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STABLY FREE MODULES OVER $\mathbf{R}[X]$ OF RANK $> \dim \mathbf{R}$ ARE FREE

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ABSTRACT. We prove that for any finite-dimensional ring $\mathbf{R}$ and $n \geq \dim \mathbf{R}+2$, the group $\mathrm{E}_n(\mathbf{R}[X])$ acts transitively on $\mathrm{Um}_n(\mathbf{R}[X])$. In particular, we obtain that for any finite-dimensional ring $\mathbf{R}$, all finitely generated stably free modules over $\mathbf{R}[X]$ of rank $> \dim \mathbf{R}$ are free. This result was only known for Noetherian rings. The proof we give is short, simple, and constructive.

INTRODUCTION

In 1955, J.-P. Serre remarked [15] that it was not known whether there exist finitely generated projective modules over $\mathbf{A} = \mathbf{K}[X_1, \ldots, X_n]$, $\mathbf{K}$ a field, which are not free. This remark turned into the "Serre conjecture", stating that indeed there were no such modules. Proven independently by Quillen [12] and Suslin [17], it became known subsequently as the Quillen-Suslin theorem. The book of Lam [5] is a nice exposition about Serre's conjecture which has been updated recently in [6]. An important related fact worth mentioning is that it was known [16] well before Serre's conjecture was settled that finitely generated projective modules over $\mathbf{A}$ are stably free; i.e., every finitely generated projective $\mathbf{A}$-module is isomorphic to the kernel of an $\mathbf{A}$-epimorphism $T : \mathbf{A}^n \to \mathbf{A}^\ell$. In that situation the matrix T is unimodular; that is, the maximal minors of T generate the unit ideal in $\mathbf{A}$.

Quillen's and Suslin's proof had a big effect on the subsequent development of the study of projective modules. Nevertheless many old conjectures and open questions still await solutions. Our concern here is the following two equivalent conjectures.

Hermite ring conjecture (1972) [5, 6]**.**

Conjecture 1. *If $\mathbf{R}$ is a commutative Hermite ring, then $\mathbf{R}[X]$ is also Hermite.*

Conjecture 2. *If $\mathbf{R}$ is a commutative ring and $v = {}^{\mathrm{t}}(v_0(X), \ldots, v_n(X))$ is a unimodular vector over $\mathbf{R}[X]$ such that $v(0) = {}^{\mathrm{t}}(1, 0, \ldots, 0)$, then v can be completed to a matrix in $\mathrm{GL}_{n+1}(\mathbf{R}[X])$.*

Recall that a vector ${}^{\mathrm{t}}(b_1, \ldots, b_n)$ over a ring $\mathbf{R}$ is said to be *unimodular* if $\langle b_1, \ldots, b_n \rangle = \mathbf{R}$. The set of such unimodular vectors will be denoted by $\mathrm{Um}_n(\mathbf{R})$. A ring $\mathbf{R}$ is said to be *Hermite* if any finitely generated stably free $\mathbf{R}$-module is free, or equivalently (the equivalence holds constructively as well; see [9]) if any unimodular vector over $\mathbf{R}$ can be completed to an invertible matrix [5, 6]. Examples of

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Hermite rings are local rings, rings of Krull dimension ≤ 1 [2, 6], univariate polynomial rings over rings of Krull dimension ≤ 1 [19], polynomial rings over Bézout domains [4, 7], and polynomial rings over zero-dimensional rings [4, 6]. Here it is worth pointing out the following precise criterion for the freeness of finitely generated stably free modules over a ring $\mathbf{R}$ in terms of unimodular vectors (this also holds constructively as above):

For any ring $\mathbf{R}$ *and integer* $d \geq 0$, *the following are equivalent*:

(*i*) *Any finitely generated stably free* $\mathbf{R}$*-module of rank* $> d$ *is free.*

(*ii*) *Any unimodular vector over* $\mathbf{R}$ *of length* $\geq d+2$ *can be completed to an invertible matrix over* $\mathbf{R}$.

(*iii*) *For* $n \geq d+2$, $\mathrm{GL}_n(\mathbf{R})$ *acts transitively on* $\mathrm{Um}_n(\mathbf{R})$.

In this paper, we give a positive partial answer to the Hermite ring conjecture. We prove that for any ring $\mathbf{R}$ with Krull dimension $\leq d$, any unimodular vector over $\mathbf{R}[X]$ of length $\geq d+2$ can be completed to an invertible matrix over $\mathbf{R}[X]$ and, more importantly, the completion can be made using elementary matrices. In particular, we obtain, without any Noetherian hypothesis, that for any finite-dimensional ring $\mathbf{R}$, all finitely generated stably free modules over $\mathbf{R}[X]$ of rank $> \dim \mathbf{R}$ are free. More particularly, this gives a positive answer to the Hermite ring conjecture for rings with Krull dimension ≤ 1. The proof we give is short, simple, and constructive. It relies heavily on the very nice paper [14] of Roitman. This result was known only for Noetherian rings (see, for example, Theorem III.3.1 of [6]).

Although the proof we give is constructive, we will not use the "constructive" vocabulary as in [9] or [10] in order to lighten the exposition. For instance, in Lemma 2, for the purpose of obtaining a "fully" constructive formulation, one has to replace "degree" by "formal degree". Recall that the *formal degree* of a polynomial $f(X) \in \mathbf{R}[X]$ is ℓ if $f(X)$ has no coefficient of degree $> \ell$ (note that one does not guarantee that $\deg f(X) = \ell$ as it is not necessary to suppose $\mathbf{R}$ to be *discrete*; i.e., there is a zero test inside $\mathbf{R}$).

Let us fix some notation. We call an $n \times n$ matrix elementary if it has 1's on the diagonal and at most one nonzero off-diagonal entry. More precisely, if $a \in \mathbf{A}$ and $i \neq j$, $1 \leq i, j \leq n$, we define the elementary matrix $E_{i,j}(a)$ to be the $n \times n$ matrix with 1's on the diagonal, with a in the (i, j)-slot, and with 0's elsewhere. In other words, $E_{i,j}(a)$ is the matrix corresponding to the elementary operation $L_i \to L_i + aL_j$. $\mathrm{E}_n(\mathbf{A})$ will denote the subgroup of $\mathrm{SL}_n(\mathbf{A})$ generated by elementary matrices.

All rings considered are unitary and commutative. The undefined terminology is standard as in [6], and, for constructive algebra in [9, 10].

Elementary completion of unimodular vectors over $\mathbf{R}[X]$ of length $\geq \dim \mathbf{R} + 2$

Let us begin by a first lemma, which has been proven constructively in [19].

Lemma 1. *Let* $\mathbf{R}$ *be a ring, and* I *an ideal in* $\mathbf{R}[X]$ *that contains a monic polynomial. Let* J *be an ideal in* $\mathbf{R}$ *such that* $I + J[X] = \mathbf{R}[X]$. *Then* $(I \cap \mathbf{R}) + J = \mathbf{R}$.

The following three lemmas were already proved constructively by their authors.

Lemma 2 (Roitman, Lemma 3 of [14])**.** *Let* $\mathbf{R}$ *be a ring, and* $f(X) \in \mathbf{R}[X]$ *be of degree* $n > 0$, *such that* $f(0) \in \mathbf{R}^{\times}$. *Then for any* $g(X) \in \mathbf{R}[X]$ *and* $k \geq$

$\deg g(X) - \deg f(X) + 1$ *there exists* $h_k(X) \in \mathbf{R}[X]$ *of degree* $< n$ *such that* $g(X) \equiv X^k h_k(X) \bmod \langle \mathrm{f(X)} \rangle$.

Proof. This can easily be done by way of division by increasing powers using the fact that $f(0) \in \mathbf{R}^\times$. □

Lemma 3 (Vaserstein, Proposition 6.1.3.b page 125 of [6])**.** *Let* $\mathbf{R}$ *be a ring, and* ${}^{\mathrm{t}}(x_0, \ldots, x_r) \in \mathrm{Um}_{r+1}(\mathbf{R})$, $r \geq 2$, *and let* t *be an element of* $\mathbf{R}$ *which is invertible* mod $\langle \mathrm{x_0, \ldots, x_{r-2}} \rangle$. *Then there exists* $E \in \mathrm{E}_{r+1}(\mathbf{R})$ *such that* $E\, {}^{\mathrm{t}}(x_0, \ldots, x_r) = {}^{\mathrm{t}}(x_0, \ldots, x_{r-1}, t^2 x_r)$.

Lemma 4 (Bass, Lemma 4.1.b of [1])**.** *Let* $k \in \mathbb{N}$, $\mathbf{R}$ *be a ring,* $f_1, \ldots, f_r \in \mathbf{R}[X]$ *have degrees* $\leq k-1$, *and* $f_{r+1} \in \mathbf{R}[X]$ *be monic with degree* k. *If the coefficients of* $f_1, \ldots, f_r$ *generate the ideal* $\mathbf{R}$ *of* $\mathbf{R}$, *then* $\langle f_1, \ldots, f_r, f_{r+1} \rangle$ *contains a monic with degree* $k-1$.

Recall that the Krull boundary ideal of an element a of a ring $\mathbf{R}$ is the ideal $\mathcal{I}(a)$ of $\mathbf{R}$ generated by a and all the $y \in \mathbf{R}$ such that ay is nilpotent. Moreover we have the following equivalence:

$$\dim \mathbf{R} \leq d \Leftrightarrow \ \forall\, a \in \mathbf{R},\ \dim(\mathbf{R}/\mathcal{I}(a)) \leq d-1$$

(this defines the Krull dimension recursively initializing with "$\dim \mathbf{R} \leq -1 \Leftrightarrow \mathbf{R}$ is trivial") [3].

Now we reach our main result.

Theorem 5. *Let* $\mathbf{R}$ *be a ring of dimension* $\leq d$, $n \geq d+1$, *and let* $v(X) = {}^{\mathrm{t}}(v_0(X), \ldots, v_n(X)) \in \mathrm{Um}_{n+1}(\mathbf{R}[X])$. *Then there exists* $E \in \mathrm{E}_{n+1}(\mathbf{R}[X])$ *such that* $E\, v(X) = {}^{\mathrm{t}}(1, 0, \ldots, 0)$.

Proof. By the stable range theorem (see [2], Theorem 2.4 for a constructive proof), for any $w \in \mathrm{Um}_{n+1}(\mathbf{R})$, there exists $M \in \mathrm{E}_{n+1}(\mathbf{R})$ such that $M\, w = {}^{\mathrm{t}}(1, 0, \ldots, 0)$. So, it suffices to prove that there exists $E \in \mathrm{E}_{n+1}(\mathbf{R}[X])$ such that $E\, v(X) = v(0)$. For this, by the local-global principle for unimodular vectors over $\mathbf{R}[X]$ under the action of the elementary group $\mathrm{E}_{n+1}(\mathbf{R}[X])$ for $n \geq 2$ (see E_n-Patching Theorem for Unimodular Rows 2.3 [6] (page 211) of which a constructive version can be obtained using the same ideas as in Lemma 22 and Concrete local-global Principle 23 of [8]), we can suppose that $\mathbf{R}$ is local. Here it is worth mentioning that the constructive local-global principal says that it suffices to prove the result constructively in the case of a "residually discrete" local ring $\mathbf{R}$, i.e., a local ring $\mathbf{R}$ in which the disjunction

$$a \in \mathbf{R}^\times \ \vee \ a \in \mathrm{Rad}(\mathbf{R})$$

is explicit. Besides, it is clear that we can suppose that $\mathbf{R}$ is reduced.

Let N be the (formal) degree of one of the v_i's such that $v_i(0) \in \mathbf{R}^\times$. We prove the claim by double induction on N and d, starting with $N = 0$ (in that case the result is immediate) and $d = 0$ (in that case the result is well known).

We will first prove by induction on N that $v(X)$ can be transformed by elementary operations into a vector with one constant entry.

Let $N \geq 1$ and denote by $m_i := \deg v_i$ (formal degree). For the sake of simplicity, we may assume that $v_0(0) \in \mathbf{R}^\times$ and $N = m_0$. Let us denote by a the leading coefficient of v_0. If $a \in \mathbf{R}^\times$, then the result follows from a lemma of Suslin [17] (Lemma 2.3). A constructive proof of this lemma is given in [11, 18]. So we may assume $a \in \mathrm{Rad}(\mathbf{R})$. By the induction hypothesis on N applied to the ring $\mathbf{R}/\langle a\rangle$, we can assume that $v(X) \equiv {}^{\mathrm{t}}(1,0,\dots,0) \bmod (\mathrm{a}\mathbf{R}[\mathrm{X}])^{\mathrm{n+1}}$ (this hypothesis will be called $\mathcal{H}$).

By Lemma 2 (considering a sufficiently large even integer $2k$), we now assume $v_i = X^{2k}w_i$ for some $k \in \mathbb{N}^*$, where $\deg w_i < m_0$ for $1 \le i \le n$. By Lemma 3 (X being invertible modulo v_0 since $v_0(0) \in \mathbf{R}^\times$), we assume $m_i < m_0$ (note that hypothesis $\mathcal{H}$ is preserved when applying Lemma 2 and Lemma 3).

If $m_0 \le 1$, our first claim is established. Assume now that $m_0 \ge 2$. Let $(c_1,\dots,c_{m_0(n-1)})$ be the coefficients of $1, X, \dots, X^{m_0-1}$ in the polynomials $v_2(X),\dots,v_n(X)$. By Lemma 1 (taking $I = v_0\mathbf{R}_a[X] + v_1\mathbf{R}_a[X]$ and $J = c_1\mathbf{R}_a + \cdots + c_{m_0(n-1)}\mathbf{R}_a$), the ideal generated in $\mathbf{R}_a$ by $\mathbf{R}_a \cap (v_0\mathbf{R}_a[X] + v_1\mathbf{R}_a[X])$ and the c_i's is $\mathbf{R}_a$. As $m_0(n-1) \ge 2d > \dim \mathbf{R}_a$, by the stable range theorem there exists

$$(c'_1,\dots,c'_{m_0(n-1)}) \equiv (c_1,\dots,c_{m_0(n-1)}) \bmod (\mathrm{v_0}\mathbf{R}[\mathrm{X}] + \mathrm{v_1}\mathbf{R}[\mathrm{X}]) \cap \mathbf{R}$$

such that $c'_1\mathbf{R}_a + \cdots + c'_{m_0(n-1)}\mathbf{R}_a = \mathbf{R}_a$ (here it is worth pointing out that despite reasoning with the ring $\mathbf{R}_a[X]$, all the elementary operations are performed using the ring $\mathbf{R}[X]$). Assume that we have already $c_1\mathbf{R}_a + \cdots + c_{m_0(n-1)}\mathbf{R}_a = \mathbf{R}_a$. By Lemma 4, the ideal $\langle v_0, v_2, \dots, v_n\rangle$ of $\mathbf{R}[X]$ contains a polynomial $w(X)$ of degree $m_0 - 1$ which is unitary in $\mathbf{R}_a$. Let us denote the leading coefficient of w by a^k where $k \in \mathbb{N}$ and that of v_1 by b. Using Lemma 3, as a is invertible modulo $\langle v_0, v_1\rangle$ (because $v(X) \equiv {}^{\mathrm{t}}(1,0,\dots,0) \bmod (\mathrm{a}\mathbf{R}[\mathrm{X}])^{\mathrm{n+1}}$), we can by elementary operations make the following transformations:

$${}^{\mathrm{t}}(v_0, v_1, \dots, v_n) \;\to\; {}^{\mathrm{t}}(v_0, a^{2k}v_1, \dots, v_n) \;\to\; {}^{\mathrm{t}}(v_0, a^{2k}v_1 + (1-a^kb)w, v_2, \dots, v_n).$$

Now, $a^{2k}v_1 + (1-a^kb)w$ is unitary in $\mathbf{R}_a$, so assume v_1 is unitary in $\mathbf{R}_a$ and $m_1 < m_0$. By Lemma 3, performing some elementary operations, ${}^{\mathrm{t}}(v_0, v_1, v_2, \dots, v_n)$ can be transformed into ${}^{\mathrm{t}}(v_0, v_1, a^\ell v_2, \dots, a^\ell v_n)$ for a suitable $\ell \in \mathbb{N}$ so that we can divide (as in Euclidean division) all $a^\ell v_2, \dots, a^\ell v_n$ by v_1, and thus we can assume that $m_i < m_1$ for $2 \le i \le n$ (note that after this series of elementary operations, hypothesis $\mathcal{H}$ is preserved).

Repeating the argument above we lower the degree of v_1 until reaching the desired form of our first claim.

The rest of the proof will be by induction on d. Let $d > 0$. Now assume that $v_1 = c \in \mathbf{R}$. Let us consider the ring $\mathbf{T} := \mathbf{R}/\mathcal{I}(c)$. Since $\dim \mathbf{T} \le d-1$ and ${}^{\mathrm{t}}(\overline{v}_0, \overline{v}_2, \dots, \overline{v}_n) \in \mathrm{Um}_n(\mathbf{T}[X])$, there exists $E_1 \in \mathrm{E}_n(\mathbf{R}[X])$ such that

$$E_1\,{}^{\mathrm{t}}(v_0, v_2, \dots, v_n) = {}^{\mathrm{t}}(1 + c\,h_0 + y_0\tilde{h}_0, c\,h_2 + y_2\tilde{h}_2, \dots, c\,h_n + y_n\tilde{h}_n),$$

where $h_i, \tilde{h}_i \in \mathbf{R}[X]$, $y_i \in \mathbf{R}$ with $c\,y_i = 0$. Denoting by $E_2 = \left(\begin{smallmatrix}1 & 0\\ 0 & E_1\end{smallmatrix}\right) \in \mathrm{E}_{n+1}(\mathbf{R}[X])$ and $\tilde{v} := {}^{\mathrm{t}}(v_1, v_0, v_2, \dots, v_n)$, we have

$$E_2\,\tilde{v} = {}^{\mathrm{t}}(c, 1 + c\,h_0 + y_0\tilde{h}_0, c\,h_2 + y_2\tilde{h}_2, \dots, c\,h_n + y_n\tilde{h}_n).$$

Thus,

$$E_{1,2}(-c)E_{2,1}(-h_0)E_{3,1}(-h_2)\cdots E_{n+1,1}(-h_n)\,E_2\,\tilde{v} = {}^{\mathrm{t}}(0, 1 + y_0\tilde{h}_0, y_2\tilde{h}_2, \dots, y_n\tilde{h}_n) =: \hat{v},$$

and we can easily find $E_3 \in \mathrm{E}_{n+1}(\mathbf{R}[X])$ such that $E_3\,\hat{v} = {}^{\mathrm{t}}(1,0,\dots,0)$. □

Corollary 6. *For any ring $\mathbf{R}$ with Krull dimension d, all finitely generated stably free modules over $\mathbf{R}[X]$ of rank $> d$ are free.*

Proof. This is an immediate consequence of Theorem 5 and the result $(\mathbf{4.5})_d$ of [6] (page 31). □

Corollary 7. *The Hermite ring conjecture is true for rings with Krull dimension* ≤ 1.

Corollary 6 encourages us to set the following conjecture.

Conjecture 3. *For any ring* $\mathbf{R}$ *with Krull dimension* $\leq d$*, all finitely generated stably free modules over* $\mathbf{R}[X_1, \ldots, X_k]$ *of rank* $> d$ *are free.*

Also, Corollary 6 raises the analogous question for Laurent polynomial rings.

Question 8. Is it true that for any ring $\mathbf{R}$ of Krull dimension $\leq d$, all finitely generated stably free modules over $\mathbf{R}[X, X^{-1}]$ of rank $> d$ are free?

References

[1] H. Bass, *Libération des modules projectifs sur certains anneaux de polynômes*, Sém. Bourbaki 1973/74, exp. 448, Lecture Notes in Math., vol. 431, Springer-Verlag, Berlin and New York, 1975, 228-254. MR0472826 (57:12516)

[2] T. Coquand, H. Lombardi, C. Quitté, *Generating non-Noetherian modules constructively*, Manuscripta Mathematica 115 (2004) 513-520. MR2103665 (2005h:13014)

[3] T. Coquand, H. Lombardi, M.-F. Roy, *An elementary characterization of Krull dimension*, From sets and types to analysis and topology: towards practicable foundations for constructive mathematics (L. Crosilla, P. Schuster, eds.), Oxford University Press, 2005. MR2188647 (2007a:54023)

[4] A. Ellouz, H. Lombardi, I. Yengui, *A constructive comparison between the rings* $\mathbf{R}(X)$ *and* $\mathbf{R}\langle X\rangle$ *and application to the Lequain-Simis induction theorem*, J. Algebra 320 (2008) 521-533. MR2422305 (2009e:13022)

[5] T.Y. Lam, *Serre's conjecture*, Lecture Notes in Mathematics, vol. 635, Springer-Verlag, Berlin-New York, 1978. MR0485842 (58:5644)

[6] T.Y. Lam, *Serre's Problem on Projective Modules*, Springer Monographs in Mathematics, 2006. MR2235330 (2007b:13014)

[7] Y. Lequain, A. Simis, *Projective modules over* $\mathbf{R}[X_1, ..., X_n]$*,* $\mathbf{R}$ *a Prüfer domain*, J. Pure Appl. Algebra 18 (2) (1980) 165–171. MR585221 (82e:13010)

[8] H. Lombardi, C. Quitté, *Constructions cachées en algèbre abstraite (2) Le principe local-global*, dans: Commutative ring theory and applications. Eds: Fontana M., Kabbaj S.-E., Wiegand S. Lecture notes in pure and applied mathematics, vol. 131, M. Dekker, 2002, 461–476. MR2029844 (2005b:13015)

[9] H. Lombardi, C. Quitté, *Algèbre commutative, Méthodes Constructives*, to appear, available at `http://hlombardi.free.fr/publis/A---PTFCours.html`. An english version is to be published by Springer.

[10] R. Mines, F. Richman, W. Ruitenburg, *A Course in Constructive Algebra*, Universitext. Springer-Verlag, 1988. MR919949 (89d:03066)

[11] A. Mnif, I. Yengui, *An algorithm for unimodular completion over Noetherian rings*, J. Algebra 316 (2007) 483-498. MR2356840 (2008h:13016)

[12] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. 36 (1976) 167–171. MR0427303 (55:337)

[13] R.A. Rao, *The Bass-Quillen conjecture in dimension three but characteristic* $\neq 2, 3$ *via a question of A. Suslin*, Invent. Math. 93 (1988) 609-618. MR952284 (89d:13011)

[14] M. Roitman, *On stably extended projective modules over polynomial rings*, Proc. Amer. Math. Soc. 97 (1986) 585-589. MR845969 (87f:13007)

[15] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) 61 (1955) 191–278. MR0068874 (16:953c)

[16] J.-P. Serre, *Modules projectifs et espaces fibrés à fibre vectorielle*, Sém. Dubreil-Pisot, no. 23, Paris (1957/58). MR0177011 (31:1277)

[17] A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR-Izv. 11 (1977) 221-238. MR0472792 (57:12482)

[18] I. Yengui, *Making the use of maximal ideals constructive*, Theoretical Computer Science 392 (2008) 174-178. MR2394992 (2009a:13012)

[19] I. Yengui, *The Hermite ring conjecture in dimension one*, J. Algebra 320 (2008) 437-441. MR2417998 (2009g:16019)

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