hp-OPTIMAL DISCONTINUOUS GALERKIN METHODS FOR LINEAR ELLIPTIC PROBLEMS

BENJAMIN STAMM AND THOMAS P. WIHLER

ABSTRACT. The aim of this paper is to present and analyze a class of hp-version discontinuous Galerkin (DG) discretizations for the numerical approximation of linear elliptic problems. This class includes a number of well-known DG formulations. We will show that the methods are stable provided that the stability parameters are suitably chosen. Furthermore, on (possibly irregular) quadrilateral meshes, we shall prove that the schemes converge all optimally in the energy norm with respect to both the local element sizes and polynomial degrees provided that homogeneous boundary conditions are considered.

1. INTRODUCTION

In this paper, we will propose and analyze a class of hp-version discontinuous Galerkin (DG) methods for the numerical approximation of linear elliptic partial differential equations. The focus of this work is to prove that the methods under consideration are stable and converge optimally in the energy norm with respect to both the local element sizes as well as the local polynomial degrees.

We will consider the model problem

$$(1) \qquad\qquad -\Delta u = f \qquad \text{in } \Omega,$$

$$(2) \qquad\qquad u = 0 \qquad \text{on } \partial\Omega,$$

with a unique solution $u \in H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz polygonal domain with straight edges, and $f \in L^2(\Omega)$. Here and throughout, we shall use the following standard notation: for a domain $D \subset \mathbb{R}^n$ ($n = 1$ or $n = 2$), we denote by $L^2(D)$ the space of all square-integrable functions on D. Furthermore, for an integer $k \in \mathbb{N}$, $H^k(D)$ signifies the usual Sobolev space of order k on D, with norm $\|\cdot\|_{k,D}$ and semi-norm $|\cdot|_{k,D}$. The space $H_0^1(D)$ is defined as the subspace of $H^1(D)$ with zero trace on ∂D.

The numerical approximation of second-order linear elliptic PDEs by DG methods was first studied in [1, 3, 11, 19, 28]. Later, additional DG formulations were proposed in the literature; see, e.g., [2] for an overview and a unified analysis. In recent years, some of the existent DG methods have been analyzed further within an hp context; see, e.g., [17, 21, 25, 26, 30] (cf. also [15, 16] for a posteriori results and hp-adaptive DG schemes). Here, the possibility of dealing with discontinuous finite element functions of possibly varying local approximation orders (even on

irregular meshes containing hanging nodes), results in a notable degree of flexibility and computational convenience. For example, for smooth problems with local singularities, the hp-spaces can be quite effectively adjusted to the behavior of the solution, and high-order algebraic or even exponential convergence rates can be attained; see, e.g., [29, 30].

The presence of discontinuous functions in DG finite element spaces is typically accounted for by introducing suitable penalty terms in the corresponding DG finite element formulation. They control the nonconformity of the numerical solution and ensure convergence to the exact solution. In many DG methods the penalty term is expressed in terms of jump operators (and/or related lifting operators; see, e.g., [2, 4, 25]), which might be, for reasons of stability, suboptimally scaled with respect to the local polynomial degree. It seems reasonable that this may result in suboptimal error estimates in an hp-DG error analysis. In this paper, it shall be shown that the use of continuous (H^1-conforming) interpolants of the exact solution overcomes this difficulty. This idea has been mentioned previously for interior penalty DG methods (cf. [12, Theorem 8.2]), and for LDG methods based on regular meshes in [21, Remark 3.1]. Furthermore, supposing that the exact solution u of (1)–(2) belongs to an augmented Sobolev space, optimal error bounds have been proved for interior penalty DG methods in [14].

The aim of this paper is to present a relatively general class of DG methods that contains various well-known DG schemes including the hp-interior penalty DG methods [17, 24, 30], the hp-LDG [21] and the hp-versions of the methods by Bassi et al. [5] and Brezzi et al. [8]. We will prove that the hp-DG methods in this paper are coercive and continuous (with explicitly described constants that are independent of h and of p). Furthermore, we will prove that, on quadrilateral meshes (with possibly one hanging node on the edges), the schemes converge all optimally (in the energy norm) with respect to h and p without any additional requirements on the regularity of the solution. Here, the use of a continuous hp-interpolant (see, e.g., [27]) of the exact solution of (1)–(2) will be essential. Indeed, due to the fact that a globally H^1-conforming interpolant belongs to the kernel of the stabilization terms, the error analysis can be carried out so that hp-optimal error bounds are achieved. In addition, the homogeneous Dirichlet boundary conditions in (2) are also crucial. In fact, for general Dirichlet boundary conditions the p-optimality of DG methods is typically not available; see the recent article [13] for a counterexample (featuring a singularity in the exact solution at the boundary).

This paper is organized as follows: In Section 2, we will present the hp-DG methods discussed in this paper, and their stability and well-posedness. Furthermore, the hp-optimal error analysis will follow in Section 3. A few numerical experiments shall be given in Section 4. Finally, we shall add some concluding remarks in Section 5.

2. hp-DISCONTINUOUS GALERKIN FEM

In this section, we shall present a class of hp-version discontinuous Galerkin (DG) finite element methods for the discretization of (1)–(2). Furthermore, we will discuss the well-posedness of these schemes, and prove some standard stability properties with respect to a suitable DG energy norm.

2.1. **Meshes, spaces, and element edge operators.** Let us first consider shape-regular meshes $\mathcal{T}_h$ that partition $\Omega \subset \mathbb{R}^2$ into open disjoint parallelograms $\{K\}_{K \in \mathcal{T}_h}$,

i.e. $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$. Each element $K \in \mathcal{T}_h$ can then be affinely mapped onto the reference square $\widehat{S} = (-1, 1)^2$. We allow the meshes to be 1-irregular, i.e., elements may contain hanging nodes. By h_K, we denote the diameter of an element $K \in \mathcal{T}_h$. We assume that these quantities are of bounded variation, i.e., there is a constant $\rho_1 \geq 1$ such that

$$
(3) \qquad \rho_1^{-1} \leq h_{K_\sharp}/h_{K_\flat} \leq \rho_1,
$$

whenever $K_\sharp$ and $K_\flat$ share a common edge. We store the elemental diameters in a vector $\boldsymbol{h}$ given by $\boldsymbol{h} = \{h_K : K \in \mathcal{T}_h\}$. Similarly, to each element $K \in \mathcal{T}_h$ we assign a polynomial degree $p_K \geq 1$ and define the degree vector $\boldsymbol{p} = \{p_K : K \in \mathcal{T}_h\}$. We suppose that $\boldsymbol{p}$ is also of bounded variation, i.e., there is a constant $\rho_2 \geq 1$ such that

$$
(4) \qquad \rho_2^{-1} \leq p_{K_\sharp}/p_{K_\flat} \leq \rho_2,
$$

whenever $K_\sharp$ and $K_\flat$ share a common edge.

Moreover, we shall define some suitable element edge operators that are required for the DG method. To this end, we denote by $\mathcal{E}_\mathcal{I}$ the set of all interior edges of the partition $\mathcal{T}_h$ of Ω, and by $\mathcal{E}_\mathcal{B}$ the set of all boundary edges of $\mathcal{T}_h$. In addition, let $\mathcal{E} = \mathcal{E}_\mathcal{I} \cup \mathcal{E}_\mathcal{B}$. The boundary ∂K of an element K and the sets $\partial K \setminus \partial \Omega$ and $\partial K \cap \partial \Omega$ will be identified in a natural way with the corresponding subsets of $\mathcal{E}$.

Let $K_\sharp$ and $K_\flat$ be two adjacent elements of $\mathcal{T}_h$, and $\boldsymbol{x}$ an arbitrary point on the interior edge $e \in \mathcal{E}_\mathcal{I}$ given by $e = (\partial K_\sharp \cap \partial K_\flat)^\circ$. Furthermore, let v and $\boldsymbol{q}$ be scalar- and vector-valued functions, respectively, that are sufficiently smooth inside the elements $K_\sharp$ and $K_\flat$. By $(v_{\sharp/\flat}, \boldsymbol{q}_{\sharp/\flat})$, we denote the element boundary traces of $(v, \boldsymbol{q})$ taken from within the interior of $K_\sharp$ and $K_\flat$, respectively. Then, the averages of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by

$$
\langle\!\langle v \rangle\!\rangle = \frac{1}{2}(v_\sharp + v_\flat), \qquad \langle\!\langle \boldsymbol{q} \rangle\!\rangle = \frac{1}{2}(\boldsymbol{q}_\sharp + \boldsymbol{q}_\flat),
$$

respectively. Similarly, the jumps of v and $\boldsymbol{q}$ at $\boldsymbol{x} \in e$ are given by

$$
[\![v]\!] = v_\sharp \, \boldsymbol{n}_{K_\sharp} + v_\flat \, \boldsymbol{n}_{K_\flat}, \qquad [\![\boldsymbol{q}]\!] = \boldsymbol{q}_\sharp \cdot \boldsymbol{n}_{K_\sharp} + \boldsymbol{q}_\flat \cdot \boldsymbol{n}_{K_\flat},
$$

respectively, where we denote by $\boldsymbol{n}_{K_{\sharp/\flat}}$ the unit outward normal vector on $\partial K_{\sharp/\flat}$, respectively. On a boundary edge $e \in \mathcal{E}_\mathcal{B}$, we set $\langle\!\langle v \rangle\!\rangle = v$, $\langle\!\langle \boldsymbol{q} \rangle\!\rangle = \boldsymbol{q}$, and $[\![v]\!] = v\boldsymbol{n}$, $[\![\boldsymbol{q}]\!] = \boldsymbol{q} \cdot \boldsymbol{n}$, with $\boldsymbol{n}$ denoting the unit outward normal vector on the boundary $\partial \Omega$.

Given a finite element mesh $\mathcal{T}_h$ and an associated polynomial degree vector $\boldsymbol{p} = (p_K)_{K \in \mathcal{T}_h}$, with $p_K \geq 1$ for all $K \in \mathcal{T}_h$, consider the *hp*-discretization space

$$
(5) \qquad V(\mathcal{T}_h, \boldsymbol{p}) = \{v \in L^2(\Omega) : v|_K \in Q_{p_K}(K), K \in \mathcal{T}_h\},
$$

for the DG method. Here, for $K \in \mathcal{T}_h$, $Q_{p_K}(K)$ is the space of all polynomials of degree at most p_K in each variable on K.

Let us define the discontinuity scaling function

$$
(6) \qquad \sigma = \frac{\mathrm{p}^2}{\mathrm{h}} \in L^\infty(\mathcal{E}),
$$

which is expressed with the two functions $\mathtt{h}, \mathtt{p} \in L^\infty(\mathcal{E})$ given by

$$
\mathtt{h}(\boldsymbol{x}) = \begin{cases} \min(h_{K_\sharp}, h_{K_\flat}) & \text{for } \boldsymbol{x} \in (\partial K_\sharp \cap \partial K_\flat)^\circ \in \mathcal{E}_\mathcal{I}, \\ h_K & \text{for } \boldsymbol{x} \in (\partial K \cap \partial \Omega)^\circ \in \mathcal{E}_\mathcal{B}, \end{cases}
$$

$$
\mathtt{p}(\boldsymbol{x}) = \begin{cases} \max(p_{K_\sharp}, p_{K_\flat}) & \text{for } \boldsymbol{x} \in (\partial K_\sharp \cap \partial K_\flat)^\circ \in \mathcal{E}_\mathcal{I}, \\ p_K & \text{for } \boldsymbol{x} \in (\partial K \cap \partial \Omega)^\circ \in \mathcal{E}_\mathcal{B}. \end{cases}
$$

In addition, consider the space

$$
H^{\boldsymbol{s}}(\Omega, \mathcal{T}_h) = \left\{ v \in L^2(\Omega) : v|_K \in H^{s_K}(K), K \in \mathcal{T}_h \right\}.
$$

Here, $\boldsymbol{s} = (s_K)_{K \in \mathcal{T}_h}$, $s_K \geq 1$ for all $K \in \mathcal{T}_h$, is an integer vector. Let $\boldsymbol{1} = (s_K = 1)_{K \in \mathcal{T}_h}$ be the vector containing only ones. Then, for $e \in \mathcal{E}$, we shall define the lifting operator

$$
\mathsf{L}_e : H^{\boldsymbol{1}}(\Omega, \mathcal{T}_h) \to V(\mathcal{T}_h, \boldsymbol{p})^2
$$

by

(7) $$
\int_\Omega \mathsf{L}_e(w) \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} = \int_e [\![w]\!] \cdot \langle\!\langle \boldsymbol{\phi} \rangle\!\rangle \, \mathrm{d}s \qquad \forall \boldsymbol{\phi} \in V(\mathcal{T}_h, \boldsymbol{p})^2;
$$

see, e.g., [2, 4, 25]. Furthermore, we set

(8) $$
\mathsf{L} = \sum_{e \in \mathcal{E}} \mathsf{L}_e.
$$

We remark that there exist constants $C_\mathsf{L}, D_\mathsf{L}, E_\mathsf{L} > 0$ independent of $\boldsymbol{h}, \boldsymbol{p}$ such that

(9) $$
\|\mathsf{L}(v)\|_{0,\Omega}^2 \leq C_\mathsf{L} \left\| \sqrt{\sigma} [\![v]\!] \right\|_{0,\mathcal{E}}^2,
$$

(10) $$
\|\mathsf{L}(v)\|_{0,\Omega}^2 \leq D_\mathsf{L} \sum_{e \in \mathcal{E}} \|\mathsf{L}_e(v)\|_{0,\Omega}^2,
$$

(11) $$
\left\| \sqrt{\sigma} [\![v]\!] \right\|_{0,\mathcal{E}}^2 \leq E_\mathsf{L} \sum_{e \in \mathcal{E}} \|\mathsf{L}_e(v)\|_{0,\Omega}^2,
$$

for any $v \in V(\mathcal{T}_h, \boldsymbol{p})$; cf. [25]. Note that, for $e \in \mathcal{E}$, the lifting operator L_e has local support. Furthermore, we mention that the constants $C_\mathsf{L}, D_\mathsf{L}, E_\mathsf{L}$ can be computed numerically by means of suitable (generalized) eigenvalue problems.

2.2. hp-DG discretization. We will now present the hp-DG methods to be analyzed in this paper. Consider the following hp-DG formulation which is to find a numerical solution $u_{\mathrm{DG}} \in V(\mathcal{T}_h, \boldsymbol{p})$ of (1)–(2) such that

(12) $$
a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(u_{\mathrm{DG}}, v) = \ell_{\mathrm{DG}}(v) \qquad \forall v \in V(\mathcal{T}_h, \boldsymbol{p}).
$$

Here, for $w, v \in V(\mathcal{T}_h, \boldsymbol{p})$, let

$$
a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(w, v)
$$

(13) $$
= \int_\Omega \nabla_h w \cdot \nabla_h v \, \mathrm{d}\boldsymbol{x} - \int_\mathcal{E} \langle\!\langle \nabla_h w \rangle\!\rangle \cdot [\![v]\!] \, \mathrm{d}s - \theta \int_\mathcal{E} [\![w]\!] \cdot \langle\!\langle \nabla_h v \rangle\!\rangle \, \mathrm{d}s
$$

$$
+ \gamma \int_\mathcal{E} \sigma [\![w]\!] \cdot [\![v]\!] \, \mathrm{d}s + \delta \sum_{e \in \mathcal{E}} \int_\Omega \mathsf{L}_e(w) \cdot \mathsf{L}_e(v) \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega \mathsf{L}(w) \cdot \mathsf{L}(v) \, \mathrm{d}\boldsymbol{x},
$$

with $\theta \in \mathbb{R}$, and

(14) $$
\ell_{\mathrm{DG}}(v) = \int_\Omega f v \, \mathrm{d}\boldsymbol{x}.
$$

TABLE 1. hp-DG methods for different parameter choices.

	stability parameters			DG method
$\theta = 1$	$\gamma = 0$	$\delta = 0$	$\varepsilon = 1$	Bassi-Rebay [4]
	$\gamma = 0$	$\delta > 0$	$\varepsilon = 1$	Brezzi et al. [8]
	$\gamma = 0$	$\delta > 0$	$\varepsilon = 0$	Bassi et al. [5]
	$\gamma > 0$	$\delta = 0$	$\varepsilon = 0$	SIPG [1, 11, 30]
	$\gamma > 0$	$\delta = 0$	$\varepsilon = 1$	LDG (with $\beta = 0$ in [2]) [9, 21]
$\theta = -1$	$\gamma = 0$	$\delta = 0$	$\varepsilon = 0$	Baumann-Oden [6, 18, 20, 24]
	$\gamma > 0$	$\delta = 0$	$\varepsilon = 0$	NIPG [17, 24, 30]
$\theta = 0$	$\gamma > 0$	$\delta = 0$	$\varepsilon = 0$	IIPG [10]

We note, recalling the definition (7) of the lifting operator $\mathbf{L}$, that

$$
a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,v)
$$

$$
(15) \quad = \int_\Omega \nabla_h w \cdot \nabla_h v \, \mathrm{d}\boldsymbol{x} - \int_\Omega \nabla_h w \cdot \mathbf{L}(v) \, \mathrm{d}\boldsymbol{x} - \theta \int_\Omega \mathbf{L}(w) \cdot \nabla_h v \, \mathrm{d}\boldsymbol{x}
$$

$$
+ \gamma \int_{\mathcal{E}} \sigma [\![w]\!] \cdot [\![v]\!] \, \mathrm{d}s + \delta \sum_{e \in \mathcal{E}} \int_\Omega \mathbf{L}_e(w) \cdot \mathbf{L}_e(v) \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega \mathbf{L}(w) \cdot \mathbf{L}(v) \, \mathrm{d}\boldsymbol{x}
$$

holds for all $w, v \in V(\mathcal{T}_h, \boldsymbol{p})$.

Remark 2.1. The methods discussed in this paper are closely related to other DG schemes in the literature; see, e.g., [2]. In Table 1, we display how different choices of the parameters $\gamma, \delta, \varepsilon$ and θ appearing in the DG form $a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}$ from (15) result in previously known DG bilinear forms (considered in their hp-version primal form).

We conclude this section by introducing an energy norm for the DG method (12):

$$
(16) \quad \|\|w\|\|_{\gamma,\delta,\varepsilon}^2 = \|\nabla_h w\|_{0,\Omega}^2 + \gamma \left\| \sqrt{\sigma} [\![w]\!] \right\|_{0,\mathcal{E}}^2 + \delta \sum_{e \in \mathcal{E}} \|\mathbf{L}_e(w)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(w)\|_{0,\Omega}^2 .
$$

We remark that the two stabilization terms involving the parameters γ and δ have the same scaling properties with respect to $\boldsymbol{h}$ and $\boldsymbol{p}$; cf. [25].

Henceforth, we shall always suppose that

$$
(17) \quad \gamma, \delta, \varepsilon \geq 0, \qquad \gamma + \delta > 0,
$$

i.e., $\|\| \cdot \|\|_{\gamma,\delta,\varepsilon}$ is indeed a norm. Note that these two conditions include all DG formulations in Table 1 except for the Baumann-Oden and the Bassi-Rebay method.

2.3. **Stability.** The aim of this section is to discuss the stability and well-posedness of the DG method (12). We define the constants

$$
(18) \quad \chi_{\gamma,\delta,\varepsilon} = \sqrt{\frac{\gamma}{C_\mathbf{L}} + \frac{\delta}{D_\mathbf{L}} + \varepsilon}, \qquad \Upsilon_{\gamma,\delta} = \sqrt{\gamma + \frac{\delta}{E_\mathbf{L}}},
$$

where $C_\mathbf{L}, D_\mathbf{L}, E_\mathbf{L} > 0$ are the constants from (9)–(11). Notice that these constants are positive provided that (17) holds.

Let us first consider the following auxiliary result:

Lemma 2.2. *If $\gamma, \delta, \varepsilon$ satisfy* (17), *then it holds that*

$$\|\mathbf{L}(v)\|_{0,\Omega}^2 \leq \frac{1}{\chi_{\gamma,\delta,\varepsilon}^2} \left(\gamma \left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 + \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(v)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(v)\|_{0,\Omega}^2 \right)$$

and

$$\left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 \leq \frac{1}{\Upsilon_{\gamma,\delta}^2} \left(\gamma \left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 + \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(v)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(v)\|_{0,\Omega}^2 \right)$$

for all $v \in V(\mathcal{T}_h, \boldsymbol{p})$.

Proof. We write

$$\|\mathbf{L}(v)\|_{0,\Omega}^2 = \frac{1}{\chi_{\gamma,\delta,\varepsilon}^2} \left(\frac{\gamma}{C_\mathbf{L}} \|\mathbf{L}(v)\|_{0,\Omega}^2 + \frac{\delta}{D_\mathbf{L}} \|\mathbf{L}(v)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(v)\|_{0,\Omega}^2 \right).$$

Applying (9)–(10) leads to

$$\|\mathbf{L}(v)\|_{0,\Omega}^2 \leq \frac{1}{\chi_{\gamma,\delta,\varepsilon}^2} \left(\gamma \left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 + \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(v)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(v)\|_{0,\Omega}^2 \right).$$

This completes the proof of the first bound.

In order to prove the second estimate, we write

$$\left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 = \frac{1}{\Upsilon_{\gamma,\delta}^2} \left(\gamma \left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 + \frac{\delta}{E_\mathbf{L}} \left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 \right).$$

Using (11), implies that

$$\left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 \leq \frac{1}{\Upsilon_{\gamma,\delta}^2} \left(\gamma \left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}}^2 + \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(v)\|_{0,\Omega}^2 \right),$$

and adding the positive quantity $\frac{\varepsilon}{\Upsilon_{\gamma,\delta}^2} \|\mathbf{L}(v)\|_{0,\Omega}^2$ yields the second inequality. $\square$

The ensuing result gives the coercivity and continuity of the bilinear form $a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}$ from (15).

Proposition 2.3. *Suppose that* (17) *is satisfied. Moreover, let*

$$C_{\mathrm{coercivity}} := 1 - \frac{|1+\theta|}{2\chi_{\gamma,\delta,\varepsilon}} > 0.$$

Then the form $a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}$ is coercive on $V(\mathcal{T}_h, \boldsymbol{p})$ with respect to the norm $\|\cdot\|_{\gamma,\delta,\varepsilon}$. More precisely, we have

$$(19) \qquad a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,w) \geq C_{\mathrm{coercivity}} \|w\|_{\gamma,\delta,\varepsilon}^2 \qquad \forall w \in V(\mathcal{T}_h, \boldsymbol{p}).$$

Additionally, the form $a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}$ is continuous on $V(\mathcal{T}_h, \boldsymbol{p})$, i.e.,

$$(20) \qquad \left| a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,v) \right| \leq \max(1, |\theta|) \left(1 + \frac{1}{\chi_{\gamma,\delta,\varepsilon}} \right) \|w\|_{\gamma,\delta,\varepsilon} \|v\|_{\gamma,\delta,\varepsilon}$$

for any $w, v \in V(\mathcal{T}_h, \boldsymbol{p})$. Here, $\chi_{\gamma,\delta,\varepsilon} > 0$ is the constant from (18).

Proof. Let $w, v \in V(\mathcal{T}_h, \boldsymbol{p})$.

Proof of the coercivity (19). Using (15) and applying the Cauchy-Schwarz inequality, we have

$$a_{\text{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,w) = \|\nabla_h w\|_{0,\Omega}^2 - (1+\theta)\int_\Omega \mathbf{L}(w) \cdot \nabla_h w \, \mathrm{d}\boldsymbol{x} + \gamma \left\|\sqrt{\sigma}[\![w]\!]\right\|_{0,\mathcal{E}}^2$$
$$+ \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(w)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(w)\|_{0,\Omega}^2$$
$$\geq \|\nabla_h w\|_{0,\Omega}^2 - |1+\theta| \, \|\mathbf{L}(w)\|_{0,\Omega} \, \|\nabla_h w\|_{0,\Omega} + \gamma \left\|\sqrt{\sigma}[\![w]\!]\right\|_{0,\mathcal{E}}^2$$
$$+ \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(w)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(w)\|_{0,\Omega}^2 \,.$$

Moreover, it holds that

$$a_{\text{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,w) \geq \left(1 - \frac{|1+\theta|}{2\chi_{\gamma,\delta,\varepsilon}}\right) \|\nabla_h w\|_{0,\Omega}^2 - \frac{\chi_{\gamma,\delta,\varepsilon}|1+\theta|}{2} \|\mathbf{L}(w)\|_{0,\Omega}^2$$
$$+ \gamma \left\|\sqrt{\sigma}[\![w]\!]\right\|_{0,\mathcal{E}}^2 + \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(w)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(w)\|_{0,\Omega}^2 \,.$$

Using Lemma 2.2, it follows that

$$a_{\text{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,w)$$
$$\geq \left(1 - \frac{|1+\theta|}{2\chi_{\gamma,\delta,\varepsilon}}\right) \|\nabla_h w\|_{0,\Omega}^2$$
$$+ \left(1 - \frac{|1+\theta|}{2\chi_{\gamma,\delta,\varepsilon}}\right) \left(\gamma \left\|\sqrt{\sigma}[\![w]\!]\right\|_{0,\mathcal{E}}^2 + \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(w)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(w)\|_{0,\Omega}^2\right).$$

This is the desired inequality (19).

Proof of the continuity (20). Using again the Cauchy-Schwarz inequality, we obtain that

$$\left|a_{\text{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,v)\right|$$
$$\leq \int_\Omega |\nabla_h w| \, |\nabla_h v| \, \mathrm{d}\boldsymbol{x} + \int_\Omega |\nabla_h w||\mathbf{L}(v)| \, \mathrm{d}\boldsymbol{x} + |\theta| \int_\Omega |\mathbf{L}(w)| \, |\nabla_h v| \, \mathrm{d}\boldsymbol{x}$$
$$+ \gamma \int_\mathcal{E} \sigma |[\![w]\!]||[\![v]\!]| \, \mathrm{d}s + \delta \sum_{e\in\mathcal{E}} \int_\Omega |\mathbf{L}_e(w)| \, |\mathbf{L}_e(v)| \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega |\mathbf{L}(w)||\mathbf{L}(v)| \, \mathrm{d}\boldsymbol{x}$$
$$\leq \max(1, |\theta|) \Bigg(\|\nabla_h w\|_{0,\Omega} \|\nabla_h v\|_{0,\Omega} + \|\nabla_h w\|_{0,\Omega} \|\mathbf{L}(v)\|_{0,\Omega} + \|\mathbf{L}(w)\|_{0,\Omega} \|\nabla_h v\|_{0,\Omega}$$
$$+ \gamma \left\|\sqrt{\sigma}[\![w]\!]\right\|_{0,\mathcal{E}} \left\|\sqrt{\sigma}[\![v]\!]\right\|_{0,\mathcal{E}} + \delta \sum_{e\in\mathcal{E}} \|\mathbf{L}_e(w)\|_{0,\Omega} \|\mathbf{L}_e(v)\|_{0,\Omega}$$
$$+ \varepsilon \|\mathbf{L}(w)\|_{0,\Omega} \|\mathbf{L}(v)\|_{0,\Omega} \Bigg).$$

Hence, it holds that

$$
\frac{\left| a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,v) \right|}{\max(1,|\theta|)} \leq \Bigg(\left(1 + \frac{1}{\chi_{\gamma,\delta,\varepsilon}} \right) \|\nabla_h w\|_{0,\Omega}^2 + \chi_{\gamma,\delta,\varepsilon} \|\mathbf{L}(w)\|_{0,\Omega}^2
$$

$$
+ \gamma \left\| \sqrt{\sigma} [\![w]\!] \right\|_{0,\mathcal{E}}^2 + \delta \sum_{e \in \mathcal{E}} \|\mathbf{L}_e(w)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(w)\|_{0,\Omega}^2 \Bigg)^{\frac{1}{2}}
$$

$$
\times \Bigg(\left(1 + \frac{1}{\chi_{\gamma,\delta,\varepsilon}} \right) \|\nabla_h v\|_{0,\Omega}^2 + \chi_{\gamma,\delta,\varepsilon} \|\mathbf{L}(v)\|_{0,\Omega}^2
$$

$$
+ \gamma \left\| \sqrt{\sigma} [\![v]\!] \right\|_{0,\mathcal{E}}^2 + \delta \sum_{e \in \mathcal{E}} \|\mathbf{L}_e(v)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(v)\|_{0,\Omega}^2 \Bigg)^{\frac{1}{2}}.
$$

Applying Lemma 2.2, we obtain

$$
\frac{\left| a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(w,v) \right|}{\max(1,|\theta|)}
$$

$$
\leq \Bigg(\left(1 + \frac{1}{\chi_{\gamma,\delta,\varepsilon}} \right) \|\nabla_h w\|_{0,\Omega}^2
$$

$$
+ \left(1 + \frac{1}{\chi_{\gamma,\delta,\varepsilon}} \right) \left(\gamma \left\| \sqrt{\sigma} [\![w]\!] \right\|_{0,\mathcal{E}}^2 + \delta \sum_{e \in \mathcal{E}} \|\mathbf{L}_e(w)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(w)\|_{0,\Omega}^2 \right) \Bigg)^{\frac{1}{2}}
$$

$$
\times \Bigg(\left(1 + \frac{1}{\chi_{\gamma,\delta,\varepsilon}} \right) \|\nabla_h v\|_{0,\Omega}^2
$$

$$
+ \left(1 + \frac{1}{\chi_{\gamma,\delta,\varepsilon}} \right) \left(\gamma \left\| \sqrt{\sigma} [\![v]\!] \right\|_{0,\mathcal{E}}^2 + \delta \sum_{e \in \mathcal{E}} \|\mathbf{L}_e(v)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(v)\|_{0,\Omega}^2 \right) \Bigg)^{\frac{1}{2}}.
$$

This implies the continuity (20). □

Moreover, we shall discuss the continuity of the linear form ℓ_{DG} from (14) with respect to the DG energy norm.

Proposition 2.4. *Suppose that* (17) *is satisfied. Then, the linear form ℓ_{DG} is continuous on $V(\mathcal{T}_h, \boldsymbol{p})$, i.e., there exists a constant $C > 0$ independent of $\boldsymbol{h}$, $\boldsymbol{p}$, γ, δ, and ε such that*

$$
|\ell_{\mathrm{DG}}(v)| \leq C \max\left(1, \Upsilon_{\gamma,\delta}^{-1} \right) \|f\|_{0,\Omega} \|v\|_{\gamma,\delta,\varepsilon}
$$

for all $v \in V(\mathcal{T}_h, \boldsymbol{p})$, where $\Upsilon_{\gamma,\delta} > 0$ is the constant from (18).

Proof. Equation (1.8) in [7] (see also Theorem 6.2) implies that there exists a constant $C > 0$ independent of $\boldsymbol{h}$ and $\boldsymbol{p}$ such that the following discrete Poincaré-Friedrichs inequality is satisfied:

$$
\|v\|_{0,\Omega}^2 \leq C \left(\|\nabla_h v\|_{0,\Omega}^2 + \int_{\mathcal{E}_{\mathcal{I}}} \mathbf{h}^{-1} \left| [\![v]\!] \right|^2 \, \mathrm{d}s + \int_{\partial\Omega} |v|^2 \, \mathrm{d}s \right)
$$

for all $v \in H^{\mathbf{1}}(\Omega, \mathcal{T}_h)$. Since $p_K \geq 1$ for all $K \in \mathcal{T}_h$, we have

$$\|v\|_{0,\Omega} \leq C \left(\|\nabla_h v\|_{0,\Omega}^2 + \left\| \sqrt{\sigma} [\![v]\!] \right\|_{0,\mathcal{E}}^2 \right)^{\frac{1}{2}}.$$

Applying the second bound in Lemma 2.2, implies

$\|v\|_{0,\Omega}$

$$\leq C \left(\|\nabla_h v\|_{0,\Omega}^2 + \frac{1}{\Upsilon_{\gamma,\delta}^2} \left(\gamma \left\| \sqrt{\sigma} [\![v]\!] \right\|_{0,\mathcal{E}}^2 + \delta \sum_{e \in \mathcal{E}} \|\mathbf{L}_e(v)\|_{0,\Omega}^2 + \varepsilon \|\mathbf{L}(v)\|_{0,\Omega}^2 \right) \right)^{\frac{1}{2}}.$$

Hence, it follows that

$$|\ell_{\mathrm{DG}}(v)| \leq \|f\|_{0,\Omega} \|v\|_{0,\Omega} \leq C \max \left(1, \Upsilon_{\gamma,\delta}^{-1} \right) \|f\|_{0,\Omega} \|v\|_{\gamma,\delta,\varepsilon}$$

for all $v \in V(\mathcal{T}_h, \boldsymbol{p})$. $\qquad\square$

The above results, Propositions 2.3–2.4, imply the well-posedness of the DG discretization (12).

Theorem 2.5. *Let* (17) *and* (19)–(20) *be satisfied. Then, the hp-DG method* (12) *has a unique solution* $u_{\mathrm{DG}} \in V(\mathcal{T}_h, \boldsymbol{p})$, *and it holds that*

$$\|u_{\mathrm{DG}}\|_{\gamma,\delta,\varepsilon} \leq C \frac{\max \left(1, \Upsilon_{\gamma,\delta}^{-1} \right)}{1 - \frac{|1+\theta|}{2\chi_{\gamma,\delta,\varepsilon}}} \|f\|_{0,\Omega},$$

where $C > 0$ *is a constant independent of* $\boldsymbol{h}$, $\boldsymbol{p}$, γ, δ, *and* ε.

We round off this section by noticing that the DG schemes (12) satisfy Galerkin orthogonality.

Proposition 2.6. *Suppose that the exact solution* u *of* (1)–(2) *belongs to* $H^2(\Omega) \cap H_0^1(\Omega)$. *Then, it holds that*

$$a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(u - u_{\mathrm{DG}}, v) = 0 \qquad \forall v \in V(\mathcal{T}_h, \boldsymbol{p}),$$

where $u_{\mathrm{DG}} \in V(\mathcal{T}_h, \boldsymbol{p})$ *is the DG solution from* (12).

This can be proved in a similar way as for IP methods; for example, see [2, 17, 29, 30].

3. hp-ERROR ANALYSIS

The goal of this section is to show that the DG method (12) converges optimally with respect to both the local element sizes $\boldsymbol{h}$ and the polynomial degrees $\boldsymbol{p}$. To do so, we shall briefly collect some hp-approximation results that will play an important role in the subsequent error analysis. Furthermore, later on in this section, the main result of this paper will be given.

3.1. hp**-approximations.** The first of the following two lemmas says that the elementwise L^2-projection on $V(\mathcal{T}_h, \boldsymbol{p})$ remains optimal on the edges of affine quadrilateral elements (in the corresponding L^2-norm). The second result is an optimal (with respect to the H^1-norm) conforming hp-interpolant in $V(\mathcal{T}_h, \boldsymbol{p})$.

Lemma 3.1. *Let $K \in \mathcal{T}_h$ and suppose that $u \in H^s(K)$ for some integer $s \geq 1$. Then, for $1 \leq \widetilde{s} \leq \min(p+1, s)$, and $p \geq 0$, we have that*

$$\|u - \pi_p u\|_{0, \partial K} \leq C \left(\frac{h_K}{p+1}\right)^{\widetilde{s} - \frac{1}{2}} |u|_{H^{\widetilde{s}}(K)}.$$

Here, $C > 0$ is a constant independent of h_K and p, and $\pi_p : L^2(K) \to Q_p(K)$ is the L^2-projection of degree p on K.

Proof. See [17, Lemma 3.9 and Remark 3.10]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Lemma 3.2. *Given $u \in H^s(\Omega, \mathcal{T}_h) \cap H^2(\Omega) \cap H^1_0(\Omega)$, there exists a continuous interpolant $\mathsf{P}_p(u) \in V(\mathcal{T}_h, p) \cap H^1_0(\Omega)$ of u such that*

$$\|u - \mathsf{P}_p(u)\|^2_{H^1(\Omega)} \leq C \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p_K}\right)^{2\widetilde{s}_K - 2} |u|^2_{H^{\widetilde{s}_K}(K)},$$

for $2 \leq \widetilde{s}_K \leq \min(p_K + 1, s_K)$, $K \in \mathcal{T}_h$. Here, $C > 0$ is a constant independent of h and p.

Proof. See, e.g., [27, Theorem 4.72 and Remark 4.73]. $\qquad\qquad\qquad\qquad\qquad$ □

3.2. hp-optimal error estimates.

Let us analyze the error

$$e_{\mathrm{DG}} = u - u_{\mathrm{DG}},$$

of the DG method in the energy norm $\|\!|\cdot|\!\|_{\gamma, \delta, \varepsilon}$. Here, u is the exact solution of (1)–(2), and u_{DG} is the DG approximation from (12). We will proceed in a similar way as in [17, 29], for example. More precisely, we split the DG error into two parts,

$$e_{\mathrm{DG}} = \eta + \xi,$$

where $\eta = u - \mathsf{P}_p(u)$ and $\xi = \mathsf{P}_p(u) - u_{\mathrm{DG}}$, and $\mathsf{P}_p(u) \in V(\mathcal{T}_h, p) \cap H^1_0(\Omega)$ is the conforming hp-interpolant of u from Lemma 3.2. We note that $\eta \in H^1_0(\Omega)$ and $\xi \in V(\mathcal{T}_h, p)$. Then, applying the triangle inequality, it holds that

$$(21) \qquad\qquad \|\!|e|\!\|_{\gamma, \delta, \varepsilon} \leq \|\!|\eta|\!\|_{\gamma, \delta, \varepsilon} + \|\!|\xi|\!\|_{\gamma, \delta, \varepsilon}.$$

We first analyze the term $\|\!|\xi|\!\|_{\gamma, \delta, \varepsilon}$ and aim at bounding it in terms of η; this will make it possible to estimate the DG error by the interpolation error η only. Applying the Galerkin orthogonality, Proposition 2.6, of the DG method (12) and the coercivity of $a_{\mathrm{DG}}^{\gamma, \delta, \varepsilon, \theta}$, Proposition 2.3, leads to

$$(22) \qquad \begin{aligned} C\|\!|\xi|\!\|^2_{\gamma, \delta, \varepsilon} &\leq a_{\mathrm{DG}}^{\gamma, \delta, \varepsilon, \theta}(\xi, \xi) = a_{\mathrm{DG}}^{\gamma, \delta, \varepsilon, \theta}(e_{\mathrm{DG}} - \eta, \xi) = -a_{\mathrm{DG}}^{\gamma, \delta, \varepsilon, \theta}(\eta, \xi) \\ &\leq \left|a_{\mathrm{DG}}^{\gamma, \delta, \varepsilon, \theta}(\eta, \xi)\right|. \end{aligned}$$

Because $\eta \in H^1_0(\Omega)$ we have that

$$(23) \quad [\![\eta]\!] = \mathbf{0} \quad \text{on } \mathcal{E}, \qquad \mathsf{L}(\eta) = \mathbf{0} \quad \text{on } \Omega, \qquad \mathsf{L}_e(\eta) = \mathbf{0} \quad \text{on } \Omega \quad \forall e \in \mathcal{E}.$$

Hence, using the definition (13) of the bilinear form $a_{\mathrm{DG}}^{\gamma, \delta, \varepsilon, \theta}$, results in

$$\left|a_{\mathrm{DG}}^{\gamma, \delta, \varepsilon, \theta}(\eta, \xi)\right| = \left|\int_\Omega \nabla \eta \cdot \nabla_h \xi \, \mathrm{d}\boldsymbol{x} - \int_{\mathcal{E}} \langle\!\langle \nabla \eta \rangle\!\rangle \cdot [\![\xi]\!] \, \mathrm{d}s\right|.$$

Applying the definition (7)–(8) of the lifting operator $\mathbf{L}$, it holds that

$$
\begin{aligned}
-\int_{\mathcal{E}}\langle\!\langle\nabla\eta\rangle\!\rangle\cdot[\![\xi]\!]\,\mathrm{d}s &= -\int_{\mathcal{E}}\langle\!\langle\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\cdot[\![\xi]\!]\,\mathrm{d}s-\int_{\mathcal{E}}\langle\!\langle\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\cdot[\![\xi]\!]\,\mathrm{d}s \\
&= -\int_{\mathcal{E}}\langle\!\langle\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\cdot[\![\xi]\!]\,\mathrm{d}s-\int_{\Omega}\boldsymbol{\Pi_p}(\nabla\eta)\cdot\mathbf{L}(\xi)\,\mathrm{d}\boldsymbol{x} \\
&= -\int_{\mathcal{E}}\langle\!\langle\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\cdot[\![\xi]\!]\,\mathrm{d}s-\int_{\Omega}\nabla\eta\cdot\mathbf{L}(\xi)\,\mathrm{d}\boldsymbol{x},
\end{aligned}
$$

where $\boldsymbol{\Pi_p}$ is the elementwise L^2-projection on $V(\mathcal{T}_h,\boldsymbol{p})^2$. Therefore, it follows that

$$
\begin{aligned}
\left|a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(\eta,\xi)\right| &= \left|\int_{\Omega}\nabla\eta\cdot\nabla_h\xi\,\mathrm{d}\boldsymbol{x}-\int_{\mathcal{E}}\langle\!\langle\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\cdot[\![\xi]\!]\,\mathrm{d}s-\int_{\Omega}\nabla\eta\cdot\mathbf{L}(\xi)\,\mathrm{d}\boldsymbol{x}\right| \\
&\leq \|\nabla\eta\|_{0,\Omega}\|\nabla_h\xi\|_{0,\Omega}+\left\|\mathrm{p}^{-1}\mathrm{h}^{\frac{1}{2}}\langle\!\langle\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\right\|_{0,\mathcal{E}}\left\|\sqrt{\sigma}[\![\xi]\!]\right\|_{0,\mathcal{E}} \\
&\quad+\|\nabla\eta\|_{0,\Omega}\|\mathbf{L}(\xi)\|_{0,\Omega} \\
&\leq \left(\|\nabla\eta\|_{0,\Omega}^2+\left\|\mathrm{p}^{-1}\mathrm{h}^{\frac{1}{2}}\langle\!\langle\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\right\|_{0,\mathcal{E}}^2\right)^{\frac{1}{2}} \\
&\quad\times\left(2\|\nabla_h\xi\|_{0,\Omega}^2+\left\|\sqrt{\sigma}[\![\xi]\!]\right\|_{0,\mathcal{E}}^2+2\|\mathbf{L}(\xi)\|_{0,\Omega}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus, applying Lemma 2.2, we conclude that

$$
\begin{aligned}
\left|a_{\mathrm{DG}}^{\gamma,\delta,\varepsilon,\theta}(\eta,\xi)\right| &\leq \left(\|\nabla\eta\|_{0,\Omega}^2+\left\|\mathrm{p}^{-1}\mathrm{h}^{\frac{1}{2}}\langle\!\langle\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\right\|_{0,\mathcal{E}}^2\right)^{\frac{1}{2}} \\
&\quad\times\left(2\|\nabla_h\xi\|_{0,\Omega}^2+\left(\frac{1}{\Upsilon_{\gamma,\delta}^2}+\frac{2}{\chi_{\gamma,\delta,\varepsilon}^2}\right)\|\xi\|_{\gamma,\delta,\varepsilon}^2\right)^{\frac{1}{2}} \\
&\leq m_{\gamma,\delta,\varepsilon}\left(\|\nabla\eta\|_{0,\Omega}^2+\left\|\mathrm{p}^{-1}\mathrm{h}^{\frac{1}{2}}\langle\!\langle\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\rangle\!\rangle\right\|_{0,\mathcal{E}}^2\right)^{\frac{1}{2}}\|\xi\|_{\gamma,\delta,\varepsilon},
\end{aligned}
$$

where

$$
m_{\gamma,\delta,\varepsilon}=\sqrt{2+\frac{1}{\Upsilon_{\gamma,\delta}^2}+\frac{2}{\chi_{\gamma,\delta,\varepsilon}^2}}.
$$

Therefore, due to (22) and (3), (4), we have

$$
\|\xi\|_{\gamma,\delta,\varepsilon}^2\leq Cm_{\gamma,\delta,\varepsilon}\left(\|\nabla\eta\|_{0,\Omega}^2+\sum_{K\in\mathcal{T}_h}\frac{h_K}{p_K^2}\|\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\|_{0,\partial K}^2\right)^{\frac{1}{2}}\|\xi\|_{\gamma,\delta,\varepsilon}.
$$

Dividing both sides of the above inequality by $\|\xi\|_{\gamma,\delta,\varepsilon}$ leads to

$$
\|\xi\|_{\gamma,\delta,\varepsilon}\leq Cm_{\gamma,\delta,\varepsilon}\left(\|\nabla\eta\|_{0,\Omega}^2+\sum_{K\in\mathcal{T}_h}\frac{h_K}{p_K^2}\|\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)\|_{0,\partial K}^2\right)^{\frac{1}{2}}.
$$

Furthermore, since $\nabla(\mathsf{P}_{\boldsymbol{p}}(u))\in V(\mathcal{T}_h,\boldsymbol{p})^2$ and because the L^2-projection preserves polynomials, it follows that

$$
\nabla\eta-\boldsymbol{\Pi_p}(\nabla\eta)=\nabla u-\nabla(\mathsf{P}_{\boldsymbol{p}}(u))-\boldsymbol{\Pi_p}(\nabla u)+\boldsymbol{\Pi_p}(\nabla(\mathsf{P}_{\boldsymbol{p}}(u)))=\nabla u-\boldsymbol{\Pi_p}(\nabla u).
$$

Hence,

$$\|\xi\|_{\gamma,\delta,\varepsilon} \le Cm_{\gamma,\delta,\varepsilon} \left(\|\nabla\eta\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K}{p_K^2} \|\nabla u - \mathbf{\Pi}_{\boldsymbol{p}}(\nabla u)\|_{0,\partial K}^2 \right)^{\frac{1}{2}},$$

and inserting this into (21), implies

$$\|e\|_{\gamma,\delta,\varepsilon} \le \|\eta\|_{\gamma,\delta,\varepsilon} + Cm_{\gamma,\delta,\varepsilon} \left(\|\nabla\eta\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K}{p_K^2} \|\nabla u - \mathbf{\Pi}_{\boldsymbol{p}}(\nabla u)\|_{0,\partial K}^2 \right)^{\frac{1}{2}}.$$

Then, using (23), we obtain

$$(24) \qquad \|e\|_{\gamma,\delta,\varepsilon} \le Cm_{\gamma,\delta,\varepsilon} \left(\|\nabla\eta\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K}{p_K^2} \|\nabla u - \mathbf{\Pi}_{\boldsymbol{p}}(\nabla u)\|_{0,\partial K}^2 \right)^{\frac{1}{2}}.$$

Finally, using the approximation properties of the interpolants $\mathsf{P}_{\boldsymbol{p}}$ and $\mathbf{\Pi}_{\boldsymbol{p}}$ (cf. Lemmas 3.1 and 3.2) leads to the main result of this paper.

Theorem 3.3. *Suppose that* (17) *and the assumptions of Proposition 2.3 are satisfied. Furthermore, let the exact solution u of* (1)–(2) *belong to $H^{\boldsymbol{s}}(\Omega,\mathcal{T}_h) \cap H^2(\Omega) \cap H_0^1(\Omega)$, with $s_K \ge 2$ for all $K \in \mathcal{T}_h$. Then, there exists a constant $C > 0$ independent of $\boldsymbol{h}$ and $\boldsymbol{p}$ such that there holds the hp-a priori error estimate*

$$(25) \qquad \|u - u_{\mathrm{DG}}\|_{\gamma,\delta,\varepsilon}^2 \le C \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p_K} \right)^{2\tilde{s}_K - 2} |u|_{H^{\tilde{s}_K}(K)}^2,$$

for $2 \le \tilde{s}_K \le \min(p_K + 1, s_K)$, $K \in \mathcal{T}_h$ and where u_{DG} is the hp-DG solution from (12).

Remark 3.4. We note that the error estimate in Theorem 3.3 is optimal with respect to both the local element sizes and polynomial degrees. The main ingredients for the proof are the use of a continuous interpolant of the exact solution which, by conformity, belongs to the kernel of the possibly suboptimally scaled stabilization terms, and the fact that the boundary conditions in the model problem (1)–(2) are homogeneous. Note that for general inhomogeneous Dirichlet boundary conditions, the p-optimality is typically not guaranteed as has been displayed in [13, Example 3.3].

Remark 3.5. Furthermore, we remark that the optimality of the L^2-projection on the boundary of the elements (cf. Lemma 3.1) is not needed for the optimality result due to the factor of p_K^{-2} in (24) (indeed, p_K^{-1} would be sufficient).

4. Numerical tests

Let us consider problem (1)–(2) on the unit square $\Omega = (-1,1)^2$ with right-hand side f such that the solution becomes

$$u(x_1, x_2) = (1 - x_1^2)(1 - x_2^2)(x_1^2 + x_2^2)^{\frac{3}{2}} \in H^{4-\varepsilon}(\Omega) \cap H_0^1(\Omega),$$

for arbitrarily small $\varepsilon > 0$. We remark that this function features the same singularity (at the origin $(0,0)$) as in [13, Example 3.3]; however, here, it is located in the interior of the computational domain, and homogeneous boundary conditions have been imposed along $\partial\Omega$.
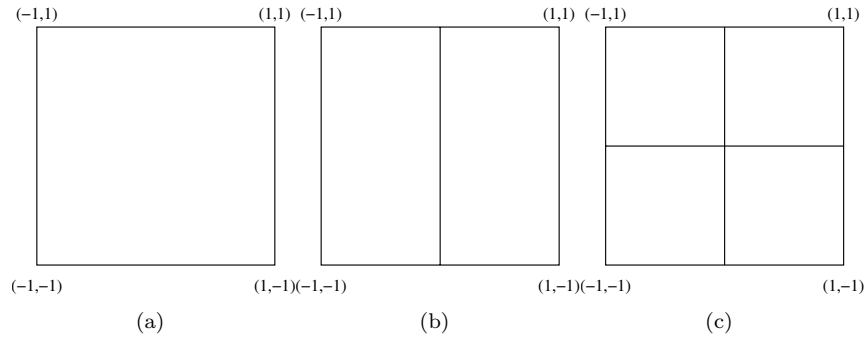
FIGURE 1. Meshes used for the computations.

For this example, the error estimate from Theorem 3.3 becomes

$$(26) \qquad \|u - u_{\mathrm{DG}}\|_{\gamma, \delta, \varepsilon} \leq C \left(\frac{h}{p} \right)^{\widetilde{s}-1} |u|_{H^{\widetilde{s}}(\Omega)},$$

for $2 \leq \widetilde{s} < \min(p+1, 4)$.

We consider the three meshes displayed in Figure 1 for uniform polynomial degrees $p = 1, \dots, 24$. Here, we focus on the dependence of the error on p, i.e., no h-refinement is performed during the computations. Observe that, due to the three different choices of meshes, the singularity lies either in the interior of the element, or on the interior of a face, or coincides with a vertex. For the computations the C++ library *life* has been used; see [22, 23].

We give numerical results for the p-IP DG methods. More precisely, we let $\gamma > 0$, $\delta = 0$, and $\varepsilon = 0$. Figures 2 and 3 show the convergence of the error measured in the energy norm with respect to increasing polynomial degrees for $\theta \in \{1, 0, -1\}$. Observe the optimal convergence for the first two meshes (a) and (b) and the superoptimality in the case of the third mesh (c).

Tables 2–4 display the convergence rates of the SIPG method, i.e., more specifically, $\theta = 1$, $\gamma = 10$, $\delta = 0$, and $\varepsilon = 0$, on the meshes (a), (b) and (c). The IIPG ($\theta = 0$) and NIPG ($\theta = -1$) behave similarly; see Figures 2 and 3. Since the errors show an even/odd alternating regime with respect to p, we present the errors in four different tables for each mesh. We notice that the convergence rates tend to a rate of at least 3 for meshes (a) and (b), and to a rate of 5.5 for mesh (c). The bound (26) is respected in all three cases. However, for mesh (c), we notice that the order is nearly doubled. This can be explained by a sharper analysis using tensorized weighted Sobolev spaces; see [17, Remark 3.8]. Indeed, the anisotropic weight function has some smoothing properties for singular functions on the element boundary, and therefore, better approximation results can be obtained. This explains the behavior of the energy error on mesh (c), where the singularity lies on a vertex of each element. On the other hand, when using mesh (b) such a behavior is not observed, although the singularity lies on the boundary of each element. In this case, the isotropic singularity cannot be smoothed out by the anisotropic weight function.

TABLE 2. Energy error for p-refinement on mesh (a), for $\theta = 1$, $\gamma = 10$, $\delta = 0$ and $\varepsilon = 0$ (SIPG). The polynomial degrees are grouped by periodicity of 4, and the quantities in brackets show the convergence rates.

p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$
1	2.51E+00	2	1.79E+00	3	1.50E+00	4	3.05E-01
5	2.95E-01 (1.33)	6	4.49E-02 (3.36)	7	4.45E-02 (4.16)	8	1.60E-02 (4.25)
9	1.57E-02 (5.00)	10	7.53E-03 (3.49)	11	7.54E-03 (3.93)	12	4.31E-03 (3.23)
13	4.19E-03 (3.58)	14	2.61E-03 (3.15)	15	2.64E-03 (3.39)	16	1.79E-03 (3.06)
17	1.73E-03 (3.31)	18	1.21E-03 (3.05)	19	1.23E-03 (3.21)	20	9.16E-04 (3.00)
21	8.78E-04 (3.20)	22	6.62E-04 (3.02)	23	6.78E-04 (3.14)	24	5.32E-04 (2.98)

TABLE 3. Energy error for p-refinement on mesh (b), for $\theta = 1$, $\gamma = 10$, $\delta = 0$ and $\varepsilon = 0$ (SIPG). The polynomial degrees are grouped by periodicity of 4, and the quantities in brackets show the convergence rates.

p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$
1	2.44E+00	2	1.53E+00	3	1.07E+00	4	2.10E-01
5	2.05E-01 (1.54)	6	3.15E-02 (3.53)	7	3.11E-02 (4.18)	8	1.10E-02 (4.26)
9	1.11E-02 (4.95)	10	5.37E-03 (3.46)	11	5.27E-03 (3.93)	12	2.95E-03 (3.25)
13	3.00E-03 (3.57)	14	1.87E-03 (3.13)	15	1.83E-03 (3.41)	16	1.22E-03 (3.08)
17	1.24E-03 (3.30)	18	8.71E-04 (3.04)	19	8.51E-04 (3.24)	20	6.19E-04 (3.03)
21	6.31E-04 (3.19)	22	4.76E-04 (3.01)	23	4.65E-04 (3.16)	24	3.58E-04 (3.00)

TABLE 4. Energy error for p-refinement on mesh (c), for $\theta = 1$, $\gamma = 10$, $\delta = 0$ and $\varepsilon = 0$ (SIPG). The polynomial degrees are grouped by periodicity of 4, and the quantities in brackets show the convergence rates.

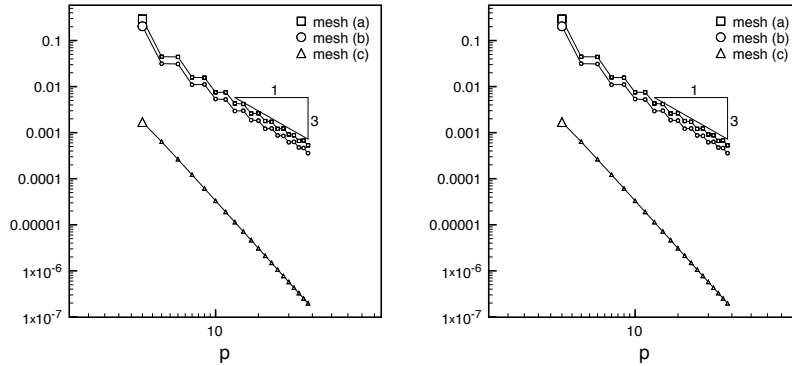| p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$	p	$\|\|u - u_{\mathrm{DG}}\|\|_{\gamma,\delta,\varepsilon}$
1	2.29E+00	2	1.27E+00	3	4.12E-01	4	5.54E-02
5	1.70E-03 (4.48)	6	6.38E-04 (6.91)	7	2.65E-04 (8.67)	8	1.22E-04 (8.82)
9	6.14E-05 (5.65)	10	3.32E-05 (5.79)	11	1.90E-05 (5.83)	12	1.14E-05 (5.84)
13	7.15E-06 (5.85)	14	4.64E-06 (5.85)	15	3.09E-06 (5.85)	16	2.12E-06 (5.85)
17	1.49E-06 (5.86)	18	1.06E-06 (5.86)	19	7.75E-07 (5.86)	20	5.73E-07 (5.86)
21	4.31E-07 (5.86)	22	3.28E-07 (5.87)	23	2.52E-07 (5.87)	24	1.97E-07 (5.87)

FIGURE 2. Energy error for p-refinement on the three meshes from Figure 1, for $\gamma = 10$, $\delta = 0$ and $\varepsilon = 0$. Left $\theta = 1$ (SIPG), right $\theta = 0$ (IIPG).
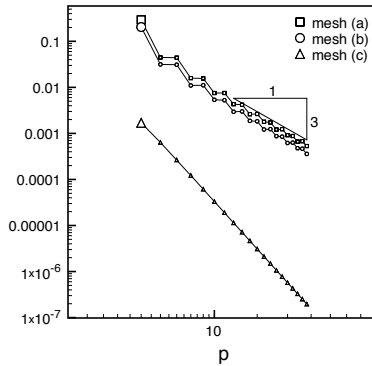

FIGURE 3. Energy error for p-refinement on the three meshes from Figure 1, for $\gamma = 10$, $\delta = 0$ and $\varepsilon = 0$, and $\theta = -1$ (NIPG).

5. CONCLUDING REMARKS

In this paper we have presented an hp-a priori error analysis of a class of discontinuous Galerkin methods for the numerical solution of linear elliptic partial differential equations. The schemes under consideration include well-known DG methods such as the IP or LDG methods. We have shown that the schemes are stable (provided that the involved parameters are chosen appropriately) and, in particular, optimally convergent with respect to both the local element sizes and polynomial degrees on quadrilateral meshes (possibly containing hanging nodes). In our analysis, both the availability of a continuous interpolant of the exact solution as well as the homogeneous Dirichlet boundary conditions (2) have played an important role.

Acknowledgments

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DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, 182 GEORGE STREET, BOX F, PROVIDENCE, RI 02912

E-mail address: `Benjamin_Stamm@Brown.edu`

MATHEMATICS INSTITUTE, UNIVERSITY OF BERN, SIDLERSTRASSE 5, CH-3012 BERN, SWITZERLAND

E-mail address: `wihler@math.unibe.ch`