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ODD HARMONIC NUMBERS EXCEED 10^{24}

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Abstract. A number $n > 1$ is harmonic if $\sigma(n) \mid n\tau(n)$, where $\tau(n)$ and $\sigma(n)$ are the number of positive divisors of n and their sum, respectively. It is known that there are no odd harmonic numbers up to 10^{15}. We show here that, for any odd number $n > 10^6$, $\tau(n) \le n^{1/3}$. It follows readily that if n is odd and harmonic, then $n > p^{3a/2}$ for any prime power divisor p^a of n, and we have used this in showing that $n > 10^{18}$. We subsequently showed that for any odd number $n > 9 \cdot 10^{17}$, $\tau(n) \le n^{1/4}$, from which it follows that if n is odd and harmonic, then $n > p^{8a/5}$ with p^a as before, and we use this improved result in showing that $n > 10^{24}$.

1. Introduction

We write $\tau(n)$ for the number of positive divisors of the natural number n and $\sigma(n)$ for their sum. It is well known that, if n has prime factor decomposition $n = \prod_{i=1}^{t} p_i^{a_i}$, then

$$\tau(n) = \prod_{i=1}^{t}(a_i + 1) \quad \text{and} \quad \sigma(n) = \prod_{i=1}^{t} \frac{p_i^{a_i+1} - 1}{p_i - 1},$$

from which it follows that these are multiplicative functions. That is,

$$\tau(mn) = \tau(m)\tau(n) \quad \text{and} \quad \sigma(mn) = \sigma(m)\sigma(n)$$

when $(m, n) = 1$.

The number $n > 1$ is harmonic if

$$h(n) = \frac{n\tau(n)}{\sigma(n)}$$

is an integer. The function h is also multiplicative. Harmonic numbers are of interest because it is easily shown that every perfect number (satisfying $\sigma(n) = 2n$) is harmonic, yet no odd harmonic numbers have been found. If it can be proved that there are none, then it will follow that there are no odd perfect numbers, solving perhaps the oldest problem in mathematics.

Harmonic numbers were introduced by Ore [11], and named (some 15 years later) by Pomerance [12]. For recent results in this area, see Goto and Shibata [9] and Sorli [14]. In the latter, it was shown that there are no odd harmonic numbers up to 10^{15}. The method required the determination of all harmonic seeds (see Cohen and Sorli [6]) to 10^{15} and the observation that all are even. This extended the result announced in [6], determined similarly, that there are no odd harmonic numbers less than 10^{12}. Previous approaches used a straightforward incremental

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search, such as in [11] and Cohen [4], giving bounds of 10^5 and $2\cdot 10^9$, respectively. In this paper, we use specific estimates for the divisor function τ and some basic results from cyclotomy to improve the bound, first to 10^{18} and then to 10^{24}.

We use the following notation. We write $\omega(n)$ for the number of distinct prime factors of n and we write $m \parallel n$ to mean that $m \mid n$ and $(n, n/m) = 1$. In particular, $p^a \parallel n$, p prime, if $p^a \mid n$ and $p^{a+1} \nmid n$; in this case, p^a is called a component of n. We use p, q and r exclusively to denote primes.

Many details and computational results are omitted from this paper. They may be found in the superseded paper [5], available from the second author or from the Department of Mathematical Sciences, University of Technology, Sydney.

2. Estimates for the divisor function

Hardy and Wright [7] showed that, given $\epsilon > 0$, there is a constant A_ϵ, dependent only on ϵ, such that $\tau(n) < A_\epsilon n^\epsilon$ for all n. They gave an expression for A_ϵ, but it is of no use for our purposes.

Taking $\epsilon = \frac{1}{2}$, Sierpiński [13] (Exercise 1, page 168) gave, as an exercise, the result $\tau(n) < 2\sqrt{n}$ for all n. His hint to its solution leads readily to the following.

Theorem 1. *If n is an odd number, not 3 or 15, then $\tau(n) \le \sqrt{n}$.*

There are two proofs given in [5]. The first uses Sierpiński's hint, but this gives no suggestion for similar results with $\epsilon < \frac{1}{2}$. The second proof gives that suggestion and is the basis for the following result.

Theorem 2. *If $n > 10^6$ and is odd, then $\tau(n) \le \sqrt[3]{n}$.*

Proof. We say n has property T if $\tau(n) \le \sqrt[3]{n}$, and property $\widetilde{T}$ otherwise. We first wrote a program that checked each odd number in $[3, 10^8]$ for property T. There are 267 numbers in this interval with property $\widetilde{T}$, the largest three being $765765 = 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17$, $855855 = 3^2 5\cdot 7\cdot 11\cdot 13\cdot 19$ and $883575 = 3^3 5^2 7\cdot 11\cdot 17$. The gap between these and 10^8 is so large that it was likely all larger numbers had property T, and this was verified as follows.

First, it is easy to see that if $\omega(n) = 1$, then n has property $\widetilde{T}$ only in the seven cases

$$n = 3,\ 5,\ 7,\ 3^2,\ 5^2,\ 3^3,\ 3^4.$$

A program was written and run in which each of these was multiplied in turn by 3, 5, 7, 11, ..., and then in turn by 3^2, 5^2, 7^2, ..., and if necessary by 3^3, 5^3, 7^3, ..., and so on, until in each case property T held. In this way, all 36 odd numbers n with $\omega(n) = 2$ and property $\widetilde{T}$ were found. In the order implied by this algorithm and after removing duplications, those numbers are

$$3\cdot 5,\ 3\cdot 7;\ 5\cdot 7,\ 5\cdot 3^2,\ 5\cdot 3^3,\ 5\cdot 3^4;\ 7\cdot 3^2,\ 7\cdot 3^3,\ 7\cdot 3^4;$$
$$11\cdot 3,\ 11\cdot 5,\ 11\cdot 3^2,\ 11\cdot 3^3,\ 11\cdot 3^4;\ 13\cdot 3,\ 13\cdot 3^2,\ 13\cdot 3^3;$$
$$17\cdot 3,\ 17\cdot 3^2,\ 17\cdot 3^3;\ 19\cdot 3,\ 19\cdot 3^2;\ 23\cdot 3^2;\ 3^2\cdot 5^2;$$
$$5^2\cdot 3,\ 5^2\cdot 7,\ 5^2\cdot 3^3,\ 5^2\cdot 3^4;\ 7^2\cdot 3,\ 7^2\cdot 3^2,\ 7^2\cdot 3^3;$$
$$5^3\cdot 3^2,\ 5^3\cdot 3^3;\ 3^4\cdot 5^2;\ 3^5\cdot 5,\ 3^5\cdot 7.$$

Then these 36 numbers were treated similarly to find all 89 odd numbers n with $\omega(n) = 3$ and property $\widetilde{T}$, and the process was repeated to find all 96 with $\omega(n) = 4$

and property $\widetilde{T}$, all 36 with $\omega(n) = 5$ and property $\widetilde{T}$, and the three with $\omega(n) = 6$ and property $\widetilde{T}$. On the next run of the program all numbers had property T. The total found this way with property $\widetilde{T}$ was 267, as above.

Hence if n is odd and $n \geq 883577$, then $\tau(n) \leq \sqrt[3]{n}$. □

The second author used the same bootstrap approach to give the following improvement.

Theorem 3. *If $n > 9 \cdot 10^{17}$ and is odd, then $\tau(n) \leq \sqrt[4]{n}$.*

In proving this, Sorli found that there are 2372091 odd numbers n with $\tau(n) > \sqrt[4]{n}$. The largest is $n = 883086389887727025 = 3^4 5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43$, for which $\omega(n) = 13$. No odd n with $\tau(n) > \sqrt[4]{n}$ has a greater number of prime factors; the largest prime factor encountered is 2011 and the largest exponent on a prime factor is 11. The modal number of prime factors is 8; there are 645321 odd numbers n with $\tau(n) > \sqrt[4]{n}$ and $\omega(n) = 8$, of which the largest is $2637919811401875 = 3^4 5^4 7^3 11^2 13^2 17 \cdot 19 \cdot 23$.

There is the following attractive consequence of Theorem 3: for any odd number $n > 9 \cdot 10^{17}$, its positive divisors have harmonic mean at most $n^{1/4}$, geometric mean equal to $n^{1/2}$, and arithmetic mean at least $n^{3/4}$.

3. Application to odd harmonic numbers

We can quickly give the following applications of Theorem 2.

Lemma 1. *If n is an odd harmonic number, then $\tau(n) \leq \sqrt[3]{n}$.*

Proof. We need only note that certainly there are no odd harmonic numbers less than 10^6. □

Lemma 2. *If n is an odd harmonic number and $p^a \mid n$, then $n > p^{3a/2}$.*

Proof. We may assume $p^a \parallel n$. Then $\sigma(p^a) \mid n\tau(n)$ and, of course, $p^a \mid n\tau(n)$. Then $p^a\sigma(p^a) \mid n\tau(n)$ since $(p^a, \sigma(p^a)) = 1$, and we have, using Lemma 1,

$$n\sqrt[3]{n} \geq n\tau(n) \geq p^a\sigma(p^a) > p^{2a},$$

and the result follows. □

This result is an important tool for our first improvement on the lower bound of the set of odd harmonic numbers, 10^{18} in place of 10^{15}. It corresponds to a similar tool in the derivation of a lower bound for the set of odd perfect numbers. In that field it is known that if n is an odd perfect number and $p^a \mid n$, then $n > p^{2a}$. The bound is currently 10^{300}. See Brent and Cohen [1], and Brent, Cohen and te Riele [2]. In the latter paper, a conditional improvement on the result $n > p^{2a}$ is discussed and used.

Lemma 2 could have been used to good effect by Mills, and his 1972 paper [10] shows that he had the machinery to obtain such a result. In [10], he proved that an odd harmonic number n must have a component greater than 10^7. Then, by Lemma 2, $n > 10^{10.5}$. Furthermore, Mills indicated that he had extended his result to show that there must be a component greater than 65551^2, so $n > 65551^3 > 2 \cdot 10^{14}$, a far greater bound than was known until Sorli's 2003 result.

We also require the following well-known results concerned with odd harmonic numbers.

Lemma 3. *If n is an odd harmonic number and $p^a \parallel n$, then $p^a \equiv 1 \pmod 4$.*

Lemma 4. *If n is an odd harmonic number, then $p \mid \tau(n)$ for some prime $p \geq 3$.*

Lemma 5. *If n is an odd harmonic number, then $\omega(n) \geq 3$.*

The first two results are due to Garcia [8] (his Theorems 2 and 3, respectively); the second generalizes a result of Ore [11], that an odd harmonic number cannot be squarefree. The third result is due to Pomerance [12] and Callan [3], independently.

From standard cyclotomy theory, as applied also in [1] and [2], we have the following.

Lemma 6. *For any prime p, $\sigma(p^a) \mid \sigma(p^b)$ if $(a+1) \mid (b+1)$. If $a > 1$ and $q = a+1$ is prime, then the possible prime divisors of $\sigma(p^a)$ are primes $r \equiv 1 \pmod q$, and q itself if and only if $p \equiv 1 \pmod q$. In the latter case, $q \parallel \sigma(p^a)$.*

At the end of the paper, we will describe the possible use of the following result, which is Lemma 1 in [2]:

Lemma 7. *If p and q are odd primes with $q \mid \sigma(p^k)$ and $p^a \mid (q+1)$, then $k \geq 3a$.*

The full details of the derivation of the next result are given in [5].

Lemma 8. *If n is an odd harmonic number, then $n > 10^{18}$.*

The proof in [5] is essentially a manual one, but computer assisted for factorizations and many intermediate results. The result of Theorem 3, which was obtained some time after that of Lemma 8, then fortuitously allows the following theorems, proved in the same manner as Lemmas 1 and 2.

Theorem 4. *If n is an odd harmonic number, then $\tau(n) \leq \sqrt[4]{n}$.*

Theorem 5. *If n is an odd harmonic number and $p^a \mid n$, then $n > p^{8a/5}$.*

These prompted the search for a more fully programmed approach to an improvement of Lemma 8, based on the earlier approach. This led to the following.

Theorem 6. *If n is an odd harmonic number, then $n > 10^{24}$.*

4. Proof of the main result

We give the proof of Theorem 6 as the result of a number of propositions, the first of which follows immediately from Theorem 5.

Proposition 1. *Let n be an odd harmonic number and a a positive integer. If $p > 10^{15/a}$ and $p^a \mid n$, then $n > 10^{24}$.*

For example, taking $a = 4$ and noting that the smallest prime greater than $10^{15/4}$ is 5639, we have, for any odd harmonic number n: If $p^4 \mid n$ for $p \geq 5639$, then $n > 10^{24}$.

In the proof of subsequent propositions, there will be tacit use of the first statement of Lemma 6, as we now describe.

Cyclotomy tells us that

$$\sigma(p^b) = \prod_{\substack{d>1 \\ d \mid (b+1)}} F_d(p),$$

where F_d is the cyclotomic polynomial of order d. If $b > 1$, then $F_{a+1}(p) = 1 + p + \cdots + p^a = \sigma(p^a)$ divides $\sigma(p^b)$, where $a+1$ is the smallest prime factor of $b+1$. From Lemma 6 it may be inferred that if it has been shown that $p^a \parallel n$ implies $n > 10^{24}$ where $a+1$ is prime, and this has been done through consideration of the prime factors of $\sigma(p^a)$, then also $p^b \parallel n$ implies $n > 10^{24}$ when $(a+1) \mid (b+1)$. Therefore, only components p^a where $a+1$ is prime need be considered.

Proposition 2. *Let n be an odd harmonic number. If $p \geq 5$ and $p \mid \tau(n)$, then $n > 10^{24}$.*

Proof. If $p \geq 37$, then $q^{36} \mid n$ for some odd prime q, so, by Theorem 5, $n > q^{8\cdot 36/5} \geq 3^{57.6} > 10^{27}$, and the result is proved.

A single run of a program allowed the proof to be completed, and only a few highlights will be described here. The algorithm may be summarised as:

$*$ $\qquad p^a : \sigma(p^a) \qquad$ B partial calculation of $[\log_{10} n]$.

Here, p^a is an assumed component of n, which gives rise to $\sigma(p^a)$ (as suggested by the colon), in factored form. Further, p^a is the root of a tree. Later branches of the tree are shown (in examples below) by indented lines and each is based similarly on the largest available prime factor of n, which will be the largest prime factor of the current $\sigma(p^a)$ or that of $\sigma(p^a)$ from an earlier branch, provided it has not previously been used for the same purpose in the same tree. If this largest prime factor, r say, has already been shown to be such that, if $r \mid \tau(n)$, then $n > 10^{24}$ (for example, if $r \geq 37$ by virtue of the preceding paragraph), then indeed it may be taken as a factor of n, rather than of $\tau(n)$. (There will be instances below where there is no such largest prime, and this matter will be considered then.) In $*$, "B" precedes a number which is the partial calculation of $[\log_{10} n]$ (where $[\cdot]$ is the floor function), based on the product of those factors of n that are known to that point, with prime factors of each $\sigma(p^a)$ taken to power 1 if 1 (mod 4) and power 2 if 3 (mod 4), in accordance with Lemma 3. Calculations are continued within a tree until $[\log_{10} n] \geq 24$ (so $n > 10^{24}$, as required) and branches are continued for given p by incrementing a until Proposition 1 is applicable, but having regard for Lemma 6.

With respect to Lemma 5, in all cases where a chain of calculations involves three or more distinct primes and the partial calculation of $[\log_{10} n]$ does not exceed 24, a simultaneous partial calculation of $h(n)$ has been carried out to ensure that no harmonic number less than 10^{24} has occurred.

A similar algorithm was used in [1] and [2].

Since the result is known to be true for $p \geq 37$, the program runs through the possibilities $p = 31, \ldots, p = 5$ (in decreasing order of primes).

For example, take $p = 13$. Then $q^{13k+12} \mid n$ for some prime q and $k \geq 0$. By Lemma 6, we need consider only $k = 0$ (and thus $q^{12} \parallel n$) and, by Proposition 1 with $a = 12$, we need consider only $3 \leq q \leq 17$. Then, as follows, there are six trees to process, with roots $17^{12}, \ldots, 3^{12}$. Further comments on the algorithm follow the sixth tree.

$17^{12} : 212057 \cdot 2919196853$ $\qquad$ B 29
$13^{12} : 53 \cdot 264031 \cdot 1803647$ $\qquad$ B 38
$11^{12} : 1093 \cdot 3158528101$ $\qquad$ B 24
$7^{12} : 16148168401$ $\qquad$ B 20
$\quad 16148168401^1 : 2 \cdot 103 \cdot 709 \cdot 110563$ $\qquad$ B 37

5^{12} : 305175781 B 16
  $305175781^1 : 2 \cdot 3499 \cdot 43609$ B 28
3^{12} : 797161 B 11
  $797161^1 : 2 \cdot 398581$ B 17
    $398581^1 : 2 \cdot 17 \cdot 19 \cdot 617$ B 23
$*$ $617^1 : 2 \cdot 3 \cdot 103$ B 27
      617^2 B 26
    $398581^2 : 3 \cdot 1621 \cdot 32668561$ B 33
  $797161^2 : 3 \cdot 61 \cdot 151 \cdot 22996651$ B 36

By way of further illustration, when we assume that $3^{12} \parallel n$ we have $\sigma(3^{12}) = 797161 \mid \sigma(n) \mid n\tau(n)$, so we may assume that $797161 \mid n$ since $797161 \geq 37$. Then $n > 3^{12}797161 > 10^{11}$, as indicated by "B 11". We continue the tree by assuming that $797161 \parallel n$ (and later, in another branch, that $797161^2 \parallel n$), and obtain $n > 3^{12}398581 \cdot 797161 > 10^{17}$. At the line marked $*$ we have $3^{12}617 \cdot 398581 \cdot 797161 \parallel n$ and $n > 3^{12}17 \cdot 19^2 103^2 617 \cdot 398581 \cdot 797161 > 10^{27} > 10^{24}$. This uses the fact that at this stage the proposition has been established already for $p \geq 17$, so it may also be assumed that $17 \cdot 19 \cdot 103 \mid n$, and $19 \equiv 103 \equiv 3 \pmod 4$, so in fact $19^2 103^2 \mid n$. That $n > 10^{27}$ is indicated in $*$ by "B 27", and this branch need be carried no further. In the following line, there is no need to factorize $\sigma(617^2)$, since we have $n > 3^{12}17 \cdot 19^2 617^2 398581 \cdot 797161 > 10^{26}$.

As dictated by Proposition 1 with $a = 10$, 6 and 4, the program runs through, respectively, $q^{10} \parallel n$ for $3 \leq q \leq 31$, $q^6 \parallel n$ for $3 \leq q \leq 313$ and $q^4 \parallel n$ for $3 \leq q \leq 5623$. We give one further illustration, arising when $p = 5$ and $163^4 \parallel n$, to show the choice of the largest available prime for each new branch:

$163^4 : 11 \cdot 31 \cdot 1301 \cdot 1601$ B 20
  $1601^1 : 2 \cdot 3^2 89$ B 22
    $1301^1 : 2 \cdot 3 \cdot 7 \cdot 31$ B 23
      $89^1 : 2 \cdot 3^2 5$ B 23
        $31^2 : 3 \cdot 331$ B 28
        31^4 B 26
      89^2 B 25
    1301^2 B 25
  $1601^2 : 37 \cdot 103 \cdot 673$ B 31
  1601^4 B 29 □

In proving Theorem 6, it follows now that we may assume $p \mid n$ when $p \mid \sigma(n)$ and $p \geq 5$, and that $\tau(n) = 2^a 3^b$. By Lemma 4, the theorem will follow once it has been shown that $b = 0$. That is, in principle, the theorem will follow once Proposition 2 has been extended to $p = 3$, and it is plausible to do this by applying the algorithm above to $q^2 \parallel n$ for $3 \leq q \leq 31622743$ (although complications will arise, corresponding to primes in the set S_2, below). Partly for historical reasons (the approach adopted in [5]) and partly as a template for later more ambitious attempts to improve the bound, we proceed in a manner that does not require such an extensive run.

Proposition 3. *Let n be an odd harmonic number. Except for the* 44 *primes q in Table* 1*, if $3 \leq p < 10^4$ and $p \mid n$, then $n > 10^{24}$. If q is one of the* 44 *primes in Table* 1 *and $q^2 \parallel n$, then $n > 10^{24}$.*

TABLE 1. Primes $q \in S_2$, $q < 10^4$, with factorizations of $\sigma(q)/2$.

q	$(q+1)/2$	q	$(q+1)/2$	q	$(q+1)/2$	q	$(q+1)/2$
5	3	809	$3^4 5$	2861	$3^3 53$	6337	3169
17	3^2	953	$3^2 53$	2957	$3 \cdot 17 \cdot 29$	7829	$3^3 5 \cdot 29$
29	$3 \cdot 5$	1009	$5 \cdot 101$	3169	$5 \cdot 317$	8009	$3^2 5 \cdot 89$
53	3^3	1601	$3^2 89$	3433	$17 \cdot 101$	8069	$3 \cdot 5 \cdot 269$
89	$3^2 5$	1613	$3 \cdot 269$	4373	3^7	8089	$5 \cdot 809$
101	$3 \cdot 17$	1801	$17 \cdot 53$	5189	$3 \cdot 5 \cdot 173$	9161	$3^2 509$
173	$3 \cdot 29$	1901	$3 \cdot 317$	5209	$5 \cdot 521$	9221	$3 \cdot 29 \cdot 53$
269	$3^3 5$	2017	1009	5717	$3 \cdot 953$	9341	$3^3 173$
317	$3 \cdot 53$	2609	$3^2 5 \cdot 29$	5857	$29 \cdot 101$	9377	$3^2 521$
509	$3 \cdot 5 \cdot 17$	2689	$5 \cdot 269$	5881	$17 \cdot 173$	9433	$53 \cdot 89$
521	$3^2 29$	2753	$3^4 17$	6053	$3 \cdot 1009$	9677	$3 \cdot 1613$

Proof. The proof relies on deriving a number of results of the form "if $p \mid n$, then $n > 10^{24}$" in a particularly useful order until finally obtaining them for all primes in $[3, 10^4]$, apart from the tabled exceptions. The set of primes for which the results are obtained is denoted by S_1 and the set $\{5,\ 17,\ 29,\ \ldots,\ 9677\}$ of exceptions by S_2. The set S_1 is developed largely to provide primes up to 10^4 that assist in the treatment of other, usually larger, primes, and the set S_2 to specify those primes up to 10^4 that cannot be used this way.

The results are obtained with the same algorithm as before, with one addition: a notation such as "$p^a : q \ldots$ D" (in the examples below) means that $q \mid \sigma(p^a)$ and $q \mid n$ has previously been shown to imply $n > 10^{24}$. That is, $q \in S_1$ implies that if p^a is a component of n, then $n > 10^{24}$. In proving $p \in S_1$, where $p^a \parallel n$, it follows from the above that we need consider only $a = 1$ when $p \equiv 1 \pmod 4$, and $a = 2$, with a similar comment for branches that may arise.

The proof begins by showing as follows that 127, 19, 11, 7, 331, $31 \in S_1$:

$127^2 : 3 \cdot 5419$ B 11
  $5419^2 : 3 \cdot 31 \cdot 313 \cdot 1009$ B 20
    $1009^1 : 2 \cdot 5 \cdot 101$ B 22
      $101^1 : 2 \cdot 3 \cdot 17$ B 24
    $1009^2 : 3 \cdot 37 \cdot 9181$ B 28

$19^2 : 127 \ldots$ D

$11^2 : 19 \ldots$ D

$7^2 : 19 \ldots$ D

$331^2 : 7 \ldots$ D

$31^2 : 331 \ldots$ D

It continues in a straightforward fashion to show that 67, 37, 47, 433, 631, 43, 79, 23, $307 \in S_1$. The order of treatment of the first few primes is the order found to be convenient for odd perfect numbers in [1] and it is also similar to the approach adopted in [10].

Whichever order is adopted, there will be difficulties associated with the set S_2. The primes $p \in S_2$ are such that $p \equiv 1 \pmod 4$ and $\sigma(p) = p + 1$ has no prime factors greater than 5 or is a product $2 \cdot 3^a \prod_{q_i \in S_2} q_i$, where the q_i have already been shown to belong to S_2. For such primes, we are content to show that $p^2 \parallel n$ implies $n > 10^{24}$ (and then, by Proposition 2 and Lemma 6, $p^a \mid n$ implies $n > 10^{24}$ for a not of the form $2^b - 1$, $b \geq 1$). In the algorithm, when it is useful as a reminder (when results are to be displayed) an underlined prime q on the right of $p^a : \sigma(p^a)$ indicates that $q \in S_2$; also, p on the left is underlined on its first occurrence in a chain if it is being demonstrated that $p \in S_2$.

For example, 5, 17, 29, $53 \in S_2$ since $\sigma(5) = 2 \cdot 3$, $\sigma(17) = 2 \cdot 3^2$, $\sigma(29) = 2 \cdot 3 \cdot 5$, $\sigma(53) = 2 \cdot 3^3$, and

$\underline{5}^2 : 31 \ldots$ D

$\underline{17}^2 : 307 \ldots$ D

$\underline{29}^2 : 67 \ldots$ D

$\underline{53}^2 : 7 \ldots$ D

We show now that $73 \in S_1$; note the subsequent comment concerning the line marked $*$.

$73^1 : 37 \ldots$ D
$73^2 : 3 \cdot \underline{1801}$ B 6
  $1801^1 : 2 \cdot \underline{17} \cdot \underline{53}$ B 9
    $53^1 : 2 \cdot 3^3$ B 9
$*$ $17^1 : 2 \cdot 3^2$ ($13^2 41^2 59^2 61^2 71^2$ B 26)
        $3^2 : 13$ B 12
          $13^1 : 7 \ldots$ D
          $13^2 : 3 \cdot 61$ B 14
            $61^1 : 31 \ldots$ D
            $61^2 : 3 \cdot 13 \cdot 97$ B 16
              $97^1 : 7 \ldots$ D
              $97^2 : 3 \cdot 3169$ B 22
                $3169^1 : 2 \cdot 5 \cdot 317$ B 25
                3169^2 B 26
  $1801^2 : 7 \ldots$ D

From the four lines up to and including that marked $*$, we observe that when $17 \cdot 53 \cdot 73^2 1801 \parallel n$ we have $3^6 \mid \sigma(n)$. Then we must have either $3^5 \mid \tau(n/73^2)$, or $3^2 \parallel n$ and $3^2 \mid \tau(n/3^2 73^2)$. (Possibilities such as $3^8 \parallel n$ are covered as usual by Lemma 6.) In the former case, since the five smallest primes greater than 3 and not yet shown to belong to S_1 or S_2 are 13, 41, 59, 61, 71, we have $n \geq 13^2 17 \cdot 41^2 53 \cdot 59^2 61^2 71^2 73^2 1801 > 10^{26}$. This is implied in parentheses in $*$. (Multipliers such as $13^8 41^2 59^2 61^2$ are clearly covered by this.) In the latter case, the algorithm continues with node 3^2.

We proceed similarly to show that $97 \in S_1$:

$97^1 : 7 \ldots$ D
$97^2 : 3 \cdot \underline{3169}$ B 7
  $3169^1 : 2 \cdot \underline{5} \cdot \underline{317}$ B 10
    $317^1 : 2 \cdot 3 \cdot \underline{53}$ B 12
      $53^1 : 2 \cdot 3^3$ B 12
        $5^1 : 2 \cdot 3$ ($13^2 41^2 59^2 61^2 71^2$ B 28)

$3^2 : 13$ B 14
$13^1 : 7 \ldots$ D
$13^2 : 3 \cdot 61$ B 17
$61^1 : 31 \ldots$ D
$61^2 : 3 \cdot 13 \cdot 97$ ($41^2 59^2$ B 25)
$317^2 : 7 \ldots$ D
$3169^2 : 3 \cdot 3348577$ B 17
$3348577^1 : 2 \cdot 1674289$ B 23
$1674289^1 : 2 \cdot 5 \cdot 167429$ B 29
1674289^2 B 29
3348577^2 B 24

It is then straightforward to show, in order, that 61, 13, $3 \in S_1$, and to then complete the proof of Proposition 3 by running through all remaining primes to 10^4. □

With regard to the proofs that $73 \in S_1$ and $97 \in S_1$, we remark that the program easily identified these cases, and no others, for special treatment, and then that treatment was handled manually, as above. Future improvements of the lower bound for the set of odd harmonic numbers, beyond 10^{24}, will have corresponding difficulties with these cases and with similar cases that may arise. Indeed, they produce the odd numbers which are "closest" to being harmonic; for example, if $m = 3^2 5 \cdot 13^2 53 \cdot 61^2 97^2 317 \cdot 3169$, then $h(m) = 61 \cdot 97/3^2$.

Now we give our final preliminary result.

Proposition 4. *Let n be an odd harmonic number. If $p^a \parallel n$ where a is not of the form $2^b - 1$, then $n > 10^{24}$.*

Proof. According to Proposition 3 and Proposition 1 with $a = 2$, it remains to consider those primes p, $10^4 < p < 31622743$, for which possibly $p^2 \parallel n$. We give first the following improvement of Proposition 1 when $a = 2$:

† If $p^2 \parallel n$ and $p \geq 1316099$, then $n > 10^{24}$.

To prove this, we consider the two cases $p \equiv 2 \pmod 3$ and $p \equiv 1 \pmod 3$. By Lemma 6, we may write $\sigma(p^2) = u$ in the first case and $\sigma(p^2) = 3u$ in the second, where $u \equiv 1 \pmod 3$. Certainly, $3 \nmid u$, so in both cases we must have $u \mid n$. In the first case, since $p > 10^6$ then $n > p^2 u > p^4 > 10^{24}$, and in the second case

$$n > p^2 u > \frac{p^4}{3} \geq \frac{1316099^4}{3} > 10^{24}.$$

Then the usual algorithm considered all possibilities $p^2 \parallel n$, with $10^4 < p < 1316099$. In each case, it was shown that $p^2 \parallel n$ implies $n > 10^{24}$. □

In [5], the two cases leading to the analogue of † were further subdivided to allow still shorter computer runs. The approach, in which u is considered separately to be prime and composite, makes use of Lemma 7 and would be useful for improving the bound in Theorem 6 beyond 10^{24}.

Completion of the proof of Theorem 6. From Proposition 4, if n is an odd harmonic number less than 10^{24} and $p \mid n$, then $p^a \parallel n$ for some a of the form $2^b - 1$, $b \geq 1$. (This might include members of S_2 and primes $p \equiv 1 \pmod 4$ with $p > 10^4$.) But then Lemma 4 is contradicted. □

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