 1$, *the estimate*

$$|U_{n,m}(x) - V_{n,m}(x)| < \frac{1}{n^{s-t}} \frac{\Gamma(s)}{\Gamma(t)} \frac{(s-t)(s+t-1)}{2}$$

holds (cf. [4, Prop. 1]*).*

(4) *Let*

$$\Lambda(x) := x \log\left(\frac{\Gamma(x+t)}{\Gamma(x+s)} x^{s-t}\right)$$

and

$$K(x) := \psi'(x+t) - \psi'(x+s) + \frac{2}{x}[\psi(x+t) - \psi(x+s)] + \frac{s-t}{x^2}.$$

If $(s,t) \in S_1$, *then the function* $K(x) = \frac{1}{x}\Lambda''(x)$ *is completely monotonic of order* 2 *on* $(0,\infty)$ *and the function* $-\Lambda'(x)$ *is completely monotonic on* $(0,\infty)$. *In particular, the function* $\Lambda(x)$ *is strictly decreasing and convex on* $(0,\infty)$, *so that*

$$-\frac{(s-t)(s+t-1)}{2} < x \log\left(\frac{\Gamma(x+t)}{\Gamma(x+s)} x^{s-t}\right) < 0, \quad \text{for all} \quad x > 0.$$

In particular, when $s = 0$, *then the above results hold for* $t \leq 0$ *and* $\frac{1}{2} \leq t \leq 1$, *but not for any other* $t \in \mathbb{R}$ *(cf.* [4, Prop. 2]*).*

Special cases of Theorem 1.2 (3) were the main tools in [7] and [8] for the estimation of certain trigonometric sums arising in the context of starlike functions. Moreover, it is perhaps interesting to note that in [5] the exact range of t for which inequality (1.2) holds when $s = 1$ has been determined to be $[\frac{1}{3}, 1)$.

An application of [6, Thm. 1.3] leads to the following.

Theorem 1.3. (1) *There is an analytically defined* $a^* < 0$ *with numerical value* $a^* = -0.0741\ldots$, *such that the function*

$$\xi(x) = a^2[\psi'(x)]^2 + a\psi''(x) + \frac{2a(1-a)}{x}\psi'(x) - \frac{a(1-a)}{x^2}, \quad a \neq 0,$$

is completely monotonic of order 3 *in* $(0,\infty)$ *if, and only if,* $a \in (-\infty, a^*] \cup [\frac{2}{3}, \infty)$ *(cf.* [4, Thm. 2 (1)]*).*

(2) *The functions*

$$\begin{aligned} f_1(x) &:= [\psi'(x)]^2 + \psi''(x), \\ f_2(x) &:= \frac{2}{3}\left(\psi'(x) + \frac{1}{2x}\right)^2 + \psi''(x) - \frac{1}{2x^2} \end{aligned}$$

are completely monotonic of order 3 *on* $(0,\infty)$. *On the other hand, while being completely monotonic of order* 2, *the function*

$$f_3(x) = -\psi''(x) - \frac{2}{x}\psi'(x) + \frac{1}{x^2}$$

is not completely monotonic of order 3 *on* $(0,\infty)$ *(cf.* [4, Cor. 3]*).*

2. Proofs of Theorems 1.2 and 1.3

We shall first show that in the case $s = 0$ the function K defined in Theorem 1.2 (4) is not completely monotonic for $t \in (0, \frac{1}{2}) \cup (1, \infty)$. To that end observe that by the asymptotic formula

$$\psi(x+t) - \psi(x) = \frac{t}{x} - \frac{t(t-1)}{2x^2} + \frac{t(1-3t+2t^2)}{6x^3} + O(x^{-4}), \quad x \to \infty,$$

we have

$$6x^4 K(x) \to -2t^3 + 3t^2 - t, \quad x \to \infty.$$

Since $-2t^3 + 3t^2 - t$ takes negative values for $t \in (0, \frac{1}{2}) \cup (1, \infty)$, K cannot be completely monotonic on $(0, \infty)$ for those t.

The remaining statements of Theorem 1.2 follow from the proofs of Thm. 1, Cor. 1, Prop. 1 and Prop. 2 in [4], since $f_{1-t,1-s}(x)$ is convex on $(0, \infty)$ for all $(s,t) \in S_1$ by Theorem 1.1 and convex and monotonic on $(0, \infty)$ for all $(s,t) \in S_2$ by Theorem 1.1 and [9, Thm. 1.1 (3), (4)].

In order to prove Theorem 1.3 (1) note that in the proof of [4, Thm. 2] it is shown that ξ can be written as the Laplace transform of the function

$$G_a(u) := a^2 \int_0^u \delta(u-v)\delta(v)dv + 2a(1-a)\int_0^u \delta(v)dv - au\delta(u) - a(1-a)u,$$

where

$$\delta(u) := \frac{ue^u}{e^u - 1}, \quad \delta(0) := 1.$$

In the proof of [4, Thm. 2] it is also shown that $G_a^{(n)}(u) \geq 0$ for $n = 0, 1, 2$, $u > 0$ and $a \in (-\infty, 0] \cup [\frac{2}{3}, \infty)$ and that

$$G_a''(u) = a^2 \int_0^u \delta'(u-v)\delta'(v)dv - au\delta''(u).$$

Hence, it follows from [6, Thm. 1.3] that $\xi(x)$ will be completely monotonic of order 3 on $(0, \infty)$ if, and only if, $G_a'''(u) \geq 0$ for $u \geq 0$. From the above formula for $G_a''(u)$ we calculate

$$G_a'''(u) = a^2 \left(\int_0^u \delta''(u-v)\delta'(v)dv + \frac{\delta'(u)}{2} \right) - a\left(u\delta'''(u) + \delta''(u)\right).$$

From the proof of [4, Thm. 2] we obtain

$$\delta'(u) > 0, \quad \delta''(u) > 0, \quad \delta'''(u) < 0 \qquad \text{for all} \quad u \geq 0, \tag{2.1}$$

and thus

$$\frac{d}{dv}\delta''(u-v)\delta'(v) = -\delta'''(u-v)\delta'(v) + \delta''(u-v)\delta''(v) > 0 \quad \text{for all} \quad u \geq v. \tag{2.2}$$

For $a < 0$ this means that $G_a'''(u) \geq 0$ for $u \geq 0$ if, and only if,

$$a \leq a^* := \min_{u \geq 0} \frac{u\delta'''(u) + \delta''(u)}{\int_0^u \delta''(u-v)\delta'(v)dv + \frac{\delta'(u)}{2}}.$$

A numerical computation shows that $a^* = -0.0741\ldots$.

For $a \geq \frac{2}{3}$ it follows from (2.2) that

$$\frac{1}{a}G_a'''(u) \geq \frac{1}{3}\left(u\delta''(u) + \delta'(u)\right) - u\delta'''(u) - \delta''(u).$$

Elementary considerations show that the right-hand side of this inequality is positive for all $u \geq 0$. Therefore $G_a'''(u) > 0$ in $(0, \infty)$ for $a \in (-\infty, a^*] \cup [\frac{2}{3}, \infty)$, but not for $a \in (a^*, 0)$. Since it has been shown in [4, Thm. 2] that $\xi(x)$ is not completely monotonic for $a \in (0, \frac{2}{3})$, the proof of Theorem 1.3 (1) is complete.

Finally, for the proof of Theorem 1.3 (2), note that the functions $f_1(x)$ and $f_2(x)$ are merely the function ξ in the special cases $a = \frac{2}{3}$ and $a = 1$ and that, as shown in the proof of [4, Cor. 2], the function $f_3(x)$ is the Laplace transform of a function $\rho_3(u)$, for which $\rho_3''(u) = u\delta''(u)$, $u \in [0, \infty)$, with $\delta(u)$ as defined above. Since it was shown above that $(u\delta''(u))' = u\delta'''(u) + \delta''(u)$ changes sign in $[0, \infty)$, it follows from [6, Thm. 1.3] that f_3 cannot be completely monotonic of order 3 on $[0, \infty)$.

3. Proof of Theorem 1.1

First, note that $f_{\alpha,\beta}(x) = -f_{\beta,\alpha}(x)$ and therefore it will be enough to examine the curvature of $f_{\alpha,\beta}(x)$ in $(0, \infty)$ for $(\alpha, \beta) \in H$.

Next, observe that
$$f''_{\alpha,\beta}(x) = \frac{e^x g_{\alpha,\beta}(e^x)}{(e^x - 1)^3}, \quad x \geq 0.$$
Hence, the curvature of $f_{\alpha,\beta}(x)$ in $(0, \infty)$ is completely determined by the sign of $g_{\alpha,\beta}(t)$ in $(1, \infty)$.

Theorem 1.1 now follows from the following four lemmas.

Lemma 3.1. *For $\alpha < 0$ the function $g_\alpha(t)$ is negative in $(1, \infty)$ and for $0 < \alpha \leq \frac{1}{2}$ and $\alpha > 1$ the function $g_\alpha(t)$ is positive in $(1, \infty)$. For $\frac{1}{2} < \alpha < 1$ the function $g_\alpha(t)$ changes sign in $(1, \infty)$.*

Proof. For all $\alpha \in \mathbb{R}$ we have $g_\alpha(1) = 0$ and
$$\begin{aligned}
g'_\alpha(t) &= (\alpha - 1)^2(\alpha + 1)t^\alpha + (-2\alpha^3 + 2\alpha^2 + \alpha)t^{\alpha-1} \\
&\quad + \alpha^2(\alpha - 1)t^{\alpha-2} - 1, \\
g''_\alpha(t) &= \alpha(\alpha - 1)t^{\alpha-3}(t - 1)((\alpha^2 - 1)t - \alpha(\alpha - 2)), \\
g'''_\alpha(1) &= \alpha(\alpha - 1)(2\alpha - 1).
\end{aligned}$$
Consequently, for all $\alpha \in \mathbb{R}$, $g'_\alpha(1) = g''_\alpha(1) = 0$.

Case $-1 \leq \alpha \leq \frac{1}{2}$ *and* $1 < \alpha$. In this case g''_α does not vanish in $(1, \infty)$ and thus the sign of g'_α in $(1, \infty)$ will be equal to the sign of $g'''_\alpha(1)$. For $-1 \leq \alpha < 0$ we have $g'''_\alpha(1) < 0$, whereas $g'''_\alpha(1) > 0$ for $0 < \alpha < \frac{1}{2}$ or $1 < \alpha$. Therefore, g_α is negative in $(1, \infty)$ if $-1 \leq \alpha < 0$ and positive if $0 < \alpha \leq \frac{1}{2}$ or $1 < \alpha$.

Case $\alpha < -1$. In this case g''_α has exactly one zero t_α in $(1, \infty)$. Since $g'''_\alpha(1) < 0$, it follows that $g''_\alpha < 0$ in $(1, \infty)$ if and only if $1 < t < t_\alpha$. Since $g'_\alpha(t) \to -1$ as $t \to \infty$, this shows that g'_α is negative in $(1, \infty)$. Hence, for $\alpha < -1$, g_α is negative in $(1, \infty)$.

Case $\frac{1}{2} < \alpha < 1$. In this case we have $g'''_\alpha(1) < 0$ and thus $g_\alpha(t) < 0$ for $t > 1$ close to 1. Since obviously $g_\alpha(t) \to \infty$ as $t \to \infty$, the proof of the lemma is complete. □

Lemma 3.2. *For $(\alpha, \beta) \in B \setminus \Gamma_1$, the sign of $g_{\alpha,\beta}(t)$ changes on $(1, \infty)$.*

Proof. Since $g_{1,\beta}(t) = -g_\beta(t)$, the case $\alpha = 1$ of our assertion follows from Lemma 3.1. For the other (α, β) in question we have $\alpha \neq 1$ and $\alpha > \beta$ and thus
$$\lim_{t \to \infty} t^{-(1+\alpha)} g_{\alpha,\beta}(t) = (1 - \alpha)^2 > 0.$$

It will therefore be enough to show that $g_{\alpha,\beta}(t)$ takes negative values in $(1,\infty)$ for $(\alpha,\beta)\in B$ with $\alpha\neq 1$.

We have $g^{(n)}_{\alpha,\beta}(1)=0$ for $n=0,1,2$ and

$$\frac{g^{(3)}_{\alpha,\beta}(1)}{\alpha-\beta}=\varepsilon_1(\alpha,\beta).$$

Consequently, for

$$(\alpha,\beta)\in\{(\alpha,\beta):\beta<\alpha,\varepsilon_1(\alpha,\beta)<0\},$$

$g_{\alpha,\beta}(t)$ takes negative values in $(1,\infty)$ and it only remains to show that the same is true for (α,β) in the triangle $\{(\alpha,\beta):\frac{1}{2}<\beta<\alpha<1\}$.

To that end, fix a $\beta\in(\frac{1}{2},1)$ and observe that by Lemma 3.1 there is a $t^*\in(1,\infty)$ such that $g_\beta(t^*)=0$. Our claim will follow once we have shown that the function $h(\alpha):=g_\alpha(t^*)$, $\alpha\in(\frac{1}{2},1)$, is negative for all $\alpha\in(\beta,1)$, since

$$g_{\alpha,\beta}(t^*)=g_\alpha(t^*)-g_\beta(t^*)=h(\alpha).$$

We calculate

$$(t^*)^{1-\alpha}h'(\alpha)=2(t^*-1)(\alpha(t^*-1)-t^*)+\log t^*(\alpha^2(t^*-1)^2-2t^*\alpha(t^*-1)+t^*(t^*+1)),$$

and thus $h'(\alpha)$ vanishes for those α for which the rational function

$$r(\alpha):=\frac{\alpha^2(t^*-1)^2-2t^*\alpha(t^*-1)+t^*(t^*+1)}{\alpha(t^*-1)-t^*}$$

cuts the horizontal $\alpha\mapsto 2(1-t^*)/\log t^*$. It is straightforward to verify that, in $(\frac{1}{2},1)$, $r(\alpha)$ has no pole and $r'(\alpha)$ has exactly one zero and hence h can have at most two local extrema in $(\frac{1}{2},1)$.

Now, suppose that $h'(\beta)>0$. Then, since $h(1)=0$ and

$$h'(1)=\log t^*(t^*+1)+2(1-t^*)>0$$

for all $t^*\in(1,\infty)$, h must have at least two local extrema in $(\beta,1)$. On the other hand, $4\sqrt{t^*}h(\frac{1}{2})=(\sqrt{t^*}-1)^4>0$ and hence $h(\alpha)$ has to have at least one local minimum in $(\frac{1}{2},\beta)$. But $h(\alpha)$ can have at most two local extrema in $(\frac{1}{2},1)$ and therefore $h'(\beta)<0$. If now $h(\alpha)>0$ would hold for an $\alpha\in(\beta,1)$, then, since $h(1)=0$ and $h'(1)>0$, $h(\alpha)$ would have to have more than two local extrema in $(\beta,1)$. Thus, we must have $h(\alpha)<0$ for all $\alpha\in(\beta,1)$, and the proof of the lemma is complete. □

Lemma 3.3. *For $(\alpha,\beta)\in H\setminus D$ we have $g_{\alpha,\beta}(t)\geq 0$ in $(1,\infty)$, and for $0\leq\beta\leq\frac{1}{2}$ the function $g_{1,\beta}(t)$ is nonpositive in $(1,\infty)$.*

Proof. The case $\alpha=\beta$ is trivial and since $g_{\alpha,1}(t)=g_\alpha(t)$ and $g_{1,\beta}(t)=-g_\beta(t)$, the cases $\alpha=1$ and $\beta=1$ of our assertion follow from Lemma 3.1.

In order to prove the lemma also for the other (α,β) in question, set

$$h_\alpha(t):=(1-\alpha)^2t^2+(1+2\alpha-2\alpha^2)t+\alpha^2,\quad \alpha,\ t\in\mathbb{R}.$$

The parabola $h_\alpha(t)$ opens upward for all $\alpha\in\mathbb{R}$, its discriminant is nonnegative exactly for $\frac{1}{2}(1-\sqrt{2})\leq\alpha\leq\frac{1}{2}(1+\sqrt{2})<\frac{3}{2}$, and $h'_\alpha(1)>0$ exactly if $\alpha<\frac{3}{2}$. Hence, $h_\alpha(t)>0$ for all $\alpha\in\mathbb{R}$ and $t>1$, and therefore $g_{\alpha,\beta}(t)\geq 0$ is equivalent to

$$\log h_\alpha(t)-\log h_\beta(t)\geq(\beta-\alpha)\log t.$$

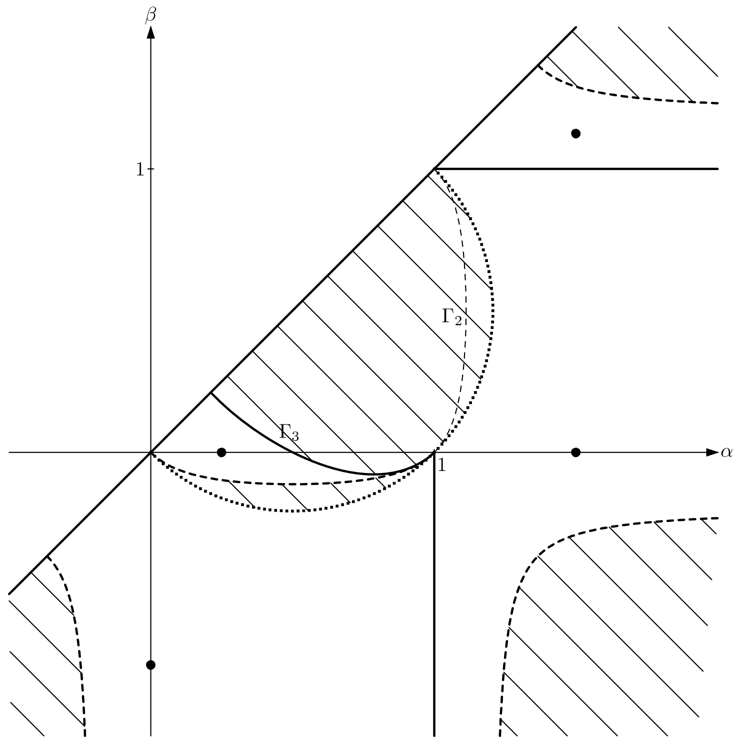

FIGURE 3. The sets P and N. The interior of P is hatched. The bold curves are ∂N. The dotted arc describes the set $H \cap \{(\alpha,\beta) : \varepsilon_3(\alpha,\beta) = 0\}$. The dashed curves describe the set $H \cap \{(\alpha,\beta) : \varepsilon_2(\alpha,\beta) = 0\}$. The four dots describe the location of the points $q_1, \ldots, q_4$.

Since this inequality holds for all $(\alpha, \beta) \in \mathbb{R}$ when $t = 1$, it will suffice to prove that

$$\frac{h'_\alpha(t)}{h_\alpha(t)} - \frac{h'_\beta(t)}{h_\beta(t)} \geq \frac{\beta-\alpha}{t} \tag{3.1}$$

or

$$th'_\alpha(t)h_\beta(t) - th'_\beta(t)h_\alpha(t) + (\alpha-\beta)h_\alpha(t)h_\beta(t) \geq 0 \quad \text{for } t > 1.$$

The left-hand side of the latter inequality is equal to $(\alpha-\beta)(1-t)^2 p_{\alpha,\beta}(t)$, where

$$p_{\alpha,\beta}(t) = t^2(1-\alpha)^2(1-\beta)^2 + t(\beta(\beta-1) + (\alpha^2-\alpha)(1-2\beta(\beta-1))) + \alpha^2\beta^2,$$

so that (3.1) is equivalent to

$$p_{\alpha,\beta}(t) \geq 0 \quad \text{for } t > 1.$$

The discriminant of the parabola (in the following we will always assume that $\alpha \neq 1$ and $\beta \neq 1$) $p_{\alpha,\beta}(t)$ is given by $-\varepsilon_2(\alpha,\beta)\varepsilon_3(\alpha,\beta)$. Since $p_{\alpha,\beta}(t)$ opens upward, it therefore follows that the nonnegativity of $p_{\alpha,\beta}(t)$ in $(1,\infty)$ only remains to be verified for $(\alpha,\beta) \in N := H \setminus P$, where (cf. Figure 3)

$$P := D \cup \{(\alpha,\beta) : \varepsilon_2(\alpha,\beta)\varepsilon_3(\alpha,\beta) \geq 0\} \cup \{(1,\beta) : \beta \in \mathbb{R}\} \cup \{(\alpha,1) : \alpha \in \mathbb{R}\}.$$

A straightforward computation shows that if $p_{\alpha,\beta}(t)$ has a zero at 1, then $\varepsilon_1(\alpha,\beta) = 0$ must hold. Since $\{(\alpha,\beta) : \varepsilon_1(\alpha,\beta) = 0\} \cap H$ is contained in $\overline{D}$, a continuity argument yields that the number of zeros of $p_{\alpha,\beta}(t)$ in $[1,\infty)$ is constant in each component of N.

It is easy to verify that N consists of exactly 4 components and that no two of the points $q_1 := (\frac{1}{4}, 0)$, $q_2 := (\frac{3}{2}, 0)$, $q_3 := (0, -\frac{3}{4})$ and $q_4 := (\frac{3}{2}, \frac{9}{8})$ lie in the same component of N (cf. Figure 3). Since one readily sees that $p_{q_j}(t)$ has no zeros in $(1,\infty)$ for $j = 1,\dots,4$, it follows that $p_{\alpha,\beta}(t)$ is positive in $(1,\infty)$ for all $(\alpha,\beta) \in N$.

The proof of the lemma is complete. $\square$

Lemma 3.4. *The function $f_{\alpha,\beta}(x)$ is convex on $(0,\infty)$ if $(\alpha,\beta) \in C_{conv}$. For $(\alpha,\beta) \in C \setminus C_{conv}$ and $(\alpha,\beta) \in \Gamma_1$ the sign of $f''_{\alpha,\beta}(x)$ changes on $(0,\infty)$.*

Proof. It follows from the proof of Lemma 3.3 that $g_{\alpha,\beta}(t)$ has a critical point t in $(1,\infty)$ if, and only if, t is a zero of $p_{\alpha,\beta}(t)$. Furthermore, since one easily checks that α, $\beta \neq 1$ and $\varepsilon_1(\alpha,\beta) \neq 0$ for all (α,β) in the connected set C, the proof of Lemma 3.3 also shows that the number of zeros of $p_{\alpha,\beta}(t)$ in $[1,\infty)$ is constant in C. It is readily verified that $q_5 := (\frac{11}{10}, \frac{1}{2}) \in C$ (cf. Figure 2) and that $p_{q_5}(t)$ has exactly two zeros in $(1,\infty)$. Hence, for $(\alpha,\beta) \in C$, $g_{\alpha,\beta}(t)$ has exactly two local extrema in $(1,\infty)$. Since $\beta < \alpha \neq 1$ in C, it follows from the proof of Lemma 3.2 that $g_{\alpha,\beta}(t)$ is positive for all t large enough. Since moreover $g_{\alpha,\beta}(1) = 0$ for all $(\alpha,\beta) \in \mathbb{R}^2$, the largest one, say t^*, of the critical points of $g_{\alpha,\beta}(t)$ in $[1,\infty)$ must be a local minimum of $g_{\alpha,\beta}(t)$, and $g_{\alpha,\beta}(t)$ will be nonnegative in $(1,\infty)$ if, and only if, $g_{\alpha,\beta}(t^*) \geq 0$. t^* must be the largest zero of $p_{\alpha,\beta}(t)$ and can thus be calculated to be $t^*(\alpha,\beta)$.

The set Γ_1 belongs to the boundary of both C and E, where E is the open bounded set that has the Jordan curve $\Gamma_1 \cup \{(1,\beta) : 0 \leq \beta \leq \frac{1}{2}\}$ as its boundary (E is shaded in Figure 2). The point $q_6 := (\frac{101}{100}, \frac{1}{4})$ lies in E and $p_{q_6}(t)$ has exactly one zero in $(1,\infty)$. Hence, on Γ_1, at least one of the zeros of $p_{\alpha,\beta}(t)$ is equal to 1. Since $\varepsilon_2(\alpha,\beta)\varepsilon_3(\alpha,\beta) \neq 0$ for $(\alpha,\beta) \in \Gamma_1$, the function $p_{\alpha,\beta}(t)$ cannot have a double zero at $t^*(\alpha,\beta)$ on Γ_1. Therefore $t^*(\alpha,\beta)$ is the only critical point of $g_{\alpha,\beta}(t)$ in $(1,\infty)$ when $(\alpha,\beta) \in \Gamma_1$. Since $g_{\alpha,\beta}(1) = 0$ and $g_{\alpha,\beta}(t) > 0$ for large t, this means that we must have $g_{\alpha,\beta}(t^*(\alpha,\beta)) < 0$ for $(\alpha,\beta) \in \Gamma_1$. $\square$

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