

CONVERGENCE ANALYSIS FOR FINITE ELEMENT DISCRETIZATIONS OF THE HELMHOLTZ EQUATION WITH DIRICHLET-TO-NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. A rigorous convergence theory for Galerkin methods for a model Helmholtz problem in \mathbb{R}^d , $d \in \{1, 2, 3\}$ is presented. General conditions on the approximation properties of the approximation space are stated that ensure quasi-optimality of the method. As an application of the general theory, a full error analysis of the classical hp -version of the finite element method (hp -FEM) is presented for the model problem where the dependence on the mesh width h , the approximation order p , and the wave number k is given explicitly. In particular, it is shown that quasi-optimality is obtained under the conditions that kh/p is sufficiently small and the polynomial degree p is at least $O(\log k)$.

1. INTRODUCTION

Helmholtz boundary value problems appear in various applications, for example, in the context of inverse and scattering problems. When such problems are solved numerically, the questions of stability and convergence arise. Of particular interest is how critical parameters such as the discretization parameters (e.g., mesh size, approximation order) and the wave number k affect the performance of the method.

Many discretization techniques for Helmholtz problems have been proposed and discussed in the literature. In the context of Galerkin methods, which is the setting of the present paper, these include both standard and non-standard finite element methods. Although significant progress in the understanding of the behavior of numerical methods for Helmholtz problems has been made in the past, a general, full analysis that is explicit in the wave number k and discretization parameters is still not available. Partial results such as sharp estimates for the inf-sup constant of the continuous equations, lower estimates for the convergence rates, one-dimensional analysis by using the discrete Green's function as well as a dispersion analysis for finite element discretizations and generalizations thereof have been derived by many researchers in the past decades (see, e.g., [2, 4, 6, 7, 9, 10, 11, 15, 17, 18, 19, 22, 23, 24, 25, 26, 27, 28, 33, 38, 43, 44]).

The goal of the present and the companion paper [32] is to derive fairly general stability and convergence estimates for Helmholtz problems that are:

- explicit in the wave number, the mesh width, and the polynomial degree of the hp -FEM space;

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- valid for problems in d spatial dimensions, $d \in \{1, 2, 3\}$;
- only based on approximation properties of the (generalized) finite element space; the rationale behind this requirement is that it is easier to verify such an approximation property than to perform a full-fledged convergence analysis for a given approximation space.

These estimates require the development of new analytical tools and cannot be achieved in one stroke. As a first step, therefore, the present paper focuses on the Helmholtz equation in a bounded d -dimensional domain Ω with transparent boundary conditions, which we assume to be realized exactly with a Dirichlet-to-Neumann map (DtN map) T_k . We will place special attention on the case where Ω is a ball since then the DtN map T_k can be analyzed fairly explicitly. In this specific setting, we provide stability and convergence estimates of finite element discretizations that are explicit in the wave number, the mesh width, and the polynomial degree of the finite element space. The companion paper [32] will build upon the results of the present paper and will address more general situations such as the Helmholtz equation with Robin boundary conditions on smooth bounded domains or in convex polygons.

The outline of this paper is as follows: Section 2 formulates the model problem. Section 3 provides an analysis of the model problem. In particular, the k -dependence of the solution is made explicit (Lemmata 3.9, 3.5). Section 4 analyzes the discrete stability and states conditions on the properties of the approximation space to ensure quasi-optimality of the Galerkin scheme. For the case where Ω is a circle or a sphere, the conditions for stability and quasi-optimality are made fully explicit (Theorems 4.2, 4.3). Section 5 applies the results of Section 4 to the hp -version of the FEM. In particular, for the setting of Theorem 4.2 we show in Corollary 5.6 that quasi-optimality of the hp -FEM can be achieved under the assumption that

$$(1.1) \quad \frac{kh}{p} + k \left(\frac{kh}{\sigma p} \right)^p \leq C$$

where the constants C , $\sigma > 0$ are sufficiently small but independent of h , p , and k . Several appendices conclude the paper: Appendix A provides detailed properties of Bessel functions that are needed in Section 3. Appendix B is concerned with hp -approximation of functions in the Sobolev spaces H^s ; the novel feature of our results is its focus on simultaneous approximation in L^2 and H^1 , which is an essential ingredient in our k -explicit bounds. Appendix C finally provides hp -approximation results for functions that are analytic. These latter approximation results are tailored to regularity properties of solutions of Helmholtz-type problems.

2. FORMULATION OF THE MODEL HELMHOLTZ PROBLEM

The Helmholtz problem in the full space \mathbb{R}^d with Sommerfeld radiation condition is given by: Find $U \in H_{\text{loc}}^1(\mathbb{R}^d)$ such that

$$(2.1) \quad \begin{aligned} (-\Delta - k^2) U &= f && \text{in } \mathbb{R}^d, \\ \left| \frac{\partial U}{\partial r} - i k U \right| &= o \left(\|\mathbf{x}\|^{\frac{1-d}{2}} \right) && \|\mathbf{x}\| \rightarrow \infty \end{aligned}$$

is satisfied in a weak sense (cf. [35]). Here, $\partial/\partial r$ denotes the derivative in radial direction $\mathbf{x}/\|\mathbf{x}\|$. We assume throughout the paper that the wave number is positive

and bounded away from zero, i.e.,

$$(2.2) \quad k \geq k_0 > 0.$$

We assume that f is local in the sense that there exists a bounded, simply connected domain $\Omega \subset \mathbb{R}^d$ that satisfies $\text{supp } f \subset \Omega$. The complement of Ω is denoted by $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}$ and the interface by $\Gamma := \overline{\Omega} \cap \overline{\Omega^+}$. Then (2.1) can be formulated in an equivalent way as a transmission problem by seeking functions $u \in H^1(\Omega)$ and $u^+ \in H^1_{\text{loc}}(\Omega^+)$ such that

$$(2.3) \quad \begin{aligned} & \begin{cases} (-\Delta - k^2) u = f & \text{in } \Omega, \\ (-\Delta - k^2) u^+ = 0 & \text{in } \Omega^+, \\ u = u^+ \text{ and } \partial u / \partial n = \partial u^+ / \partial n & \text{on } \partial\Omega, \end{cases} \\ & \left| \frac{\partial u^+}{\partial r} - i k u^+ \right| = o\left(\|x\|^{\frac{1-d}{2}}\right) \quad \|x\| \rightarrow \infty. \end{aligned}$$

Here, n denotes the normal vector pointing into the exterior domain Ω^+ .

It can be shown that, for given $g \in H^{1/2}(\partial\Omega)$, the problem

$$(2.4) \quad \text{Find } w \in H^1_{\text{loc}}(\Omega^+) \text{ such that } \begin{cases} (-\Delta - k^2) w = 0 & \text{in } \Omega^+, \\ w = g & \text{on } \partial\Omega, \\ \left| \frac{\partial w}{\partial r} - i k w \right| = o\left(\|x\|^{\frac{1-d}{2}}\right) & \|x\| \rightarrow \infty \end{cases}$$

has a unique weak solution. The mapping $g \mapsto w$ is called the *Steklov-Poincaré operator* and is denoted by $S_P : H^{1/2}(\partial\Omega) \rightarrow H^1_{\text{loc}}(\Omega^+)$. The *Dirichlet-to-Neumann map* is given by $T_k := \gamma_1 S_P : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$, where $\gamma_1 := \partial/\partial n$ is the normal trace operator. Hence, problem (2.3) can be reformulated as: Find $u \in H^1(\Omega)$ such that

$$(2.5) \quad \begin{aligned} & \begin{cases} (-\Delta - k^2) u = f & \text{in } \Omega, \\ \partial u / \partial n = T_k u & \text{on } \partial\Omega. \end{cases} \end{aligned}$$

The weak formulation of this equation is given by: Find $u \in H^1(\Omega)$ such that

$$(2.6) \quad a(u, v) := \int_{\Omega} \langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v} - \int_{\partial\Omega} (T_k u) \bar{v} = \int_{\Omega} f \bar{v} \quad \forall v \in H^1(\Omega).$$

The exact solution of (2.1) can be written as the acoustic volume potential. Let $G_k : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{C}$ denote the fundamental solution to the operator $\mathcal{L}_k := -\Delta - k^2$, i.e., $G_k(z) = g_k(\|z\|)$, where

$$g_k(r) := \begin{cases} -\frac{e^{i k r}}{2 i k} & d = 1, \\ \frac{i}{4} H_0^{(1)}(k r) & d = 2, \\ \frac{e^{i k r}}{4 \pi r} & d = 3. \end{cases}$$

Then, the solution of (2.1) is given by

$$(2.7) \quad U(x) := (N_k f)(x) := \int_{\Omega} G_k(x - y) f(y) dy \quad \forall x \in \mathbb{R}^d.$$

Consequently, the solution of (2.5) and (2.6) is given by

$$u(x) := (N_k f)(x) := \int_{\Omega} G_k(x - y) f(y) dy \quad \forall x \in \Omega.$$

Finally, we recall that a Galerkin method for (2.6) is given as follows: For a (typically finite dimensional) space $S \subset H^1(\Omega)$, the Galerkin approximation $u_S \in S$ to the exact solution u is given by:

$$(2.8) \quad \text{Find } u_S \in S \text{ s.t. } a(u_S, v) = \int_{\Omega} f \bar{v} \quad \forall v \in S.$$

3. ANALYSIS OF THE CONTINUOUS PROBLEM

The analysis of the continuous problem is split into three parts. First, we provide some estimates for the Dirichlet-to-Neumann map T_k . Then, we prove some mapping properties of the solution operator and, finally, state the existence and uniqueness of the continuous problem.

3.1. Estimates for the DtN operator T_k . We equip the space $H^1(\Omega)$ with the norm

$$\|u\|_{\mathcal{H}} := \left(|u|_{H^1(\Omega)}^2 + k^2 \|u\|_{L^2(\Omega)}^2 \right)^{1/2},$$

which is obviously equivalent to the $H^1(\Omega)$ -norm. For $d = 1$, the boundary $\partial\Omega$ consists of the two endpoints of Ω and the $L^2(\partial\Omega)$ - and $H^{1/2}(\partial\Omega)$ -scalar product and norm are understood as

$$(u, v)_{L^2(\partial\Omega)} := \sum_{x \in \partial\Omega} u(x) \overline{v(x)} \quad \text{and} \quad \|u\|_{L^2(\partial\Omega)} = \|u\|_{H^{1/2}(\partial\Omega)} = \sqrt{\sum_{x \in \partial\Omega} |u(x)|^2}.$$

For Lipschitz domains, it is well known that a trace estimate holds.

Lemma 3.1. *There exists a constant C_{tr} depending only on Ω and k_0 such that for all $u \in H^1(\Omega)$,*

$$(3.1a) \quad \|u\|_{H^{1/2}(\partial\Omega)} \leq C_{\text{tr}} \|u\|_{\mathcal{H}},$$

$$(3.1b) \quad \|u\|_{L^2(\partial\Omega)} \leq C_{\text{tr}} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2}.$$

Corollary 3.2. *For $u \in H^1(\Omega)$, we have*

$$\sqrt{k} \|u\|_{L^2(\partial\Omega)} \leq \tilde{C}_{\text{tr}} \|u\|_{\mathcal{H}} \quad \text{with} \quad \tilde{C}_{\text{tr}} := \frac{C_{\text{tr}}}{\sqrt{2}} \frac{\sqrt{1+k_0^2}}{k_0},$$

where k_0 is as in (2.2).

Proof. There holds

$$(3.2) \quad \begin{aligned} k \|u\|_{L^2(\partial\Omega)}^2 &\leq C_{\text{tr}}^2 k \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \leq \frac{C_{\text{tr}}^2}{2} \left(k^2 \|u\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right) \\ &= \frac{C_{\text{tr}}^2}{2} \left((1+k^2) \|u\|_{L^2(\Omega)}^2 + |u|_{H^1(\Omega)}^2 \right) \leq \tilde{C}_{\text{tr}}^2 \|u\|_{\mathcal{H}}^2. \quad \square \end{aligned}$$

Let $B_r(x)$ denote the open ball with radius r about x . For $x = 0$, we write B_r short for $B_r(0)$. Since the right-hand side f in (2.3) has compact support, we may always choose Ω as some ball B_R . In the following analysis we will always restrict our attention to this case and assume that

$$(3.3) \quad R \geq R_0 > 0.$$

Lemma 3.3. *Let (3.3) and (2.2) be satisfied. For $d = 2$, we assume additionally that $k_0 \geq 1$. Then, there exist constants $c, C > 0$ that depend solely on R_0 and k_0 such that the following is true:*

(1)

$$(3.4a) \quad \left| (T_k u, v)_{L^2(\partial B_R)} \right| \leq C \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \quad \forall u, v \in H^1(B_R).$$

(2) For $d \in \{2, 3\}$ and all $u \in H^{1/2}(\partial B_R)$ the real and imaginary parts of $(T_k u, u)_{L^2(\partial B_R)}$ satisfy

$$(3.4b) \quad -\operatorname{Re} (T_k u, u)_{L^2(\partial B_R)} \geq c \frac{\|u\|_{L^2(\partial B_R)}^2}{R},$$

$$(3.4c) \quad \operatorname{Im} (T_k u, u)_{L^2(\partial B_R)} > 0 \quad \text{for } u \neq 0.$$

For $d = 1$, instead of (3.4b), (3.4c), there holds

$$(3.4d) \quad -\operatorname{Re} (T_k u, u)_{L^2(\partial B_R)} = 0,$$

$$(3.4e) \quad \operatorname{Im} (T_k u, u)_{L^2(\partial B_R)} \geq k \|u\|_{L^2(\partial B_R)}^2.$$

Before proving Lemma 3.3, we note the following corollary.

Corollary 3.4. *There exists $C_c > 0$ which depend only on k_0 and R_0 (cf. (2.2), (3.3)) such that for all $u, v \in H^1(B_R)$,*

$$|a(u, v)| \leq C_c \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.$$

Proof. The estimate

$$|a(u, v)| \leq |u|_{H^1(B_R)} |v|_{H^1(B_R)} + k^2 \|u\|_{L^2(B_R)} \|v\|_{L^2(B_R)} + \left| \int_{\partial B_R} (T_k u) \bar{v} \right|$$

is obvious. Hence, the assertion follows from Lemma 3.3. □

Proof of Lemma 3.3. Case $d = 3$.

The Dirichlet data on ∂B_R can be expanded according to

$$(3.5) \quad u(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell}^m Y_{\ell}^m(\theta, \phi),$$

where (R, θ, ϕ) are the spherical coordinates for $x \in \partial B_R$ and the functions Y_{ℓ}^m are the standard spherical harmonics. The solution to the exterior homogeneous Helmholtz problem with Sommerfeld radiation conditions at infinity and prescribed Dirichlet data at ∂B_R can be expanded in the form

$$(3.6) \quad u(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell}^m Y_{\ell}^m(\theta, \phi) \frac{h_{\ell}^{(1)}(kr)}{h_{\ell}^{(1)}(kR)},$$

where (r, θ, ϕ) are the spherical coordinates of $x \in \mathbb{R}^3 \setminus \overline{B_R}$. By taking the normal derivative at the boundary we end up with a representation of the Dirichlet-to-Neumann map

$$(3.7) \quad T_k u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell}^m Y_{\ell}^m(\theta, \phi) \frac{z_{\ell}(kR)}{R}$$

with the functions $z_{\ell}(r) := r \frac{(h_{\ell}^{(1)})'(r)}{h_{\ell}^{(1)}(r)}$. These functions have been analyzed in [35, Theorem 2.6.1] where it is shown that

$$(3.8) \quad 1 \leq -\operatorname{Re}(z_{\ell}(r)) \leq \ell + 1 \quad \text{and} \quad 0 < \operatorname{Im}(z_{\ell}(r)) \leq r.$$

(In [35, Theorem 2.6.1], only $\text{Im } z_\ell(r) \geq 0$ is stated, while the strict positivity follows from the positivity of the function q_ℓ in [35, (2.6.34)].) It follows from (3.7) that

$$\int_{\partial B_R} (T_k u) \bar{v} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{z_\ell(kR)}{R} u_\ell^m \bar{v}_\ell^m$$

and from (3.8) we conclude that

$$\begin{aligned} \left| \text{Re} \int_{\partial B_R} (T_k u) \bar{v} \right| &= \left| \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \frac{\text{Re } z_\ell(kR)}{R} \text{Re}(u_\ell^m \bar{v}_\ell^m) - \frac{\text{Im } z_\ell(kR)}{R} \text{Im}(u_\ell^m \bar{v}_\ell^m) \right\} \right| \\ &\leq \frac{1}{R} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \{ |\text{Re } z_\ell(kR)| + |\text{Im } z_\ell(kR)| \} |u_\ell^m| |v_\ell^m| \\ &\leq \frac{1}{R} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \{ |\ell + 1| + kR \} |u_\ell^m| |v_\ell^m| \\ &\leq C \left(R^{-1} \|u\|_{H^{1/2}(\partial B_R)} \|v\|_{H^{1/2}(\partial B_R)} + k \|u\|_{L^2(\partial B_R)} \|v\|_{L^2(\partial B_R)} \right). \end{aligned}$$

Using Corollary 3.2 we get

$$\left| \text{Re} \int_{\partial B_R} (T_k u) \bar{v} \right| \leq C \tilde{C}_{\text{tr}}^2 \left(1 + \frac{1}{R_0 k_0} \right) \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.$$

Repeating these steps for the imaginary part results in the same upper bound, and we get for some C that depends only on R_0 and k_0 the estimate

$$\left| \int_{\partial B_R} (T_k u) \bar{v} \right| \leq C \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.$$

The lower estimate of the real part follows from

$$-\text{Re} \int_{\partial B_R} (T_k u) \bar{u} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{-\text{Re } z_\ell(kR)}{R} |u_\ell^m|^2 \geq \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{R} |u_\ell^m|^2 = \frac{\|u\|_{L^2(B_R)}^2}{R}.$$

The upper estimate for the imaginary part is just a repetition of the previous arguments.

For the lower estimate of the imaginary part, we consider $u \in H^{1/2}(\partial B_R) \setminus \{0\}$. Hence, there exists (m_\star, ℓ_\star) in the expansion (3.5) so that $u_{\ell_\star}^{m_\star} \neq 0$. This leads to

$$\text{Im} \int_{\partial B_R} (T_k u) \bar{u} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\text{Im } z_\ell(kR)}{R} |u_\ell^m|^2 \geq C |u_{\ell_\star}^{m_\star}|^2 > 0,$$

and the lower bound is proved.

Case $d = 2$.

We expand the Dirichlet data on ∂B_R in polar coordinates

$$(3.9) \quad u(x) = \sum_{\ell \in \mathbb{Z}} u_\ell e^{i\ell\theta},$$

where (R, θ) are the polar coordinates of $x \in \partial B_R$. It follows (see, e.g., [12, (2.10)]) that

$$(3.10) \quad T_k u = \sum_{\ell \in \mathbb{Z}} u_\ell \frac{w_\ell(kR)}{R} e^{i\ell\theta} \quad \text{with} \quad w_\ell(r) := r \frac{\left(H_{|\ell|}^{(1)}\right)'(r)}{H_{|\ell|}^{(1)}(r)}.$$

Obviously, it is sufficient to analyze w_ℓ only for $\ell \in \mathbb{N}_0$. By decomposing w_ℓ into its real and imaginary part we get

$$w_\ell = r \frac{J'_\ell J_\ell + Y'_\ell Y_\ell + i(Y'_\ell J_\ell - J'_\ell Y_\ell)}{J_\ell^2 + Y_\ell^2}.$$

For the imaginary part, we obtain

$$Y'_\ell J_\ell - J'_\ell Y_\ell \stackrel{[1, 9.1.27]}{=} Y_{\ell-1} J_\ell - J_{\ell-1} Y_\ell \stackrel{[1, 9.1.16]}{=} \frac{2}{\pi r}.$$

We set $M_\ell := \left|H_\ell^{(1)}\right|$ and obtain

$$(3.11) \quad w_\ell = r \frac{J'_\ell J_\ell + Y'_\ell Y_\ell}{M_\ell^2} + i \frac{2}{\pi M_\ell^2} = \frac{r}{2} \frac{d}{dr} \frac{M_\ell^2}{M_\ell^2} + i \frac{2}{\pi M_\ell^2}.$$

In the next step, we derive estimates for the coefficients w_ℓ .

Case $d = 2$ and $\ell \in \mathbb{N}_{\geq 2}$.

Let

$$(3.12) \quad M_{\ell,n}^2(r) := \frac{2}{\pi r} \sum_{m=0}^n \frac{\delta_{\ell,m}}{r^{2m}} \quad \text{with} \quad \delta_{\ell,m} := \frac{(2m)! \gamma_{\ell,m}}{(m!)^2 16^m} \quad \text{and}$$

$$\gamma_{\ell,m} := \prod_{k=1}^m \left(4\ell^2 - (2k-1)^2\right)$$

and define $R_{\ell,n}^M := M_\ell^2 - M_{\ell,n}^2$. Note that

$$(3.13) \quad \gamma_{\ell,\ell} = \frac{(4\ell)!}{2^{2\ell} (2\ell)!} \geq 0 \quad \text{and} \quad \gamma_{\ell,\ell+1} = -(4\ell+1) \gamma_{\ell,\ell} < 0.$$

We conclude from [46, §13.75] that, for the choice $n = \ell - 1 \geq 0$, there holds $R_{\ell,\ell-1}^M(r) \geq 0$. Thus,

$$(3.14) \quad M_\ell^2(r) \geq M_{\ell,\ell-1}^2(r) \quad \forall r \geq 0.$$

Let K_ν be the modified Bessel function of order ν . From [46, §13.75] we obtain

$$N_\ell^2 := \frac{d}{dr} M_\ell^2 = -\frac{16}{\pi^2} \int_0^\infty K_1(2r \sinh t) \sinh t \cosh(2\ell t) dt$$

and

$$\frac{\cosh(2\ell t)}{\cosh t} = \sum_{m=0}^n \frac{\gamma_{\ell,m}}{(2m)!} \sinh^{2m} t + \tilde{R}_{\ell,n}^2.$$

If $n > \ell - 3/2$, the remainder $\tilde{R}_{\ell,n}^2$ satisfies

$$(3.15) \quad \tilde{R}_{\ell,n}^2 \in \begin{cases} \left[0, \frac{\gamma_{\ell,n+1}}{(2n+2)!} \sinh^{2(n+1)} t\right] & \text{if } \gamma_{\ell,n+1} > 0, \\ \left[\frac{\gamma_{\ell,n+1}}{(2n+2)!} \sinh^{2(n+1)} t, 0\right] & \text{otherwise.} \end{cases}$$

We introduce

$$\begin{aligned} N_{\ell,n}^2 &:= -\frac{16}{\pi^2} \sum_{m=0}^n \frac{\gamma_{\ell,m}}{(2m)!} \int_0^\infty K_1(2r \sinh t) (\cosh t) (\sinh^{2m+1} t) dt \\ &= -\frac{16}{\pi^2} \sum_{m=0}^n \frac{\gamma_{\ell,m}}{(2m)! (2r)^{2m+2}} \int_0^\infty K_1(z) z^{2m+1} dz \\ &= -\frac{2}{\pi r^2} \sum_{m=0}^n (2m+1) \frac{\delta_{\ell,m}}{r^{2m}} = \frac{d}{dr} M_{\ell,n}^2. \end{aligned}$$

Note that $M_\ell^2(r)$ is monotone decreasing for $r > 0$ (cf. [37, §9-7.3]) and hence $N_\ell^2(r) < 0$ for $r > 0$. Thus,

$$|N_\ell^2(r)| = -N_{\ell,n}^2(r) + R_{\ell,n}^N \quad \text{with} \quad R_{\ell,n}^N := -N_\ell^2(r) + N_{\ell,n}^2(r)$$

and $R_{\ell,n}^N$ has the explicit representation

$$R_{\ell,n}^N(r) = \frac{16}{\pi^2} \int_0^\infty K_1(2r \sinh t) (\sinh t) (\cosh t) \tilde{R}_{\ell,n}^2(t) dt.$$

Note that \sinh , \cosh , and K_1 are positive on the positive real axes (cf. [1, 9.6.23]). We choose $n = \ell$ and obtain from (3.13) and (3.15) that $\tilde{R}_{\ell,\ell}(t)$ is negative for $t > 0$ and hence

$$(3.16) \quad |N_\ell^2(r)| \leq -N_{\ell,\ell}^2(r) \quad \forall r > 0.$$

In summary, we have proved that

$$\begin{aligned} (3.17) \quad |\operatorname{Re} w_\ell| &\leq -\frac{r}{2} \frac{N_{\ell,\ell}^2}{M_{\ell,\ell-1}^2} = \frac{1}{2} \frac{\sum_{m=0}^\ell (2m+1) \frac{\delta_{\ell,m}}{r^{2m}}}{\sum_{m=0}^{\ell-1} \frac{\delta_{\ell,m}}{r^{2m}}} \leq \frac{2\ell-1}{2} + \frac{2\ell+1}{2} \frac{\frac{\delta_{\ell,\ell}}{r^{2\ell}}}{\frac{\delta_{\ell,\ell-1}}{r^{2\ell-2}}} \\ &= \frac{2\ell-1}{2} + \frac{(4\ell-1)(4\ell^2-1)}{16\ell r^2}. \end{aligned}$$

Hence, for $\ell \geq 2$ and $r \geq C_1 \sqrt{\ell}$ we arrive at

$$|\operatorname{Re} w_\ell| \leq \frac{2\ell-1}{2} \left(1 + \frac{9}{8C_1^2} \right).$$

It remains to consider the case

$$(3.18) \quad r \leq C_1 \sqrt{\ell}.$$

We derive from (3.11) and [1, 9.1.27]

$$\left| \frac{r}{2} N_\ell^2(r) \right| = -\frac{r}{2} N_\ell^2(r) = \ell M_\ell^2(r) - r(J_{\ell-1} J_\ell + Y_\ell Y_{\ell-1}),$$

and this leads to

$$(3.19) \quad |\operatorname{Re} w_\ell| = \frac{\left| \frac{r}{2} N_\ell^2(r) \right|}{M_\ell^2(r)} = \ell - \frac{r(J_\ell J_{\ell-1} + Y_\ell Y_{\ell-1})}{M_\ell^2(r)}.$$

We deduce from [1, 9.5.2, 9.1.7, 9.1.9] that

$$J_\ell(r) > 0 \quad \text{and} \quad Y_\ell(r) < 0 \quad \forall 0 \leq r \leq \ell$$

and thus

$$J_\ell J_{\ell-1} + Y_\ell Y_{\ell-1} > 0 \quad \forall 0 \leq r \leq \ell - 1.$$

If $C_1 \leq 2^{-1/2}$, there holds $C_1 \sqrt{\ell} \leq \ell - 1$ for all $\ell \geq 2$, and we have proved $|\operatorname{Re} w_\ell| \leq \ell$.

To derive a lower bound for $(-\operatorname{Re} w_\ell)$, we proceed as for (3.17) and obtain, for $r \geq k_0$,

$$(3.20) \quad -\operatorname{Re} w_\ell(r) \geq -\frac{r N_{\ell,\ell-1}^2(r)}{2 M_{\ell,\ell}^2(r)} = \frac{1}{2} \frac{\sum_{m=0}^{\ell-1} (2m+1) \frac{\delta_{\ell,m}}{r^{2m}}}{\sum_{m=0}^{\ell} \frac{\delta_{\ell,m}}{r^{2m}}} \geq \frac{1}{2} \frac{1}{1 + \frac{\frac{\delta_{\ell,\ell}}{r^{2\ell}}}{(2\ell-1) \frac{\delta_{\ell,\ell-1}}{r^{2\ell-2}}}}$$

$$= \frac{1}{2} \frac{1}{1 + \frac{4\ell-1}{8\ell r^2}} \geq \frac{1}{2} \frac{1}{1 + \frac{1}{2k_0^2}}.$$

For the imaginary part of w_ℓ we get

$$(3.21) \quad \operatorname{Im} w_\ell(r) = \frac{2}{\pi M_\ell^2(r)} > 0 \quad \forall \ell \in \mathbb{N}_0 \quad \forall r \geq k_0$$

because M_ℓ^2 is non-negative and decreasing for $r > 0$ (cf. [37, §9-7.3]). For the upper bound, we combine [20, 8.479] with the fact that M_ℓ^2 is decreasing to obtain for $\ell \in \mathbb{N}_{\geq 1}$,

$$(3.22a) \quad M_\ell^2(r) \geq \frac{2}{\pi r} \quad \forall r \geq 1.$$

Hence, the upper bound

$$(3.23) \quad \operatorname{Im} w_\ell(r) = \frac{2}{\pi M_\ell^2(r)} \leq r$$

follows.

Case $d = 2$ and $\ell = 0, 1$.

For $\ell = 0$, we use [46, §13.75] and get

$$M_0^2(r) \geq M_{0,1}^2(r) = \frac{2}{\pi r} \left(1 - \frac{1}{8r^2}\right).$$

For $d = 2$, there holds $k_0 > 1/2$ by our assumptions and, thus, for $r \geq k_0$ we get

$$(3.22b) \quad M_0^2(r) \geq \frac{1}{\pi r}.$$

The combination of (3.11) and (3.22) implies

$$|\operatorname{Re} w_\ell(r)| \leq \frac{\pi r^2}{2} |N_\ell^2(r)|.$$

We deduce from (3.16) (which is also valid for $\ell = 0, 1$) that

$$|N_\ell^2(r)| \leq |N_{\ell,\ell}^2(r)| \leq \frac{2}{\pi r^2} \sum_{m=0}^{\ell} (2m+1) \frac{\delta_{\ell,m}}{r^{2m}} \leq \frac{2}{\pi r^2} \begin{cases} 1 & \ell = 0, \\ 1 + \frac{9}{8r^2} & \ell = 1. \end{cases}$$

This implies, for $r \geq k_0$ (cf. (2.2)),

$$|N_\ell^2(r)| \leq C \frac{2}{\pi r^2},$$

where C depends solely on k_0 . Thus, for $\ell = 0, 1$,

$$|\operatorname{Re} w_\ell| \leq C \leq C(\ell + 1).$$

Since M_ℓ^2 is monotone decreasing (see [37, §9-7.3]), it follows from (3.10) that $\operatorname{Re} w_\ell(r) < 0$ for all $r > 0$.

In (3.20) we have derived a lower bound for $(-\operatorname{Re} w_\ell)$ provided $\ell \geq 1$. It remains to consider the case $\ell = 0$. The assumption on k_0 implies $r \geq k_0 \geq \frac{1}{2}\sqrt{3}$ so that

$$-\operatorname{Re} w_0(r) \geq -\frac{r N_{0,1}^2(r)}{2 M_{0,0}^2(r)} = \frac{1}{2} \left(1 - \frac{3}{8r^2} \right) \geq \frac{1}{4}.$$

To summarize both cases, we have proved that

$$(3.24a) \quad 0 < c \leq -\operatorname{Re} w_\ell(r) \leq C(\ell + 1) \quad \forall r \geq k_0 \quad \forall \ell \in \mathbb{N}_0,$$

where c, C only depends on k_0 .

For the imaginary part, it remains (cf. (3.22a), (3.23)) to prove the upper bound for $(\operatorname{Im} w_0)$ and employ (3.11) and (3.22b) to obtain

$$(3.24b) \quad \operatorname{Im} w_0 = \frac{2}{\pi M_0^2} \leq 2r.$$

By proceeding as for $d = 3$ (after (3.8)) the estimates (3.4) follow from (3.24).

Case $d = 1$.

For boundary values $\psi : \{-R, R\} \rightarrow \mathbb{R}$, the Dirichlet-to-Neumann operator is given by

$$(3.25) \quad T_k \psi = i k \psi.$$

The trace theorem (in one dimension) leads to

$$\begin{aligned} \left| \operatorname{Re} \int_{\partial B_R} (T_k u) \bar{v} \right| &= \left| \operatorname{Re} \left(i k \sum_{r=\pm R} u(r) \bar{v}(r) \right) \right| \\ &\leq k \left| \operatorname{Im} \sum_{r=\pm R} u(r) \bar{v}(r) \right| \leq k \sum_{r=\pm R} |u(r)| |\bar{v}(r)| \\ &\stackrel{\text{Cor. 3.2}}{\leq} C \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \end{aligned}$$

where C only depends on R_0 and k_0 . By the same techniques we can estimate the imaginary part and, thus, obtain (3.4a). The lower bounds (3.4d), (3.4e) follow from

$$\begin{aligned} -\operatorname{Re} \int_{\partial B_R} (T_k u) \bar{u} &= -\operatorname{Re} \left(i k \sum_{r=\pm R} |u(r)|^2 \right) = 0, \\ \operatorname{Im} \int_{\partial B_R} (T_k u) \bar{u} &= k \sum_{r=\pm R} |u(r)|^2 \geq k \|u\|_{L^2(\partial B_R)}^2. \quad \square \end{aligned}$$

3.2. Analysis of the solution operator N_k . In this section, we derive some explicit bounds for the solution operator N_k under the assumption that the right-hand side is in $L^2(\Omega)$. These estimates will be the basic tool for proving the discrete stability of the finite element discretization and the convergence. The key ingredient of the analysis of the hp -FEM in Section 5 is the following decomposition result:

Lemma 3.5 (decomposition lemma). *Let Ω be contained in a ball of radius $R > 0$. Then there exists a constant $C > 0$ depending only on R and k_0 such that for $f \in L^2(\Omega)$ the function v given by*

$$v(x) = N_k f(x) = \int_{\Omega} G_k(x - y) f(y) dy, \quad x \in \Omega,$$

satisfies

$$k^{-1} \|v\|_{H^2(\Omega)} + \|v\|_{H^1(\Omega)} + k \|v\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Furthermore, for every $\lambda > 1$, there exists a λ - and k -dependent splitting $v = v_{H^2} + v_{\mathcal{A}}$ with

$$(3.26a) \quad \|\nabla^p v_{H^2}\|_{L^2(\Omega)} \leq C \left(1 + \frac{1}{\lambda^2 - 1}\right) (\lambda k)^{p-2} \|f\|_{L^2(\Omega)} \quad \forall p \in \{0, 1, 2\},$$

$$(3.26b) \quad \|\nabla^p v_{\mathcal{A}}\|_{L^2(\Omega)} \leq C \lambda \left(\sqrt{d} \lambda k\right)^{p-1} \|f\|_{L^2(\Omega)} \quad \forall p \in \mathbb{N}_0.$$

Here, $\nabla^p v_{\mathcal{A}}$ stands for a sum over all derivatives of order p (see (5.1) for details).

Remark 3.6. 1) For $f \in L^2(\Omega)$ the function $v = N_k(f)$ cannot be expected to have more Sobolev regularity than H^2 . The decomposition $v = v_{H^2} + v_{\mathcal{A}}$ of Lemma 3.5 splits v into an H^2 -regular part v_{H^2} and an analytic part $v_{\mathcal{A}}$. The essential feature of this splitting is that the H^2 -part v_{H^2} has a better H^2 -regularity constant in terms of k than v itself, namely, (3.26a), (3.26b), and the triangle inequality $\|\nabla^2 v\|_{L^2(\Omega)} \leq \|\nabla^2 v_{H^2}\|_{L^2(\Omega)} + \|\nabla^2 v_{\mathcal{A}}\|_{L^2(\Omega)}$ imply

$$\|\nabla^2 v_{H^2}\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \quad \text{versus} \quad \|\nabla^2 v\|_{L^2(\Omega)} \leq C k \|f\|_{L^2(\Omega)}.$$

The fact that $\|v_{H^2}\|_{H^2} \leq C \|f\|_{L^2}$ for a $C > 0$ independent of k will be essential for the stability and convergence analysis below.

2) Inspection of the proof shows that the mappings $f \mapsto v_{H^2}$ and $f \mapsto v_{\mathcal{A}}$ are linear maps.

Proof of Lemma 3.5. The estimates for v follow directly from those for v_{H^2} and $v_{\mathcal{A}}$ by fixing a parameter $\lambda > 1$. In order to construct the splitting $v = v_{H^2} + v_{\mathcal{A}}$, we start by recalling the definition of the Fourier transform for functions with compact support

$$\hat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle} u(x) dx \quad \forall \xi \in \mathbb{R}^d$$

and the inversion formula

$$u(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} \hat{u}(\xi) d\xi \quad \forall x \in \mathbb{R}^d.$$

Let $B_\Omega \subset \mathbb{R}^d$ be a ball of radius R containing Ω . Extend f by zero outside of Ω and denote this extended function again by f . Let $\mu \in C^\infty(\mathbb{R}_{\geq 0})$ be a cutoff function such that

$$(3.27) \quad \begin{aligned} \text{supp } \mu &\subset [0, 4R], & \mu|_{[0, 2R]} &= 1, & |\mu|_{W^{1, \infty}(\mathbb{R}_{\geq 0})} &\leq \frac{C}{R}, \\ \forall x \in \mathbb{R}_{\geq 0} : 0 &\leq \mu(x) \leq 1, & \mu|_{[4R, \infty[} &= 0, & |\mu|_{W^{2, \infty}(\mathbb{R}_{\geq 0})} &\leq \frac{C}{R^2}. \end{aligned}$$

Define $M(z) := \mu(\|z\|)$ and

$$v_\mu(x) := \int_{B_\Omega} G_k(x-y) M(x-y) f(y) dy \quad \forall x \in \mathbb{R}^d.$$

The properties of μ guarantee $v_\mu|_{B_\Omega} = v|_{B_\Omega}$ so that we may restrict our attention to the function v_μ . Since $\text{supp } f \subset B_\Omega$ we may write

$$(3.28) \quad v_\mu = (G_k M) \star f,$$

where “ \star ” denotes the convolution in \mathbb{R}^d . We will define a decomposition of v_μ (which will determine the decomposition of v on B_Ω) by decomposing its Fourier transform, i.e.,

$$(3.29) \quad \widehat{v}_\mu = \widehat{v}_{H^2} + \widehat{v}_A.$$

In order to define the two terms on the right-hand side of (3.29), we let $B_{\lambda k}(0)$ denote the ball of radius λk centered at the origin where $\lambda > 1$ is the fixed constant (independent of k) selected in the statement of the lemma. The characteristic function of $B_{\lambda k}(0)$ is denoted by $\chi_{\lambda k}$. The Fourier transform of f is then decomposed as

$$\widehat{f} = \widehat{f}\chi_{\lambda k} + (1 - \chi_{\lambda k})\widehat{f} =: \widehat{f}_k + \widehat{f}_k^c.$$

By the inverse Fourier transformation, this decomposition of \widehat{f} entails a decomposition of f into f_k and f_k^c given by

$$(3.30) \quad f_k(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} \chi_{\lambda k}(\xi) \widehat{f}(\xi) d\xi \quad \text{and} \quad f_k^c(x) := f - f_k.$$

Accordingly, we define the decomposition of v_μ by

$$(3.31) \quad v_{\mu, H^2} := (G_k M) \star f_k^c \quad \text{and} \quad v_{\mu, A} := (G_k M) \star f_k.$$

The functions v_{H^2} and v_A in (3.29) are then obtained by setting $v_{H^2} := v_{\mu, H^2}|_\Omega$ and $v_A := v_{\mu, A}|_\Omega$. We will obtain the desired estimates by showing the following, stronger estimates:

$$(3.32a) \quad \|v_{\mu, H^2}\|_{H^2(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)},$$

$$(3.32b) \quad \|D^\alpha v_{\mu, A}\|_{L^2(\mathbb{R}^d)} \leq C \lambda (\lambda k)^{|\alpha|-1} \|f\|_{L^2(\mathbb{R}^d)}, \quad \forall \alpha \in \mathbb{N}_0^d.$$

The estimates (3.32) are obtained by Fourier techniques. To that end, we compute the Fourier transform of $G_k M$:

$$(3.33) \quad \begin{aligned} \widehat{(G_k M)}(\xi) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle} G_k(x) M(x) dx \\ &= (2\pi)^{-d/2} \int_0^\infty g_k(r) \mu(r) r^{d-1} \left(\int_{\mathbb{S}_{d-1}} e^{-i r \langle \xi, \zeta \rangle} dS_\zeta \right) dr \\ &= (2\pi)^{-d/2} I(\xi). \end{aligned}$$

The inner integral in $I(\xi)$ can be evaluated analytically¹ and $I(\xi) = \iota(\|\xi\|)$ with

$$(3.34) \quad \iota(s) = \begin{cases} 2 \int_0^\infty g_k(r) \mu(r) \cos(sr) dr & d = 1, \\ 2\pi \int_0^\infty g_k(r) \mu(r) r J_0(rs) dr & d = 2, \\ 4\pi \int_0^\infty g_k(r) \mu(r) r^2 \frac{\sin(rs)}{(rs)} dr & d = 3. \end{cases}$$

¹This is trivial for $d = 1$ and follows for $d = 2$ from [20, (3.338)(4)]. For $d = 3$, we use the formula

$$\int_{\mathbb{S}^2} e^{-i\langle x, \hat{x} \rangle} Y_\ell^m(x) dx = g_\ell(\|\hat{x}\|) Y_\ell^m\left(\frac{\hat{x}}{\|\hat{x}\|}\right) \quad \text{with} \quad g_\ell(r) = (-i)^\ell 4\pi j_\ell(r)$$

(which follows by a comparison of [35, Section 3.2.4, formula (3.2.44) and (3.2.54)]) for $m = \ell = 0$, where $Y_0^0 = \text{const}$ and $g_0(r) = 4\pi \sin(r)/r$.

Applying the Fourier transform to the convolutions (3.31) leads to

$$\begin{aligned} \widehat{v}_{\mu,H^2} &= (2\pi)^{d/2} \widehat{G_k M f_k^c} = (2\pi)^{d/2} \widehat{G_k M f} (1 - \chi_{\lambda k}), \\ \widehat{v}_{\mu,\mathcal{A}} &= (2\pi)^{d/2} \widehat{G_k M f_k} = (2\pi)^{d/2} \widehat{G_k M f} \chi_{\lambda k}. \end{aligned}$$

To estimate higher order derivatives of v_{μ,H^2} and $v_{\mu,\mathcal{A}}$ we define for a multi-index $\alpha \in \mathbb{N}_0^d$ the function $P_\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $P_\alpha(\xi) := \xi^\alpha$ and obtain, by using standard properties of the Fourier transformation and the support properties of $\chi_{\lambda k}$, for all $|\alpha| \leq 2$,

$$\begin{aligned} (3.35) \quad \|\partial^\alpha v_{\mu,H^2}\|_{L^2(\mathbb{R}^d)} &= (2\pi)^{d/2} \left\| P_\alpha \widehat{G_k M} (1 - \chi_{\lambda k}) \widehat{f} \right\|_{L^2(\mathbb{R}^d)} \\ &\leq (2\pi)^{d/2} \left(\max_{\xi \in \mathbb{R}^d: |\xi| \geq \lambda k} |P_\alpha I(\xi)| \right) \left\| (1 - \chi_{\lambda k}) \widehat{f} \right\|_{L^2(\mathbb{R}^d)} \\ &\leq (2\pi)^{d/2} \left(\max_{s \geq \lambda k} |s^{|\alpha|} \iota(s)| \right) \|f\|_{L^2(\Omega)}. \end{aligned}$$

Lemma 3.7, (iv) implies for $|\alpha| \in \{0, 1, 2\}$

$$\max_{s \geq \lambda k} |s^{|\alpha|} \iota(s)| \leq C (\lambda k)^{|\alpha|-2} \left(1 + \frac{1}{\lambda^2 - 1} \right).$$

Thus,

$$\|\partial^\alpha v_{H^2}\|_{L^2(B_\Omega)} \leq C (\lambda k)^{|\alpha|-2} \left(1 + \frac{1}{\lambda^2 - 1} \right) \|f\|_{L^2(\Omega)}$$

and (3.26a) follows.

Completely analogously, we derive for all $\alpha \in \mathbb{N}_0^d$,

$$(3.36) \quad \|\partial^\alpha v_{\mu,\mathcal{A}}\|_{L^2(\mathbb{R}^d)} \leq (2\pi)^{d/2} \left(\max_{0 \leq s \leq \lambda k} |s^{|\alpha|} \iota(s)| \right) \|f\|_{L^2(\Omega)}.$$

We can complete the proof of the lemma using the bounds on the function ι given in Lemma 3.7, (v) below and using (5.1), (5.2). \square

Lemma 3.7. *For the function ι defined in (3.34) the quantity $s^m \iota(s)$ can be estimated:*

(i) for $m = 0$ by

$$|\iota(s)| \leq C \frac{R}{k};$$

(ii) for $m = 1$ by

$$|s \iota(s)| \leq CR \begin{cases} 1 + (Rk)^{-1} & d = 1, \\ |\log kR| & d = 2 \text{ and } 4Rk \leq 1, \\ 1 & d = 2 \text{ and } 4Rk > 1, \\ 1 & d = 3; \end{cases}$$

(iii) and for $m = 2$ by

$$s^2 |\iota(s)| \leq C \begin{cases} Rk + \frac{1}{Rk} & d = 1, \\ |\log(kR)| & d = 2 \text{ and } 4Rk \leq 1, \\ Rk & d = 2 \text{ and } 4Rk > 1, \\ 1 + kR & d = 3. \end{cases}$$

(iv) For fixed $R_0, R_1 > 0$ there exists $C > 0$ (depending only on R_0, R_1, k_0, d , and the constant appearing in (3.27)) such that for any $R \in [R_0, R_1]$ and any $\lambda > 1$,

$$\sup_{|s| \geq \lambda k} s^2 |\iota(s)| \leq C \left(1 + \frac{1}{\lambda^2 - 1} \right).$$

(v) For any $\lambda > 0$ and all $m \in \mathbb{N}_0$ we have

$$\sup_{|s| \leq \lambda k} |s|^m |\iota(s)| \leq C \lambda R (\lambda k)^{m-1}.$$

Proof. In this proof, C denotes a generic constant which may vary from term to term. It suffices to prove the estimates (3.7)–(3.7) because (3.7) follows directly from (3.7). We discuss the cases $d = 3, d = 1$, and $d = 2$ in turn.

Case 1: $d = 3$.

There holds

$$|s\iota(s)| = C \left| \int_0^\infty e^{ikr} \mu(r) \sin(rs) \, dr \right| \leq CR.$$

Applying integration by parts we obtain

$$\begin{aligned} |\iota(s)| &= \frac{C}{k} \left| \int_0^\infty e^{ikr} \left(\mu'(r) \frac{\sin(rs)}{s} + \mu(r) \cos(rs) \right) \, dr \right| \\ &\leq \frac{C}{k} \int_0^{4R} \left(\frac{C}{R} r + 1 \right) \, dr = C \frac{R}{k}. \end{aligned}$$

For the product $s^2\iota(s)$, we get

$$\begin{aligned} |s^2\iota(s)| &= C \left| \int_0^\infty e^{ikr} \mu(r) s \sin(rs) \, dr \right| = C \left| \int_0^\infty e^{ikr} \mu(r) \partial_r \cos(rs) \, dr \right| \\ &\leq C \left(\left| \int_0^\infty \cos(rs) \partial_r (e^{ikr} \mu(r)) \, dr \right| + 1 \right) \\ &\leq Ck \left| \int_0^\infty \cos(rs) e^{ikr} \mu(r) \, dr \right| + C \left(\left| \int_0^\infty \cos(rs) e^{ikr} \mu'(r) \, dr \right| + 1 \right) \\ &=: T^I + T^{II}. \end{aligned}$$

The estimates $T^I \leq C^I kR$ and $T^{II} \leq C^{II}$ follows from the properties of μ (cf. (3.27)). For $|s| \geq \lambda k$, the estimate of T^I can be refined by using integration by parts:

$$\begin{aligned} T^I &\leq Ck \left| \int_0^\infty \cos(rs) e^{ikr} \mu(r) \, dr \right| = C \frac{k}{2} \left| \int_0^\infty \left(e^{i(k+s)r} + e^{i(k-s)r} \right) \mu(r) \, dr \right| \\ &\leq C' \left(\frac{k^2}{s^2 - k^2} + \int_0^\infty \frac{k^2}{s^2 - k^2} |\mu'(r)| \, dr \right) \leq \frac{C'(1+C)}{\lambda^2 - 1}. \end{aligned}$$

Case 2: $d = 1$.

There holds

$$|\iota(s)| \leq \frac{1}{k} \int_0^\infty \mu(r) \, dr \leq C \frac{R}{k}.$$

To estimate $s\iota(s)$, we apply integration by parts to obtain

$$\begin{aligned} |s\iota(s)| &\leq \left| \int_0^\infty \frac{e^{ikr}}{k} \mu(r) \partial_r \sin(sr) \, dr \right| \\ &= \left| \int_0^\infty \sin(sr) \partial_r \left(\frac{e^{ikr}}{k} \mu(r) \right) \, dr \right| \leq C \frac{1+Rk}{k}. \end{aligned}$$

Similarly, we get by two-fold integration by parts:

$$\begin{aligned} |s^2\iota(s)| &\leq \left| \int_0^\infty \frac{e^{ikr}}{k} \mu(r) \partial_r^2 \cos(sr) \, dr \right| = \left| \int_0^\infty \{\partial_r \cos(sr)\} \left\{ \partial_r \left(\frac{e^{ikr}}{k} \mu(r) \right) \right\} \, dr \right| \\ &\leq \left| \int_0^\infty \cos(sr) \left\{ \partial_r^2 \left(\frac{e^{ikr}}{k} \mu(r) \right) \right\} \, dr + 1 \right| \\ &\leq k \left| \int_0^\infty \cos(sr) e^{ikr} \mu(r) \, dr \right| \\ &\quad + \left| \int_0^\infty \cos(sr) \left(2i e^{ikr} \mu'(r) + \frac{e^{ikr}}{k} \mu''(r) \right) \, dr + 1 \right| \\ &=: T^I + T^{III}. \end{aligned}$$

The estimate $T^{III} \leq C(1 + \frac{1}{kR})$ directly follows from the properties of the cutoff function μ (3.27). The term T^I was estimated already in Case 1 so that the proof of the case $d = 1$ is complete.

Case 3a: $d = 2$ and $4R \leq 1/k$.

For brevity, we write

$$h_k(r) := H_0^{(1)}(kr) \quad \text{and} \quad j_{\nu,s}(r) := J_\nu(sr).$$

Estimate (A.3c) implies

$$\forall 0 < r < 4R \leq 1/k : |h_k(r)| \leq C(1 + |\log kr|) \quad \text{and} \quad \forall r \geq 0 : |J_0(r)| \stackrel{[1, 9.1.60]}{\leq} 1.$$

Hence,

$$|\iota(s)| \leq C \int_0^{4R} (1 + |\log kr|) r \, dr = CR^2(1 + |\log(4kR)|).$$

For the estimate of $s^m\iota(s)$, $m \in \{1, 2\}$, we employ the relations (see [1, 9.1.30], [1, 9.1.1])

$$(3.37) \quad (rj_{1,s}(r))' = rsj_{0,s}(r) \quad \text{and} \quad (rj'_{0,s}(r))' = -rs^2j_{0,s}(r).$$

Integration by parts results in

$$\begin{aligned} |s\iota(s)| &\leq C \left| \int_0^\infty h_k \mu (rj_{1,s})' \, dr \right| = C \left| \int_0^\infty rj_{1,s} (\mu' h_k + \mu h'_k) \, dr \right| \\ &\stackrel{(A.8), (A.3c), (A.11)}{\leq} C \int_0^{4kR} r \left\{ \frac{(1 + |\log kr|)}{R} + \frac{1}{r} + k^2 r \right\} \, dr \\ &\leq CR \{1 + |\log kR| + k^2 R^2\} \leq CR(1 + |\log kR|) \leq CR |\log kR|. \end{aligned}$$

Finally, we estimate $s^2\iota(s)$ by two-fold integration by parts

$$(3.38) \quad |s^2\iota(s)| = C \left| \int_0^\infty h_k \mu (rj'_{0,s})' \right| \leq C \left(\left| \int_0^{4R} j_{0,s}(r) (h_k \mu)' \right| + \left| \lim_{r \rightarrow 0} (r h'_k(r)) \right| \right).$$

Note that $\lim_{r \rightarrow 0} r h'_k(r) = 2i/\pi$. For the first term, we use

$$(3.39) \quad (r(h_k \mu)')' = \mu(rh'_k)' + 2r\mu'h'_k + (r\mu)'h_k.$$

We employ (A.12) for the first, (3.27), (A.11) for the second, (3.27), and (A.3c) for the third term on the right-hand side in (3.39) to obtain

$$\left| (r(h_k \mu)')' \right| \leq Ck^2r(1 + |\log(kr)|) + \frac{1}{R} + \frac{R+r}{R^2}(1 + |\log(kr)|).$$

Hence,

$$\begin{aligned} |s^2 \iota(s)| &\leq C \left((kR)^2(1 + |\log(kR)|) + 1 + |\log(kR)| \right) \\ &\leq C(1 + |\log(kR)|). \end{aligned}$$

Case 3b: $d = 2$ and $4Rk > 1$.

We define $\varphi_k(r) := h_k(r)\mu(r)r$ and denote its antiderivative by $\Phi_k(r) := \int_{1/k}^r \varphi_k(t) dt$. We use the splitting

$$\iota(s) = \frac{\pi i}{2} \int_0^{1/k} \varphi_k j_{0,s} + \frac{\pi i}{2} \int_{1/k}^{4R} \varphi_k j_{0,s} =: \iota_I(s) + \iota_{II}(s).$$

For $\iota_I(s)$, we employ the estimates as in Case 3a (with $4R$ replaced by $1/k$ therein) to obtain

$$|\iota_I(s)| \leq \frac{C}{k^2}.$$

It remains to estimate $\iota_{II}(s)$. Note that $j'_{0,s} = -sj_{1,s}$. There holds

$$(3.40) \quad \iota_{II}(s) = \frac{\pi i}{2} \int_{1/k}^\infty \varphi_k j_{0,s} = \frac{\pi i}{2} \int_{1/k}^{4R} \Phi_k s j_{1,s} + \frac{\pi i}{2} \Phi_k j_{0,s} \Big|_{r=1/k}^{4R}.$$

In the next step, we will estimate Φ_k . Let $\tilde{\varphi}_k(r) := e^{-ikt} \varphi_k(r)$ so that Φ_k can be written as

$$\begin{aligned} \Phi_k(r) &:= \int_{1/k}^r e^{ikt} \tilde{\varphi}_k(t) dt = - \int_{1/k}^r \frac{e^{ikt}}{ik} \tilde{\varphi}'_k(t) dt + \frac{\varphi_k(t)}{ik} \Big|_{t=1/k}^r \\ &= - \underbrace{\int_{1/k}^r \frac{e^{ikt}}{ik} \tilde{\varphi}'_k(t) dt}_{=: \Phi_k^I(r)} + \underbrace{\frac{1}{ik} th_k \mu|_{t=1/k}^r}_{=: \Phi_k^{II}(r)}. \end{aligned}$$

By using (A.6) and $\sup_{t>0} |(t\mu(t))'| \leq C$ we obtain

$$\begin{aligned} |\Phi_k^I(r)| &\leq \frac{1}{k} \int_{1/k}^r |\tilde{\varphi}'_k| dt = \frac{1}{k} \int_{1/k}^r \left| t\mu(e^{-ikt}h_k)' + (t\mu)'e^{-ikt}h_k \right| dt \\ &\leq \frac{C}{k} \int_{1/k}^r \frac{1}{\sqrt{kt}} dt \leq \frac{C}{k} \sqrt{\frac{r}{k}}. \end{aligned}$$

The function Φ_k^{II} can be estimated by using (A.3a)

$$|\Phi_k^{II}(r)| = \left| \frac{1}{ik} th_k \mu|_{t=1/k}^r \right| \leq C \left(\frac{1}{k} \sqrt{\frac{r}{k}} + \frac{1}{k^2} \right) \stackrel{\sqrt{\frac{1}{k}} \leq \sqrt{r}}{\leq} \frac{C}{k} \sqrt{\frac{r}{k}}.$$

In summary we have proved

$$|\Phi_k(r)| \leq \frac{C}{k} \sqrt{\frac{r}{k}}.$$

By inserting this estimate and (A.3b) into (3.40) we get

$$\begin{aligned} |\iota_{II}(s)| &\leq C \frac{\sqrt{|s|}}{k^{3/2}} \int_0^{4R} \sqrt{\frac{|s|r}{1+r|s|}} dr + \frac{C}{k} \sqrt{\frac{r}{k}} \Big|_{r=4R} + \frac{C}{k} \sqrt{\frac{r}{k}} \Big|_{r=1/k} \\ &\leq C \left(\frac{4R\sqrt{|s|}}{k^{3/2}} + \frac{1}{k} \sqrt{\frac{4R}{k}} + \frac{1}{k^2} \right). \end{aligned}$$

This leads to

$$(3.41) \quad |\iota(s)| \leq C \frac{R}{k} \sqrt{\frac{|s|}{k}} + \frac{R}{k}.$$

Next, we estimate $s^2\iota(s)$. As in the Case 3a, our starting point is (3.38). Recalling $|\lim_{r \rightarrow 0} rh'_k(r)| = 2/\pi$, we are left with estimating

$$(3.42) \quad \left| \int_0^{4R} j_{0,s}(r(h_k\mu)')' \right| \leq \underbrace{\left| \int_0^{1/k} j_{0,s}(r(h_k\mu)')' \right|}_{=:I_1} + \underbrace{\left| \int_{1/k}^{4R} j_{0,s}(r(h_k\mu)')' \right|}_{=:I_2}.$$

We conclude from Case 3a that $|I_1| \leq C$ holds. For the second integral, we employ (A.3a), (3.27), (A.5), (A.7) to get

$$(3.43) \quad \begin{aligned} \left| (r(h_k\mu)')' \right| &= |h_k(\mu' + r\mu'') + h'_k(\mu + 2r\mu') + rh''_k\mu| \\ &\leq C \left(\frac{1}{r\sqrt{kr}} + \sqrt{\frac{k}{r}} + rk\sqrt{\frac{k}{r}} \right) \leq C \left(\frac{1}{r\sqrt{kr}} + k\sqrt{kr} \right). \end{aligned}$$

The combination of (3.42), (3.43), and (A.3b) leads to

$$I_2 \leq CkR \sqrt{\frac{Rk}{1+R|s|}}.$$

Thus, we have proved

$$(3.44) \quad |s^2\iota(s)| \leq CkR \sqrt{\frac{Rk}{1+R|s|}}.$$

For $0 \leq |s| \leq k$, we employ (3.41) and for $|s| > k$ we use (3.44) to obtain for $m \in \{0, 1, 2\}$,

$$|s|^m |\iota(s)| \leq CRk^{m-1}.$$

For the case $d = 2$, we now show (iv), i.e., we consider the case $|s| \geq \lambda k$. The assumptions $R \geq R_0$ and $k \geq k_0$ imply for the case $Rk \leq 1/4$ immediately $\sup_{|s|>0} s^2|\iota(s)| \leq C$. For $Rk > 1/4$, we take, as in the Case 3b, the estimate (3.42) as our starting point. The integral I_1 in (3.42) is already seen to be bounded independent of k . Since, by [1, 9.1.1],

$$(rh'_k)' = -k^2rh_k$$

we can write the integral I_2 as

$$I_2 = \left| \int_{1/k}^{4R} j_{0,s}(-k^2rh_k\mu + 2rh'_k\mu' + (r\mu')'h_k) \right|.$$

Recalling that $\mu' \equiv 0$ on $(0, 2R)$, we can estimate I_2 by

$$I_2 \leq \underbrace{\int_{1/k}^{4R} j_{0,s} k^2 r h_k \mu}_{=: I_2^I} + CR \sup_{r \in (2R, 4R)} \{|j_{0,s} h'_k| + |j_{0,s} h_k|\}.$$

We conclude from (A.3), (A.5), and (A.1) together with (A.2)

$$CR \sup_{r \in (2R, 4R)} \{|j_{0,s} h'_k| + |j_{0,s} h_k|\} \leq CR \frac{1}{\sqrt{|s|R}} \left(\frac{1}{R\sqrt{Rk}} + \sqrt{\frac{k}{R}} + \frac{1}{\sqrt{Rk}} \right) \leq C,$$

where we used $|s| \geq \lambda k \geq k$ and the fact that $k \geq k_0$. It remains to bound I_2^I . Lemma A.1 allows us to write

$$I_2^I = \frac{2k^2}{\pi \sqrt{k|s|}} \left| \int_{1/k}^{4R} g^I(kr) \mu(r) \left\{ e^{i(|s|+k)r} f^I(|s|r) + e^{i(-|s|+k)r} f^{II}(|s|r) \right\} \right|.$$

Since f^I, f^{II}, g^I are bounded functions by Lemma A.1, an integration by parts leads to

$$I_2^I \leq Ck \left(\frac{1}{|s|+k} + \frac{1}{|s|-k} \right) + Ck \left| \int_{1/k}^{4R} \frac{e^{i(|s|+k)r}}{|s|+k} \partial_r (f^I(|s|r)g^I(kr)\mu(r)) + \frac{e^{i(k-|s|)r}}{k-|s|} \partial_r (f^{II}(|s|r)g^I(kr)\mu(r)) \right|.$$

Since $|s| \geq \lambda k$, Lemma A.1 provides the estimates

$$|\partial_r (f^I(sr)g^I(kr)\mu(r))| + |\partial_r (f^{II}(sr)g^I(kr)\mu(r))| \leq C \left(\frac{1}{R} + \frac{1}{kr^2} \right), \quad 1/k \leq r.$$

Combining these results, we arrive at

$$I_2^I \leq C \frac{1}{\lambda - 1}.$$

Observing $1 + (\lambda - 1)^{-1} \leq 2 + (\lambda^2 - 1)^{-1}$ allows us to conclude the proof. \square

3.3. Existence and uniqueness. Existence, uniqueness, and well-posedness of problem (2.6) has been studied in much more generality (concerning the assumption on the domain Ω) in [12] by using different techniques.

The main goal of the estimates which we have derived in the previous sections is their application to the proof of the discrete stability for the finite element discretization and the convergence rates. However, since existence, uniqueness, and well-posedness for our model problem are simple by-products we state them in passing.

Theorem 3.8. *Let B_R be a ball of radius $R > 0$. Then, there exists a constant $C(R, k) > 0$ such that for all $f \in (H^1(B_R))'$ the unique solution u of problem (2.6) satisfies*

$$\|u\|_{\mathcal{H}} \leq C(R, k) \|f\|_{H^1(B_R)'}$$

Proof. The coercivity of the bilinear form $a(u, v)$ follows from the compact embedding $H^1(B_R) \xrightarrow{c} L^2(B_R)$ and (3.4b), (3.4d):

$$\operatorname{Re} a(u, u) \geq \|u\|_{\mathcal{H}}^2 - 2k^2 \|u\|_{L^2(B_R)}^2 - \operatorname{Re} \int_{\partial B_R} (T_k u) \bar{u} \geq \|u\|_{\mathcal{H}}^2 - 2k^2 \|u\|_{L^2(B_R)}^2.$$

Next, we show uniqueness of the adjoint problem (see, e.g., [29, p. 43]):

$$\overline{a(v, u)} = 0 \quad \forall v \in H^1(B_R) \implies u = 0.$$

Let $u \in H^1(B_R)$ be a solution of the homogeneous adjoint problem. We choose $v = u$ and consider the imaginary part:

$$0 = \operatorname{Im} \overline{a(u, u)} = -\operatorname{Im} \int_{\partial B_R} (T_k u) \bar{u} = \operatorname{Im} \int_{\partial B_R} (T_k u) \bar{u}.$$

Lemma 3.3 implies $u = 0$ on ∂B_R . Hence, $u \in H_0^1(B_R)$ and satisfies

$$(3.45) \quad \int_{B_R} \langle \nabla u, \nabla \bar{v} \rangle = k^2 \int_{B_R} u \bar{v} \quad \forall v \in H^1(B_R).$$

This means in particular that $u \in H_0^1(B_R)$ is an eigenfunction of $(-\Delta)^{-1}$ with eigenvalue k^{-2} . However, for any domain $\tilde{\Omega} \supset B_R$, equation (3.45) implies that the extension

$$\tilde{u}(x) := \begin{cases} u(x) & x \in B_R, \\ 0 & x \notin B_R, \end{cases}$$

satisfies (3.45) with B_R replaced by $\tilde{\Omega}$, i.e., \tilde{u} is also an eigenfunction of $(-\Delta)^{-1}$ with eigenvalue k^{-2} on any domain $\tilde{\Omega} \supset B_R$. A simple scaling argument shows that this is impossible.

Thus, the assertion follows from the theory of Fredholm operators (see, e.g., [29, Theorem 2.4]). \square

Note that the proof of Theorem 3.8 does not provide how the constant $C(R, k)$ depends on the wave number. In [12], this question has been investigated in much more generality and, hence, will not be discussed here. The Fourier analysis which we developed in Section 3.2 gives explicit bounds on this constant provided the right-hand side is in $L^2(\Omega)$.

Lemma 3.9. *Let Ω be a bounded domain which is contained in the ball B_R for some R satisfying (3.3). For any $f \in L^2(\Omega)$ and $v := N_k f$, there holds*

$$\|v\|_{\mathcal{H}} \leq C \|f\|_{L^2(\Omega)},$$

where C only depends on k_0 and R_0 (cf. (2.2), (3.3)).

Proof. The radius of the minimal ball that contains Ω is denoted by R_Ω . If $4kR_\Omega > 1$, the estimate

$$\begin{aligned} \|v\|_{L^2(\Omega)} &\stackrel{(3.28)}{=} \|v_\mu\|_{L^2(\Omega)} \stackrel{(3.35),(3.36)}{\leq} (2\pi)^{\frac{d}{2}} \left(\max_{s \in \mathbb{R}_{\geq 0}} |s^{|\alpha|} \iota(s)| \right) \|f\|_{L^2(\Omega)} \\ &\stackrel{\text{Lemma 3.7(i)}}{\leq} (2\pi)^{\frac{d}{2}} \frac{R_\Omega}{k} \|f\|_{L^2(\Omega)} \end{aligned}$$

follows. The estimate

$$\|\nabla v\|_{L^2(\Omega)} \leq C \left(\frac{1}{k_0} + R_\Omega \right) \|f\|_{L^2(\Omega)}$$

follows by the same reasoning. If $\alpha < 4kR_\Omega \leq 1$, then $|\log kR_\Omega| \leq |\log \alpha|$. Hence, both estimates remain valid (cf. Lemma 3.7), possibly with a different constant C which, in addition, depend on α . \square

3.4. An adjoint problem. The operator N_k and the DtN operator T_k introduced in Section 2 are associated with the *outgoing* radiation condition. Adopting the notation Ω and Ω^+ of Section 2 and assuming $\text{supp } f \subset \Omega$, one can define a problem with *incoming* radiation conditions: Find $u \in H^1(\Omega)$ and $u^+ \in H^1_{loc}(\Omega^+)$ such that

$$(3.46) \quad \begin{aligned} & \begin{cases} (-\Delta - k^2) u = \bar{f} & \text{in } \Omega, \\ (-\Delta - k^2) u^+ = 0 & \text{in } \Omega^+, \end{cases} \\ & u = u^+ \quad \text{and} \quad \partial u / \partial n = \partial u^+ / \partial n \quad \text{on } \partial \Omega, \\ & \left| \frac{\partial u^+}{\partial r} + i k u^+ \right| = o\left(\|\mathbf{x}\|^{\frac{1-d}{2}}\right) \quad \|\mathbf{x}\| \rightarrow \infty. \end{aligned}$$

For $k > 0$, we see that the complex conjugate \bar{u} and $\overline{u^+}$ of the solution satisfy (2.3). By uniqueness, this allows us to identify the solution operator $N_k^* : L^2(\Omega) \rightarrow H^1(\Omega)$ for the u -component of the solution of (3.46), namely,

$$(3.47) \quad u = N_k^*(f) = \overline{N_k(\bar{f})} = \int_{\Omega} \overline{G_k(x-y)} \bar{f}(y) dy.$$

The solution component u^+ is related to a Dirichlet-to-Neumann map. For the incoming radiation condition, this operator is given by $T_k^* g := \gamma_1 w$, where $w \in H^1_{loc}(\Omega^+)$ solves:

$$(3.48) \quad \text{Find } w \in H^1_{loc}(\Omega^+) \text{ such that } \begin{cases} (-\Delta - k^2) w = 0 & \text{in } \Omega^+, \\ w = g & \text{on } \partial \Omega, \\ \left| \frac{\partial w}{\partial r} + i k w \right| = o\left(\|\mathbf{x}\|^{\frac{1-d}{2}}\right) & \|\mathbf{x}\| \rightarrow \infty. \end{cases}$$

Again by using $k > 0$ and complex conjugation, we note (again by uniqueness) the representation $T_k^* g = \overline{T_k \bar{g}}$. We employed the notation T_k^* since the operator T_k^* is the adjoint of T_k with respect to the $L^2(\partial B_R)$ inner product in the case of a ball:

Lemma 3.10. *Let $\Omega = B_R \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$. Then $\int_{\partial B_R} T_k u \bar{v} = \int_{\partial B_R} u \overline{T_k^* v}$ for all $u, v \in H^{1/2}(\partial B_R)$.*

Proof. We will only consider the case $d = 2$. We expand u and v as in (3.9) with coefficients $(u_\ell)_{\ell \in \mathbb{Z}}, (v_\ell)_{\ell \in \mathbb{Z}}$. For the calculations below, we assume that only finitely many coefficients u_ℓ, v_ℓ are non-zero; the generalization to $u, v \in H^{1/2}(\partial B_R)$ then follows by a density argument.

We see immediately from (3.10) that $w_\ell(r) = w_{-\ell}(r)$. From the orthogonality properties of functions $e^{i\ell\theta}$ with the representation of T_k in (3.10) we get

$$\int_{\partial B_R} u \overline{T_k^* v} = \int_{\partial B_R} u T_k \bar{v} = 2\pi \sum_{\ell \in \mathbb{Z}} u_\ell \bar{v}_\ell w_{-\ell}(kR) = 2\pi \sum_{\ell \in \mathbb{Z}} u_\ell \bar{v}_\ell w_\ell(kR) = \int_{\partial B_R} T_k u \bar{v}.$$

□

4. STABILITY AND CONVERGENCE ANALYSIS

This section is devoted to the analysis of the discrete problem (2.8) for the finite-dimensional space $S \subset H^1(\Omega)$; we will provide conditions on S under which unique solvability and quasi-optimality of (2.8) can be guaranteed.

We employ the generalization of the theory of [33] that has been developed in [39]. There, a measure of “almost invariance”² of the approximation space S under the solution operator of an adjoint Helmholtz problem has been introduced.

Adjoint Problem: The weak formulation of problem (2.5) corresponds to the sesquilinear form $a(\cdot, \cdot)$ as in (2.6), where Ω may be chosen as a ball B_R with sufficiently large radius R . The adjoint sesquilinear form $a^*(\cdot, \cdot)$ is defined by (see, e.g., [29, p. 43])

$$a^*(u, v) = \overline{a(v, u)}.$$

For given $f \in L^2(B_R)$, the corresponding adjoint problem is given by finding $z \in H^1(B_R)$ such that

$$(4.1) \quad a^*(z, v) = (v, f)_{L^2(B_R)} \quad \forall v \in H^1(B_R).$$

Explicitly, we have

$$a^*(z, v) = \int_{B_R} \langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v} - \int_{\partial B_R} u \overline{(T_k v)}.$$

From Lemma 3.10 we conclude that

$$a^*(z, v) = \int_{B_R} \langle \nabla u, \nabla \bar{v} \rangle - k^2 u \bar{v} - \int_{\partial B_R} T_k^* u \bar{v}.$$

The strong formulation of the adjoint problem is: Find z such that

$$(4.2) \quad -\Delta z - k^2 z = \bar{f} \quad \text{in } B_R, \quad \frac{\partial z}{\partial n} = T_k^* z \quad \text{on } \partial B_R.$$

Recalling the definition of T_k^* , we see that the solution z of this problem is given by the solution u of (3.46); the solution formula (3.47) therefore allows us to write the solution of (4.1) as

$$(4.3) \quad z = N_k^* f = \int_{\Omega} \overline{G_k}(x - y) \bar{f}(y) dy.$$

In view of $\bar{z} = N_k f$ and $\|z\|_{\mathcal{H}} = \|\bar{z}\|_{\mathcal{H}}$, we obtain from Lemma 3.9 the following observation:

Lemma 4.1. *Let Ω be a bounded Lipschitz domain and $k \geq k_0$. Then the constant*

$$(4.4) \quad \sup_{f \in L^2(\Omega) \setminus \{0\}} \frac{\|N_k^* f\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}} =: C_{\text{stab}} < \infty$$

is independent of k and depends solely on Ω .

For the stability of the discrete problem, the following adjoint approximation property plays a crucial role:

$$(4.5) \quad \eta(S) := \sup_{f \in L^2(\Omega) \setminus \{0\}} \inf_{v \in S} \frac{\|N_k^* f - v\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}}.$$

(Note that the quantity $\eta(S)$ was denoted in [39] by $\tilde{\eta}(S)$.)

²We slightly changed the definition here and denote the new quantity by “adjoint approximation property”.

4.1. Discrete stability. In this section, we will prove the discrete stability in the form of an inf-sup condition.

Theorem 4.2. *Let $\Omega = B_R$ be a ball with radius R and let the assumptions of Lemma 3.3 be satisfied. Assume that the space S is chosen such that*

$$(4.6) \quad k\eta(S) \leq \frac{1}{4C_c},$$

where C_c is defined in Corollary 3.4. Then, with C_{stab} defined in Lemma 4.1, the discrete inf-sup constant satisfies

$$\inf_{u \in S} \sup_{v \in S \setminus \{0\}} \frac{|a(u, v)|}{\|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}} \geq \frac{1}{2 + C_c^{-1} + 4kC_{\text{stab}}},$$

and this ensures existence and uniqueness of the discrete problem (2.8).

Proof. Let $u \in S$ and set $z := 2k^2 N_k^* u$. Then,

$$\begin{aligned} a(u, u + z) &= \left(\int_{B_R} \langle \nabla u, \nabla \bar{u} \rangle + k^2 |u|^2 - \int_{\partial B_R} (T_k u) \bar{u} \right) + a(u, z) - 2k^2 \int_{B_R} |u|^2 \\ &= \int_{B_R} \langle \nabla u, \nabla \bar{u} \rangle + k^2 |u|^2 - \int_{\partial B_R} (T_k u) \bar{u}. \end{aligned}$$

We derive from Lemma 3.3 that

$$\operatorname{Re} a(u, u + z) \geq \|u\|_{\mathcal{H}}^2.$$

Let $z_S \in S$ denote the best approximation of z with respect to the $\|\cdot\|_{\mathcal{H}}$ -norm. Then,

$$\begin{aligned} \operatorname{Re} a(u, u + z_S) &\geq \operatorname{Re} a(u, u + z) - |a(u, z - z_S)| \stackrel{\text{Cor. 3.4}}{\geq} \|u\|_{\mathcal{H}}^2 - C_c \|u\|_{\mathcal{H}} \|z - z_S\|_{\mathcal{H}} \\ &\geq \|u\|_{\mathcal{H}} \left(\|u\|_{\mathcal{H}} - 2k^2 C_c \eta(S) \|u\|_{L^2(B_R)} \right) \geq (1 - 2kC_c \eta(S)) \|u\|_{\mathcal{H}}^2. \end{aligned}$$

The stability of the continuous problem (cf. Lemma 4.1) implies

$$\begin{aligned} \|u + z_S\|_{\mathcal{H}} &\leq \|u\|_{\mathcal{H}} + \|z - z_S\|_{\mathcal{H}} + \|z\|_{\mathcal{H}} \\ &\leq \|u\|_{\mathcal{H}} + 2k^2 \eta(S) \|u\|_{L^2(B_R)} + 2k^2 C_{\text{stab}} \|u\|_{L^2(B_R)} \\ &\leq (1 + 2k\eta(S) + 2kC_{\text{stab}}) \|u\|_{\mathcal{H}} \end{aligned}$$

so that

$$\operatorname{Re} a(u, u + z_S) \geq \frac{1 - 2C_c k \eta(S)}{1 + 2k\eta(S) + 2kC_{\text{stab}}} \|u\|_{\mathcal{H}} \|u + z_S\|_{\mathcal{H}}.$$

Therefore, in view of the assumption (4.6), we have proved

$$\inf_{u \in S} \sup_{v \in S \setminus \{0\}} \frac{|a(u, v)|}{\|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}} \geq \frac{1}{2 + C_c^{-1} + 4kC_{\text{stab}}}. \quad \square$$

4.2. Convergence analysis. The convergence of the finite element discretization is proved by applying the theory as developed in [39] (see also [7, 33, 40], [8, Sec. 5.7]).

Theorem 4.3. *Let the assumptions of Theorem 4.2 be satisfied. Let u denote the solution of (2.6) and u_S its Galerkin approximation (cf. (2.8)).*

Then

$$(4.7) \quad \|u - u_S\|_{\mathcal{H}} \leq 2C_c \inf_{v \in S} \|u - v\|_{\mathcal{H}}.$$

The L^2 -error satisfies

$$\|u - u_S\|_{L^2(B_R)} \leq C_c \eta(S) \|u - u_S\|_{\mathcal{H}}.$$

Proof. In the first step, we will estimate the L^2 -error by the H^1 -error and employ the Aubin-Nitsche technique. The Galerkin error is denoted by $e = u - u_S$. We set $\psi := N_k^* e$ (cf. (4.3)) and denote by $\psi_S \in S$ the best approximation of ψ with respect to the \mathcal{H} -norm.

The L^2 -error can be estimated by

$$\begin{aligned} \|e\|_{L^2(B_R)}^2 &= a(e, \psi) \leq a(e, \psi - \psi_S) \leq C_c \|e\|_{\mathcal{H}} \|\psi - \psi_S\|_{\mathcal{H}} \\ (4.8) \qquad &\leq C_c \eta(S) \|e\|_{\mathcal{H}} \|e\|_{L^2(B_R)}, \end{aligned}$$

i.e.,

$$(4.9) \qquad \|e\|_{L^2(B_R)} \leq C_c \eta(S) \|e\|_{\mathcal{H}}.$$

To estimate the \mathcal{H} -norm of the error we proceed as follows. Note that (3.4b), (3.4d) imply

$$(4.10) \qquad \operatorname{Re} (T_k u, u)_{L^2(\partial B_R)} \leq 0.$$

Hence, for any $v_S \in S$,

$$\begin{aligned} \|e\|_{\mathcal{H}}^2 &= \operatorname{Re} (a(e, e)) + \left\{ \|e\|_{\mathcal{H}}^2 - \operatorname{Re} a(e, e) \right\} \\ &= \operatorname{Re} a(e, u - v_S) + 2k^2 \|e\|_{L^2(B_R)}^2 + \operatorname{Re} \int_{\partial B_R} (T_k e) \bar{e} \\ (4.9), (4.10), k\|e\|_{L^2} \leq \|e\|_{\mathcal{H}} &\leq C_c \|e\|_{\mathcal{H}} \|u - v_S\|_{\mathcal{H}} + 2k C_c \eta(S) \|e\|_{\mathcal{H}}^2. \end{aligned}$$

Noting that (4.6) implies $2k C_c \eta(S) \leq 1/2$ we arrive at the final estimate $\|e\|_{\mathcal{H}} \leq 2C_c \|u - v_S\|_{\mathcal{H}}$. \square

5. EXAMPLE: hp -FEM

Theorems 4.2, 4.3 show quasi-optimality of arbitrary approximation spaces under the assumption (4.6) on the adjoint approximation property $\eta(S)$. However, for concrete finite element spaces, or generalizations thereof, the verification of condition (4.6) is far from trivial. The purpose of this section is two-fold: First, we show that for classical higher order FEM spaces the assumption (4.6) can be met under a relatively mild condition on the local polynomial order of the classical FEM space; in particular, we will demonstrate that for spaces consisting of piecewise polynomials of degree p on quasi-uniform meshes that satisfy the side condition $p \geq c \ln k$, the key condition (4.6) is satisfied. Second, we derive conditions on the approximation space that may be easier to ascertain in practice than the condition (4.6).

In view of the fact that the circle (in 2D) and the sphere (in 3D) are relevant geometries for our theory (recall that Theorems 4.2, 4.3 have been shown for circles/spheres), we consider triangulations with curved elements that permit inclusion of these geometries. Before formulating the conditions on the mesh in an abstract way, we give an example of a typical construction.

Example 5.1 (Patchwise construction of FE mesh.). Let Ω denote a bounded domain.

1. We assume that there exists a polyhedral (polygonal in 2D) domain $\tilde{\Omega}$ along with a bi-Lipschitz mapping $\chi : \tilde{\Omega} \rightarrow \Omega$. Let $\tilde{\mathcal{T}}^{\text{macro}} = \{\tilde{K}_i^{\text{macro}} : 1 \leq i \leq q\}$ denote a conforming finite element mesh for $\tilde{\Omega}$ consisting of simplices which are regular in the sense of [13]. $\tilde{\mathcal{T}}^{\text{macro}}$ is considered as a coarse partition of $\tilde{\Omega}$, i.e., the diameters of the elements in $\tilde{\mathcal{T}}^{\text{macro}}$ are of order 1. We assume that the restrictions $\chi_i := \chi|_{\tilde{K}_i^{\text{macro}}}$ are *analytic* for all $1 \leq i \leq q$.
2. The finite element mesh with step size h is generated by refining the mesh $\tilde{\mathcal{T}}^{\text{macro}}$ in a standard way (e.g., in 2D, by connecting the midpoints of the triangle edges) and denoted by $\tilde{\mathcal{T}}_h = \{\tilde{K}_i : 1 \leq i \leq N\}$. The corresponding finite element mesh for Ω is then defined by $\mathcal{T}_h = \{K = \chi(\tilde{K}) : \tilde{K} \in \tilde{\mathcal{T}}_h\}$.

Note that, for any $K = \chi(\tilde{K}) \in \mathcal{T}_h$, there exists an affine bijection $A_K : \hat{K} \rightarrow \tilde{K}$ which maps the *reference element* $\hat{K} := \{x \in (\mathbb{R}_{\geq 0})^d : \sum_{i=1}^d x_i \leq 1\}$ to the simplex \tilde{K} . A parametrization $F_K : \hat{K} \rightarrow K$ can be chosen by $F_K := R_K \circ A_K$, where $R_K := \chi|_{\tilde{K}}$ is independent of the mesh width $h := \max\{\text{diam } K : K \in \mathcal{T}_h\}$.

To formulate the smoothness and scaling assumptions on R_K and A_K in an abstract way we have to introduce some notation first. For a function $u : \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, we write

$$(5.1) \quad |\nabla^n u(x)|^2 = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha|=n} \frac{n!}{\alpha!} |D^\alpha u(x)|^2.$$

For later purposes, we recall the multinomial formula and a simple fact that follows from the Cauchy-Schwarz inequality for sums:

$$(5.2) \quad \frac{d^n}{n!} = \sum_{\alpha \in \mathbb{N}_0^d : |\alpha|=n} \frac{1}{\alpha!},$$

$$(5.3) \quad \sum_{\alpha \in \mathbb{N}_0^d : |\alpha|=n} \frac{1}{\alpha!} |D^\alpha u(x)| \leq \frac{1}{n!} d^{n/2} |\nabla^n u(x)|.$$

Assumption 5.2 (quasi-uniform regular triangulation). Each element map F_K can be written as $F_K = R_K \circ A_K$, where A_K is an affine map and the maps R_K and A_K satisfy for constants $C_{\text{affine}}, C_{\text{metric}}, \gamma > 0$ independent of h :

$$\begin{aligned} \|A'_K\|_{L^\infty(\hat{K})} &\leq C_{\text{affine}} h, & \|(A'_K)^{-1}\|_{L^\infty(\hat{K})} &\leq C_{\text{affine}} h^{-1}, \\ \|(R'_K)^{-1}\|_{L^\infty(\tilde{K})} &\leq C_{\text{metric}}, & \|\nabla^n R_K\|_{L^\infty(\tilde{K})} &\leq C_{\text{metric}} \gamma^n n! \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

Here, $\tilde{K} = A_K(\hat{K})$.

For meshes \mathcal{T}_h satisfying Assumption 5.2 with element maps F_K we denote the usual space of piecewise (mapped) polynomials by

$$S^{p,1}(\mathcal{T}_h) := \{u \in H^1(\Omega) \mid \forall K \in \mathcal{T}_h : u|_K \circ F_K \in \mathcal{P}_p\},$$

where \mathcal{P}_p denotes the space of polynomials of degree p . It is desirable to construct an approximant $Iu \in S^{p,1}(\mathcal{T}_h)$ of a given (sufficiently smooth) function u in an elementwise fashion. The C^0 -continuity of an elementwise defined approximant Iu is most conveniently ensured if Iu is defined in such a way that for every topological

entity E of the mesh (i.e., E is an element K , a face f , an edge e , or a vertex V) the restriction $(Iu)|_E$ is fully determined by $u|_{\overline{E}}$. There are many ways of realizing this construction principle. The construction employed in the present paper is based on the following concept:

Definition 5.3 (element-by-element construction). Let \widehat{K} be the reference simplex in \mathbb{R}^d , $d \in \{2, 3\}$. A polynomial π is said to permit an *element-by-element* construction of polynomial degree p for $u \in H^s(\widehat{K})$, $s > d/2$, if:

- (i) $\pi(V) = u(V)$ for all $d + 1$ vertices V of \widehat{K} ,
- (ii) for every edge e of \widehat{K} , the restriction $\pi|_e \in \mathcal{P}_p$ is the unique minimizer of

$$(5.4) \quad \pi \mapsto p^{1/2} \|u - \pi\|_{L^2(e)} + \|u - \pi\|_{H_0^{1/2}(e)}$$

under the constraint that π satisfies (5.3); here the Sobolev norm $H_0^{1/2}$ is defined in (B.1).

- (iii) (for $d = 3$) for every face f of \widehat{K} , the restriction $\pi|_f \in \mathcal{P}_p$ is the unique minimizer of

$$(5.5) \quad \pi \mapsto p \|u - \pi\|_{L^2(f)} + \|u - \pi\|_{H^1(f)}$$

under the constraint that π satisfies (5.3), (5.3) for all vertices and edges of the face f .

Remark 5.4. The conditions of Definition 5.3 are a variation of similar proposals in the literature, e.g., [14] and [21]. For example, the effective difference between the projection-based interpolation of [14] and the present construction lies in the choice of the norms employed in the minimization process in Definition 5.3. Our motivation for formulating the conditions in Definition 5.3 is that they permit us in Appendix B to construct approximation operators with optimal *simultaneous* approximation properties in L^2 and H^1 . Previously, the literature had focused on H^1 -approximation alone.

We are now in position to show that the solution $v = N_k^* f$ can be approximated well by the FEM space $S^{p,1}(\mathcal{T}_h)$ provided that kh/p is sufficiently small and $p \geq c \ln k$.

Theorem 5.5. *Let $d \in \{1, 2, 3\}$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain. Then there exist constants $C, \sigma > 0$ that depend solely on the constants appearing in Assumption 5.2 such that for every $f \in L^2(\Omega)$ the function $v := N_k^* f$ satisfies*

$$\inf_{w \in S^{p,1}(\mathcal{T}_h)} k \|v - w\|_{\mathcal{H}} \leq C \|f\|_{L^2(\Omega)} \left(1 + \frac{kh}{p} \right) \left\{ \frac{kh}{p} + k \left(\frac{kh}{\sigma p} \right)^p \right\}.$$

Proof. We will only prove the cases $d \in \{2, 3\}$. The case $d = 1$ follows by similar arguments where the appeal to Theorem B.4 and Lemma C.3 is replaced with that to [41, Thm. 3.17].

We note $v = N_k^* f = \overline{N_k f}$, fix $\lambda > 1$ in Lemma 3.5, and split with its aid $v = v_{H^2} + v_A$ with $v_{H^2} \in H^2(\Omega)$ and v_A analytic; we have the following bounds:

$$\|v_{H^2}\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}, \quad \|\nabla^p v_A\|_{L^2(\Omega)} \leq C (\lambda k)^{p-1} \|f\|_{L^2(\Omega)} \quad \forall p \in \mathbb{N}_0.$$

We approximate v_{H^2} and v_A separately. Theorem B.4 and a scaling argument provides an approximant $w_{H^2} \in S^{p,1}(\mathcal{T}_h)$ such that for every $K \in \mathcal{T}_h$ we have, for

$q = 0, 1,$

$$\|v_{H^2} - w_{H^2}\|_{H^q(K)} \leq C \left(\frac{h}{p}\right)^{2-q} \|v_{H^2}\|_{H^2(K)} \quad \forall K \in \mathcal{T}_h.$$

Hence, by summation over all elements, we arrive at

$$k\|v_{H^2} - w_{H^2}\|_{\mathcal{H}} \leq C \left(\frac{kh}{p} + \left(\frac{kh}{p}\right)^2\right) \|f\|_{L^2(\Omega)}.$$

We now turn to the approximation of $v_{\mathcal{A}}$. Again, we construct the approximation $w_{\mathcal{A}} \in S^{p,1}(\mathcal{T}_h)$ in an element-by-element fashion. We start by defining for each element $K \in \mathcal{T}_h$ the constant C_K by

$$(5.6) \quad C_K^2 := \sum_{p \in \mathbb{N}_0} \frac{\|\nabla^p v_{\mathcal{A}}\|_{L^2(K)}^2}{(2\lambda k)^{2p}}$$

and we note

$$(5.7) \quad \|\nabla^p v_{\mathcal{A}}\|_{L^2(K)} \leq (2\lambda k)^p C_K \quad \forall p \in \mathbb{N}_0,$$

$$(5.8) \quad \sum_{K \in \mathcal{T}_h} C_K^2 \leq \frac{4}{3} \left(\frac{C}{\lambda k}\right)^2 \|f\|_{L^2(\Omega)}^2.$$

Let the element map for K be $F_K = R_K \circ A_K$. Lemma C.1 gives that the function $\tilde{v} := v_{\mathcal{A}}|_K \circ R_K$ satisfies, for suitable constants C, \tilde{C} (which depend additionally on the constants describing the analyticity of the element maps R_K),

$$\|\nabla^p \tilde{v}\|_{L^2(\tilde{K})} \leq C \tilde{C}^p \max\{p, k\}^p C_K \quad \forall p \in \mathbb{N}_0.$$

Since A_K is affine, the function $\hat{v} := v_{\mathcal{A}}|_K \circ F_K = \tilde{v} \circ A_K$ therefore satisfies

$$\|\nabla^p \hat{v}\|_{L^2(\hat{K})} \leq Ch^{-d/2} \tilde{C}^p h^p \max\{p, k\}^p C_K \quad \forall p \in \mathbb{N}_0.$$

Hence, the assumptions of Lemma C.3 (with $R = 1$ there) are satisfied, and we get an approximation w on the element K by lifting an element-by-element construction on \hat{K} to K via F_K which satisfies for $q \in \{0, 1\}$,

$$\|v_{\mathcal{A}} - w\|_{H^q(K)} \leq Ch^{d/2-q} h^{-d/2} C_K \left\{ \left(\frac{h}{h+\sigma}\right)^{p+1} + \left(\frac{kh}{\sigma p}\right)^{p+1} \right\}.$$

Summation over all elements $K \in \mathcal{T}_h$ gives

$$(5.9) \quad \|v_{\mathcal{A}} - w\|_{\mathcal{H}}^2 \leq \left[\left(\frac{h}{h+\sigma}\right)^{2p} + k^2 \left(\frac{h}{h+\sigma}\right)^{2p+2} + \frac{k^2}{p^2} \left(\frac{kh}{\sigma p}\right)^{2p} + k^2 \left(\frac{kh}{\sigma p}\right)^{2p+2} \right] \sum_{K \in \mathcal{T}_h} C_K^2.$$

The combination of (5.9) and (5.8) yields

$$k\|v_{\mathcal{A}} - w\|_{\mathcal{H}} \leq C \left[\left(\frac{h}{h+\sigma}\right)^p \left(1 + \frac{hk}{h+\sigma}\right) + k \left(\frac{kh}{\sigma p}\right)^p \left(\frac{1}{p} + \frac{kh}{\sigma p}\right) \right] \|f\|_{L^2(\Omega)}.$$

Furthermore, we estimate using $h \leq \text{diam } \Omega$ and $\sigma > 0$ (independent of h)

$$\begin{aligned} \left(\frac{h}{h+\sigma}\right)^p \left(1 + \frac{kh}{\sigma+h}\right) &\leq Ch(1+kh) \left(\frac{h}{\sigma+h}\right)^{p-1} \\ &\leq Ch(1+kh)p^{-2} \leq C\frac{h}{p} \left(\frac{1}{p} + \frac{kh}{p}\right). \end{aligned}$$

We therefore arrive at

$$k\|v_{\mathcal{A}} - w\|_{\mathcal{H}} \leq C \left(\frac{1}{p} + \frac{kh}{p}\right) \left[\frac{kh}{p} + k \left(\frac{kh}{\sigma p}\right)^p\right] \|f\|_{L^2(\Omega)},$$

which completes the proof of the theorem. □

Combining Theorems 5.5, 4.3 produces the condition (1.1) for quasi-optimality of the hp -FEM announced in the Introduction. We extract from Theorem 5.5 that quasi-optimality of the h -version FEM can be achieved under the side condition that $p \geq C \log k$:

Corollary 5.6. *Let $\Omega = B_R$ be a ball of radius R and assume (3.3), (2.2) with the additional condition $k_0 \geq 1$ in the case $d = 2$. Let Assumption 5.2 be valid. Then there exist constants $c_1, c_2 > 0$ independent of k, h , and p such that (4.6) is implied by the following condition:*

$$(5.10) \quad \frac{kh}{p} \leq c_1 \quad \text{together with} \quad p \geq c_2 \ln k.$$

Alternatively, the discrete stability follows from

$$(5.11) \quad p = O(1) \text{ fixed independent of } k \text{ and } kh + k(kh)^p \leq C,$$

which is understood as a condition on the maximal step size h .

Proof. Theorem 5.5 implies

$$k\eta(S) \leq C \left(1 + \frac{kh}{p}\right) \left(\frac{kh}{p} + k \left(\frac{kh}{\sigma p}\right)^p\right).$$

The right-hand side needs to be bounded by $1/C_c$. It is now easy to see that we can select c_1, c_2 such that this can be ensured. □

An easy consequence of the stability result Corollary 5.6 is:

Corollary 5.7. *Let the assumptions of Corollary 5.6 be satisfied and let (5.10) or (5.11) hold. Then, the Galerkin solution u_S exists and satisfies the error estimate*

$$\|u - u_S\|_{\mathcal{H}} \leq C_c \left(\frac{h}{p} + \left(\frac{kh}{\sigma p}\right)^p\right) \|f\|_{L^2(\Omega)}.$$

Remark 5.8. To the best of the authors' knowledge, discrete stability in 2D and 3D has only been shown under much more restrictive conditions than (5.10), e.g., the condition $k^2h \lesssim 1$. Even in one dimension, condition (5.10) improves the stability condition $kh \lesssim 1$ that was required in [27].

Finally, we reformulate Theorem 5.5 by deriving the statement under some conditions on abstract approximation spaces that may be easier to verify than a direct proof of (4.6).

The key step in Theorem 5.5 is the ability to decompose $v = N_k^* f$ into an analytic, but oscillatory part and an H^2 -regular part and to approximate each part separately. This gives rise to the definition of two types of approximation properties.

Definition 5.9. For given $\gamma > 0$ and $k > 0$ let

$$\mathcal{H}^{\text{osc}}(\gamma, k) := \left\{ v \in H^1(\Omega) \mid \|\nabla^p v\|_{L^2(\Omega)} \leq (\gamma k)^{p-1} \quad \forall p \in \mathbb{N}_0 \right\},$$

$$\mathcal{H}^{H^2} := \left\{ v \in H^2(\Omega) \mid \|v\|_{H^2(\Omega)} \leq 1 \right\}.$$

Let $S \subset H^1(\Omega)$ be the (possibly k -dependent) finite dimensional approximation space for the Galerkin method. The approximation properties for the oscillatory and the H^2 -part are

$$(5.12) \quad \eta_{\mathcal{A}}(S, k, \gamma) := \sup_{v \in \mathcal{H}^{\text{osc}}(\gamma, k)} \inf_{w \in S} \|v - w\|_{\mathcal{H}},$$

$$\eta_{H^2}(S) := \sup_{v \in \mathcal{H}^{H^2}} \inf_{w \in S} \|v - w\|_{\mathcal{H}}.$$

The decomposition Lemma 3.5 allows us to recast $\eta(S)$ in terms of $\eta_{\mathcal{A}}(S, k, \gamma)$ and $\eta_{H^2}(S)$:

Lemma 5.10. Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a bounded domain and select $\lambda > 1$. Set $\gamma := \sqrt{d}\lambda$ and define

$$C_{H^2} := \sup_{f \in L^2(\Omega)} \frac{\|v_{H^2}\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}}, \quad C_{\mathcal{A}} := \sup_{f \in L^2(\Omega)} \sup_{p \in \mathbb{N}_0} \frac{\|\nabla^p v_{\mathcal{A}}\|_{L^2(\Omega)}}{(\gamma k)^{p-1} \|f\|_{L^2(\Omega)}}$$

where, for each $f \in L^2(\Omega)$ we employ the λ -dependent decomposition $N_k^* f = v_{H^2} + v_{\mathcal{A}}$ according to Lemma 3.5. Let $S \subset H^1(\Omega)$ be a finite dimensional approximation space. Then, the adjoint approximability $\eta(S)$ is bounded by

$$\eta(S) \leq C_{\mathcal{A}} \eta_{\mathcal{A}}(S, k, \gamma) + C_{H^2} \eta_{H^2}(S).$$

Before proving this statement, we stress that the scaling in Definition 5.9 has been chosen such that, according to Lemma 3.5, the constants $C_{\mathcal{A}}$ and C_{H^2} are bounded uniformly in k .

Proof. For $f \in L^2(\Omega)$, we employ the splitting $v = N_k^* f = v_{H^2} + v_{\mathcal{A}}$ as in Lemma 3.5 for the selected $\lambda > 1$. We set

$$\widetilde{v}_{H^2} := \begin{cases} 0 & \text{if } f = 0, \\ \frac{v_{H^2}}{C_{H^2} \|f\|_{L^2(\Omega)}} & \text{if } f \neq 0, \end{cases} \quad \text{and} \quad \widetilde{v}_{\mathcal{A}} := \begin{cases} 0 & \text{if } f = 0, \\ \frac{v_{\mathcal{A}}}{C_{\mathcal{A}} \|f\|_{L^2(\Omega)}} & \text{if } f \neq 0, \end{cases}$$

and note $\widetilde{v}_{H^2} \in \mathcal{H}^{H^2}$ and $\widetilde{v}_{\mathcal{A}} \in \mathcal{H}^{\text{osc}}(\gamma, k)$. Then,

$$\begin{aligned} \eta(S) &= \sup_{f \in L^2(\Omega) \setminus \{0\}} \inf_{w \in S} \frac{\|v_{\mathcal{A}} + v_{H^2} - w\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}} \\ &\leq \sup_{f \in L^2(\Omega) \setminus \{0\}} \inf_{w \in S} \frac{\|v_{\mathcal{A}} - w\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}} + \sup_{f \in L^2(\Omega) \setminus \{0\}} \inf_{w \in S} \frac{\|v_{H^2} - w\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}} \\ &\leq \sup_{f \in L^2(\Omega) \setminus \{0\}} \frac{\|v_{\mathcal{A}}\|_{\mathcal{H}} / \|\widetilde{v}_{\mathcal{A}}\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}} \inf_{w \in S} \|\widetilde{v}_{\mathcal{A}} - w\|_{\mathcal{H}} \\ &\quad + \sup_{f \in L^2(\Omega) \setminus \{0\}} \frac{\|v_{H^2}\|_{\mathcal{H}} / \|\widetilde{v}_{H^2}\|_{\mathcal{H}}}{\|f\|_{L^2(\Omega)}} \inf_{w \in S} \|\widetilde{v}_{H^2} - w\|_{\mathcal{H}} \\ &\leq C_{\mathcal{A}} \eta_{\mathcal{A}}(S) + C_{H^2} \eta_{H^2}(S). \end{aligned} \quad \square$$

APPENDIX A. ESTIMATE OF BESSEL FUNCTIONS

In this appendix we derive some estimates for the Hankel and Bessel functions that are used in Subsection 3.2. First, we will consider the case of large arguments $z > 1$ and then the case $0 < z \leq 1$.

Case 1: $z = kr > 1$.

From [1, 9.2.5-9.2.16], we conclude that the Hankel functions $H_\ell^{(1)}$ and Bessel functions J_ℓ , $\ell \in \mathbb{N}_0$, can be written in the form

$$(A.1a) \quad J_\ell(z) \stackrel{[1, 9.2.5]}{=} \sqrt{\frac{2}{\pi z}} (P_\ell(z) \cos \chi - Q_\ell(z) \sin \chi),$$

$$(A.1b) \quad H_\ell^{(1)}(z) \stackrel{[1, 9.2.7]}{=} \sqrt{\frac{2}{\pi z}} (P_\ell(z) + i Q_\ell(z)) e^{i\chi},$$

where $\chi := z - \pi/4$. The functions P_ℓ, Q_ℓ have the following property: Upon defining

$$(A.1c) \quad P_{\ell,m}(z) := \sum_{k=0}^m \frac{\beta_{\ell,2k}}{z^{2k}},$$

$$(A.1d) \quad Q_{\ell,m}(z) = -i \sum_{k=0}^m \frac{\beta_{\ell,2k+1}}{z^{2k+1}}$$

with

$$\beta_{\ell,k} := \frac{i^k \gamma_{\ell,k}}{2^{3k} k!} \quad \text{and} \quad \gamma_{\ell,m} \text{ as in (3.12)}$$

there holds

$$\forall z > 0 \quad \forall m > \frac{\ell}{2} - \frac{1}{4} \quad \begin{cases} |(P_\ell - P_{\ell,m-1})(z)| \leq \frac{|\gamma_{\ell,2m}|}{2^{6m} (2m)!} \frac{1}{z^{2m}}, \\ |(Q_\ell - Q_{\ell,m-1})(z)| \leq \frac{\gamma_{\ell,2m+1}}{2^{6m+2} (2m+1)!} \frac{1}{z^{2m+1}}. \end{cases}$$

Note that in Subsection 3.2 the order ℓ is always small, i.e., $\ell \in \{0, 1\}$ and, hence, we do not analyze the dependence of the constants on ℓ in the following estimates.

We conclude that

$$(A.2a) \quad \forall z \geq 1 : |P_\ell(z)| \leq \left| P_{\ell, \lceil \frac{\ell}{2} \rceil - 1}(z) \right| + \frac{|\gamma_{\ell, 2\lceil \frac{\ell}{2} \rceil}|}{2^{6\lceil \frac{\ell}{2} \rceil} (2\lceil \frac{\ell}{2} \rceil)!} \frac{1}{z^{2\lceil \frac{\ell}{2} \rceil}} \leq C,$$

and similarly,

$$(A.2b) \quad \forall z \geq 1 : |Q_\ell(z)| \leq \frac{C}{z}, \quad |P'_\ell(z)| \leq \frac{C}{z^3}, \quad |Q'_\ell(z)| \leq \frac{C}{z^2}.$$

Hence, for $f \in \{J_\ell, H_\ell^{(1)}\}$, $\ell \in \mathbb{N}_0$, there holds

$$(A.3a) \quad \forall z \geq 1 : |f(z)| \leq \frac{C}{\sqrt{z}},$$

and the combination with $|J_\ell(z)| \stackrel{[1, 9.1.60]}{\leq} C$ for all $z \geq 0$ yields

$$(A.3b) \quad \forall z \geq 0 : |J_\ell(z)| \leq C \sqrt{\frac{1}{1+z}}.$$

We need an estimate of the derivative at the argument $z = kr$ for $z \geq 1$. The derivative of (A.1b) can be written in the form

$$(A.4) \quad \frac{d}{dr} H_0^{(1)}(kr) \stackrel{[1, 9.2.7]}{=} C e^{ikr} \sqrt{\frac{1}{kr}} \frac{d}{dr} (P_0(kr) + i Q_0(kr)) + C (P_0(kr) + i Q_0(kr)) \frac{d}{dr} \left(e^{ikr} \sqrt{\frac{1}{kr}} \right).$$

The combination of (A.4) and (A.2) leads to

$$(A.5) \quad \left| \frac{d}{dr} H_0^{(1)}(kr) \right| \leq C \left(\frac{1}{r\sqrt{kr}} + \sqrt{\frac{k}{r}} \right).$$

We also need an estimate of $\partial_r (e^{-ikr} H_0^{(1)}(kr))$. Employing (A.1b) we obtain

$$\frac{d}{dr} (e^{-ikr} H_0^{(1)}(kr)) = \sqrt{\frac{2}{\pi}} e^{-\pi/4} \frac{d}{dr} \left(\sqrt{\frac{1}{kr}} (P_0(kr) + i Q_0(kr)) \right).$$

Thus, for $kr \geq 1$, we get

$$(A.6) \quad \left| \partial_r (e^{-ikr} H_0^{(1)}(kr)) \right| \leq \frac{C}{r\sqrt{kr}}.$$

An estimate of the second derivative of $H_0^{(1)}$ is derived by using [1, 9.1.27, 9.1.28]:

$$(A.7) \quad \left| \frac{d^2}{dr^2} H_0^{(1)}(kr) \right| = k^2 \left| -H_0^{(1)}(kr) + \frac{H_1^{(1)}(kr)}{kr} \right| \stackrel{(A.3a)}{\leq} Ck\sqrt{\frac{k}{r}}.$$

Case 2: $z = kr \in (0, 1)$.

To estimate $H_0^{(1)}(z)$ in the range $(0, 1)$ we employ

$$H_0^{(1)}(z) = J_0(z) + iY_0(z)$$

and use for $Y_0(z)$ the expansion

$$Y_0(z) = \frac{2}{\pi} \left(\log \frac{z}{2} \right) J_0(z) - \frac{2}{\pi} \sum_{k=0}^{\infty} \psi(k+1) \frac{\left(-\frac{z^2}{4} \right)^k}{(k!)^2},$$

where

$$\psi(n) := -\gamma + \sum_{k=1}^{n-1} k^{-1} \quad \text{and} \quad \gamma := 0.57721566\dots \text{ is Euler's constant.}$$

For $0 \leq z \leq 1$, we have

$$|Y_0(z)| \leq \frac{2}{\pi} \left| \log \frac{z}{2} \right| + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\psi(k+1)}{4^k (k!)^2}.$$

Furthermore,

$$|\psi(k+1)| \leq \gamma + 1 + \sum_{s=2}^k \frac{1}{s} \leq \gamma + 1 + \int_1^k \frac{1}{x} dx = \gamma + 1 + \log k =: \gamma' + \log k.$$

Thus, for $0 \leq z \leq 1$, we have

$$|Y_0(z)| \leq \frac{2}{\pi} \left| \log \frac{z}{2} \right| + \frac{2}{\pi} \left(\gamma + \sum_{k=1}^{\infty} \frac{\gamma' + \log k}{4^k (k!)^2} \right)$$

Since $\frac{\gamma' + \log k}{4^k k!} \leq 1$ we get

$$|Y_0(z)| \leq \frac{2}{\pi} |\log z| + C.$$

This leads to the estimate:

$$(A.3c) \quad \forall z \in]0, 1] : \quad |H_0^{(1)}(z)| \leq \frac{2}{\pi} |\log z| + C.$$

The combination with (A.3a) finally results in

$$(A.3d) \quad |H_0^{(1)}(z)| \leq \min \left\{ \frac{2}{\pi} |\log z| + C, \frac{C}{\sqrt{z}} \right\}.$$

We will need further estimates of J_1 and $\partial_r H_0^{(1)}$. From [1, 9.1.60], we conclude that

$$(A.8) \quad \forall z \geq 0 : \quad J_1(z) \leq 1/\sqrt{2}.$$

For the derivative of Y_0 , we obtain (by using $J_0' = -J_1$)

$$(A.9) \quad Y_0'(z) = \frac{2}{\pi} \left(\frac{J_0(z)}{z} - J_1(z) \log \frac{z}{2} \right) + \frac{z}{\pi} \sum_{k=0}^{\infty} \psi(k+2) \frac{\left(-\frac{z^2}{4}\right)^k}{k!(k+1)!}.$$

For $0 \leq z \leq 1$, we obtain

$$\frac{z}{\pi} \sum_{k=0}^{\infty} \psi(k+2) \frac{\left(-\frac{z^2}{4}\right)^k}{k!(k+1)!} \leq \frac{z}{\pi} \sum_{k=0}^{\infty} \frac{\gamma' + \log(k+1)}{k!4^k(k+1)!} \leq \frac{z}{\pi} \sum_{k=0}^{\infty} \frac{1}{k!} = Cz.$$

Now,

$$(A.10) \quad |J_1(z)| \stackrel{[1, 9.1.62]}{\leq} z/2,$$

and we get

$$Y_0'(z) \leq \frac{2}{\pi} \left(z^{-1} + \frac{z}{2} \log \frac{z}{2} + \frac{ez}{2} \right) \leq \frac{2}{\pi z} + C.$$

Hence, for $0 \leq r \leq 1/k$, we arrive at

$$(A.11) \quad \left| \partial_r H_0^{(1)}(kr) \right| = k (|J_0'(kr)| + |Y_0'(kr)|) \leq \frac{2}{\pi r} + Cr + \frac{k^2 r}{2} \leq C \left(\frac{1}{r} + \frac{k^2 r}{2} \right).$$

In addition, we need some weighted estimates for second order derivatives of $H_0^{(1)}$. From (A.9) we obtain

$$\partial_r (r \partial_r Y_0(kr)) = \frac{2k}{\pi} \left(-2J_1(kr) - kr \log \frac{kr}{2} J_0(kr) + kr \sum_{k=0}^{\infty} \psi(k+2) \frac{\left(-\frac{(kr)^2}{4}\right)^k}{(k!)^2} \right).$$

This leads to the estimate, for $0 < z \leq 1$,

$$|\partial_r (r \partial_r Y_0(kr))| \leq \frac{2}{\pi} k^2 r \left(C + \left| \log \frac{kr}{2} \right| \right).$$

Note that

$$\partial_r (r \partial_r H_0^{(1)}(kr)) = -rk^2 J_0(kr) + i \partial_r (r \partial_r Y_0(kr)),$$

and, hence,

$$(A.12) \quad \forall 0 \leq kr \leq 1 : \quad \left| \partial_r \left(r \partial_r H_0^{(1)}(kr) \right) \right| \leq \frac{2}{\pi} k^2 r \left(C + \left| \log \frac{kr}{2} \right| \right).$$

We finally state a lemma required for the proof of Lemma 3.7:

Lemma A.1. *Let $|s| \geq k$ and $k \geq k_0 > 0$. Then*

$$r J_0(sr) H_0^{(1)}(kr) = \frac{2}{\pi \sqrt{k|s|}} \left\{ e^{i(|s|+k)r} f^I(|s|r) + e^{i(-|s|+k)r} f^{II}(|s|r) \right\} g^I(kr),$$

where the functions f^I, f^{II}, g^I satisfy for $r \geq 1$ and a $C > 0$ depending solely on k_0 :

$$\begin{aligned} |f^I(r)| + |f^{II}(r)| + |g^I(r)| &\leq C, \\ r^2 \left(\left| \frac{d}{dr} f^I(r) \right| + \left| \frac{d}{dr} f^{II}(r) \right| + \left| \frac{d}{dr} g^I(r) \right| \right) &\leq C. \end{aligned}$$

Proof. By symmetry of J_0 , we may assume $s > 0$. Formulas (A.1a), (A.1b) imply the stated representation with $f^I(sr) = \frac{1}{2} (P_0(sr) + i Q_0(sr)) e^{-i\pi/4}$, $f^{II}(sr) = \frac{1}{2} (P_0(sr) - i Q_0(sr)) e^{i\pi/4}$, and $g^I(kr) = (P_0(kr) + i Q_0(kr)) e^{i\pi/4}$. The estimates for f^I, f^{II}, g^I now follow from the bounds for P_0, Q_0, P'_0, Q'_0 given in (A.2a), (A.2b). \square

APPENDIX B. APPROXIMATION BY *hp*-FINITE ELEMENTS.
CASE I: FINITE REGULARITY

The purpose of the present section is the proof of Theorem B.4, which constructs a polynomial approximation to a function $u \in H^s, s > d/2$, in an element-by-element fashion (see Definition 5.3). The novelty of the present construction over existing operators such as those of [3], [14], [34] is that we obtain optimal rates (in p) *simultaneously* in the H^1 -norm and the L^2 -norm. Closely related results can be found in the recent paper [21], where similar duality arguments are employed to obtain estimates in L^2 .

B.1. Lifting operators. In the p -FEM, globally continuous, piecewise polynomial approximations to a function u are typically constructed in two steps: in a first step, discontinuous approximations are constructed element by element. In a second step, the jumps across the element interfaces are corrected by suitable lifting operators. The construction of these lifting operators is the purpose of the present section; the ensuing Section B.2 is devoted to the analysis of polynomial approximation.

Before proceeding we recall the definition of the Sobolev space $H_{00}^{1/2}(\Omega)$. If Ω is an edge or a face of a triangle or a tetrahedron, then the Sobolev norm $\|\cdot\|_{H_{00}^{1/2}(\Omega)}$ is defined by

$$(B.1) \quad \|u\|_{H_{00}^{1/2}(\Omega)}^2 := \|u\|_{H^{1/2}(\Omega)}^2 + \left\| \frac{u}{\sqrt{\text{dist}(\cdot, \partial\Omega)}} \right\|_{L^2(\Omega)}^2,$$

and the space $H_{00}^{1/2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under this norm.

Lemma B.1. *Let \widehat{K}^{2D} be the reference triangle in 2D. Vertex and edge lifting operators can be constructed with the following properties:*

- (1) For each vertex V of \widehat{K}^{2D} there exists a polynomial $L_{V,p} \in \mathcal{P}_p$ that attains the value 1 at the vertex V and vanishes on the edge of \widehat{K}^{2D} opposite to V . Additionally, for every $s \geq 0$, there exists $C_s > 0$ such that $\|L_{V,p}\|_{H^s(\widehat{K}^{2D})} \leq C_s p^{-1+s}$.
- (2) For every edge e of \widehat{K}^{2D} there exists a bounded linear operator $\pi_e : H_{00}^{1/2}(e) \rightarrow H^1(\widehat{K}^{2D})$ with the following properties:
 - (a) $\forall u \in \mathcal{P}_p \cap H_{00}^{1/2}(e) : \pi_e u \in \mathcal{P}_p$,
 - (b) $\forall u \in H_{00}^{1/2}(e) : \pi_e u|_{\partial\widehat{K}^{2D} \setminus e} = 0$,
 - (c) $\forall u \in H_{00}^{1/2}(e) : p\|\pi_e u\|_{L^2(\widehat{K}^{2D})} + \|\pi_e u\|_{H^1(\widehat{K}^{2D})} \leq C \left(\|u\|_{H_{00}^{1/2}(e)} + p^{1/2}\|u\|_{L^2(e)} \right)$.

Proof. Let $\widehat{K}^{2D} = \{(x, y) \mid 0 < x < 1, 0 < y < 1 - x\}$. The vertex function $L_{V,p}$ for the vertex $V = (0, 0)$ is defined as $L_{V,p}(x, y) = (1 - (x + y))^p$. A simple calculation then shows the result. The functions $L_{V,p}$ for the remaining 2 vertices are obtained by suitable affine transformations.

For the edge lifting, let e be the edge $e = \{(x, 0) \mid 0 < x < 1\}$. By [3] there exists a bounded linear operator $E : H_{00}^{1/2}(e) \rightarrow H^1(\widehat{K}^{2D})$ with the following properties: $Eu|_e = u$, $Eu|_{\partial\widehat{K}^{2D} \setminus e} = 0$, and $Eu \in \mathcal{P}_p$ if $u \in \mathcal{P}_p \cap H_{00}^{1/2}(e)$. Introduce the auxiliary operator $(Gu)(x, y) := (1 - y)^p (Eu)(x, y)$. By [31, Lemma B.5], we have

$$\begin{aligned} p\|Gu\|_{L^2(\widehat{K}^{2D})} + \|Gu\|_{H^1(\widehat{K}^{2D})} &\leq C \left(\|Eu\|_{H^1(\widehat{K}^{2D})} + p^{1/2}\|u\|_{L^2(e)} \right) \\ &\leq C \left(\|u\|_{H_{00}^{1/2}(e)} + p^{1/2}\|u\|_{L^2(e)} \right). \end{aligned}$$

Denote by $\Pi_p^{H^1} : H_0^1(\widehat{K}^{2D}) \rightarrow H_0^1(\widehat{K}^{2D}) \cap \mathcal{P}_p$ the H^1 -projection and set $\pi_e u := Eu + \Pi_p^{H^1}(Gu - Eu)$. Then by the stability of $\Pi_p^{H^1}$ and E we have

$$\begin{aligned} \|\pi_e u\|_{H^1(\widehat{K}^{2D})} &\leq \|Gu\|_{H^1(\widehat{K}^{2D})} + 2\|Gu - Eu\|_{H^1(\widehat{K}^{2D})} \\ &\leq C \left(\|u\|_{H_{00}^{1/2}(e)} + p^{1/2}\|u\|_{L^2(e)} \right), \end{aligned}$$

which is the desired H^1 -stability result. For the L^2 -bound, we use a duality argument as in [21]:

$$\|Gu - Eu - \Pi_p^{H^1}(Gu - Eu)\|_{L^2(\widehat{K}^{2D})} \leq Cp^{-1}\|(Gu - Eu) - \Pi_p^{H^1}(Gu - Eu)\|_{H^1(\widehat{K}^{2D})}.$$

The H^1 -stability of $\Pi_p^{H^1}$ together with stability properties of E and G produce the desired L^2 -bound. □

Lemma B.2. Let \widehat{K}^{3D} be the reference tetrahedron in 3D. Vertex, edge, and face lifting operators can be constructed with the following properties:

- (i) For each vertex V of \widehat{K}^{3D} there exists a polynomial $L_{V,p} \in \mathcal{P}_p$ that attains the value 1 at the vertex V and vanishes on the face opposite V . Additionally, for every $s \geq 0$ there exists $C_s > 0$ such that $\|L_{V,p}\|_{H^s(\widehat{K}^{3D})} \leq C_s p^{-3/2+s}$.
- (ii) For every edge e of \widehat{K}^{3D} there exists a bounded linear operator $\pi_e : H_{00}^{1/2}(e) \rightarrow H^1(\widehat{K}^{3D})$ with the following properties:
 - (a) $\pi_e u \in \mathcal{P}_p$ if $u \in \mathcal{P}_p \cap H_{00}^{1/2}(e)$,

- (b) $(\pi_e u)|_f = 0$ for the two faces f with $\bar{f} \cap e = \emptyset$,
- (c) for the two faces f adjacent to e (i.e., $\bar{f} \cap e = e$),

$$p \|\pi_e u\|_{L^2(f)} + \|\pi_e u\|_{H^1(f)} \leq C \|u\|_{H_{00}^{1/2}(e)} + p^{1/2} \|u\|_{L^2(e)},$$

$$\|\pi_e u\|_{H^{1/2}(\partial \widehat{K}^{3D})} \leq C \left(p^{-1/2} \|u\|_{H_{00}^{1/2}(e)} + \|u\|_{L^2(e)} \right),$$

$$p \|\pi_e u\|_{L^2(\widehat{K}^{3D})} + \|\pi_e u\|_{H^1(\widehat{K}^{3D})} \leq C \left(p^{-1/2} \|u\|_{H_{00}^{1/2}(e)} + \|u\|_{L^2(e)} \right).$$

(iii) For every face f of \widehat{K}^{3D} there exists a bounded linear operator $\pi_f : H_{00}^{1/2}(f) \rightarrow H^1(\widehat{K}^{3D})$ with the following properties:

- (a) $\pi_f u \in \mathcal{P}_p$ if $u \in \mathcal{P}_p \cap H_{00}^{1/2}(f)$,
- (b) $(\pi_e u)|_{f'} = 0$ for the faces $f' \neq f$,

$$p \|\pi_f u\|_{L^2(\widehat{K}^{3D})} + \|\pi_f u\|_{H^1(\widehat{K}^{3D})} \leq C \left(\|u\|_{H_{00}^{1/2}(f)} + p^{1/2} \|u\|_{L^2(f)} \right).$$

Proof. We take the reference tetrahedron \widehat{K}^{3D} to be $\widehat{K}^{3D} = \{(x, y, z) \mid 0 < x < 1, 0 < y < 1 - x, 0 < z < 1 - x - y\}$.

Proof of (i). For the vertex $V = (0, 0, 0)$ we select $L_{V,p}(x, y, z) := (1 - (x + y + z))^p$. A calculation shows that $L_{V,p}$ has the desired properties. The functions $L_{V,p}$ for the remaining 3 vertices are obtained by affine transformations.

Proof of (iii). [34, Lemma 8] exhibits a bounded linear operator $F : H_{00}^{1/2}(f) \rightarrow H^1(\widehat{K}^{3D})$ with the additional property that $Fu \in \mathcal{P}_p$ if $u \in \mathcal{P}_p \cap H_{00}^{1/2}(f)$. Without loss of generality, let $f = \partial \widehat{K}^{3D} \cap \{z = 0\}$. Define the auxiliary operator $(Gu)(x, y, z) := (1 - z)^p (Fu)(x, y, z)$. This operator satisfies (see [31, Lemma B.5] where the analogous arguments are worked out in the 2D setting)

$$\begin{aligned} p \|Gu\|_{L^2(\widehat{K}^{3D})} + \|Gu\|_{H^1(\widehat{K}^{3D})} &\leq C \left(|Fu|_{H^1(\widehat{K}^{3D})} + p^{1/2} \|Fu\|_{L^2(f)} \right) \\ &\leq C \|u\|_{H_{00}^{1/2}(f)} + p^{1/2} \|u\|_{L^2(f)}. \end{aligned}$$

Again, letting $\Pi_p^{H^1} : H_0^1(\widehat{K}^{3D}) \rightarrow H_0^1(\widehat{K}^{3D}) \cap \mathcal{P}_p$ be the H^1 -projection, we can set $\pi_f u := Fu + \Pi_p^{H^1}(Gu - Fu)$. The desired properties of π_f are then seen in exactly the same way as in the 2D case of Lemma B.1.

Proof of (ii). Set $f_{e,1} = \partial \widehat{K}^{3D} \cap \{z = 0\}$ and $f_{e,2} = \partial \widehat{K}^{3D} \cap \{1 - x - y - z = 0\}$. The edge shared by the faces $f_{e,1}$ and $f_{e,2}$ is $e = \{(x, 1 - x, 0) \mid 0 < x < 1\}$. By Lemma B.1 a function $u \in H_{00}^{1/2}(e)$ can be lifted to a function $Eu \in H^1(f_{e,1})$ such that $Eu|_{\partial f_{e,1} \setminus e} = 0$ and

$$p \|Eu\|_{L^2(f_{e,1})} + \|Eu\|_{H^1(f_{e,1})} \leq C \left(\|u\|_{H_{00}^{1/2}(e)} + p^{1/2} \|u\|_{L^2(e)} \right).$$

Additionally, if $u \in \mathcal{P}_p$, then $Eu \in \mathcal{P}_p$. Since the same lifting can be done for the face $f_{e,2}$, we can find a function, again denoted $Eu \in H^1(\partial \widehat{K}^{3D})$, that vanishes on $\partial \widehat{K}^{3D} \setminus (f_{e,1} \cup f_{e,2} \cup e)$, such that $p \|Eu\|_{L^2(\partial \widehat{K}^{3D})} + \|Eu\|_{H^1(\partial \widehat{K}^{3D})} \leq C \left(\|u\|_{H_{00}^{1/2}(e)} + p^{1/2} \|u\|_{L^2(e)} \right)$. Additionally, Eu is a piecewise polynomial of degree p if $u \in \mathcal{P}_p$. An interpolation inequality gives

$$\begin{aligned} \|Eu\|_{H^{1/2}(\partial \widehat{K}^{3D})} &\leq C \|Eu\|_{L^2(\partial \widehat{K}^{3D})}^{1/2} \|Eu\|_{H^1(\partial \widehat{K}^{3D})}^{1/2} \\ &\leq Cp^{-1/2} \left(\|u\|_{H_{00}^{1/2}(\partial \widehat{K}^{3D})} + p^{1/2} \|u\|_{L^2(\partial \widehat{K}^{3D})} \right). \end{aligned}$$

For this function Eu , [34, Lemma 8] provides a lifting $Fu \in H^1(\widehat{K}^{3D})$ with $\|Fu\|_{H^1(\widehat{K}^{3D})} \leq C\|Eu\|_{H^{1/2}(\widehat{K}^{3D})}$. To get a better L^2 -bound, we introduce the distance functions $d_1(\cdot) := \text{dist}(\cdot, f_{e,1})$ and $d_2(\cdot) := \text{dist}(\cdot, f_{e,2})$ as well as $d(\cdot) := \text{dist}(\cdot, f_{e,1} \cup f_{e,2}) = \min\{d_1(\cdot), d_2(\cdot)\}$ and set $w := (1 - d)^p$. Define $Gu := wFu$. Then $(Gu)|_{\partial\widehat{K}^{3D}} = (Fu)|_{\partial\widehat{K}^{3D}}$ since $w|_{f_{e,1} \cup f_{e,2}} \equiv 1$ and $Fu|_{\partial\widehat{K}^{3D} \setminus (f_{e,1} \cup f_{e,2})} = 0$. Additionally, $Gu \in H^1(\widehat{K}^{3D})$ since w is Lipschitz continuous. Furthermore, we have

$$(B.2) \quad p\|Gu\|_{L^2(\widehat{K}^{3D})} + \|Gu\|_{H^1(\widehat{K}^{3D})} \leq C \left(\|Fu\|_{H^1(\widehat{K}^{3D})} + p^{1/2}\|Fu\|_{L^2(f_{e,1} \cup f_{e,2})} \right).$$

To see this, we adapt the proof given in [31, Lemma B.5]. We split $\widehat{K}^{3D} = K_1 \cup K_2$ with $K_i = \{(x, y, z) \in \widehat{K}^{3D} \mid d(x, y, z) \leq d_i(x, y, z)\}$, $i \in \{1, 2\}$. We note that on K_1 , we have $d(x, y, z) = d_1(x, y, z) = z$. Hence, by the arguments given in [31, Lemma B.5], we get

$$p\|Gu\|_{L^2(K_1)} + \|Gu\|_{H^1(K_1)} \leq C \left(\|Fu\|_{H^1(K_1)} + p^{1/2}\|Fu\|_{L^2(f_{e,1})} \right).$$

Proceeding completely analogously for K_2 gives us (B.2). Since $Fu|_{\partial K^{3D}}$ coincides with Eu , we conclude that Gu satisfies

$$(B.3) \quad p\|Gu\|_{L^2(K_1)} + \|Gu\|_{H^1(K_1)} \leq Cp^{-1/2} \left(\|u\|_{H_0^{1/2}(e)} + p^{1/2}\|u\|_{L^2(e)} \right).$$

We recall that $\Pi_p^{H^1} : H_0^1(\widehat{K}^{3D}) \rightarrow H_0^1(\widehat{K}^{3D}) \cap \mathcal{P}_p$ denotes the H^1 -projection and we define

$$\pi_e u := Fu + \Pi_p^{H^1}(Gu - Fu).$$

If u is a polynomial of degree p , then $\pi_e u$ is a polynomial of degree p . Additionally, $\pi_e u = Fu$ on $\partial\widehat{K}^{3D}$ so that the estimates for π_e on the faces of \widehat{K}^{3D} are satisfied. To see the $H^1(\widehat{K}^{3D})$ - and $L^2(\widehat{K}^{3D})$ -bounds we note that the stability of $\Pi_p^{H^1}$ together with (B.3) and the stability of F gives us the H^1 -bound. The L^2 -bound follows as in the proof of Lemma B.1 and in [21] from Nitsche’s trick: $\|\pi_e u\|_{L^2(\widehat{K}^{3D})} \leq \|\pi_e u - Gu\|_{L^2(\widehat{K}^{3D})} + \|Gu\|_{L^2(\widehat{K}^{3D})} \leq Cp^{-1}\|Fu - Gu\|_{H^1(\widehat{K}^{3D})} + \|Gu\|_{L^2(\widehat{K}^{3D})}$. \square

B.2. Approximation operators. Lemma B.3 provides polynomial approximation results on triangles and tetrahedra. The lifting operators of the preceding subsection are employed in Theorem B.4 to modify the approximations of Lemma B.3 such that approximations are obtained that permit an element-by-element construction in the sense of Definition 5.3; that is, the approximation πu of a function u satisfies the following: For every vertex V , edge e , face f of \widehat{K} , the restrictions $(\pi u)(V)$, $(\pi u)|_e$, $(\pi u)|_f$ are completely determined by $u(V)$, $u|_e$, $u|_f$, respectively.

Lemma B.3. *Let \widehat{K} be the reference triangle or the reference tetrahedron. Let $s > d/2$. Then there exists for every p a bounded linear operator $\pi_p : H^s(\widehat{K}) \rightarrow \mathcal{P}_p$ and for each $t \in [0, s]$ a constant $C > 0$ (depending only on s and t) such that*

$$(B.4) \quad \|u - \pi_p u\|_{H^t(\widehat{K})} \leq Cp^{-(s-t)}|u|_{H^s(\widehat{K})}, \quad p \geq s - 1.$$

Additionally, we have $\|u - \pi_p u\|_{L^\infty(\widehat{K})} \leq Cp^{-(s-d/2)}|u|_{H^s(\widehat{K})}$. Furthermore, for the case $d = 2$ we have $\|u - \pi_p u\|_{H^t(e)} \leq Cp^{-(s-1/2-t)}|u|_{H^s(\widehat{K})}$ for $0 \leq t \leq s - 1/2$ for every edge. For the case $d = 3$ we have $\|u - \pi_p u\|_{H^t(f)} \leq Cp^{-(s-1/2-t)}|u|_{H^s(\widehat{K})}$

for $0 \leq t \leq s - 1/2$ for every face f and $\|u - \pi_p u\|_{H^t(e)} \leq Cp^{-(s-1-t)}|u|_{H^k(\widehat{K})}$ for $0 \leq t \leq s - 1$ for every edge.

Proof. The construction of π_p with the property (B.4) is fairly classical (see, e.g., [5]). One possible construction is worked out in [31, Appendix A] first for integers s, t and, then, interpolation arguments remove this restriction. Next, we consider the L^∞ -bound, for which we need the assumption $s > d/2$: We recall that for a Lipschitz domain $K \subset \mathbb{R}^d$ and $s > d/2$ there exists $C > 0$ such that

$$(B.5) \quad \|u\|_{L^\infty(K)} \leq C \|u\|_{L^2(K)}^{1-d/(2s)} \|u\|_{H^s(K)}^{d/(2s)} \quad \forall u \in H^s(K).$$

From this, the desired L^∞ -bound follows easily. The inequality (B.5) can be seen as follows: First, using an extension operator for K (e.g., the one given in [42, Chap. VI]) it suffices to show this estimate with K replaced with the full space \mathbb{R}^d . Next, [45, Thm. 4.6.1] asserts the embedding $B_{2,1}^{d/2}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. Finally, the Besov space $B_{2,1}^{d/2}(\mathbb{R}^d)$ is recognized as an interpolation space between $L^2(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d)$: $B_{2,1}^{d/2}(\mathbb{R}^d) = (L^2(\mathbb{R}^d), H^s(\mathbb{R}^d))_{d/(2s),1}$. The interpolation inequality then produces the desired result. The remaining estimates on the edges and faces follow from appropriate trace inequalities. Specifically, let $\omega \subset \partial\widehat{K}$ be an edge (for $d = 2$) or a face (for $d = 3$). By [45, Thm. 2.9.3] the trace operator γ is a continuous mapping in the following spaces:

$$\gamma : B_{2,1}^{1/2}(\widehat{K}) \rightarrow L^2(\omega) \quad \text{and} \quad \gamma : H^t(\widehat{K}) \rightarrow H^{t-1/2}(\omega), \quad t > 1/2.$$

Together with the observation $B_{2,1}^{1/2}(\widehat{K}) = (L^2(\widehat{K}), H^s(\widehat{K}))_{1/(2s),1}$ the desired estimates can be inferred. It remains to see the case of traces on an edge e of the tetrahedron in the case $d = 3$. In this case [45, Thm. 2.9.4] asserts the continuity of the trace operator in the following spaces:

$$\gamma : B_{2,1}^1(\widehat{K}) \rightarrow L^2(e) \quad \text{and} \quad \gamma : H^t(\widehat{K}) \rightarrow H^{t-1}(e), \quad t > 1.$$

Again, these continuity properties are sufficient to establish the desired error estimates. □

We conclude this section with the construction of an approximation operator that permits an easy element-by-element construction.

Theorem B.4. *Let $\widehat{K} \subset \mathbb{R}^d$ be the reference triangle or the reference tetrahedron. Let $s > d/2$. Then there exists $C > 0$ (depending only on s and d) and for every p a linear operator $\pi : H^s(\widehat{K}) \rightarrow \mathcal{P}_p$ that permits an element-by-element construction in the sense of Definition 5.3 such that*

$$(B.6) \quad p \|u - \pi u\|_{L^2(\widehat{K})} + \|u - \pi u\|_{H^1(\widehat{K})} \leq Cp^{-(s-1)}|u|_{H^s(\widehat{K})} \quad \forall p \geq s - 1.$$

Proof. We discuss only the case $d = 3$; the case $d = 2$ is treated very similarly. Also, we will construct πu for a given u ; inspection of the construction shows that $u \mapsto \pi u$ is in fact a linear operator.

Let $\pi^1 \in \mathcal{P}_p$ be given by Lemma B.3. Then, for $p \geq s - 1$ there holds

$$(B.7) \quad \|u - \pi^1\|_{H^t(\widehat{K})} \leq Cp^{-(s-t)}|u|_{H^s(\widehat{K})}, \quad 0 \leq t \leq s,$$

$$(B.8) \quad \|u - \pi^1\|_{H^t(f)} \leq Cp^{-(s-t-1/2)}|u|_{H^s(\widehat{K})}, \quad \forall \text{ faces } f, \quad 0 \leq t \leq s - 1/2,$$

$$(B.9) \quad \|u - \pi^1\|_{H^t(e)} \leq Cp^{-(s-t-1)}|u|_{H^s(\widehat{K})}, \quad \forall \text{ edges } e, \quad 0 \leq t \leq s - 1,$$

$$(B.10) \quad \|u - \pi^1\|_{L^\infty(\widehat{K})} \leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}.$$

From (B.10) and the vertex-lifting properties given in Lemma B.2, we may adjust π^1 by vertex liftings to obtain a polynomial π^2 satisfying (B.7)–(B.9) and additionally the condition (5.3) of Definition 5.3. We next adjust the edge values. The polynomial π^2 coincides with u in the vertices and satisfies (B.9). By fixing a $t \in (1/2, s - 1)$, we get from an interpolation inequality:

$$\begin{aligned} p^{1/2}\|u - \pi^2\|_{L^2(e)} + \|u - \pi^2\|_{H_{00}^{1/2}(e)} &\leq p^{1/2}\|u - \pi^2\|_{L^2(e)} + C\|u - \pi^2\|_{L^2(e)}^{1-1/(2t)}\|u - \pi^2\|_{H^t(e)}^{1/(2t)} \\ &\leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}. \end{aligned}$$

Hence, for an edge e , the minimizer π^e of the functional (5.4) satisfies $p^{1/2}\|u - \pi^e\|_{L^2(e)} + \|u - \pi^e\|_{H_{00}^{1/2}(e)} \leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}$; the triangle inequality therefore gives that the correction $\pi^e - \pi^2$ needed to obtain condition (5.3) of Definition 5.3 likewise satisfies $p^{1/2}\|\pi^e - \pi^2\|_{L^2(e)} + \|\pi^e - \pi^2\|_{H_{00}^{1/2}(e)} \leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}$. We conclude that the edge lifting of Lemma B.2 allows us to adjust π^2 to get a polynomial $\pi^3 \in \mathcal{P}_p$ that satisfies the conditions (5.3) and (5.3) of Definition 5.3. Additionally, we have

$$\begin{aligned} p\|u - \pi^3\|_{L^2(\widehat{K})} + \|u - \pi^3\|_{H^1(\widehat{K})} &\leq Cp^{-(s-1)}|u|_{H^s(\widehat{K})}, \\ p\|u - \pi^3\|_{L^2(f)} + \|u - \pi^3\|_{H^1(f)} &\leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})} \quad \text{for all faces } f. \end{aligned}$$

Since $\pi^3|_e = \pi^e$ for the edges, the minimizer π^f of the functional (5.5) for each face f has to satisfy $p\|u - \pi^f\|_{L^2(f)} + \|u - \pi^f\|_{H^1(f)} \leq p\|u - \pi^3\|_{L^2(f)} + \|u - \pi^3\|_{H^1(f)} \leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})}$. From the triangle inequality, we conclude that

$$p\|\pi^3 - \pi^f\|_{L^2(f)} + \|\pi^3 - \pi^f\|_{H^1(f)} \leq Cp^{-(s-3/2)}|u|_{H^s(\widehat{K})},$$

together with $\pi^3 - \pi^f \in H_0^1(f)$.

Hence, the face lifting of Lemma B.2 allows us to correct the face values to achieve also condition (5.3) of Definition 5.3. Lemma B.2 also implies that the correction is such that (B.6) is true. \square

APPENDIX C. APPROXIMATION BY *hp*-FINITE ELEMENTS.
CASE II: ANALYTIC REGULARITY

In this section, we construct a polynomial approximation operator for analytic functions that permits element-by-element construction in the sense of Definition 5.3 and leads to exponential rates of convergence.

Lemma C.1. *Let $d \in \{2, 3\}$. Let $G_1, G \subset \mathbb{R}^d$ be bounded open sets. Assume that $g : \overline{G_1} \rightarrow \mathbb{R}^d$ satisfies $g(G_1) \subset G$. Assume additionally that g is injective on $\overline{G_1}$,*

analytic on G_1 and satisfies

$$\|\nabla^p g\|_{L^\infty(G_1)} \leq C_g \gamma_g^p p! \quad \forall p \in \mathbb{N}_0, \quad |\det(g')| \geq c_0 > 0 \quad \text{on } G_1.$$

Let f be analytic on G and satisfy, for some $C_f, \gamma_f, \kappa > 0$,

$$(C.1) \quad \|\nabla^p f\|_{L^2(G)} \leq C_f \gamma_f^p \max\{p, \kappa\}^p \quad \forall p \in \mathbb{N}_0.$$

Then, the function $f \circ g$ is analytic on G_1 and there exist constants $C, \gamma_1 > 0$ that depend solely on γ_g, C_g, c_0 , and γ_f such that

$$\|\nabla^p (f \circ g)\|_{L^2(G)} \leq CC_f \gamma_1^p \max\{p, \kappa\}^p \quad \forall p \in \mathbb{N}_0.$$

Proof. This is essentially proved in [30, Lemma 4.3.1]; specifically, [30, Lemma 4.3.1] analyzes the case $d = 2$ and states that C, γ_1 depends on the function g . Inspection of the proof shows that the case $d = 3$ can be handled analogously and shows that the dependence on the function g can be reduced to a dependence on C_g, γ_g , and γ_f . \square

Lemma C.2. Let $d \in \{1, 2, 3\}$, and let $\widehat{K} \subset \mathbb{R}^d$ be the reference simplex. Let $\bar{\gamma}, \tilde{C} > 0$ be given. Then there exist constants $C, \sigma > 0$ that depend solely on $\bar{\gamma}$ and \tilde{C} such that the following is true: For any function u that satisfies for some $C_u, h, R > 0, \kappa \geq 1$ the conditions

$$(C.2) \quad \|\nabla^n u\|_{L^2(\widehat{K})} \leq C_u (\bar{\gamma}h)^n \max\{n/R, \kappa\}^n \quad \forall n \in \mathbb{N}, \quad n \geq 2,$$

and for any polynomial degree $p \in \mathbb{N}$ that satisfies

$$(C.3) \quad h/R + \kappa h/p \leq \tilde{C}$$

there holds

$$(C.4) \quad \inf_{\pi \in \mathcal{P}_p} \|u - \pi\|_{W^{2,\infty}(\widehat{K})} \leq CC_u \left[\left(\frac{h/R}{\sigma + h/R} \right)^{p+1} + \left(\frac{\kappa h}{\sigma p} \right)^{p+1} \right].$$

Proof. Let $\Pi_1 u \in \mathcal{P}_1$ be the L^2 -projection of u onto the space \mathcal{P}_1 . Set $\tilde{u} := u - \Pi_1 u$. It suffices to approximate \tilde{u} from \mathcal{P}_p . By the lemma of Deny and Lions and (C.3) we have

$$\begin{aligned} \|\tilde{u}\|_{L^2(\widehat{K})} &\leq C \|\nabla^2 u\|_{L^2(\widehat{K})} \leq CC_u (1 + (h/R)^2 + (h\kappa)^2) \leq CC_u p^2, \\ \|\nabla \tilde{u}\|_{L^2(\widehat{K})} &\leq C \|\nabla^2 u\|_{L^2(\widehat{K})} \leq CC_u (1 + (h/R)^2 + (h\kappa)^2) \leq CC_u p h/R \max\{1, \kappa R\}, \\ \|\nabla^n \tilde{u}\|_{L^2(\widehat{K})} &= \|\nabla^n u\|_{L^2(\widehat{K})} \leq CC_u (\bar{\gamma}h/R)^n \max\{n, R\kappa\}^n \quad \forall n \geq 2. \end{aligned}$$

We conclude that (estimating generously $p \leq p^2$ for the case $n = 1$)

$$(C.5) \quad \|\nabla^n \tilde{u}\|_{L^2(\widehat{K})} \leq CC_u p^2 (\bar{\gamma}h/R)^n \max\{n, \kappa R\}^n \quad \forall n \in \mathbb{N}_0.$$

For the case $\kappa R \leq 1$, we estimate $\kappa R \leq 1$ and get directly from [30, Thm. 3.2.19]

$$\inf_{\pi \in \mathcal{P}_p} \|\tilde{u} - \pi\|_{W^{2,\infty}(\widehat{K})} \leq CC_u \left(\frac{h/R}{\sigma + h/R} \right)^{p+1}.$$

It remains to consider the case $\kappa R > 1$. To that end, we note that (C.5) and the Sobolev embedding theorem $H^2(\widehat{K}) \subset C(\widehat{K})$ gives us for suitable $C > 0$,

$$\begin{aligned} \|\nabla^n \tilde{u}\|_{L^\infty(\widehat{K})} &\leq C_u p^2 C [(\bar{\gamma}h/R)^{n+2} \max\{n+2, \kappa R\}^{n+2} \\ &\quad + (\bar{\gamma}h/R)^n \max\{n+2, \kappa R\}^n] \\ &\leq CC_u p^2 (\bar{\gamma}h/R)^n \max\{n+2, \kappa R\}^n (1 + \max\{(n+2)h/R, h\kappa\}^2) \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

Hence, we get for suitable constant $\gamma > 0$ in view of (C.3),

$$(C.6) \quad \|\nabla^n \tilde{u}\|_{L^\infty(\widehat{K})} \leq CC_u p^4 (\bar{\gamma} \gamma h/R)^n \max\{n + 2, \kappa R\}^n \quad \forall n \in \mathbb{N}_0.$$

Define

$$(C.7) \quad \mu := \gamma \bar{\gamma} \sqrt{d} e,$$

and let $r_0 = \text{diam}(\widehat{K})$ and $b_{\widehat{K}}$ be the barycenter of \widehat{K} . The bounds (C.6), (5.2) and Stirling’s formula in the form $n! \geq (n/e)^n$ imply that the Taylor series of \tilde{u} about $x \in \widehat{K}$ converges on a (complex) ball $B_{1/(\mu h/R)}(x) \subset \mathbb{C}^d$ of radius $1/(\mu h/R)$ and center $x \in \widehat{K}$. For the polynomial approximation of \tilde{u} , we distinguish the cases $\mu h/R \leq 1/(2r_0)$ and $\mu h/R > 1/(2r_0)$.

The case $\mu h/R \leq 1/(2r_0)$: In this case the Taylor series of \tilde{u} about $b_{\widehat{K}}$ converges on an open ball that contains the closure of \widehat{K} . We may therefore approximate \tilde{u} by its truncated Taylor series $T_p u$. The error is then given by

$$\tilde{u}(x) - T_p u(x) = \sum_{\alpha \in \mathbb{N}_0^d: |\alpha| \geq p+1} \frac{1}{\alpha!} D^\alpha \tilde{u}(b_{\widehat{K}}) (x - b_{\widehat{K}})^\alpha, \quad x \in B_{1/(\mu h/R)}(b_{\widehat{K}}) \subset \mathbb{C}^d.$$

Hence (5.2) and (C.6) imply

$$\begin{aligned} \|\tilde{u} - T_p u\|_{L^\infty(B_{r_0}(b_{\widehat{K}}))} &\leq \sum_{|\alpha| \geq p+1} \frac{1}{\alpha!} |D^\alpha \tilde{u}(b_{\widehat{K}})| r_0^{|\alpha|} \leq \sum_{n=p+1}^\infty r_0^n d^{n/2} \frac{1}{n!} \|\nabla^n \tilde{u}\|_{L^\infty(\widehat{K})} \\ &\leq CC_u p^4 \sum_{n=p+1}^\infty \frac{1}{n!} \max\{n + 2, \kappa R\}^n d^{n/2} (\gamma \bar{\gamma} h/R)^n r_0^n =: S. \end{aligned}$$

The last sum S is split further using Stirling’s formula $n! \geq (n/e)^n$ and $(1+2/n)^n \leq e^2$:

$$\begin{aligned} S &= CC_u p^4 \left(\sum_{p+1 \leq n \leq \kappa R - 2} \frac{1}{n!} (\sqrt{d} r_0 \gamma \bar{\gamma} \kappa h)^n \right. \\ &\quad \left. + \sum_{n \geq \max\{p+1, \kappa R - 2\}} (\gamma \bar{\gamma} r_0 \sqrt{d} h/R)^n \frac{(n+2)^n}{n!} \right) \\ &\leq CC_u p^4 e^2 \left(\sum_{n \geq p+1} \frac{1}{n!} (\sqrt{d} r_0 \gamma \bar{\gamma} \kappa h)^n + \sum_{n \geq p+1} \underbrace{(e \gamma \bar{\gamma} \sqrt{d} r_0 h/R)^n}_{=\mu} \right) =: S_1 + S_2. \end{aligned}$$

We estimate these two sums separately. For S_2 , we use the assumption $\mu r_0 h/R \leq 1/2$, which allows us to estimate

$$\begin{aligned} S_2 &\leq CC_u p^4 e^2 (\mu r_0 h/R)^{p+1} = CC_u p^4 e^2 \left(\frac{h/R}{\frac{1}{2\mu r_0} + \frac{1}{2\mu r_0}} \right)^{p+1} \\ &\leq CC_u p^4 e^2 \left(\frac{h/R}{\frac{1}{2\mu r_0} + h/R} \right)^{p+1}. \end{aligned}$$

For S_1 , we recall that Taylor’s formula gives, for $x > 0$,

$$\sum_{n \geq p+1} \frac{1}{n!} x^n = e^x - \sum_{n=0}^p \frac{1}{n!} x^n = \frac{1}{p!} \int_0^x (x-t)^p e^t dt \leq \frac{x^{p+1}}{p!} e^x.$$

Hence, we can estimate S_1 by (recall that $\bar{\gamma}\gamma\sqrt{d} = \mu/e$),

$$S_2 \leq CC_u p^4 \frac{((\mu/e)r_0\kappa h)^{p+1}}{p!} e^{(\mu/e)r_0\kappa h} \leq CC_u p^5 \left(\frac{e^\theta \mu r_0 \kappa h}{p+1} \right)^{p+1},$$

where, in the second inequality, we have used the assumption $h\kappa/p \leq \tilde{C}$ and Stirling’s formula $n! \geq (n/e)^n$ and have abbreviated $\theta := \tilde{C}\mu/er_0$. Combining the estimates for S_1 and S_2 we arrive at the following estimate for suitable $\sigma > 0$ (depending only on μ, r_0 , and \tilde{C}):

$$\|\tilde{u} - T_p u\|_{L^\infty(B_{r_0}(b_{\tilde{K}}))} \leq S \leq CC_u \left(\left(\frac{\kappa h}{\sigma p} \right)^{p+1} + \left(\frac{h/R}{\sigma + h/R} \right)^{p+1} \right).$$

Since $\text{dist}(\hat{K}, \partial B_{r_0}(b_{\tilde{K}})) > 0$, the Cauchy integral formula for derivatives then implies

$$\|\tilde{u} - T_p u\|_{W^{2,\infty}(\hat{K})} \leq CC_u \left(\left(\frac{\kappa h}{\sigma p} \right)^{p+1} + \left(\frac{h/R}{\sigma + h/R} \right)^{p+1} \right).$$

The case $\mu h/R > 1/(2r_0)$: We recall that for every $x \in \hat{K}$ the Taylor series of \tilde{u} about x converges on the (complex) ball $B_{1/(\mu h/R)}(x) \subset \mathbb{C}^d$. From (C.3) we get a lower bound for $1/(\mu h/R)$, namely, $1/(\mu h/R) \geq 1/(\mu \tilde{C}) =: 2r_1$. We conclude that \tilde{u} is analytic on $\hat{\mathcal{U}}_{2r_1} := \bigcup_{x \in \hat{K}} B_{2r_1}(x) \subset \mathbb{C}^d$. The estimate (C.6) and a calculation analogous to the above reveals that on $\hat{\mathcal{U}}_{r_1} := \bigcup_{x \in \hat{K}} B_{r_1}(x)$ we have

$$\|\tilde{u}\|_{L^\infty(\hat{\mathcal{U}}_{r_1})} \leq CC_u p^4 e^{\vartheta \kappa h}$$

for a constant $\vartheta > 0$ independent of p, κ, h . Approximation results for analytic functions on triangles/tetrahedra (see [30, Prop. 3.2.16] for the case $d = 2$ and [16, Thm. 1] for the case $d = 3$) imply the existence of $C, b > 0$ that depend solely on r_1 such that

$$\inf_{\pi \in \mathcal{P}_{p+1}} \|u - \pi\|_{W^{2,\infty}(\hat{K})} \leq CC_u p^4 e^{\vartheta \kappa h} e^{-bp} \quad \forall p \in \mathbb{N}_0.$$

We finally distinguish two further cases: If $\vartheta \kappa h < pb/2$, then we can estimate

$$p^4 e^{\vartheta \kappa h p} e^{-bp} \leq p^4 e^{-b/2p} \leq C \left(\frac{1/(2\mu r_0)}{\sigma + 1/(2\mu r_0)} \right)^{p+1},$$

for suitable constants $C, \sigma > 0$ depending only on b, μ , and r_0 . Since $h/R \geq 1/(2\mu r_0)$ and the function $x \mapsto x/(\sigma + x)$ is monotone increasing, we have reached the desired bound. If, on the other hand, $\vartheta \kappa h \geq pb/2$, then

$$p^4 e^{\vartheta \kappa h} e^{-bp} \leq C e^{\vartheta \kappa h} \leq C e^{\vartheta \tilde{C} p} = C \left(e^{\vartheta \tilde{C}} \right)^p \leq C \left(\frac{\kappa h}{p} \frac{2}{b} e^{\vartheta \tilde{C}} \right)^p;$$

we recognize this bound to have the desired form. □

Lemma C.3. *Assume the hypotheses of Lemma C.2. Then one can find a polynomial $\pi \in \mathcal{P}_p$ that satisfies*

$$(C.8) \quad \|u - \pi\|_{W^{1,\infty}(\widehat{K})} \leq CC_u \left[\left(\frac{h/R}{\sigma + h/R} \right)^{p+1} + \left(\frac{\kappa h}{\sigma p} \right)^{p+1} \right]$$

and additionally admits an element-by-element construction as defined in Definition 5.3.

Proof. The construction follows standard lines. We will only outline the arguments for the case $d = 3$. In order to keep the notation compact, we introduce the expression

$$E(C, \sigma) := CC_u \left[\left(\frac{h/R}{\sigma + h/R} \right)^{p+1} + \left(\frac{\kappa h}{\sigma p} \right)^{p+1} \right].$$

In what follows, the constants $C_i, \sigma_i > 0$ ($i = 1, 2, \dots$) will be independent of C_u, h, R, p , and κ . Let $\pi \in \mathcal{P}_p$ be the polynomial given by Lemma C.2. It satisfies $\|u - \pi\|_{W^{2,\infty}(\widehat{K})} \leq E(C, \sigma)$. Therefore, we may correct π by a linear polynomial without sacrificing the approximation rate to ensure $u(V) - \pi(V)$ for all vertices $V \in \mathcal{V}$. This corrected polynomial, denoted π^2 , vanishes in the vertices and still satisfies $\|u - \pi^2\|_{W^{2,\infty}(\widehat{K})} \leq E(C_2, \sigma_2)$. Next, we correct the edges. We illustrate the procedure only for one edge. Without loss of generality, we assume that $\widehat{K} = \{(x, y, z) \mid 0 < x, y, z < 1, x + y < 1 - z\}$ and that the edge e considered is $e = \{(0, 0, z) \mid z \in (0, 1)\}$. Let the univariate polynomial $\pi^e \in \mathcal{P}_p$ be the minimizer of the functional (5.4). From $\|u - \pi^2\|_{W^{2,\infty}(e)} \leq \|u - \pi^2\|_{W^{2,\infty}(\widehat{K})} \leq E(C_2, \sigma_2)$ we can conclude that $p^{1/2}\|u - \pi^e\|_{L^2(e)} + \|u - \pi^e\|_{H_0^{1/2}(e)} \leq Cp^{1/2}E(C_2, \sigma_2)$. Hence, for the required correction $\pi^c := \pi^2|_e - \pi^e$, which vanishes in the two endpoints of e , we get from a triangle inequality and standard polynomial inverse estimates $\|\frac{1}{1-z}\pi^c\|_{L^\infty(e)} + \|\pi^c\|_{L^\infty(e)} \leq E(C_3, \sigma_3)$. We may lift this univariate function to \widehat{K} by

$$\tilde{\pi}^e(x, y, z) := \frac{1 - x - y - z}{1 - z} \pi^c(z).$$

This is a polynomial of degree $\leq p$ that vanishes on all edges but the edge e ; clearly, $\|\tilde{\pi}^e\|_{L^\infty(\widehat{K})} \leq E(C_3, \sigma_3)$. The polynomial inverse estimate $\|\tilde{\pi}^e\|_{W^{1,\infty}(\widehat{K})} \leq Cp^2\|\tilde{\pi}^e\|_{L^\infty(\widehat{K})}$ shows that $\|\tilde{\pi}^e\|_{W^{2,\infty}(\widehat{K})} \leq E(C_4, \sigma_4)$. Proceeding in this fashion for all edges, we arrive at a polynomial π^3 with the desired behavior on all edges of \widehat{K} and satisfies $\|u - \pi^3\|_{W^{2,\infty}(\widehat{K})} \leq E(C_5, \sigma_5)$.

It remains to construct a correction for the faces. To that end, the key issue is again that of a lifting from a face f . Without loss of generality, this face is $f := \{(x, y, 0) \mid 0 < x, y, x + y < 1\}$. For a polynomial π^c defined on f that additionally vanishes on ∂f , we define the lifting $\tilde{\pi}^f$ by

$$\tilde{\pi}^f(x, y, z) = \frac{xy(1 - x - y - z)}{xy(1 - x - y)} \pi^c(x, y).$$

This is a polynomial that vanishes on all faces of \widehat{K} except on f . Additionally, it is a lifting, i.e., $\tilde{\pi}^f|_f = \pi^c$. As in the case of the lifting from the edge we see that if π^c is exponentially small on f , then the lifting is likewise exponentially small. To see that the required correction π^c is exponentially small, let π^f be the minimizer of the functional (5.5). Since π^3 has the desired behavior on the edges of f , we have

$\pi^3|_{\partial f} = \pi^f|_{\partial f}$ and therefore $\|u - \pi^3\|_{W^{2,\infty}(\widehat{K})} \leq E(C_5, \sigma_5)$ allows us to conclude $\|\pi^3 - \pi^f\|_{H^1(f)} \leq CE(C_5, \sigma_5)$. Polynomial inverse estimates then imply for the lifting $\tilde{\pi}^f$ that $\|\tilde{\pi}^f\|_{W^{1,\infty}(\widehat{K})} \leq E(C_6, \sigma_6)$. \square

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