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A PROOF OF DEJEAN'S CONJECTURE

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Abstract. We prove Dejean's conjecture. Specifically, we show that Dejean's conjecture holds for the last remaining open values of n, namely $15 \le n \le 26$.

1. Introduction

Repetitions in words have been studied since the beginning of the previous century [18, 19]. Recently, there has been much interest in repetitions with fractional exponent [1, 3, 6, 7, 8, 11]. For rational $1 < r \le 2$, a **fractional r-power** is a nonempty word $w = pe$ such that e is the prefix of p of length $(r-1)|p|$. We call e the **excess** of the repetition. We also say that r is the **exponent** of the repetition pe. For example, 010 is a 3/2-power, with excess 0. A basic problem is that of identifying the repetitive threshold for each alphabet of size $n > 1$:

> What is the infimum of r such that an infinite sequence on n letters exists, not containing any factor of exponent greater than r?

This infimum is called the **repetitive threshold** of an n-letter alphabet and is denoted by $RT(n)$. Dejean's conjecture [6] is that

$$RT(n) = \begin{cases} 7/4, & n = 3, \\ 7/5, & n = 4, \\ n/(n-1), & n \ne 3, 4. \end{cases}$$

Thue, Dejean and Pansiot, respectively [19, 6, 14], established the values $RT(2)$, $RT(3)$, $RT(4)$. Moulin Ollagnier [13] verified Dejean's conjecture for $5 \le n \le 11$, and Mohammad-Noori and Currie [12] proved the conjecture for $12 \le n \le 14$. Recently, Carpi [3] showed that Dejean's conjecture holds for $n \ge 33$. The present authors strengthened Carpi's construction to show that Dejean's conjecture holds for $n \ge 27$ [4, 5]. In this note we show that in fact Dejean's conjecture holds for $n \ge 2$. We will freely assume the usual notions of combinatorics on words as set forth in, for example, [9].

2. Morphisms

Given previous work, it remains only to show that Dejean's conjecture holds for $15 \le n \le 26$. This follows from the fact that the following morphisms are 'convenient' in the sense of [13]. To make our exposition self-contained, we demonstrate in the remainder of this paper how these morphisms are used to prove Dejean's conjecture for $15 \le n \le 26$. We introduce several simplifications and one correction to the work of Moulin Ollagnier [13].

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$h_{15}(0) =$ 10110110101101101101101101010110101011011011011010110110
$h_{15}(1) =$ 10101010110101101101101011010110110101101011011011010101

$h_{16}(0) =$ 10101010101010101101010101010101010110110110101101101101010 1
$h_{16}(1) =$ 1010101010101010110101010110110101011011010110110110110101 10

$h_{17}(0) =$ 1010101010101010101101101011011010101010101011010110110110101101
$h_{17}(1) =$ 1010101010101010101101101101101010101010110110110110110110110110

$h_{18}(0) =$ 1010101010110110101101010101101101101010110110101011011011010101010101
$h_{18}(1) =$ 1010101010101010101101010101101101101010110110110101011011010101010110

$h_{19}(0) =$ 10101010101010101010110110101011011010101010101010110101101010110110101101
$h_{19}(1) =$ 10101010101010101010110110101101101010101010101011011011011010110110110110

$h_{20}(0) =$ 101010101010101010101101101101010101010101101101101101101011010101011011010101
$h_{20}(1) =$ 101010101010101010101101101101010110110101101101101101011011010101011011010110

$h_{21}(0) =$ 10101010101010101010101011011010101010101011011010101010101011011011011010101010101010
10101101
$h_{21}(1) =$ 10101010101010101010101011011010101010101101101010101010101011010110101010101010101010
10110110

$h_{22}(0) =$ 10101010101010101010101011010101010101010101010101011011011011011010110110110110
11010101
$h_{22}(1) =$ 10101010101010101010101011010101010101011011010101101101101101011011011011011010110
11010110

$h_{23}(0) =$ 10101010101010101010101010101010101010101101101011011011011010101101011011010110
110110101101
$h_{23}(1) =$ 10101010101010101010101010101010101010101101101010110110110110110110110110110110110
110110110110

$h_{24}(0) =$ 10101010101010101010101010110101011011010101010101010101101010101011011010110110
1101011011010101
$h_{24}(1) =$ 10101010101010101010101010110101011011010101101101010110101010101101011011010110
1101011011010110

$h_{25}(0) =$ 101010101010101010101010101011011010101011011010110110110101101101011010101010
10101011011010110110
$h_{25}(1) =$ 10101010101010101010101010101011011010101010110110110110110101101101101101101010
10101011011010101101

$h_{26}(0) =$ 10101010101010101010101010101101010101010110110101010101011011011010110101101010
11011011011011011010101
$h_{26}(1) =$ 10101010101010101010101010101101010101010110110110110101011011011010110101010110
1101101101101101101101 10

We remark that the last letter of $h_n(0)$ is different from the last letter of $h_n(1)$ in each case. We also note that for each n, $|h_n(1)| = 4n - 4$, except for $n = 21$ where we have $|h_n(1)| = 4n$. It follows that $|h_n^m(1)|$ becomes arbitrarily large as m increases.

Let an occurrence of v in $h_n^\omega(1)$ be written $h_n^\omega(1) = xv\mathbf{y}$. Suppose that v has period q. We can write $x = x'x''$, $\mathbf{y} = y'\mathbf{y}''$ such that $x''vy'$ has period q, and $|x''vy'|$ is maximal. This is possible since none of the $h_n^\omega(1)$ is ultimately periodic. We refer to $x''vy'$ as the **maximal period q extension** of the occurrence $xv\mathbf{y}$ of v.

3. Pansiot encoding

Fix $n \geq 2$. Let $\Sigma_n = \{1, 2, \ldots, n\}$. Let $v \in \Sigma_n^*$ have length $m \geq n - 1$, and write $v = v_1 v_2 \cdots v_m$, $v_i \in \Sigma_n$. In the case where every factor of v of length $n - 1$ contains $n - 1$ distinct letters, we define the **Pansiot encoding of** v to be the word $b(v) = b_1 b_2 \cdots b_{m-(n-1)}$, where for $1 \leq i \leq m - n + 1$,

$$b_i = \begin{cases} 0, & v_i = v_{i+n-1}, \\ 1, & \text{otherwise.} \end{cases}$$

We can recover v from $b(v)$ and $v_1 v_2 \ldots v_{n-1}$. We see that if v has period $q < m - (n-1)$, then so does $b(v)$. The exponent $|v|/q$ of v corresponds to an exponent $\dfrac{|v| - n + 1}{q}$ of $b(v)$.

Let S_n denote the symmetric group on Σ_n with identity $\mathtt{id}$ and left multiplication, i.e.,

$$(fg)(i) = f(g(i)) \text{ for } f, g \in S_n, i \in \Sigma_n.$$

We use the standard two-line notation for permutations. (See Chapter 3 of [16] for example.) Let $\sigma : \{0,1\}^* \to S_n$ be the semigroup homomorphism generated by

$$\begin{aligned} \sigma(0) &= \begin{pmatrix} 1 & 2 & \cdots & (n-2) & (n-1) & n \\ 2 & 3 & \cdots & (n-1) & 1 & n \end{pmatrix}, \\ \sigma(1) &= \begin{pmatrix} 1 & 2 & \cdots & (n-2) & (n-1) & n \\ 2 & 3 & \cdots & (n-1) & n & 1 \end{pmatrix}. \end{aligned}$$

One proves by induction that

$$\sigma(b(v)) = \begin{pmatrix} 1 & 2 & \cdots & (n-2) & (n-1) & n \\ v_{m-n+2} & v_{m-n+3} & \cdots & v_{m-1} & v_m & \hat{v} \end{pmatrix}, \tag{1}$$

where $\hat{v}$ is the unique element of $\Sigma \setminus \{v_m, v_{m-1}, \ldots, v_{m-n+2}\}$.

Suppose that $PE \in \Sigma_n^*$ is a repetition of period $q = |P| > 0$ with $|E| \geq n - 1$. It follows from (1) that $\sigma(b(P)) = \mathtt{id}$, i.e., that P is in the kernel of σ. We refer to $b(PE)$ as a **kernel repetition** of period q. Conversely, if $u \in \Sigma_n^*$ and $b(u)$ is a kernel repetition of period q, then we may write $u = PE = EP'$ for some words P, P', E, where $|P| = |P'| = q$.

Suppose that for a morphism $h : \{0,1\}^* \to \{0,1\}^*$ there is a $\tau \in S_n$ such that

$$\begin{aligned} \tau \cdot \sigma(h(0)) \cdot \tau^{-1} &= \sigma(0), \\ \tau \cdot \sigma(h(1)) \cdot \tau^{-1} &= \sigma(1). \end{aligned}$$

In this case we say that h satisfies the 'algebraic condition'.

4. Kernel repetitions with markable excess

Let a uniform morphism $h : \{0,1\}^* \to \{0,1\}^*$ be given. Let $|h(0)| = r > 0$. A word $v \in \{0,1\}^*$ is **markable** (with respect to h) if whenever $h(X)xv$ and $h(Y)yv$ are prefixes of $h^\omega(1)$ with $|x|, |y| < r$, then $x = y$. If a word is markable, its extensions are markable. Let U be the set of length 2 factors of $h^\omega(1)$. A word $v \in \{0,1\}^*$ is **2-markable** (with respect to h) if whenever

(1) $u, u' \in U$,
(2) $h(X)xv$ is a prefix of $h(u)$ with $|x| < r$, and
(3) $h(Y)yv$ is a prefix of $h(u')$ with $|y| < r$,

then $x = y$.

If $|v| = r$ and v is a factor of $h^\omega(1)$, then v is a factor of $h(u)$, for some $u \in U$. It follows that if v is 2-markable, then v is markable. For each n, if $h = h_n$, we find $U = \{01, 10, 11\}$. It follows that all length r factors v are factors of $h(0110)$. A finite check shows that if $|v| = r$ and v is a factor of $h^\omega(1)$, then v is 2-markable, hence markable.

Let n be fixed, $15 \le n \le 26$ and let $h = h_n$. One checks that h satisfies the algebraic condition. Suppose that $v = pe$ is a kernel repetition with period $q = |p|$, where $h^\omega(1) = xv\mathbf{y}$. Notice that every length q factor of pe is conjugate to p, by the periodicity of pe. It follows that every length q factor of pe lies in the kernel of σ. Suppose that the excess e of v is markable. Let $V = x''vy'$ be the maximal period q extension of the occurrence $xv\mathbf{y}$ of v. Write $x = Xx'$, $\mathbf{y} = y'\mathbf{Y}$, so that $h^\omega(1) = XV\mathbf{Y}$. Write $V = PE = EP'$, where $|P| = q$. Since E is an extension of e, E is markable. Write $X = h(\chi)\chi'$, where $|\chi'| < r$, and write $XP = h(\gamma)\gamma'$, where $|\gamma'| < r$. It follows from the markability of E that $\chi' = \gamma'$. Then the maximality of V yields $|\chi'| = |\gamma'| = 0$. We may thus write $X = h(\chi)$, $E = h(\eta)\eta'$, with $|\eta'| < r$. By the maximality of V, word η' must be the longest common prefix of $h(0)$ and $h(1)$. Since E is a prefix and suffix of PE and E is markable, we know that r divides $|P|$. In total then, we may write $XPE = h(\chi\pi\eta)\eta'$, where $h(\pi) = P$, and η is a prefix of π. Also, since h satisfies the algebraic condition, $\sigma(\pi) = \mathtt{id}$. Thus $\pi\eta$ is a kernel repetition in $h^\omega(1)$. We see that $|PE| = r|\pi\eta| + |\eta'|$.

The maximality of V implies that $\pi\eta$ is maximal with respect to having period $|\pi|$. This means that if η is markable, we can repeat the foregoing construction. Eventually we obtain a kernel repetition $\mathcal{PE}$ with nonmarkable excess $\mathcal{E}$. If it takes s steps to arrive at $\mathcal{PE}$, then we find that $|PE| = r^s|\mathcal{PE}| + |\eta'|\sum_{i=0}^{s-1} r^i$ and $|P| = r^s|\mathcal{P}|$.

5. Main result

Let n be fixed, $15 \le n \le 26$ and let $h = h_n$. Suppose that u_1 is a factor of $h^\omega(1)$ with $|u_1| = \ell$. Extending u_1 by a suffix of length at most $r-1$, and a prefix of length at most $r-1$, we obtain a word $h(u_2)$, some factor u_2 of $h^\omega(1)$, where $|u_2| \le \lfloor(\ell + 2(r-1))/r\rfloor$. Repeating the argument, we find that u_1 is a factor of $h^2(u_3)$, for some factor u_3 of $h^\omega(1)$, where

$$|u_3| \le \left\lfloor \frac{\lfloor(\ell + 2(r-1))/r\rfloor + 2(r-1)}{r} \right\rfloor. \tag{2}$$

Define

$$I(\ell, r) = \left\lfloor \frac{\lfloor(\ell + 2(r-1))/r\rfloor + 2(r-1)}{r} \right\rfloor.$$

Let $\mathbf{w}$ be the ω-word over Σ_n with prefix $123\cdots(n-1)$ and Pansiot encoding $b(\mathbf{w}) = h^\omega(1)$. We will show that $\mathbf{w}$ contains no $\left(\frac{n}{n-1}\right)^+$-powers. Suppose to the contrary that pe is a repetition in $\mathbf{w}$ with $|pe|/|p| > n/(n-1)$ and e a prefix of p.

First suppose that $|e| \geq (n-1)$. Let $PE = b(pe)$. Then PE is a kernel repetition. Let η' be the longest common prefix of $h(0)$ and $h(1)$. As in the previous section, replacing pe and PE by longer repetitions of period $|P|$ if necessary, we may assume that $h^\omega(1)$ contains a kernel repetition $\mathcal{PE}$ with nonmarkable excess $\mathcal{E}$ such that $|PE| = r^s|\mathcal{PE}| + |\eta'|\sum_{i=0}^{s-1} r^i$ and $|P| = r^s|\mathcal{P}|$.

We find that

$$\begin{aligned}
1 + \frac{1}{n-1} &= \frac{n}{n-1} \\
&< \frac{|pe|}{|p|} \\
&= \frac{|PE| + n - 1}{|P|} \\
&= \frac{r^s|\mathcal{PE}| + |\eta'|\sum_{i=0}^{s-1} r^i + n - 1}{r^s|\mathcal{P}|} \\
&= \frac{r^s|\mathcal{P}| + r^s|\mathcal{E}|}{r^s|\mathcal{P}|} + \frac{|\eta'|\sum_{i=1}^{s} r^{-i}}{|\mathcal{P}|} + \frac{n-1}{r^s|\mathcal{P}|} \\
&< 1 + \frac{1}{|\mathcal{P}|}\left(|\mathcal{E}| + |\eta'|\frac{r}{r-1} + n - 1\right)
\end{aligned}$$

so that

$$|\mathcal{P}| < (n-1)\left(|\mathcal{E}| + |\eta'|\frac{r}{r-1} + n - 1\right)$$

and

$$\begin{aligned}
|\mathcal{PE}| &< |\mathcal{E}| + (n-1)\left(|\mathcal{E}| + |\eta'|\frac{r}{r-1} + n - 1\right) \\
&\leq r + (n-1)\left(r + (r-1)\frac{r}{r-1} + n - 1\right) \\
&\leq 4n + (n-1)(9n-1) \\
&= 9n^2 - 6n + 1.
\end{aligned}$$

We use that $|\mathcal{E}| < r$ (since all factors of $h^\omega(1)$ of length r or greater are markable) and $r \leq 4n$ (as observed in Section 2). Finally, since η' is a proper prefix of $h(0)$, $|\eta'| < r$.

One verifies that $I(9n^2 - 6n + 1, r) = 2$. Since every length 2 factor of $h^\omega(1)$ is a factor of 0110, word $b(PE)$ must be a factor of $h^2(0110)$. Let v be the word of Σ_n with prefix $123\cdots(n-1)$ and Pansiot encoding $h^2(0110)$. Since $b(PE)$ is a kernel repetition, word v contains a repetition $\hat{p}\hat{e}$ with $|\hat{e}| \geq n-1$. However, a computer search shows that v contains no such repetition.

We conclude that $|e| \leq n-2$. In this case,

$$\begin{aligned}\frac{n}{n-1} < \frac{|pe|}{|p|} \quad &\Longrightarrow \quad |e|n > |pe| \\ &\Longrightarrow \quad (n-2)n - (n-1) > |b(pe)| \\ &\Longrightarrow \quad n^2 - 3n + 1 > |b(pe)|.\end{aligned}$$

However, $n^2 - 3n + 1 < 9n^2 - 6n + 1$, so that again $b(pe)$ must be a factor of $h^2(0110)$, and v, defined as in the previous case, must contain a $\left(\frac{n}{n-1}\right)^+$-power. However, a computer search shows that word v is $\left(\frac{n}{n-1}\right)^+$-power free.

We have proved the following:

Main result. Let $\mathbf{w}$ be the word over Σ_n with prefix $123\cdots(n-1)$ and Pansiot encoding $b(\mathbf{w}) = h^\omega(1)$. Word $\mathbf{w}$ contains no $\left(\frac{n}{n-1}\right)^+$-powers.

6. Final remarks

Our result builds on that of [13], but uses somewhat simpler arguments, taking advantage of properties of our specific morphisms. In addition, we have specified bounds for the various computer checks, rather than invoking mere decidability.

A large simplification results from the fact that our morphisms give binary words with no kernel repetitions at all (even of small exponent). When moving from PE to $\pi\eta$ in Section 4 one can give the relationship between the exponents of these two kernel repetitions:

$$\frac{|PE|}{|P|} = \frac{|\pi\eta|}{|\pi|} + \frac{|\eta'|}{r|\pi|}.$$

If it takes s steps to arrive from repetition PE to a repetition $\pi\eta$ with nonmarkable excess, then the exponents differ by

$$\frac{|\eta'|}{|\pi|}\sum_{i=1}^{s} r^{-i}.$$

In the notation of [13], PE corresponds to $\mu^s(\pi,\eta)$ and has the largest exponent among the $\mu^i(\pi,\eta)$, $0 \leq i \leq s$. Unfortunately, [13] is marred by getting this backward, saying that for uniform morphisms the largest exponent occurs either for $i = 0$ or for $i = 1$!

In fact, for the morphisms given for $n = 5, 6, 7$, η' is empty, so the aforementioned reversal has no effect. However, for $8 \leq n \leq 11$, η' is nonempty, and a more complicated check than indicated in [13] is necessary to ensure that the constructions given by Moulin Ollagnier actually work. Happily, they do indeed work, as a more careful check shows.

Finally, we mention a few points regarding the search strategy for finding morphisms. The second step of the strategy indicated in [13] calls for enumerating all candidate morphisms of short enough length. A priori, this involves enumerating all binary words of length at most r which are Pansiot encodings of $\left(\frac{n}{n-1}\right)^+$-free words over Σ_n. Initially this was part of our strategy. Unfortunately, our experience supports the conjecture in [17], that the number of these words grows approximately as 1.24^r (independently of n).

For successive r values we looked at all possible pairs $\langle h(0), h(1)\rangle$ such that $|h(0)|, |h(1)| \leq r$, where $h(0), h(1)$ were Pansiot encodings of $\left(\frac{n}{n-1}\right)^{+}$-free words and satisfied the algebraic condition; this allowed us to verify the claim of [13] that the morphisms presented therein for $5 \leq n \leq 11$ are shortest possible 'convenient morphisms'; the uniforms are all uniform, with lengths around $4n-4$ in each case. However, storing all legal Pansiot encodings up to length $4n-4$ fills up a laptop with 2G RAM at around $n = 15$. Therefore, our search program had to migrate to computers with more and more RAM, simply to store Pansiot encodings. On the plus side, we found a great number of 'convenient morphisms' for $12 \leq n \leq 17$, not just the ones presented in this paper.

To find morphisms for n up to 26 (and indeed for various other higher values of n) we adopted a different strategy. Using backtracking, we found legal Pansiot encodings of length exactly $r = 4n-4$ (or $r = 4n$, in the case $n = 21$), but only saved encodings v for which the permutation $\sigma(v)$ was an r-cycle (and thus a candidate for $h(1)$) or an $(r-1)$-cycle (and thus a candidate for $h(0)$). As soon as a candidate for $h(i)$ was found, it was tested together with each previously found candidate for $h(1-i)$ to see whether a 'convenient morphism' could be formed, in which case the search terminated. This search used very little memory and terminated quickly. For $n = 26$, our C^{++} code found the morphism in just over 6 hours.

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We have recently been informed that Dr. Michaël Rao has also announced a proof of Dejean's conjecture [15].

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