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A SHARP REGION WHERE $\pi(x) - \mathrm{li}(x)$ IS POSITIVE

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Abstract. In this article, we study the problem of changes of sign of $\pi(x) - \mathrm{li}(x)$. We provide three improvements. First, we give better esimates of error term for Lehman's theorem. Second, we rigorously prove the positivity of this difference for a region formerly conjectured by Patrick Demichel. Third, we improve the estimates for regions of positivity by using number theoretic results.

1. Previous work

The problem of estimating the number of prime numbers goes back to Gauss. The function counting prime numbers is classically denoted π, i.e. $\pi(x) = \sum_{p \le x} 1$. In 1791, he conjectured that $\pi(x) \simeq \frac{x}{\log x}$. This result was proven in 1896 by Hadamard and de la Vallée-Poussin. In 1849, Gauss suggested that the log-integral function, defined by $\mathrm{li}(x) = \lim_{\varepsilon \to 0} \left\{ \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} \right\}$, should give a better approximation for π. Gauss also noted that the inequality $\pi(x) < \mathrm{li}(x)$ holds for any x in the interval $[2, 3000000]$. Since then, this property has been checked numerically up to 10^{14} [1].

On the other hand, Littlewood proved in 1914 that the property $\pi(x) > \mathrm{li}(x)$ holds infinitely often, but he did not give any explicit value for such an x. In 1933, Skewes proved, assuming the Riemann hypothesis, that the latter inequality occurs at least once for a value $x < 10^{10^{10^{34}}}$. A great improvement was given in 1966 by Lehman [2]. He established a theorem which enables one unconditionally to obtain much lower values. His theorem enabled him to show that there exists a region near 1.65×10^{1165} where the difference $\pi(x) - \mathrm{li}(x)$ admits positive values. Then, te Riele [3] in 1987 discovered another region near 6.65×10^{370}, and Bays and Hudson [4] exhibited a region near 1.40×10^{316} in 1999. In 2006, Chao and Plymen [5] gave an improvement on the error terms of Lehman's theorem. This enabled them to sharpen Bays and Hudson's region and established a new lower bound equal to 1.398×10^{316}. Independently, in 2005, Demichel [6] made intensive computations on this problem and conjectured that this value could be improved to 1.397×10^{316}, without rigorously establishing the result. Another point is that this latter region is a new one, i.e. it is not included in that of Bays and Hudson, contrary to the result of Chao and Plymen. Our contribution to this problem is three-fold. First, we will show that Chao and Plymen's error term can be lowered. Second, by numerical computation, we will prove the validity of Demichel's region. Third, we

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show, using some theorems in number theory, that the final results obtained by the classic approach to this subject can be improved.

2. Lehman's theorem

The main tool to deal with this problem is Lehman's theorem:

Theorem 2.1 (Lehman's Theorem)**.** *Let A be a positive number such that $\beta = \frac{1}{2}$ for all zeros $\rho = \beta + i\gamma$ of Riemann zeta function $\zeta(s)$ for which $0 < \gamma \leq A$. Let α, η and ω be positive values such that $\omega - \eta > 1$ and the following conditions hold:*

$$4A/\omega \leq \alpha \leq A^2, \tag{2.1}$$

$$2A/\alpha \leq \eta \leq \omega/2. \tag{2.2}$$

Let

$$K(y) = \sqrt{\frac{\alpha}{2\pi}} e^{-\alpha y^2/2}, \tag{2.3}$$

$$I(\omega, \eta) = \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) u e^{-u/2} \{\pi(e^u) - \mathrm{li}(e^u)\} du. \tag{2.4}$$

Then for $2\pi e < T \leq A$, we have

$$I(\omega, \eta) = -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha} + R \tag{2.5}$$

where $|R| \leq S_1 + S_2 + S_3 + S_4 + S_5 + S_6$, with

$$S_1 = \frac{3}{\omega - \eta} + 4(\omega + \eta) e^{-(\omega-\eta)/6}, \tag{2.6}$$

$$S_2 = \frac{2e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\eta}, \tag{2.7}$$

$$S_3 = 0.08\sqrt{\alpha} e^{-\alpha\eta^2/2}, \tag{2.8}$$

$$S_4 = e^{-T^2/2\alpha} \left\{ \frac{\alpha}{\pi T^2} \log \frac{T}{2\pi} + 8 \frac{\log T}{T} + \frac{4\alpha}{T^3} \right\}, \tag{2.9}$$

$$S_5 = \frac{0.05}{\omega - \eta}, \tag{2.10}$$

$$S_6 = A \log A e^{-A^2/2\alpha + (\omega+\eta)/2} \{4\alpha^{-1/2} + 15\eta\}. \tag{2.11}$$

If the Riemann hypothesis holds, then conditions (2.1) *and* (2.2) *may be omitted and the term S_6 may be omitted in the upper bound for R.*

The complete proof can be found in [2]. The application of the previous theorem makes two essential assumptions. First, the Riemann hypothesis has to be checked up to height A. Second, explicit values for the zeros of ζ have to be known up to height T. If we suppose that both conditions are met, we can estimate the integral (2.4) using the equation (2.5). Lehman's method amounts then to finding suitable values for α and ω such that the first two terms on the right-hand side of equation (2.5) sum to a positive value larger than the associated error term $|R|$. The integral (2.4) is then established to be positive and thus, by virtue of the positivity of K, the term $\{\pi(e^u) - \mathrm{li}(e^u)\}$ must admit some positive values for u in the interval $[\omega - \eta, \omega + \eta]$.

3. Improvements

Improvements on the error term R are possible. In fact, the dominating term in R is generally S_1. In his seminal work, Lehman [2] derived S_1 from an upper bound for $\pi(x)$ obtained by Rosser and Schoenfeld [7]. In their paper, Chao and Plymen derived a tighter bound by using recent results of Panaitopol [8]. Doing so, they could lower the constant 3 in the first term of S_1 to 2.1457. In this part, we use one result obtained by Dusart [9, th. 1.10] to show that this constant can be replaced by 2 (with some other terms in S_1). Moreover, this value cannot be improved.

Theorem 3.1 (Dusart's Theorem)**.** *If $x \geq 32299$, we have*

$$\frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right) \leq \pi(x). \tag{3.1}$$

If $x \geq 355991$, we have

$$\pi(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right). \tag{3.2}$$

Given this result, we prove the following theorem:

Theorem 3.2. *Under the hypothesis of Lehman's theorem and if $\omega - \eta > 25.57$, equation* (2.5) *still holds if S_1 is replaced by*

$$S_1' = \frac{2}{\omega-\eta} + \frac{10.04}{(\omega-\eta)^2} + \log 2.(\omega+\eta)e^{-(\omega-\eta)/2} + \frac{2}{\log 2}(\omega+\eta)e^{-(\omega-\eta)/6}. \tag{3.3}$$

Proof. We proceed as in Lehman's work. Let

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots, \tag{3.4}$$

and let

$$\Pi_0(x) = \lim_{\varepsilon\to 0}\frac{1}{2}\left\{\Pi(x+\varepsilon)+\Pi(x-\varepsilon)\right\}. \tag{3.5}$$

The Riemann-von Mangoldt formula states that for $x > 1$,

$$\Pi_0(x) = \text{li}(x) - \sum_\rho \text{li}(x^\rho) + \int_x^{+\infty}\frac{du}{(u^2-1)u\log u} - \log 2, \tag{3.6}$$

where ρ runs over the zeros of function ζ in the critical strip.

We have

$$\frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots \leq \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3})\left\lfloor\frac{\log x}{\log 2}\right\rfloor. \tag{3.7}$$

Then we use Dusart's theorem together with the classic bound $\pi(x) \leq \frac{2x}{\log x}$. Thus, if $x \geq 355991^2$, we have

$$\frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots \leq \frac{x^{1/2}}{\log x}\left(1+\frac{2}{\log x}+\frac{10.04}{\log^2 x}\right) + 2\left\lfloor\frac{\log x}{\log 2}\right\rfloor\frac{x^{1/3}}{\log x}, \tag{3.8}$$

and thus

$$\frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots \leq \frac{x^{1/2}}{\log x}\left(1+\frac{2}{\log x}+\frac{10.04}{\log^2 x}\right) + \frac{2}{\log 2}x^{1/3}. \tag{3.9}$$

Substituting in (3.6), we have

$$\pi(x) \geq \mathrm{li}(x) - \sum_\rho \mathrm{li}(x^\rho) - \frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{10.04}{\log^2 x}\right) - \frac{2}{\log 2}x^{1/3} - \log 2. \tag{3.10}$$

We put $x = e^u$, and then if $u > 25.57$, we have

$$\begin{aligned} &ue^{-u/2}(\pi(e^u) - \mathrm{li}(e^u)) \\ &\qquad \geq -1 - \sum_\rho ue^{-u/2}\mathrm{li}(e^{u\rho}) - \frac{2}{u} - \frac{10.04}{u^2} - \frac{2u}{\log 2}e^{-u/6} - \log 2.ue^{-u/2}. \end{aligned} \tag{3.11}$$

So, following Lehman's proof, we derive equation (2.5) with the same bounding terms S_2, S_3, S_4, S_5 and S_6. Term S_2 comes from bounding the two tail integrals $\int_{-\infty}^{\omega-\eta} K(u-\omega)du$ and $\int_{\omega+\eta}^{+\infty} K(u-\omega)du$. Terms S_3, S_4, S_5 and S_6 come from the estimate of $\sum_\rho \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)ue^{-u/2}\mathrm{li}(e^{u\rho})du$.

Finally, the term corresponding to term S_1 in our theorem comes from bounding the expression

$$J = \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(\frac{2}{u} + \frac{10.04}{u^2} + \frac{2u}{\log 2}e^{-u/6} + \log 2.ue^{-u/2}\right)du. \tag{3.12}$$

Both terms in the previous integral are positive and $\int_{-\infty}^{+\infty} K(y)dy = 1$, thus we have

$$J \leq \frac{2}{\omega-\eta} + \frac{10.04}{(\omega-\eta)^2} + \frac{2(\omega+\eta)}{\log 2}e^{-(\omega-\eta)/6} + \log 2.(\omega+\eta)e^{-(\omega-\eta)/2}. \tag{3.13}$$ □

4. Numerical results

As mentioned previously, the use of the previous theorems presupposes numerical verifications of the Riemann hypothesis up to height A. In his seminal paper, Lehman used a verification made on his own on the first 250000 zeros, giving $A = 170571.35$. Since then, a lot of work has been done to check numerically the Riemann hypothesis up to larger and larger heights. In 2001, van de Lune [10] established that the conjecture is verified for the first 10000000000 zeros up to height $A = 3293531632.415$. This value, in fact, sets an upper bound for the value of A that can be used in Lehman's theorem. However, two more recent verifications are noteworthy. The first was performed by Gourdon and Demichel [11] in 2004 using a fast multiple evaluation algorithm for ζ invented by Odlyzko. With their implementation, the conjecture has been verified up to the 10^{13}-th zero. The second is the distributed ZetaGrid project [12], managed by Wedeniwski, which was active between 2002 and 2005. The official status of these verifications is not clear: Gourdon and Demichel's work has never been independently verified and, in the case of the ZetaGrid project, it was not established that all zeros were checked.

For $0 < T \leq A$, we know that the real part of zeros $\rho = \beta + i\gamma$ of ζ such that $|\gamma| < T$ is equal to 1/2. Moreover, zeros of ζ in the critical strip occur as conjugate pairs, so the sum to evaluate is:

$$\sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho}e^{-\gamma^2/2\alpha} = \sum_{0<\gamma\leq T} \frac{\cos\gamma\omega + 2\gamma\sin\gamma\omega}{\frac{1}{4}+\gamma^2}e^{-\gamma^2/2\alpha}. \tag{4.1}$$

For our numerical computations, we computed the first 22 million zeros of ζ. This was done in two phases. First, an approximation was computed by the classic

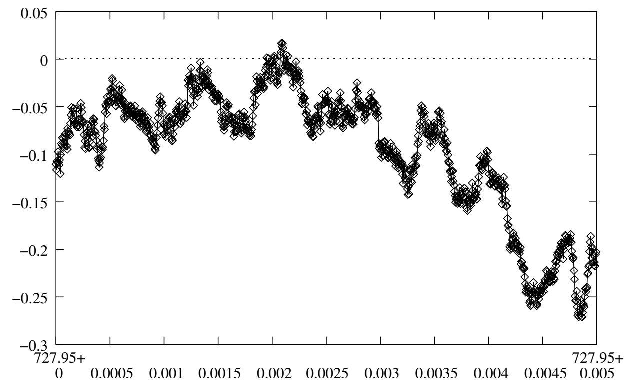

FIGURE 1. The Bays and Hudson region.

Riemann-Siegel formula and then precision was improved up to 9 decimal digits using correction terms in this formula. The three additional correction terms we used were computed by formulae given in [13]. The last zero of our database gave us $T = 10379599.727431060$. The relative precision that can be expected when computing the right-hand side of the previous equation is then bounded by

$$\Delta I = 10^{-9}. \sum_{0<\gamma\le T} \frac{\partial}{\partial\gamma}\left(\frac{\cos\gamma\omega + 2\gamma\sin\gamma\omega}{\frac{1}{4}+\gamma^2} e^{-\gamma^2/2\alpha}\right). \tag{4.2}$$

In our computations, we set $\alpha = 6 \times 10^{12}$, and in the range of our application we have $\omega \simeq 727.95$. We compute the associated precision given by equation (4.2), and it gives approximately 7×10^{-7}. We chose to make the computation explicitly, instead of using bounds as in previous work, in order to obtain the best possible precision. Figure 1 plots the value of $I(\omega, \eta)$ for ω in $[727.950, 727.955]$. Values are corrected by the error term R. This figure shows Bays and Hudson's region whose center is a bit larger than $\omega = 727.952$. In this figure, Chao and Plymen's region as well as our region are contained in the peak between 727.951 and 727.9515. At the level of magnification of this figure, it is not very clear that the curve effectively cuts the line $I(\omega, \eta) = 0$. Figure 2 depicts Chao and Plymen's region with a better level of magnification. In this figure, quite large areas of the curve are in the positive domain. Figure 3 depicts the new region with the same scale level as for Figure 2. The positive region is much sharper than the one of Figure 2. By looking at the ω axis, we notice that this region is left of Chao and Plymen's one. Finally, Figure 4 gives a closeup of the peak of the new region.

Numerically, the least value for ω giving a positive value for $I(\omega, \eta)$ is $\omega = 727.951335792$. By studying the remainder terms and especially S_6, we found that $A = 6.85 \times 10^7$ is the value minimizing the interval length. We then have $\eta = 2A/\alpha = 0.00002283333334$ and conditions (2.1) and (2.2) are met. Moreover, the condition $\omega - \eta > 25.57$ of Theorem 3.2 is also verified. By computation, we then obtain $\sum_{0<|\gamma|\le T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha} = -1.002906086981405$, thus giving $I^*(\omega, \eta) = 0.002906086981405$ as an estimate of $I(\omega, \eta)$. We then have

$$I(\omega, \eta) \geq I^*(\omega, \eta) - \Delta I - S_1' - S_2 - S_3 - S_4 - S_5 - S_6.$$

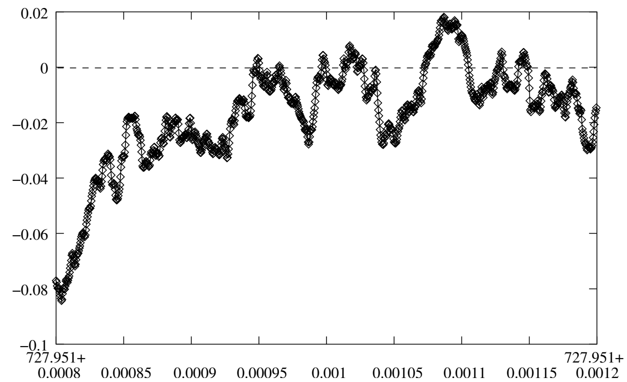

FIGURE 2. The Chao and Plymen region.

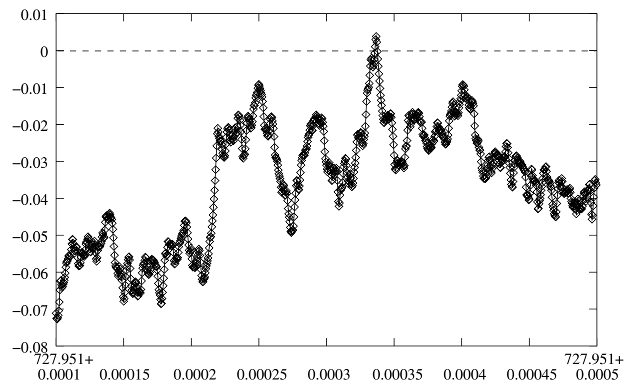

FIGURE 3. The new region on the same scale.

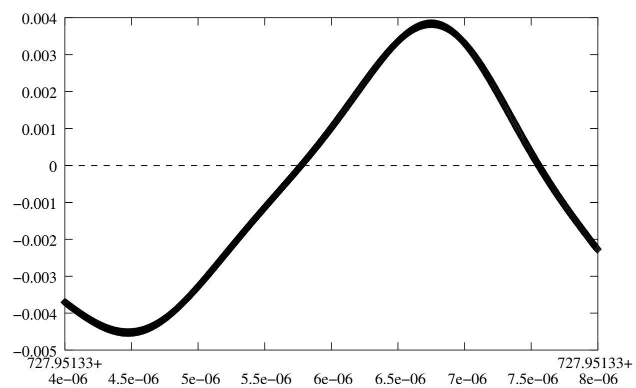

FIGURE 4. The maximum in the new region, scaled up.

Numerically, we have

$$\begin{aligned}
\Delta I &= 7.1645945511 \times 10^{-7},\\
S_1' &= 0.002766382992,\\
S_2 &= 7.612616047 \times 10^{-682},\\
S_3 &= 1.045693526 \times 10^{-674},\\
S_4 &= 0.00003202055301,\\
S_5 &= 0.00006868591225,\\
S_6 &= 7.640973098 \times 10^{-7}.
\end{aligned}$$

Thus we obtain

$$I(\omega, \eta) \geq 0.00003751696746. \tag{4.3}$$

Thus, we proved that there exists a value x in $[\exp(727.9502380), \exp(727.9524336)]$ for which the inequality $\pi(x) > \mathrm{li}(x)$ holds. More precisely, there are some values u in the interval $[727.9513130, 727.9513586]$ such that

$$\pi(e^u) - \mathrm{li}(e^u) > 0.00003751696746 * e^{u/2}/u > 6.091784490 \times 10^{150}. \tag{4.4}$$

So, we can claim:

Theorem 4.1. *There exists at least one value x in the interval* $[\exp(727.9513130), \exp(727.9513586)]$ *for which $\pi(x) > \mathrm{li}(x)$ holds. Moreover, there are more than* 6.09×10^{150} *successive integers in the vicinity of* $\exp(727.951335792)$ *where the inequality holds.*

5. Sharpening the interval

The previous theorem gives us an upper bound of $\exp(727.9513586)$ for the first crossover. This value is better than the one obtained by Chao and Plymen. Nevertheless, it is possible to reduce again the length of the interval. Indeed, the integrand function decays very fast to 0 around its center and thus the meaningful part of the integral is in fact around ω. In order to reduce the interval, we need some information about the growth of $\pi(x) - \mathrm{li}(x)$. At this point, we split the study into two cases. First, we will consider the general case and second, we will suppose that the Riemann hypothesis holds.

In the general case, we will first prove:

Theorem 5.1. *If $x \geq \exp(8)$, we have*

$$0 \leq \mathrm{li}(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 \leq \frac{12x}{\log^4 x} + C_2, \tag{5.1}$$

with $C_1 = \mathrm{li}(2) - \frac{2}{\log 2}(1 + \frac{1}{\log 2} + \frac{2}{\log^2 2})$ *and* $C_2 = \int_2^{\exp(8)} \frac{48dt}{\log^5 t} - \frac{24}{\log^4 2}$.

Proof. From the definition of $\mathrm{li}(x)$, we have for $x \geq 2$,

$$\mathrm{li}(x) = \mathrm{li}(2) + \int_2^x \frac{dt}{\log t}. \tag{5.2}$$

Then, after three successive integrations by parts, we have for $x \geq 2$,

$$\mathrm{li}(x) = \mathrm{li}(2) + \left[\frac{t}{\log t}\left(1 + \frac{1}{\log t} + \frac{2}{\log^2 t}\right)\right]_2^x + \int_2^x \frac{6dt}{\log^4 t}, \tag{5.3}$$

and thus

$$\mathrm{li}(x) - \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x}\right) - C_1 = \int_2^x \frac{6dt}{\log^4 t}. \tag{5.4}$$

At this point we obtain the first inequality of Theorem 5.1. Another integration by parts on the right-hand side gives, for $x \geq 2$,

$$\int_2^x \frac{6dt}{\log^4 t} = \left[\frac{6t}{\log^4 t}\right]_2^x + \int_2^x \frac{24dt}{\log^5 t}. \tag{5.5}$$

Now, for $x \geq \exp(8)$, we have

$$\int_2^x \frac{6dt}{\log^4 t} - \left[\frac{6t}{\log^4 t}\right]_2^x - \int_2^{\exp(8)} \frac{24dt}{\log^5 t} = \int_{\exp(8)}^x \frac{24dt}{\log^5 t}. \tag{5.6}$$

But, for $t \geq \exp(8)$, we have $\frac{24}{\log^5 t} \leq \frac{1}{2}.\frac{6}{\log^4 t}$. So we obtain, for $x \geq \exp(8)$,

$$\int_2^x \frac{6dt}{\log^4 t} - \left[\frac{6t}{\log^4 t}\right]_2^x - \int_2^{\exp(8)} \frac{24dt}{\log^5 t} \leq \frac{1}{2}\int_{\exp(8)}^x \frac{6dt}{\log^4 t} \leq \frac{1}{2}\int_2^x \frac{6dt}{\log^4 t}. \tag{5.7}$$

We obtain finally for $x \geq \exp(8)$,

$$\int_2^x \frac{6dt}{\log^4 t} \leq \frac{12x}{\log^4 x} - \frac{24}{\log^4 2} + \int_2^{\exp(8)} \frac{48dt}{\log^5 t}, \tag{5.8}$$

which establishes the theorem. □

The latter theorem could be further optimized but it will suffice for our purpose. Combined with Theorem 3.1, it gives:

Theorem 5.2. *If $x \geq 355991$, we have*

$$-\frac{0.2x}{\log^3(x)} - \frac{12x}{\log^4(x)} - 44.53131 \leq \pi(x) - \mathrm{li}(x) \leq \frac{0.51x}{\log^3(x)} + 1.80141. \tag{5.9}$$

Moreover, if $x \geq \exp(40)$, then

$$|\pi(x) - \mathrm{li}(x)| \leq \frac{0.51x}{\log^3(x)} + 1.80141. \tag{5.10}$$

Although this theorem was obtained by elementary methods, in the range of values for x where we intend to use it, it gives a finer result than earlier theorems; for instance, that of Dusart [9, th. 1.12]. With this theorem, we will now study the tail parts of the integral (2.4). Now, let η_0 be a real positive number such that $\eta_0 < \eta$. We then have, since $\omega > 40$,

$$\begin{aligned}
&\left|\int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{-u/2}\{\pi(e^u) - \mathrm{li}(e^u)\}du\right| \\
&\quad \leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{-u/2}\{|\pi(e^u) - \mathrm{li}(e^u)|\}du \\
&\quad \leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)ue^{-u/2}\left\{\frac{0.51e^u}{u^3} + 1.80141\right\}du \\
&\quad \leq \int_{\omega+\eta_0}^{\omega+\eta} K(u-\omega)\left\{\frac{0.51e^{u/2}}{u^2} + 1.80141.ue^{-u/2}\right\}du \\
&\quad \leq (\eta-\eta_0)K(\eta_0)\left\{0.51\frac{e^{(\omega+\eta)/2}}{(\omega+\eta_0)^2} + 1.80141.(\omega+\eta)e^{-(\omega+\eta_0)/2}\right\}.
\end{aligned} \tag{5.11}$$

Likewise, we obtain, since $\omega - \eta > 40$,

$$\begin{aligned} &\left| \int_{\omega-\eta}^{\omega-\eta_0} K(u-\omega) u e^{-u/2} \{\pi(e^u) - \mathrm{li}(e^u)\} du \right| \\ &\qquad \le (\eta - \eta_0) K(-\eta_0) \left\{ 0.51 \frac{e^{(\omega-\eta_0)/2}}{(\omega-\eta)^2} + 1.80141.(\omega - \eta_0) e^{-(\omega-\eta)/2} \right\}. \end{aligned} \tag{5.12}$$

We denote, respectively, T_1 and T_2, the right-hand sides of the two previous final inequalities. The sum of the two tail integrals is then bounded above by T_1+T_2. Now, numerically, with the previous values we used and obtained in our computations, if we set $\eta_0 = \eta/2.074$, we obtain

$$\begin{aligned} T_1 &= 0.00001594194397, \\ T_2 &= 0.00001594167602. \end{aligned}$$

Those numeric values, together with the estimate (4.3), gives the following result:

$$I(\omega, \eta_0) \ \ge\ 0.00000563334747.$$

This result then allows us to obtain a result finer than the one obtained in Theorem 4.1. However, as we will see in the next part, more work can still be done to improve the final result. Thus, for the moment, we will state our result in a different way:

Theorem 5.3. *There exists one value x in the interval* $[\exp(727.95132478),\ \exp(727.\ 95134681)]$ *such that* $\pi(x) - \mathrm{li}(x) > 9.1472 \times 10^{149}$.

In the following, we assume that the Riemann hypothesis holds. In this case, Theorem 5.2 is far from being optimal, even in the range of values we consider. A much finer result is given by Schoenfeld [14, p. 339]:

Theorem 5.4. *If the Riemann hypothesis holds, then for $x \ge 2657$, we have*

$$|\pi(x) - \mathrm{li}(x)| < \frac{1}{8\pi} \sqrt{x} \log x. \tag{5.13}$$

If we denote T_1' and T_2' upper bounds of the corresponding tail integrals, we then obtain

$$T_1' = \frac{1}{8\pi} K(\eta_0)(\omega + \eta)^2, \tag{5.14}$$

$$T_2' = \frac{1}{8\pi} K(-\eta_0)(\omega - \eta_0)^2. \tag{5.15}$$

Numerically, if we set $\eta_0 = \eta/6.72$, we obtain

$$\begin{aligned} T_1' &= 0.00001870458817, \\ T_2' &= 0.00001870458683, \end{aligned}$$

and thus,

$$I(\omega, \eta_0) \ \ge\ 0.000000107793.$$

We can then state:

Theorem 5.5. *If the Riemann hypothesis holds, then there exists one value x in the interval* $[\exp(727.95133239), \exp(727.95133919)]$ *such that* $\pi(x) - \mathrm{li}(x) > 1.7503 \times 10^{148}$.

6. Interval of positivity

Theorem 4.1 exhibits an interval of 6.09×10^{150} consecutive integers where $\pi(x)-\mathrm{li}(x)$ is positive. Indeed, equation (4.4) states that there exists a point x, such that $\pi(x)-\mathrm{li}(x) > 6.09\times 10^{150}$. Let b be a positive integer. Then $\mathrm{li}(x-b)\leq \mathrm{li}(x)$. Moreover, for any $x\geq 1$, we have $\pi(x-1)\geq \pi(x)-1$, thus by recurrence and with the previous inequality, we can deduce that $\pi(x-b)-\mathrm{li}(x-b) > 6.09\times 10^{150}-b$. Thus we can affirm that the 6.09×10^{150} successive integers preceding x belong to the interval of positivity. This result is obtained by considering integers inferior to x. However, as we will see, this result can be much improved by considering integers greater than x. In fact, we have:

Theorem 6.1. *Let $x > 1$ and $y > 0$, then we have*

$$\begin{aligned}\mathrm{li}(x+y)-\mathrm{li}(x) &= \int_x^{x+y}\frac{dt}{\log t}\\ &< \frac{y}{\log x}.\end{aligned}$$

With the previous theorem, we can state:

Theorem 6.2. *Let x be a real positive number such that $\pi(x)-\mathrm{li}(x) = A > 0$. Then if y is a real number such that $0 < y < A.\log x$, we have $\pi(x+y)-\mathrm{li}(x+y) > 0$.*

Proof. Let $y > 0$, since the function $\pi(x)$ is increasing, we then have

$$\begin{aligned}\pi(x+y)-\mathrm{li}(x+y) &= (\pi(x+y)-\pi(x)) + (\pi(x)-\mathrm{li}(x)) + (\mathrm{li}(x)-\mathrm{li}(x+y))\\ &> A-\frac{y}{\log x}.\end{aligned}$$ □

Then Theorem 5.3 enables us to state:

Theorem 6.3. *There are at least 6.6587×10^{152} consecutive integers x in the interval* $[\exp(727.95132478), \exp(727.\ 95134682)]$ *such that* $\pi(x)-\mathrm{li}(x) > 0$.

The value 6.6587×10^{152} obtained is a direct application of Theorem 6.2. However, we do not know where the first x lies in the interval of Theorem 5.3. We only know that its maximal value is $\exp(727.\ 95134681)$. But we have $\exp(727.\ 95134682)-\exp(727.\ 95134681)\simeq 1.397\times 10^{308}$, which is larger than 6.6587×10^{152}. Thus we can affirm that the 6.6587×10^{152} integers following x belong to the interval $[\exp(727.95132478), \exp(727.\ 95134682)]$. In a similar way, Theorem 5.5 gives us the following result:

Theorem 6.4. *If the Riemann hypothesis holds, then there are at least 1.2741×10^{151} consecutive integers x in the interval* $[\exp(727.95133239), \exp(727.95133920)]$ *such that* $\pi(x)-\mathrm{li}(x) > 0$.

A final remark is in order about Theorem 6.4. The number of consecutive integers satisfying $\pi(x)-\mathrm{li}(x) > 0$ is approximately 50 times smaller than in Theorem 6.3. This theorem might then appear weaker, which would then make assuming the Riemann hypothesis pointless. In fact, this theorem is stronger than Theorem 6.3, since the length of the interval is three times shorter. The difference in terms of consecutive integers comes from the fact that the estimate for $I(\omega,\eta_0)$ is much sharper when the Riemann hypothesis holds. In turn, this fact is a consequence of having better upper bounds for tail integrals (see equations (5.11)–(5.12) and (5.14)–(5.15)).

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