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THE SMALLEST PERRON NUMBERS

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ABSTRACT. A Perron number is a real algebraic integer α of degree $d \geq 2$, whose conjugates are α_i, such that $\alpha > \max_{2\leq i\leq d} |\alpha_i|$. In this paper we compute the smallest Perron numbers of degree $d \leq 24$ and verify that they all satisfy the Lind-Boyd conjecture. Moreover, the smallest Perron numbers of degree 17 and 23 give the smallest house for these degrees. The computations use a family of explicit auxiliary functions. These functions depend on generalizations of the integer transfinite diameter of some compact sets in $\mathbb{C}$

1. INTRODUCTION

Let $\boldsymbol{\alpha}$ be an algebraic integer of degree d, whose conjugates are $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}$, $\boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_d$ and

$$P = X^d + b_1 X^{d-1} + \cdots + b_{d-1} X + b_d,$$

its minimal polynomial. A *Perron number*, which was defined by Lind [LN1], is a real algebraic integer $\boldsymbol{\alpha}$ of degree $d \geq 2$ such that $\boldsymbol{\alpha} > \max_{2\leq i\leq d} |\boldsymbol{\alpha}_i|$. Any Pisot number or Salem number is a Perron number. From the Perron-Frobenius theorem, if A is a nonnegative integral matrix which is aperiodic, i.e. some power of A has strictly positive entries, then its spectral radius $\boldsymbol{\alpha}$ is a Perron number. Lind has proved the converse, that is to say, if $\boldsymbol{\alpha}$ is a Perron number, then there is a nonnegative aperiodic integral matrix whose spectral radius is $\boldsymbol{\alpha}$. Lind [LN2] has investigated the arithmetic of the Perron numbers. The set of Perron numbers is closed under addition and multiplication. Moreover, if $\boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_3$ are Perron numbers and $\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1\boldsymbol{\alpha}_2$, then $\boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2 \in Q(\boldsymbol{\alpha}_3)$. He has also shown that every Perron number can be factored into a finite product of irreducible Perron numbers, and that there are only finitely many such factorizations.

The Perron numbers and their applications were also studied by many people such as D. Boyd [BO1], [BO2], A. Dubickas [DU1], [DU2], D. Lind [LN1], [LN2] and A. Schinzel [SC].

Lind [LN1] conjectured that the smallest Perron number of degree $d \geq 2$ should have minimal polynomial $X^d - X - 1$. Boyd [BO1] has computed all smallest Perron numbers of degree $d \leq 12$, and Lind's conjecture turns out to be true if $d = 2, 3, 4, 6, 7, 8, 10$, but false if $d > 3$ and $d \equiv 3$ or $d \equiv 5 \bmod 6$. So in [BO1], we have

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Conjecture (Lind-Boyd). *The smallest Perron number of degree $d \geq 2$ has minimal polynomial*

$$\begin{aligned} X^d - X - 1 \ &\textit{if } d \not\equiv 3,5 \mod 6,\\ (X^{d+2} - X^4 - 1)/(X^2 - X + 1) \ &\textit{if } d \equiv 3 \mod 6,\\ (X^{d+2} - X^2 - 1)/(X^2 - X + 1) \ &\textit{if } d \equiv 5 \mod 6. \end{aligned}$$

For a given degree d, let $B > 1$ be a bound sufficiently large to assure that there exists at least one Perron number $\boldsymbol{\alpha} \leq B$. In Boyd's computations, this bound was taken to be $B = (2 + 1/d)^{1/d}$ for $d \geq 3$ because $B^d - B - 1 > 0$. Then let $S_k = \sum_{i=1}^{d} \boldsymbol{\alpha}_i^k$ for $k \geq 1$. Since all $|\boldsymbol{\alpha}_i| \leq \boldsymbol{\alpha} \leq B$, we have $|S_k| \leq dB^k$ for $k \geq 1$. These numbers are related to the coefficients of P by Newton's relation:

$$S_k + S_{k-1}b_1 + \ldots + S_1 b_{k-1} + kb_k = 0 \tag{1.1}$$

for $k \geq 1$ with $b_k = 0$ for $k > d$. The numbers S_k also satisfy the following inequalities [BO1]:

$$S_{2k} \geq \frac{2}{d} S_k^2 - dB^{2k} \tag{1.2}$$

for $k \geq 1$. The numbers S_k, with $S_1 \leq 0$, are computed for $1 \leq i \leq d$ and this gives a large set R_d of polynomials where R_d denotes, as in Boyd [BO1], the set of P satisfying $S_1 \leq 0$, (1.1) and (1.2) for $k \leq d$. Let $n > d$, then R_n denotes the set of P in R_d satisfying $b_d \neq 0$ and (1.1) for $k \leq n$. Then the numbers S_k are computed by induction for $k > d$ and for each k we keep only those polynomials with $|S_k| \leq dB^k$. Then for $n = 3d$ we get a reduced set R_{3d} of polynomials. By this method Boyd has, for $d = 12$ with $B = 1.0631 \gtrapprox (2 + 1/12)^{1/12}$,

$$\begin{aligned} |R_{12}| = 415,682,220 \quad |R_{23}| = 37,019 \quad |R_{35}| = 4,931\\ |R_{13}| = 23,746,503 \quad |R_{24}| = 28,277 \quad |R_{36}| = 4,435 \end{aligned}$$

for only 30 irreducible polynomials. The last step is to keep only irreducible polynomials which are not cyclotomic and we get all algebraic integers of degree d with $\max_{1 \leq i \leq d} |\boldsymbol{\alpha}| \leq B$ and so we have the smallest Perron number.

In this paper we compute all smallest Perron numbers of degree $d \leq 24$. We follow Boyd's strategy but we give better bounds for the numbers S_k and more efficient relations between S_k and S_{2k} than (1.2) with a family of explicit auxiliary functions. These functions are related to a generalization of the integer transfinite diameter. This hugely speeds up the search. For example, we have, for $d = 12$,

$$\begin{aligned} |R_{12}| = 950,484 \quad |R_{23}| = 7,367 \quad |R_{35}| = 1,892\\ |R_{13}| = 211,761 \quad |R_{24}| = 6,861 \quad |R_{36}| = 1,679. \end{aligned}$$

The computing time for $d = 12$ is a few seconds on a 2.8Ghz PC.

We denote the house of $\boldsymbol{\alpha}$ by

$$\overline{|\boldsymbol{\alpha}|} = \max_{1 \leq i \leq d} |\boldsymbol{\alpha}_i|.$$

Remark. In [RW], G. Rhin and the author have computed all the algebraic integers with smallest house of degree ≤ 28. Since here, we follow the same strategy, but with greater bounds B, we can use the same list of polynomials for the auxiliary functions for the bounds of $|S_k|$ to find the smallest Perron number. But for degree 22, the computing time becomes 50 hours and for degree 23 it will be more than

TABLE 1. List of all Perron numbers $\boldsymbol{\alpha}$ of degree $13 \leq d \leq 24$ with $\boldsymbol{\alpha} \leq (2+1/d)^{1/d}$ and their minimal polynomials.

d	$\boldsymbol{\alpha}$	$(2+1/d)^{1/d}$	polynomial P
13	$1.057050\ldots$	$1.057832\ldots$	$X^{13}-X-1$
14	$1.052710\ldots$	$1.053393\ldots$	$X^{14}-X-1$
15	$1.047595\ldots$	$1.049585\ldots$	$(X^{17}-X^4-1)/(X^2-X+1)$
	$1.048984\ldots$		$X^{15}-X-1$
16	$1.045751\ldots$	$1.046284\ldots$	$X^{16}-X-1$
17	$1.039302\ldots$	$1.043393\ldots$	$(X^{19}-X^2-1)/(X^2-X+1)$
	$1.042917\ldots$		$X^{17}-X-1$
18	$1.040414\ldots$	$1.040842\ldots$	$X^{18}-X-1$
19	$1.038188\ldots$	$1.038573\ldots$	$X^{19}-X-1$
20	$1.036193\ldots$	$1.036543\ldots$	$X^{20}-X-1$
21	$1.033665\ldots$	$1.034716\ldots$	$(X^{23}-X^4-1)/(X^2-X+1)$
	$1.034397\ldots$		$X^{21}-X-1$
22	$1.032770\ldots$	$1.033063\ldots$	$X^{22}-X-1$
23	$1.029320\ldots$	$1.031559\ldots$	$(X^{25}-X^2-1)/(X^2-X+1)$
	$1.031291\ldots$		$X^{23}-X-1$
24	$1.029939\ldots$	$1.030186\ldots$	$X^{24}-X-1$

800 hours of CPU time. For degree 24, the computing time would certainly be too large. So we need to improve the bounds for $|S_k|$ by using better auxiliary functions. Using the improvement of our algorithm given by V. Flammang [FL], we get many new polynomials Q_j in the auxiliary function of greater degree as we explain in Section 2c and decrease the computing time for degree 23 down to 20 hours.

We give in Table 1 all Perron numbers of degree $13 \leq d \leq 24$ with $\boldsymbol{\alpha} \leq (2+1/d)^{1/d}$ and their minimal polynomials, and we verify that all the smallest Perron numbers of degree $13 \leq d \leq 24$ satisfy the Lind-Boyd conjecture. We also get all algebraic integers $\boldsymbol{\alpha}$ of degree $d \leq 24$, which are not a root of unity, with $\overline{|\boldsymbol{\alpha}|} \leq (2+1/d)^{1/d}$. The complete list may be obtained on request to the author (for $d=24$ see Table 4 in Section 3). The computing time, for example, for $d=24$, is 358 hours on a 2.8Ghz PC.

We also had in [RW], for $d=29$, $(d \equiv 5 \mod 6)$, a small Perron number whose minimal polynomial is

$$\frac{X^{31}-X^2-1}{X^2-X-1}.$$

We expect that this provides an algebraic integer $\boldsymbol{\alpha}$ of degree 29 of smallest house ($\overline{|\boldsymbol{\alpha}|} = 1.023383\ldots$). Then it would satisfy the Lind-Boyd conjecture. So we have the following

Conjecture. *The smallest Perron number of degree* $d \geq 17$, $d \equiv 5 \mod 6$ *and* d *is a prime number, gives the smallest house.*

In Section 2, we explain how to use explicit auxiliary functions to give bounds for S_k, and relations between S_k and S_{2k}. We explain the relations between explicit auxiliary functions and integer transfinite diameter. Section 3 is devoted to the final computation to get the smallest Perron numbers of degree $d \leq 24$.

TABLE 2

k	1	2	3	4	5	6	7	8	9	13	17	23	33	48	60	72
BN_k	3	5	6	8	9	10	12	14	15	21	28	37	60	97	140	203
BC_k	24	25	26	27	27	28	29	30	31	35	39	47	64	100	142	204

2. THE BOUNDS FOR S_k WITH THE EXPLICIT AUXILIARY FUNCTIONS

a. The explicit auxiliary functions for the bounds for S_k.

Compared with Boyd's strategy, the main improvement of the calculation is to compute bounds of S_k which replace the classical bounds $|S_k| \leq d(2+1/d)^{k/d}$ for $1 \leq k \leq 3d$ by using a family of explicit auxiliary functions. For small k, this method improves drastically the classical bounds. In Table 2 we give an example of the two kinds of bounds for some values of k for degree $d = 24$. BN_k denotes the new bound of S_k and BC_k is the classical one. We define the explicit auxiliary function f by the formula

$$f(z) = -\mathrm{Re}(z) - \sum_{1\leq j\leq J} e_j \log |Q_j(z)| \tag{2.1}$$

where z is a complex number, the numbers e_j are positive real numbers and the polynomials Q_j are nonzero elements of $\mathbb{Z}[X]$. The numbers e_j are always chosen to get the best auxiliary function. We denote by m the minimum of $f(z)$ for $|z| \leq B$. Since the function f is harmonic in this disk outside the union of small disks around the roots of the polynomials Q_j, this minimum is taken on $|z| = B$.

We have

$$\sum_{1\leq i\leq d} f(\boldsymbol{\alpha}_i) \geq md$$

and

$$-S_1 \geq d\,m + \sum_{1\leq j\leq J} e_j \log \left| \prod_{1\leq i\leq d} Q_j(\boldsymbol{\alpha}_i) \right|.$$

$\prod_{1\leq i\leq d} Q_j(\boldsymbol{\alpha}_i)$ is equal to the resultant of P and Q_j. If we assume now that the polynomial P does not divide any polynomial Q_j, then this is a nonzero integer. Therefore,

$$S_1 \leq -dm. \tag{2.2}$$

We give, for example, in Table 3 the list of polynomials Q_j and the numbers e_j which are used in the auxiliary function for S_1 for $d = 24$. With this auxiliary function, we have $-S_1 \geq -3.85392590$, i.e. $S_1 \leq 3$ as in Table 2.

By symmetry, the same inequality is valid for $-S_1$. If we replace B by B^k and the numbers $\boldsymbol{\alpha}_i$ by the numbers $\pm\boldsymbol{\alpha}_i{}^k$ we get upper bounds for $\pm S_k$.

b. The explicit auxiliary functions which give relations between S_k and S_{2k}.

In Boyd [BO1], we have the classical relation (1.2) between S_k and S_{2k}. Here we exploit the relations between S_k and S_{2k} that will be given by explicit auxiliary functions of the following type:

$$f(z) = \mathrm{Re}(z^2) - e_0\mathrm{Re}(z) - \sum_{1\leq j\leq J} e_j \log |Q_j(z)| \tag{2.3}$$

TABLE 3. List of Q_j and e_j in the auxiliary function for S_1 for $d = 24$. $d_j = \deg Q_j$, and the coefficient of Q_j are written from degree 0 to d_j.

e_j	d_j	Coefficients of Q_j												
0.60641770	1	-1	1											
0.27754798	2	1	-1	1										
0.05030884	2	1	0	1										
0.10381486	4	1	-1	1	-1	1								
0.00590394	4	1	0	0	0	1								
0.01086250	4	1	0	-1	0	1								
0.04953738	6	1	-1	1	-1	1	-1	1						
0.00749535	6	1	0	0	-1	0	0	1						
0.00661499	6	1	0	0	1	0	0	1						
0.01545423	8	1	-1	0	1	-1	1	0	-1	1				
0.00246685	8	2	-4	4	-2	1	-1	2	-2	1				
0.01273706	9	1	-1	1	-1	1	0	-1	2	-2	1			
0.00767717	10	1	-2	3	-3	2	-1	2	-3	3	-2	1		
0.00052989	10	1	-1	1	-1	1	-1	2	-3	3	-2	1		
0.00102170	10	2	-4	5	-4	2	-1	2	-3	3	-2	1		
0.00481569	12	1	-2	2	-1	1	-1	1	-1	1	-1	2	-2	1
0.00148998	14	3	-10	19	-25	25	-19	12	-7	5	-5	6	-6	5
		-3	1											
0.00161833	14	2	-5	7	-7	7	-7	7	-6	5	-5	6	-6	5
		-3	1											
0.00014458	15	1	-1	1	-3	6	-8	10	-10	8	-6	4	-1	-1
		2	-2	1										
0.00014745	16	2	-6	11	-14	13	-8	2	4	-8	9	-7	3	2
		-5	5	-3	1									
0.00063289	16	4	-17	39	-63	83	-95	98	-92	79	-62	47	-37	31
		-25	17	-8	2									
0.00095988	20	-3	12	-27	45	-66	93	-132	184	-242	292	-320	314	-271
		199	-117	47	-4	-12	11	-5	1					
0.00477448	22	3	-12	23	-24	7	25	-58	78	-79	65	-42	17	4
		-16	18	-13	8	-7	9	-10	8	-4	1			

where the numbers e_j and the polynomials Q_j are as in paragraph a. If m is the minimum of $f(z)$ for $|z| \leq B$, by the same argument as in paragraph a, we get

$$S_2 - e_0 S_1 \geq md.$$

If we assume that S_1 has the value $\boldsymbol{\sigma}$, then $S_2 \geq dm + e_0\boldsymbol{\sigma}$. We optimize the numbers $e_0, \ldots, e_J$ to get a maximal value of $dm + e_0\boldsymbol{\sigma}$. Therefore, we get a lower bound for S_2 depending on the value of $\boldsymbol{\sigma}$. This gives a better bound than the one which was given in paragraph a if we take $\boldsymbol{\sigma}$ close to its upper bound. If we replace in (2.3), $e_0\mathrm{Re}(z)$ by $-e_0\mathrm{Re}(z)$, we get the same lower bound for S_2 when S_1 has the value $-\boldsymbol{\sigma}$. We may also replace $\mathrm{Re}(z^2)$ by $-\mathrm{Re}(z^2)$ and get upper bounds for S_2. Then replacing B by B^k we get bounds for S_{2k} when S_k has values which are close to its bounds.

c. Relations between explicit auxiliary functions and integer transfinite diameter.

If, inside the auxiliary function of (2.1), we replace the real numbers e_j by rational numbers, we may write

$$f(z) = -\mathrm{Re}(z) - \frac{t}{h}\log|H(z)|$$

where H is in $\mathbb{Z}[X]$ of degree h and t is a positive real number. We want to get a function f whose minimum m in $|z| \le B$ is as large as possible. That is to say that we seek a polynomial $H \in \mathbb{Z}[X]$ such that

$$\sup_{|z|\le B} |H(z)|^{t/h} e^{\mathrm{Re}(z)} \le e^{-m}.$$

Now, if we suppose that t is fixed, say $t = 1$, it is clear that we need to get an effective upper bound for the quantity

$$t_{\mathbb{Z},\boldsymbol{\varphi}}(|z| \le B) = \liminf_{\substack{h\ge 1\\ h\to\infty}} \inf_{\substack{H\in\mathbb{Z}[X]\\ \deg H = h}} \sup_{|z|\le B} |H(z)|^{t/h}\boldsymbol{\varphi}(z) \tag{2.4}$$

in which we use the weight $\boldsymbol{\varphi}(z) = e^{\mathrm{Re}(z)}$. To get an upper bound for $t_{\mathbb{Z},\boldsymbol{\varphi}}(|z| \le B)$, it is sufficient to get an explicit polynomial $H \in \mathbb{Z}[X]$ and then to use the sequence of the successive powers of H.

This is a generalization of the integer transfinite diameter. For any $h \ge 1$ we say that a polynomial H (not always unique) is an *integer Chebyshev polynomial* if the quantity $\sup_{|z|\le B} |H(z)|^{t/h}\boldsymbol{\varphi}(z)$ is minimum. With the author's algorithm [WU], we compute polynomials H of degree less than 30 and take their irreducible factors as polynomials Q_j. For example, for the bound of $|S_1|$ of smallest Perron number of degree 24, we start with the polynomials $X - 1$ and $X^2 - X + 1$. With the semi-infinite linear programming method that was introduced into number theory by C. J. Smyth [SM], we get the best e_1 and e_2. We then have $f_1(z) = -\mathrm{Re}(z) - e_1 \log|z-1| - e_2 \log|z^2 - z + 1|$. We deduce the value of $t_1 = e_1 \deg(X-1) + e_2 \deg(X^2 - X + 1)$. Now, we search for a polynomial Q of fixed degree d (say 25), such that $H(z) = (z-1)^{[de_1/t_1]}(z^2 - z + 1)^{[de_2/t_1]}Q(z)$ is small on $|z| \le B$. We take a finite set of points z_i in $|z| \le B$ containing all the local minima of $f_1(z)$. LLL will give polynomials Q_j such that all $H(z_i)$ are small. We optimize the function

$$f_2(z) = -\mathrm{Re}(z) - e_1 \log|z-1| - e_2 \log|z^2 - z + 1| - \sum_j e_j \log|Q_j|.$$

We keep only Q_j when $e_j \ne 0$. Then we have a new bound of $|S_1|$ which is better than the previous one. We repeat LLL and optimization of the auxiliary function and finally we have the function in Table 2. More details can be found in [FRSE], [RW] and [FL].

With this method, for $d = 24$, we find 143 polynomials Q_j for the family of explicit auxiliary functions for S_k for all $1 \le k \le 72$, and the largest degree of the polynomials Q_j is 22. We have used 125 polynomials for the relations (2.3) and 94 polynomials if we replace $e_0\mathrm{Re}(z)$ by $-e_0\mathrm{Re}(z)$. So we have a set of 341 different polynomials, and this set contains 35 of the 42 polynomials used in [RW].

TABLE 4. List of the minimal noncyclotomic polynomials of algebraic integers of degree 24 with small house.

house of P	polynomial P
$1.017730\ldots$	$X^{24}+X^{16}-1$
$1.022651\ldots$	$X^{24}-X^{20}+X^{12}-X^{8}+1$
$1.024177\ldots$	$X^{24}+X^{8}-1$
$1.024596\ldots$	$X^{24}+X^{21}+X^{12}-X^{6}+1$
$1.025758\ldots$	$(X^{28}+X^{2}+1)/(\boldsymbol{\Phi}_3\boldsymbol{\Phi}_6)$
$1.027157\ldots$	$(X^{30}+X^{15}-X^{3}+1)/(\boldsymbol{\Phi}_4\boldsymbol{\Phi}_{12})$
$1.027482\ldots$	$(X^{26}-X+1)/\boldsymbol{\Phi}_6$
$1.028309\ldots$	$X^{24}+X^{20}+X^{16}+1$
$1.028513\ldots$	$X^{24}-X^{6}+1$
$1.029302\ldots$	$X^{24}-X^{12}+2$
$1.029312\ldots$	$X^{24}+X^{21}+X^{18}+X^{15}+1$
$1.029354\ldots$	$(X^{27}-X^{14}+X-1)/(\boldsymbol{\Phi}_1\boldsymbol{\Phi}_4)$
$1.029841\ldots$	$X^{24}+X+1$
$1.029939\ldots$	$X^{24}-X-1$ Perron
$1.030138\ldots$	$X^{24}-X^{4}+1$

3. THE FINAL COMPUTATION

For all algebraic integers with $\boxed{\boldsymbol{\alpha}} \le (2+1/d)^{1/d}$, we have the bounds of S_k for k from 1 to $3d$. Now we follow Boyd's strategy [BO1] to compute the coefficients of the polynomial P with $b_1 \ge 0$.

As in Section 1, for $d=24$, we have $|R_{24}| = 12,132,432,709,704$ and $|R_{72}| = 33,132$. Moreover, we refine some steps as follows in our computation:

We take $|b_d| = \pm 1,\ \pm 2$, because $b_d \in \mathbb{Z}$ and

$$|b_d| = \prod_{j=1}^{d} |\boldsymbol{\alpha}_j| \le B^d \le 2 + \frac{1}{d}.$$

Moreover, by Corollary 2 of Matveev [MAT], we know that, for any algebraic integer $\boldsymbol{\alpha}$ of degree d, if

$$\boxed{\boldsymbol{\alpha}} \le \exp\left(\frac{\ln(d+\frac{1}{2})}{d^2}\right), \tag{3.1}$$

then $\boldsymbol{\alpha}$ is a root of unity. So we eliminate, by the Schur-Cohn algorithm [MAR], any polynomial P of R_{3d} which satisfies the condition (3.1). We obtain 48 polynomials for $d=24$. Then we use Pari [PA] to eliminate any polynomial with a cyclotomic factor, get all the irreducible polynomials, and compute explicitly their roots.

Then we find all irreducible polynomials of degree d which satisfy $\boxed{\boldsymbol{\alpha}} \le (2+1/d)^{1/d}$, so we have all smallest Perron numbers for $d \le 24$, and also all algebraic integers $\boldsymbol{\alpha}$ with small house ($\le (2+1/d)^{1/d}$). We list in Table 4 all the minimal polynomials of degree 24 of algebraic integers which are not a root of unity with small house that we get by our computations. We give only one polynomial if there exist several polynomials which have the same house. Here $\boldsymbol{\Phi}_1 = X-1$, $\boldsymbol{\Phi}_3 = X^2+X+1$, $\boldsymbol{\Phi}_4 = X^2+1$, $\boldsymbol{\Phi}_6 = X^2-X+1$ and $\boldsymbol{\Phi}_{12} = X^4-X^2+1$ are cyclotomic polynomials.

As in [WU], Section 2, when we use the auxiliary functions, we suppose that the polynomial P does not divide any polynomial $Q_j(\pm X^k)$ for $k \leq 3d$ (i.e. $\pm\boldsymbol{\alpha}_i^k$ is not a zero of Q_j for all j and k). So it is necessary to add to the list of Table 4 the nonprimitive polynomials P obtained as polynomials $Q_j(\pm X^k)$ whose house is less than $(2+1/d)^{1/d}$ where $d = k \deg Q_j$. For example, if $Q_j = X^2 - 2X + 2$, then all the polynomials $X^{2k} - 2X^k + 2$ whose house $\leq (2+1/2k)^{1/2k}$ have to be added to the list.

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