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FINITE DIFFERENCE SOLUTION OF A NONLINEAR KLEIN-GORDON EQUATION WITH AN EXTERNAL SOURCE

G. BERIKELASHVILI, O. JOKHADZE, S. KHARIBEGASHVILI, AND B. MIDODASHVILI

Abstract. In this paper, we consider the Darboux problem for a (1+1)-dimensional cubic nonlinear Klein-Gordon equation with an external source. Stable finite difference scheme is constructed on a four-point stencil, which does not require additional iterations for passing from one level to another. It is proved, that the finite difference scheme converges with the rate $O(h^2)$, when the exact solution belongs to the Sobolev space W_2^2.

1. Introduction

We consider cubic the nonlinear Klein-Gordon type equation

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + m^2 v + \lambda v^3 = \varphi(x,t), \tag{1.1}$$

where $\lambda > 0,\ m \geq 0$ are constants.

These equations arise in the quantum field theory [19], solid and high energy physics [3], radiation theory [7], the thermal equilibrium properties of solitary wave solutions (“kinks”) in a classical ϕ^4 field theory [1].

Many works are dedicated to the investigation of boundary value problems for these equations, among them we mention [13, 14, 16, 20, 23]. The difference schemes of some problems for nonlinear wave equations have been studied in [5, 17, 18, 24]. In particular, in [17] difference schemes are shown for semilinear wave equations, in which second-order convergence is shown for $u \in C^4$. Certain numerical methods (the symplectic and multisymplectic finite difference schemes and the spectral methods, etc.) are devoted to homogeneous nonlinear Klein-Gordon equations [26]–[30]. In [30] the authors propose a spectral method using as a basis the Legendre functions. It shows the convergence with rate $O(N^{-2})$ for the equation with cubic nonlinearity, where N denotes the maximal degree of polynomials and the exact solution belongs to $H^2(0,T;H^3(-\infty;\infty))$.

In the domain $\{(x,t)|0 < x < t,\ 0 < t < T\}$ for equation (1.1) we consider the first Darboux problem with the following boundary conditions:

$$v(0,t) = 0, \quad v(t,t) = 0, \quad 0 \leq t \leq T. \tag{1.2}$$

Problem (1.1), (1.2) represents a problem with moving boundary (see e.g. [10, 15]).

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Note that problem (1.1), (1.2) arises in mathematical modeling of various physical processes, e.g. in the study of harmonic oscillations of a chock in a supersonic flow [9], and also in the investigation of oscillation of a string with a piston beaded on it, which is in a cylinder filled with viscous liquid [22].

Corresponding to (1.1), (1.2), linear problems were considered in works of Goursat [11], Darboux [6], Hadamard [12], Gellerstedt [8], Tricomi [21] and Bitsadze [4].

The paper is organized as follows. In the following section we present a statement of the problem and main results. Then, in the section 3 we prove auxiliary statements and in the section 4 we give the proof of the main results. In the last section the numerical experiments are discussed.

2. Statement of the problem and main results

For obtaining the difference solution of problem (1.1), (1.2) we rewrite it in characteristic variables. Using transformation of independent variables $x \to x - t,\ t \to x + t$, problem (1.1), (1.2) can be rewritten in the form

$$\frac{\partial^2 u}{\partial x \partial t} + m^2 u + \lambda u^3 = f(x,t), \quad (x,t) \in D_T, \tag{2.1}$$

$$u(x,0) = 0, \quad 0 \le x \le T, \quad u(t,t) = 0, \quad 0 \le t \le T/2, \tag{2.2}$$

where

$$u(x,t) := v(x-t, x+t), f(x,t) := \varphi(x-t, x+t),$$
$$D_\tau := \{(x,t) | t < x < \tau - t, 0 < t < \tau/2\}.$$

By $W_p^m(D)$ we denote a Sobolev space with the norm defined by

$$\|u\|_{W_p^m(D)} := \Big(\sum_{k=0}^{m} |u|^p_{W_p^k(D)} \Big)^{1/p}, \quad |u|_{W_p^m(D)} := \Big(\sum_{\alpha+\beta=m} \Big\| \frac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial t^\beta} \Big\|^p_{L_p(D)} \Big)^{1/p}.$$

In particular, for $m = 0$ we have $W_p^0 = L_p$.

It is proved in [2] that if the right-hand side of equation (1.1) belongs to C^1, then problem (1.1), (1.2), and therefore (2.1), (2.2), have classical solutions. Since input data in practical problems are not always sufficiently smooth, it is expedient to substantiate corresponding difference schemes in Sobolev spaces [25]. In the investigation of the difference scheme we require only that the solution of problem (2.1), (2.2) belongs to the space W_2^2.

Denote $h := T/n$. In the domain $\bar{D}_T$ for $k \ge 2$ let us introduce the meshes

$$Q_{2k} := \{(x_i,t_j) | x_i = ih,\ t_j = jh,\ j = 1,...,k-1,\ i = j+1,...,2k-j\},$$
$$Q_{2k-1} := \{(x_i,t_j) | x_i = ih,\ t_j = jh,\ j = 1,...,k-1,\ i = j+1,...,2k-j-1\}.$$

For the grid points, laying on the same straight lines, parallel to $x + t = 0$, let us introduce the following notations:

$$\gamma_{2k} := \{(x_i,t_j) |\ i+j = 2k, \quad j = 1,...,k-1\},$$
$$\gamma_{2k}^+ := \{(x_i,t_j) |\ i+j = 2k, \quad j = 1,...,k\},$$
$$\gamma_{2k-1} := \gamma_{2k-1}^+ := \{(x_i,t_j) |\ i+j = 2k-1, \quad j = 1,...,k-1\}.$$

Moreover, let $e_{ij} := (x_{i-1},x_i) \times (t_{j-1},t_j)$,

$$\gamma_x := \{(x_i,0) | i = 1,2,...,n\}, \quad \gamma_t = \{(t_j,t_j) | j = 1,2,...,[n/2]\},$$

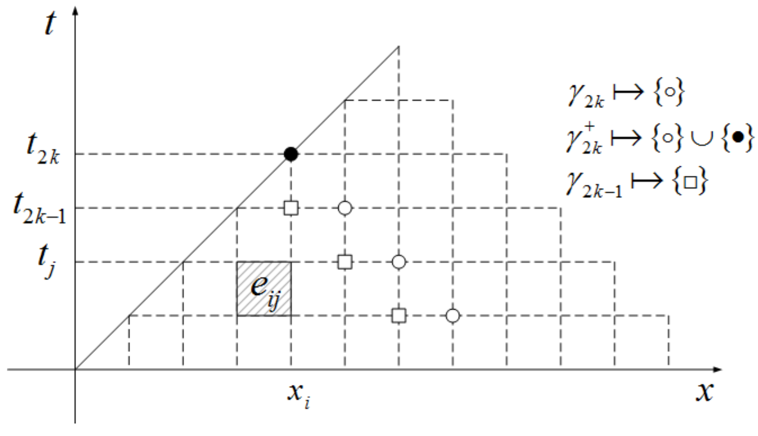

FIGURE 1. Meshes

where $[\cdot]$ denotes the integer part of a number.

For mesh functions we use the notation $U_i^j := U(x_i, t_j)$. Denote

$$(\partial U)_i^j := (U_i^j - U_{i-1}^{j-1})/h,\ U_{x,i}^j := (U_i^j - U_{i-1}^j)/h,\ U_{t,i}^j := (U_i^j - U_i^{j-1})/h.$$

Let H be the set of mesh functions satisfying conditions (2.4). Define in H the inner product and corresponding norm as follows:

$$(U,V)_{Q_s} := h^2 \sum_{(x,t)\in Q_s} U(x,t)V(x,t), \quad \|U\|^2_{Q_s} := (U,U)_{Q_s}.$$

Let

$$\|U\|_{C(\gamma)} := \max_{(x,t)\in\gamma} |U(x,t)|, \quad |U|^2_{W_2^1(\gamma_s)} := h\sum_{\gamma_s}(U_x)^2 + h\sum_{\gamma_s^+}(U_t)^2.$$

Let us approximate problem (2.1), (2.2) by the difference scheme

$$\mathcal{L}U + \lambda\Lambda U = F, \quad (x,t) \in Q_n, \tag{2.3}$$

$$U|_{\gamma_x} = U|_{\gamma_t} = 0, \tag{2.4}$$

where

$$(\mathcal{L}U)_i^j := (U_i^j - U_{i-1}^j + U_{i-1}^{j-1} - U_i^{j-1})/h^2 + 0.5m^2(U_i^j + U_{i-1}^{j-1}),$$

$$(\Lambda U)_i^j := (U_i^j + U_{i-1}^{j-1})\big((U_{i-1}^j)^2 + (U_i^{j-1})^2\big)/4, \quad F_i^j := \frac{1}{h^2}\int_{e_{ij}} f(x,t)\,dxdt.$$

Note that difference scheme (2.3), (2.4) is an explicit one: first for $j = 1$ we calculate all unknown values of U_i^j, then the same procedure is done for $j = 2, \ldots$, and each equation contains unknown values in linear form. Therefore, we conclude that the solution of difference scheme (2.3), (2.4) exists and it is unique.

Let $Z := U - u$, where u is the exact solution of problem (2.1), (2.2) and U is the solution of finite difference scheme (2.3), (2.4). For the discretization error Z we obtain the problem

$$\mathcal{L}Z := \Psi - \lambda(\Lambda U - \Lambda u), \quad Z|_{\gamma_x} = Z|_{\gamma_t} = 0, \tag{2.5}$$

where we get the truncation error

$$\Psi := F - (\mathcal{L}u + \lambda\Lambda u). \tag{2.6}$$

Theorem 2.1. *For the solution of difference scheme* (2.3), (2.4) *the following estimate is valid:*

$$\|U\|_{C(Q_n)} \le \delta, \quad \delta := T\sqrt{e/2}\|f\|_{L_2(D_T)}. \tag{2.7}$$

Theorem 2.2. *For the error of the solution of difference scheme* (2.3), (2.4) *the following estimate is valid:*

$$\big|U - u\big|_{W_2^1(\gamma_n)} \le c_1\|\Psi\|_{Q_n}, \tag{2.8}$$

where $c_1 := 2\sqrt{2T}\exp\big[2\{1 + (3\lambda\delta^2T^2/2)^2\}\big]$ *and* δ *is defined in* (2.7).

Theorem 2.3. *The solution of difference scheme* (2.3), (2.4) *converges to the solution of problem* (2.1), (2.2) *and the following estimates are valid:*

$$\big|U - u\big|_{W_2^1(\gamma_n)} \le ch^2\|u\|_{W_2^2(Q_T)}, \quad \|U - u\|_{C(Q_n)} \le ch^2\|u\|_{W_2^2(Q_T)}, \tag{2.9}$$

where the constant $c > 0$ *is independent of* h.

Corollary 2.1. *Returning to the geometry of problem* (1.1), (1.2) *we have*

$$\gamma_{2k} \to \widetilde{\gamma}_{2k} := \{(2ih, 2kh) | i = 1, 2, ..., k-1\},$$
$$\gamma_{2k-1} \to \widetilde{\gamma}_{2k-1} := \{((2i-1)h, (2k-1)h) | i = 1, 2, ..., k-1\}$$

and the corresponding difference solution V *of problem* (1.1), (1.2) *is defined as follows:*

$$V_{2i}^{2k} = U_{k+i}^{k-i}, \quad V_{2i-1}^{2k-1} = U_{k+i-1}^{k-i}, \quad i = 1, 2, ..., k-1,$$

for the convergence of which the same results are valid.

3. Auxiliary statements

Lemma 3.1. *For the solution of problem* (2.1), (2.2) *the following estimate is valid:*

$$\|u\|_{C(\bar{D}_T)} \le \delta, \tag{3.1}$$

where δ *is defined in* (2.7).

Proof. Multiplying equation (2.1) by $\partial u/\partial x + \partial u/\partial t$ and integrating in the domain D_τ, we obtain

$$\int_0^{\tau/2} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2\Big]\Big|_{x=\tau-t} dt + 2m^2 \int_0^{\tau/2} u^2(\tau - t, t)\, dt$$
$$+\lambda \int_0^{\tau/2} u^4(\tau - t, t)\, dt = 2\int_{D_\tau} \Big(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\Big) f\, dxdt,$$

hence

$$\int_0^{\tau/2} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2\Big]\Big|_{x=\tau-t} dt \le 2\int_{D_\tau} \Big(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\Big) f\, dxdt$$
$$\le \frac{2}{\varepsilon}\int_{D_\tau} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2\Big] dxdt + \varepsilon\|f\|^2_{L_2(D_\tau)}. \tag{3.2}$$

Using transformation of variables we receive

$$\int_{D_\tau} v(x,t)\, dxdt = \int_0^\tau d\sigma \int_0^{\sigma/2} v(\sigma - t, t)\, dt.$$

Therefore, by choosing $\varepsilon = 2\tau$ in (3.2), we have

$$w(\tau) \le \frac{1}{\tau}\int_0^\tau w(\sigma)\,d\sigma + 2\tau\|f\|^2_{L_2(D_\tau)},$$

where

$$w(\sigma) := \int_0^{\sigma/2}\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2\Big]\Big|_{x=\sigma-t}\,dt.$$

Thus, due to Gronwall's lemma we have

$$w(\tau) \le 2\tau e\|f\|^2_{L_2(D_\tau)}. \tag{3.3}$$

Taking into account boundary conditions (2.2) we have

$$u(\tau-\eta,\eta) = \int_0^\eta \Big(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\Big)\Big|_{x=\tau-t}\,dt,\ \ u(\tau-\eta,\eta) = \int_\eta^{\tau/2} \Big(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\Big)\Big|_{x=\tau-t}\,dt.$$

Whence

$$2|u(\tau-\eta,\eta)| \le \int_0^{\tau/2}\Big(\Big|\frac{\partial u}{\partial x}\Big| + \Big|\frac{\partial u}{\partial t}\Big|\Big)\Big|_{x=\tau-t}\,dt$$

and, also,

$$4|u(\tau-\eta,\eta)|^2 \le \tau\int_0^{\tau/2}\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2\Big]\Big|_{x=\tau-t}\,dt,$$

so

$$|u(\tau-\eta,\eta)|^2 \le (\tau/4)w(\tau). \tag{3.4}$$

Taking into account that $\|f\|^2_{L_2(D_\tau)}$ is a nondecreasing function with respect to τ, from (3.3), (3.4) we obtain inequality (3.1). □

Let us study the properties of the difference operators of scheme (2.3), (2.4).

Lemma 3.2. *For any $U \in H$ the following estimate is valid:*

$$(\mathcal{L}U, \partial U)_{Q_s} \ge 0.5|U|^2_{W_2^1(\gamma_s)}, \quad s = 3, 4, \cdots. \tag{3.5}$$

Proof. Let $(\mathcal{L}_m U)_i^j := 0.5m^2(U_i^j + U_{i-1}^{j-1})$. For $s = 2k$ even, we have

$$(\mathcal{L}U - \mathcal{L}_m U, \partial U)_{Q_{2k}} = \frac{1}{h}\sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j}(U_i^j - U_{i-1}^j + U_{i-1}^{j-1} - U_i^{j-1})(U_i^j - U_{i-1}^{j-1})$$

$$= \frac{1}{2h}(I_1 + I_2), \tag{3.6}$$

where

$$I_1 := \sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j}(U_i^j - U_{i-1}^j + U_{i-1}^{j-1} - U_i^{j-1})(U_i^j - U_{i-1}^j + U_i^{j-1} - U_{i-1}^{j-1}),$$

$$I_2 := \sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j}(U_i^j - U_{i-1}^j + U_{i-1}^{j-1} - U_i^{j-1})(U_i^j - U_i^{j-1} + U_{i-1}^j - U_{i-1}^{j-1}).$$

By simple transformations we obtain

$$I_1 = \sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j}\Big((U_i^j - U_{i-1}^j)^2 - (U_i^{j-1} - U_{i-1}^{j-1})^2\Big) = \sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j}(U_i^j - U_{i-1}^j)^2$$

$$-\sum_{j=1}^{k-2}\sum_{i=j+2}^{2k-j-1}(U_i^j - U_{i-1}^j)^2 = \sum_{i=k}^{k+1}(U_i^{k-1} - U_{i-1}^{k-1})^2$$

$$+\sum_{j=1}^{k-2}\Big((U_{2k-j}^j - U_{2k-j-1}^j)^2 + (U_{j+1}^j)^2\Big)$$

$$= \sum_{j=1}^{k-1}\Big((U_{2k-j}^j - U_{2k-j-1}^j)^2 + (U_{j+1}^j)^2\Big), \tag{3.7}$$

$$I_2 = \sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j}\Big((U_i^j - U_i^{j-1})^2 - (U_{i-1}^j - U_{i-1}^{j-1})^2\Big)$$

$$= \sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j}(U_i^j - U_i^{j-1})^2 - \sum_{j=1}^{k-1}\sum_{i=j}^{2k-j-1}(U_i^j - U_i^{j-1})^2$$

$$= \sum_{j=1}^{k-1}\Big((U_{2k-j}^j - U_{2k-j}^{j-1})^2 - (U_j^{j-1})^2\Big). \tag{3.8}$$

Furthermore, $I_1 + I_2$ can be represented as follows:

$$I_1 + I_2 = \sum_{j=1}^{k-1}\Big(h^2(U_{x,2k-j}^j)^2 + (U_{j+1}^j)^2\Big) + \sum_{j=1}^{k-1}\Big(h^2(U_{t,2k-j}^j)^2 - (U_j^{j-1})^2\Big)$$

$$= h^2\sum_{j=1}^{k-1}(U_{x,2k-j}^j)^2 + h^2\sum_{j=1}^{k-1}(U_{t,2k-j}^j)^2 + (U_k^{k-1})^2.$$

But

$$U_k^{k-1} = -(U_k^k - U_k^{k-1}) = -hU_{t,k}^k;$$

therefore,

$$I_1 + I_2 = h^2\sum_{j=1}^{k-1}(U_{x,2k-j}^j)^2 + h^2\sum_{j=1}^{k}(U_{t,2k-j}^j)^2,$$

and from (3.6) we obtain

$$(\mathcal{L}U - \mathcal{L}_m U, \partial U)_{Q_{2k}} = 0.5|U|^2_{W_2^1(\gamma_{2k})}. \tag{3.9}$$

By the analogous argument for the equality ($s = 2k-1$),

$$(\mathcal{L}U - \mathcal{L}_m U, \partial U)_{Q_{2k-1}} = 0.5|U|^2_{W_2^1(\gamma_{2k-1})} \tag{3.10}$$

is valid. It is easy to verify that

$$(\mathcal{L}_m U, \partial U)_{Q_s} = 0.5m^2\Big(h\sum_{\gamma_s}U^2 + h\sum_{\gamma_{s-1}}U^2\Big) \geq 0,$$

and from (3.9), (3.10) the validity of estimate (3.5) follows.

Lemma 3.2 is proved. □

Lemma 3.3. *For any $U \in H$ the following estimate is valid:*

$$(\Lambda U, \partial U)_{Q_s} \geq 0, \quad s = 3, 4, \cdots . \tag{3.11}$$

Proof. For $s = 2k$ even, we have

$$(\Lambda U, \partial U)_{Q_{2k}} = \frac{h}{4} \sum_{j=1}^{k-1} \sum_{i=j+1}^{2k-j} \Big((U_i^j)^2 - (U_{i-1}^{j-1})^2 \Big) \Big((U_{i-1}^j)^2 + (U_i^{j-1})^2 \Big)$$

$$= \frac{h}{4} \sum_{j=1}^{k-1} \sum_{i=j+1}^{2k-j} \Big((U_i^j U_{i-1}^j)^2 + (U_i^j U_i^{j-1})^2 - (U_{i-1}^{j-1} U_{i-1}^j)^2 - (U_{i-1}^{j-1} U_i^{j-1})^2 \Big)$$

$$= \frac{h}{4} \sum_{j=1}^{k-1} \Big((U_{2k-j}^j U_{2k-j}^{j-1})^2 + (U_{2k-j}^j U_{2k-j-1}^j)^2 \Big) \geq 0. \tag{3.12}$$

Analogously, for $s = 2k - 1$ odd, we have

$$(\Lambda U, \partial U)_{Q_{2k-1}} = \frac{h}{4} \sum_{j=1}^{k-1} \Big((U_{2k-j-1}^j U_{2k-j-1}^{j-1})^2 + (U_{2k-j-1}^j U_{2k-j-2}^j)^2 \Big) \geq 0. \tag{3.13}$$

Lemma 3.3 is proved. □

Lemma 3.4. *For any $U \in H$ the following estimate is valid:*

$$\|\partial U\|_{Q_s}^2 \leq 2h \sum_{l=3}^{s} |U|_{W_2^1(\gamma_l)}^2, \quad s = 3, 4, \cdots . \tag{3.14}$$

Proof. Let us single out the cases for even and odd s. For $s = 2k$ we have

$$\begin{aligned} \|\partial U\|_{Q_{2k}}^2 =& h^2 \sum_{j=1}^{k-1} \sum_{i=j+1}^{2k-j} (U_{x,i}^j + U_{t,i-1}^j)^2 \leq 2h^2 \sum_{j=1}^{k-1} \sum_{i=j+1}^{2k-j} (U_{x,i}^j)^2 \\ &+ 2h^2 \sum_{j=1}^{k-1} \sum_{i=j+1}^{2k-j} (U_{t,i-1}^j)^2 \\ &= 2h^2 \sum_{l=3}^{2k} \sum_{\gamma_l} (U_x)^2 + 2h^2 \sum_{l=3}^{2k-1} \sum_{\gamma_l^+} (U_t)^2. \end{aligned} \tag{3.15}$$

Analogously, for $s = 2k - 1$ we have

$$\begin{aligned} \|\partial U\|_{Q_{2k-1}}^2 &\leq 2h^2 \sum_{j=1}^{k-1} \sum_{i=j+1}^{2k-j-1} (U_{x,i}^j)^2 + 2h^2 \sum_{j=1}^{k-1} \sum_{i=j+1}^{2k-j-1} (U_{t,i-1}^j)^2 \\ &= 2h^2 \sum_{l=3}^{2k-1} \sum_{\gamma_l} (U_x)^2 + 2h^2 \sum_{l=3}^{2k-2} \sum_{\gamma_l^+} (U_t)^2. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we obtain

$$\|\partial U\|_{Q_s}^2 \leq 2h \sum_{l=3}^{s-1} |U|_{W_2^1(\gamma_l)}^2 + 2h^2 \sum_{\gamma_s} |U_x|^2,$$

and, therefore, Lemma 3.4 is valid □

Lemma 3.5. *For any $U \in H$ the following estimates are valid:*

$$\|U\|^2_{C(\gamma_s)} \le c_2 |U|^2_{W_2^1(\gamma_s)}, \quad c_2 = sh/4, \quad s = 3, 4, \cdots . \tag{3.17}$$

$$\|U\|^2_{Q_s} \le c_3 h \sum_{l=3}^{s} |U|^2_{W_2^1(\gamma_l)}, \quad c_3 := (sh)^2/8, \quad s = 3, 4, \cdots . \tag{3.18}$$

Proof. If $(x_i, t_j) \in \gamma_{2k}$, i.e., $i + j = 2k$, then

$$U_i^j = h \sum_{s=i}^{2k-1} U_{t,s}^{2k-s} - h \sum_{s=i+1}^{2k-1} U_{x,s}^{2k-s}, \quad U_i^j = h \sum_{s=k+1}^{i} U_{x,s}^{2k-s} - h \sum_{s=k}^{i-1} U_{t,s}^{2k-s}.$$

According to these equalities we have

$$2|U_i^j| \le h \sum_{s=k+1}^{2k-1} |U_{x,s}^{2k-s}| + h \sum_{s=k}^{2k-1} |U_{t,s}^{2k-s}|.$$

Whence

$$|U_i^j|^2 \le 0.5\Big(\sum_{s=k+1}^{2k-1} h|U_{x,s}^{2k-s}|\Big)^2 + 0.5\Big(\sum_{s=k}^{2k-1} h|U_{t,s}^{2k-s}|\Big)^2$$
$$\le \frac{(k-1)h}{2} \sum_{s=k+1}^{2k-1} h|U_{x,s}^{2k-s}|^2 + \frac{kh}{2} \sum_{s=k}^{2k-1} h|U_{t,s}^{2k-s}|^2.$$

Thus,

$$\|U\|^2_{C(\gamma_{2k})} \le (kh/2)|U|^2_{W_2^1(\gamma_{2k})}. \tag{3.19}$$

Analogously, if $(x_i, t_j) \in \gamma_{2k-1}$, i.e. $i + j = 2k - 1$, we obtain

$$2|U_i^j| \le h \sum_{s=k}^{2k-2} |U_{x,s}^{2k-s-1}| + h \sum_{s=k}^{2k-2} |U_{t,s}^{2k-s-1}|,$$

whence

$$\|U\|^2_{C(\gamma_{2k-1})} \le \frac{(k-1)h}{2} |U|^2_{W_2^1(\gamma_{2k-1})}. \tag{3.20}$$

Since the quantity of mesh points of γ_l is less than $l/2$, we have

$$h^2 \sum_{Q_s} U^2 = h^2 \sum_{l=3}^{s} \sum_{\gamma_l} U^2 \le h^2 (l/2) \sum_{l=3}^{s} \|U\|^2_{C(\gamma_l)}$$
$$\le h^2 (l/2)(lh/4) \sum_{l=3}^{s} \|U\|^2_{W_2^1(\gamma_l)} \le (sh)^2/8 \sum_{l=3}^{s} h\|U\|^2_{W_2^1(\gamma_l)}.$$

Lemma 3.5 is proved. □

Lemma 3.6 (Gronwall's discrete inequality)**.** *Let w_s, g_s be nonnegative sequences of numbers and g_s nondecreasing. Then from the inequalities*

$$w_s \le c \sum_{i=k}^{s-1} w_i + g_s, \; s = k+1, k+2, ..., n, \quad w_k \le g_k, \quad c > 0$$

it follows that

$$w_n \le g_n \exp(c(n-k)).$$

Lemma 3.7. *Let u be the exact solution of problem* (2.1), (2.2) *and U the solution of finite difference scheme* (2.3), (2.4). *Then*

$$\|\Lambda U - \Lambda u\|^2_{Q_s} \le c_4 h \sum_{l=3}^{s-1} |U-u|^2_{W_2^1(\gamma_l)} + (2/\lambda^2)\|\Psi\|^2_{Q_s}, \quad c_4 := (3\delta^2 sh/2)^2 \tag{3.21}$$

is valid, where Ψ is defined in (2.6).

Proof. Let $Z := U - u$. From the definition of the operator Λ we get

$$(\Lambda U - \Lambda u)^j_i = \Big(Z^j_i + Z^{j-1}_{i-1}\Big)\mathcal{M} + Z^j_{i-1}\mathcal{M}_1 + Z^{j-1}_i \mathcal{M}_2, \tag{3.22}$$

where

$$\mathcal{M} := \frac{1}{8}\Big((U^j_{i-1})^2 + (U^{j-1}_i)^2 + (u^j_{i-1})^2 + (u^{j-1}_i)^2\Big),$$
$$\mathcal{M}_1 := \frac{1}{8}(U+u)^j_{i-1}\Big((U+u)^j_i + (U+u)^{j-1}_{i-1}\Big),$$
$$\mathcal{M}_2 := \frac{1}{8}(U+u)^{j-1}_i\Big((U+u)^j_i + (U+u)^{j-1}_{i-1}\Big).$$

By virtue of (3.22) let us rewrite (2.5) in the form

$$\Big(Z^j_i + Z^{j-1}_{i-1}\Big)(1 + \lambda h^2 \mathcal{M} + m^2h^2/2) = Z^j_{i-1}(1 - \lambda h^2 \mathcal{M}_1) + Z^{j-1}_i(1 - \lambda h^2\mathcal{M}_2) + h^2\Psi^j_i. \tag{3.23}$$

By substitution of $(Z^j_i + Z^{j-1}_{i-1})$, by (3.23) into (3.22) we have

$$(\Lambda U - \Lambda u)^j_i = \frac{\mathcal{M} + \mathcal{M}_1(1 + m^2h^2/2)}{1 + \lambda h^2\mathcal{M} + m^2h^2/2} Z^j_{i-1} + \frac{\mathcal{M} + \mathcal{M}_2(1 + m^2h^2/2)}{1 + \lambda h^2\mathcal{M} + m^2h^2/2} Z^{j-1}_i + \frac{h^2\mathcal{M}}{1 + \lambda h^2\mathcal{M} + m^2h^2/2}\Psi^j_i.$$

Whence

$$|(\Lambda U - \Lambda u)^j_i| \le (\mathcal{M} + |\mathcal{M}_1|)|Z^j_{i-1}| + (\mathcal{M} + |\mathcal{M}_2|)|Z^{j-1}_i| + (1/\lambda)|\Psi^j_i|.$$

According to Lemma 3.1 and Theorem 2.1 we have $\|U\|_C,\ \|u\|_C \le \delta$. Therefore,

$$|(\Lambda U - \Lambda u)^j_i| \le (3\delta^2/2)(|Z^j_{i-1}| + |Z^{j-1}_i|) + (1/\lambda)|\Psi^j_i|.$$

Whence

$$|(\Lambda U - \Lambda u)^j_i|^2 \le 9\delta^4(|Z^j_{i-1}|^2 + |Z^{j-1}_i|^2) + (2/\lambda^2)|\Psi^j_i|^2$$

and

$$\sum_{\gamma_l} |\Lambda U - \Lambda u|^2 \le 18\delta^4 \sum_{\gamma_{l-1}} Z^2 + (2/\lambda^2)\sum_{\gamma_l}\Psi^2.$$

Thus,

$$\|\Lambda U - \Lambda u\|^2_{Q_s} = \sum_{l=3}^{s} h^2 \sum_{\gamma_l} |\Lambda U - \Lambda u|^2 \le 18\delta^4 \sum_{Q_{s-1}} h^2 Z^2 + (2/\lambda^2)\|\Psi\|^2_{Q_s}, \quad s \ge 4.$$

From this inequality, using (3.18) we obtain estimate (3.21).

Lemma 3.7 is proved. □

4. Proof of main results

Proof of Theorem 2.1. Let us show that for the solution of the difference scheme (2.3),(2.4) the following estimate is valid:

$$|U|^2_{W_2^1(\gamma_k)} \le 2khe\|F\|^2_{Q_k}, \quad k = 3, 4, \dots, n. \tag{4.1}$$

Note that the difference equation on grid Q_3, consisting of point $(2h, h)$ only, has the form

$$(1/h^2 + m^2/2)U_2^1 = F_2^1, \quad \Rightarrow (U_2^1)^2 \le h^4 (F_2^1)^2;$$

therefore,

$$|U|^2_{W_2^1(\gamma_3)} = (2/h)(U_2^1)^2 \le 2h^3 (F_2^1)^2 = 2h\|F\|^2_{Q_3} \tag{4.2}$$

and (4.1) is valid for $k = 3$.

Furthermore, due to Lemmas 3.2, 3.3 we have

$$(1/2)|U|^2_{W_2^1(\gamma_s)} \le (\mathcal{L}U + \lambda\Lambda U, \partial U)_{Q_s} = (F, \partial U)_{Q_s} \le \|F\|_{Q_s}\|\partial U\|_{Q_s}, \quad s \ge 3,$$

whence, by use of ε-inequality we receive

$$(1/2)|U|^2_{W_2^1(\gamma_s)} \le (1/2\varepsilon)\|\partial U\|^2_{Q_s} + (\varepsilon/2)\|F\|^2_{Q_s}. \tag{4.3}$$

According to Lemma 3.4 we have

$$(1/2)|U|^2_{W_2^1(\gamma_s)} \le (h/\varepsilon)\sum_{i=3}^{s} |U|^2_{W_2^1(\gamma_i)} + (\varepsilon/2)\|F\|^2_{Q_s}. \tag{4.4}$$

From (4.4) it follows that

$$\Big(\frac{1}{2} - \frac{h}{\varepsilon}\Big)|U|^2_{W_2^1(\gamma_s)} \le \frac{h}{\varepsilon}\sum_{i=3}^{s-1} |U|^2_{W_2^1(\gamma_i)} + \frac{\varepsilon}{2}\|F\|^2_{Q_s}, \quad s = 4, 5, \dots, k. \tag{4.5}$$

Let us choose $\varepsilon = 2h(k-2)$ in (4.5). We obtain

$$|U|^2_{W_2^1(\gamma_s)} \le \frac{1}{k-3}\sum_{i=3}^{s-1} |U|^2_{W_2^1(\gamma_i)} + 2hk\|F\|^2_{Q_s}, \quad s = 4, 5, \dots, k. \tag{4.6}$$

Using Lemma 3.6, from (4.2), (4.6) we obtain estimate (4.1).

Due to $kh \le T$ and $\|F\|_{Q_k} \le \|F\|_{Q_n}$, and also by virtue of Lemma 3.5, from (4.1) we receive $\|U\|^2_{C(\gamma_k)} \le (T^2 e/2)\|F\|^2_{Q_n}$ and therefore,

$$\|U\|^2_{C(Q_n)} \le (T^2 e/2)\|F\|^2_{Q_n}. \tag{4.7}$$

The following estimate is valid:

$$\|F\|_{Q_n} \le \|f\|_{L_2(D_T)}. \tag{4.8}$$

Indeed, for even n, taking into account the expression of function F, we have

$$\|F\|^2_{Q_{2k}} = \sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j} \frac{1}{h^2}\Big(\int_{e_{ij}} f(x,t)\,dxdt\Big)^2$$

$$\le \sum_{j=1}^{k-1}\sum_{i=j+1}^{2k-j} \int_{e_{ij}} f^2(x,t)\,dxdt \le \|f\|^2_{L_2(D_T)}$$

and the same estimate for odd n. The validity of Theorem 2.1 follows from (4.7), (4.8). □

Proof of Theorem 2.2. Multiplying equation (2.5) by ∂Z and summing up on Q_s we obtain

$$(\mathcal{L}Z, \partial Z)_{Q_s} = (\Psi, \partial Z)_{Q_s} - \lambda(\Lambda U - \Lambda u, \partial Z)_{Q_s},$$

whence using Lemma 3.2 and ε-inequality we receive

$$\begin{aligned} 0.5|Z|^2_{W_2^1(\gamma_s)} &\le \|\Psi\|_{Q_s}\|\partial Z\|_{Q_s} + \lambda\|\Lambda U - \Lambda u\|_{Q_s}\|\partial Z\|_{Q_s} \\ &\le T\|\Psi\|^2_{Q_s} + (2T)^{-1}\|\partial Z\|^2_{Q_s} + T\lambda^2\|\Lambda U - \Lambda u\|^2_{Q_s}. \end{aligned}$$

Due to Lemmas 3.4 and 3.7 we have

$$|Z|^2_{W_2^1(\gamma_s)} \le (2h/T)\sum_{l=3}^{s}|Z|^2_{W_2^1(\gamma_l)} + 4T\|\Psi\|^2_{Q_s} + 2T\lambda^2 c_4 h\sum_{l=3}^{s-1}|Z|^2_{W_2^1(\gamma_l)}.$$

Since $2h/T = 2h/(nh) \le 1/2$ for $4 \le s \le n$, we have

$$|Z|^2_{W_2^1(\gamma_s)} \le (1/T + T\lambda^2 c_4)4h\sum_{l=3}^{s-1}|Z|^2_{W_2^1(\gamma_l)} + 8T\|\Psi\|^2_{Q_s}, \quad s = 4, 5, \dots, n. \tag{4.9}$$

Now, let us show that

$$|Z|^2_{W_2^1(\gamma_3)} \le 6T\|\Psi\|^2_{Q_3}. \tag{4.10}$$

First note that $|Z|^2_{W_2^1(\gamma_3)} = (2/h)(Z_2^1)^2, \quad \|\Psi\|^2_{Q_3} = h^2(\Psi_2^1)^2$. Therefore, equation (2.5) on grid Q_3 (consisting on one grid point only) has the form

$$(1/h^2 + m^2/2)Z_2^1 = \Psi_2^1 \quad \Rightarrow (Z_2^1)^2 \le h^4(\Psi_2^1)^2.$$

Therefore,

$$|Z|^2_{W_2^1(\gamma_3)} \le 2h^3(\Psi_2^1)^2 = 2h\|\Psi\|^2_{Q_3} \le 6T\|\Psi\|^2_{Q_3}.$$

Applying Lemma 3.6 to inequalities (4.9), (4.10) we obtain estimate (2.8). Theorem 2.2 is proved. □

Proof of Theorem 2.3. Integrating equation (2.1) in domain e_{ij} we obtain

$$\begin{aligned} F_i^j =& (1/h^2)(u_i^j - u_i^{j-1} + u_{i-1}^{j-1} - u_{i-1}^j) + (m^2/h^2)\int_{e_{ij}} u(x,t)\,dxdt \\ &+ (\lambda/h^2)\int_{e_{ij}} u^3(x,t)\,dxdt. \end{aligned}$$

Introducing this expression into equation (2.6) and using the notation

$$l(u) := \frac{1}{h^2}\int_{e_{ij}} u(x,t)\,dxdt,$$

$$\widetilde{\psi}_1(u) := l(u) - 0.5(u_i^j + u_{i-1}^{j-1}), \quad \widetilde{\psi}_2(u) := l(u) - 0.5(u_i^{j-1} + u_{i-1}^j),$$

$$\widetilde{\psi}_3(u) := l(u^2)\widetilde{\psi}_1(u) + 0.5(u_i^j + u_{i-1}^{j-1})\widetilde{\psi}_2(u^2),$$

we can represent the truncation error in the form

$$\Psi = m^2\widetilde{\psi}_1(u) + \lambda\Big(l(u^3) - l(u)l(u^2) + \widetilde{\psi}_3(u)\Big). \tag{4.11}$$

Easy to see that

$$|\widetilde{\psi}_3(u)| \le \|u\|^2_C|\widetilde{\psi}_1(u)| + \|u\|_C|\widetilde{\psi}_2(u^2)|. \tag{4.12}$$

Let us transform expressions $\widetilde{\psi}_1(u)$ and $\widetilde{\psi}_2(u)$ as follows:

$$\widetilde{\psi}_1(u) := \frac{1}{2h^2}\int_{e_{ij}}\Big((t-t_j)(t-t_{j-1})\frac{\partial^2 u}{\partial t^2}+(x-x_i)(x-x_{i-1})\frac{\partial^2 u}{\partial x^2}$$
$$-(x-x_i)(t-t_j)\frac{\partial^2 u}{\partial x\partial t}-(x-x_{i-1})(t-t_{j-1})\frac{\partial^2 u}{\partial x\partial t}\Big)\,dxdt$$
$$\widetilde{\psi}_2(u) := \frac{1}{2h^2}\int_{e_{ij}}\Big((t-t_j)(t-t_{j-1})\frac{\partial^2 u}{\partial t^2}+(x-x_i)(x-x_{i-1})\frac{\partial^2 u}{\partial x^2}$$
$$-(x-x_i)(t-t_{j-1})\frac{\partial^2 u}{\partial x\partial t}-(x-x_{i-1})(t-t_j)\frac{\partial^2 u}{\partial x\partial t}\Big)\,dxdt,$$

whence

$$|\widetilde{\psi}_\alpha(u)| \le \frac{1}{2}\int_{e_{ij}}\Big(\Big|\frac{\partial^2 u}{\partial t^2}\Big|+\Big|\frac{\partial^2 u}{\partial x^2}\Big|+2\Big|\frac{\partial^2 u}{\partial x\partial t}\Big|\Big)\,dxdt \quad \alpha=1,2. \tag{4.13}$$

Using Cauchy-Schwarz inequality and algebraic inequality $(a+b+2c)^2 \le 4(a^2+b^2+2c^2)$ we have

$$|\widetilde{\psi}_1(u)| \le \sqrt{2}\,h|u|_{W_2^2(e_{ij})}, \quad |\widetilde{\psi}_2(u^2)| \le \sqrt{2}\,h|u^2|_{W_2^2(e_{ij})}. \tag{4.14}$$

Therefore, from (4.12) it follows that

$$|\widetilde{\psi}_3(u)| \le \sqrt{2}\,h\|u\|_C\big(\|u\|_C|u|_{W_2^2(e_{ij})}+|u^2|_{W_2^2(e_{ij})}\big). \tag{4.15}$$

The following identity is valid:

$$l(u^3)-l(u)l(u^2)$$
$$=\frac{1}{2h^4}\int_{e_{ij}}\int_{e_{ij}}\big(u(x,t)-u(\xi,\tau)\big)^2\big(u(x,t)+u(\xi,\tau)\big)\,dxdtd\xi d\tau.$$

Thus,

$$|l(u^3)-l(u)l(u^2)|$$
$$\le \frac{1}{h^4}\int_{e_{ij}}\int_{e_{ij}}\big(u(x,t)-u(\xi,\tau)\big)^2\,dxdtd\xi d\tau\,\|u\|_C$$
$$=\frac{1}{h^4}\int_{e_{ij}}\int_{e_{ij}}\Big(\int_\xi^x\frac{\partial u(\eta_1,t)}{\partial\eta_1}\,d\eta_1+\int_\tau^t\frac{\partial u(\xi,\eta_2)}{\partial\eta_2}\,d\eta_2\Big)^2\,dxdtd\xi d\tau\,\|u\|_C$$
$$\le 2|u|^2_{W_2^1(e_{ij})}\|u\|_C.$$

Therefore,

$$|l(u^3)-l(u)l(u^2)| \le 2|u|^2_{W_2^1(e_{ij})}\|u\|_C. \tag{4.16}$$

Note that

$$|u|^2_{W_2^1(e_{ij})}=\int_{e_{ij}}\Big(\Big|\frac{\partial u}{\partial x}\Big|^2+\Big|\frac{\partial u}{\partial t}\Big|^2\Big)dxdt \le h\Big(\int_{e_{ij}}\Big(\Big|\frac{\partial u}{\partial x}\Big|^2+\Big|\frac{\partial u}{\partial t}\Big|^2\Big)^2dxdt\Big)^{1/2}$$
$$\le h\Big(2\int_{e_{ij}}\Big(\Big|\frac{\partial u}{\partial x}\Big|^4+\Big|\frac{\partial u}{\partial t}\Big|^4\Big)dxdt\Big)^{1/2}=\sqrt{2}h|u|^2_{W_4^1(e_{ij})}.$$

Therefore, from (4.16) we have

$$|l(u^3)-l(u)l(u^2)| \le 2\sqrt{2}h|u|^2_{W_4^1(e_{ij})}\|u\|_C. \tag{4.17}$$

According to estimates (4.14), (4.15), (4.17) from (4.11) we obtain

$$|\Psi_i^j| \le \sqrt{2}\, m^2 h |u|_{W_2^2(e_{ij})} + \lambda h \|u\|_C \big(2\sqrt{2}\, |u|^2_{W_4^1(e_{ij})} + \sqrt{2}\, \|u\|_C |u|_{W_2^2(e_{ij})} + \sqrt{2}\, |u^2|_{W_2^2(e_{ij})}\big)$$

or

$$|\Psi_i^j| \le \sqrt{2}\, c_6 h \Big(|u|_{W_2^2(e_{ij})} + |u|^2_{W_4^1(e_{ij})} + |u^2|_{W_2^2(e_{ij})}\Big),$$

where $c_6 := m^2 + \lambda\delta^2 + 2\lambda\delta$ and δ is defined in (2.7).

Therefore,

$$\|\Psi\|^2_{Q_n} = h^2 \sum_{Q_n} |\Psi|^2 \le 6c_6^2 h^4 \Big(|u|^4_{W_4^1(D_T)} + |u|^2_{W_2^2(D_T)} + |u^2|^2_{W_2^2(D_T)}\Big),$$

or, as we note that

$$|u^2|^2_{W_2^2(D_T)} \le 8\Big(\delta^2\, |u|^2_{W_2^2(D_T)} + |u|^4_{W_4^1(D_T)}\Big),$$

we have

$$\|\Psi\|^2_{Q_n} \le 6c_6^2 h^4 \Big(9|u|^4_{W_4^1(D_T)} + (1+8\delta^2)|u|^2_{W_2^2(D_T)}\Big). \tag{4.18}$$

Since $W_2^2 \subset W_4^1$, then from (4.18) it follows that Theorem 2.3 is valid. □

5. Numerical experiments

Example 1. Consider problem (1.1), (1.2) with $\lambda = 1$, $m = 0$, $f(x,t) = x^3 \sin^3(x-t) - 2\cos(x-t)$. Table 1 gives approximate solution errors and relative errors for different values of parameter T and grid step h. The exact solution of the problem is $u(x,t) = x\sin(x-t)$. The obtained computations show convergency of the difference scheme.

Table 1

T		4	8	16	32	64
$h=$ 0.1	$z_{\max}$	$0.18 \cdot 10^{-1}$	$0.71 \cdot 10^{-1}$	$0.51 \cdot 10^{0}$	$0.30 \cdot 10^{1}$	$0.10 \cdot 10^{4}$
	$z_{\max}/u_{\max}$	$0.68 \cdot 10^{-2}$	$0.11 \cdot 10^{-1}$	$0.35 \cdot 10^{-1}$	$0.98 \cdot 10^{-1}$	$0.17 \cdot 10^{2}$
$h=$ 0.05	$z_{\max}$	$0.44 \cdot 10^{-2}$	$0.22 \cdot 10^{-1}$	$0.10 \cdot 10^{0}$	$0.53 \cdot 10^{0}$	$0.29 \cdot 10^{3}$
	$z_{\max}/u_{\max}$	$0.17 \cdot 10^{-2}$	$0.34 \cdot 10^{-2}$	$0.72 \cdot 10^{-2}$	$0.17 \cdot 10^{-1}$	$0.46 \cdot 10^{1}$
$h=$ 0.01	$z_{\max}$	$0.17 \cdot 10^{-3}$	$0.86 \cdot 10^{-3}$	$0.45 \cdot 10^{-2}$	$0.21 \cdot 10^{-1}$	$0.92 \cdot 10^{-1}$
	$z_{\max}/u_{\max}$	$0.66 \cdot 10^{-4}$	$0.13 \cdot 10^{-3}$	$0.32 \cdot 10^{-3}$	$0.68 \cdot 10^{-3}$	$0.15 \cdot 10^{-2}$
$h=$ 0.001	$z_{\max}$	$0.17 \cdot 10^{-5}$	$0.86 \cdot 10^{-5}$	$0.46 \cdot 10^{-4}$	$0.21 \cdot 10^{-3}$	$0.91 \cdot 10^{-3}$
	$z_{\max}/u_{\max}$	$0.66 \cdot 10^{-6}$	$0.13 \cdot 10^{-5}$	$0.32 \cdot 10^{-5}$	$0.67 \cdot 10^{-5}$	$0.15 \cdot 10^{-4}$

Example 2. Figures 2–4 give the graphs for the approximate solution of problem (2.1), (2.2) obtained for different values of equation parameters and a right-hand side. The graphs show some properties of the solution of the problem; interestingly, the graphs undergo high frequency oscillations in the neighborhood of moving boundary.

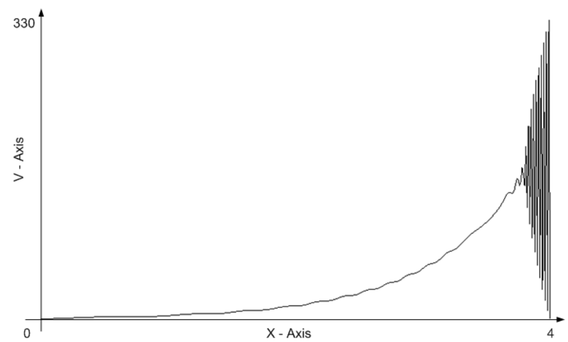

FIGURE 2. The graph of $V(x,T)$ for $\varphi(x,t) = \exp(xt), m = 0, \lambda = 1, T = 4$

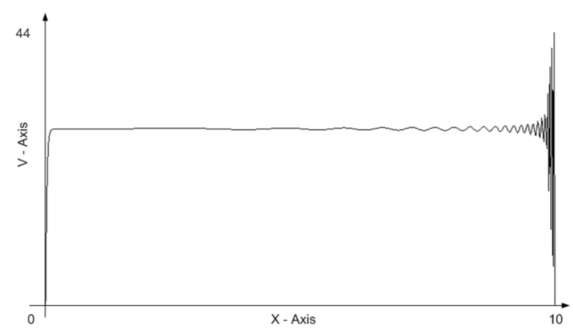

FIGURE 3. The graph of $V(x,T)$ for $\varphi(x,t) = \exp(t), m = 0, \lambda = 1, T = 10$

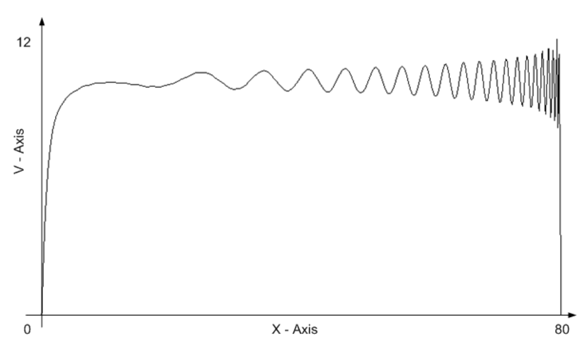

FIGURE 4. The graph of $V(x,T)$ for $\varphi(x,t) = 1, m = 0, \lambda = 1, T = 80$

Remark 1. The obtained results can be extended when: the nonlinearity of equations has the form $|u|^\alpha u$, $\alpha = \text{const} > 0$; the Dirichlet condition when $x = 0$ can be replaced with the Neumann condition; the parameter $\lambda < 0$ and the solution undergoes blow up.

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A. Razmadze Mathematical Institute, 1 M. Aleksidze str., 0193, Tbilisi, Georgia
E-mail address: bergi@rmi.acnet.ge

A. Razmadze Mathematical Institute, 1 M. Aleksidze str., 0193, Tbilisi, Georgia
E-mail address: ojokhadze@yahoo.com

A. Razmadze Mathematical Institute, 1 M. Aleksidze str., 0193, Tbilisi, Georgia
E-mail address: khar@rmi.acnet.ge

I. Javakhishvili Tbilisi State University, 2, University str., 0186, Tbilisi, Georgia
E-mail address: bidmid@hotmail.com