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SHARP BOUNDS OF THE LANDAU CONSTANTS

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ABSTRACT. The aim of this paper is to establish new bounds of the Landau constants.

1. Introduction and motivation

The Landau constants defined for all positive integers n by

$$G_n = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 + \cdots + \left(\frac{1\cdot 3\cdot \cdots \cdot (2n-1)}{2\cdot 4\cdot \cdots \cdot (2n)}\right)^2,$$

play an important role in some extremal problems in complex analysis and in the theory of Fourier series. More precisely, in 1913 Landau [6] proved that G_n is the maximum of the expression $\left|\sum_{k=1}^{n} a_k\right|$, with respect to all functions of the form $f(z) = \sum_{k=0}^{\infty} a_k z^k$ which is analytic in the unit disk and satisfies $|f(z)| < 1$, for every $|z| < 1$.

In consequence, the problem of approximation of the Landau constants have attracted the attention of many authors. In particular, Landau himself studied the asymptotic behaviour of G_n and showed that

$$G_n \sim \frac{1}{\pi}\ln n,$$

then Watson [7] established the following asymptotic formula

$$G_n = c_0 + \frac{1}{\pi}\ln(n+1) - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right) \quad (n\to\infty).$$

Here, and in what follows, $c_0 = \frac{1}{\pi}(\gamma + 4\ln 2) = 1.06627...$, and $\gamma = 0.577215...$ is the Euler-Mascheroni constant.

This problem of approximation of the Landau constants was continued in the works of Brutman [3] and Falaleev [5], who proved that for every non-negative integer n,

$$1 + \frac{1}{\pi}\ln(n+1) < G_n < 1.0663 + \frac{1}{\pi}\ln(n+1),$$

respectively

$$1.0662 + \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) < G_n < 1.0916 + \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right).$$

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We improve the upper bound in the following way, that also shows that the constant 3/4 is the best possible.

Theorem 1. *For every integer $n \geq 1$, we have*

$$c_0 + \frac{1}{\pi} \ln \left(n + \frac{3}{4} \right) < G_n < c_0 + \frac{1}{\pi} \ln \left(n + \frac{3}{4} + \frac{11}{192n} \right).$$

Very recently, Zhao [8, Corollary 1] extended the asymptotic expansion of G_n to

$$G_n = c_0 + \frac{1}{\pi} \ln (n+1) - \frac{1}{4\pi (n+1)} + \frac{5}{192\pi (n+1)^2} + O\left(\frac{1}{(n+1)^3} \right), \tag{1.1}$$

as a direct consequence of the following double inequality

$$\begin{aligned} &c_0 + \frac{1}{\pi} \ln (n+1) - \frac{1}{4\pi (n+1)} + \frac{5}{192\pi (n+1)^2} < G_n \\ &< c_0 + \frac{1}{\pi} \ln (n+1) - \frac{1}{4\pi (n+1)} + \frac{5}{192\pi (n+1)^2} + \frac{3}{128\pi (n+1)^3} \end{aligned} \tag{1.2}$$

stated in [8, Theorem 1], as the main result.

In this paper we establish the following double inequality that improves much of Zhao's results (1.1)–(1.2).

Theorem 2. *For every integer $n \geq 1$, we have*

$$\begin{aligned} &c_0 + \frac{1}{\pi} \ln (n+1) - \frac{1}{4\pi (n+1)} + \frac{5}{192\pi (n+1)^2} \\ &+ \frac{3}{128\pi (n+1)^3} - \frac{341}{122880\pi (n+1)^4} - \frac{75}{8192\pi (n+1)^5} < G_n \\ &< c_0 + \frac{1}{\pi} \ln (n+1) - \frac{1}{4\pi (n+1)} + \frac{5}{192\pi (n+1)^2} \\ &+ \frac{3}{128\pi (n+1)^3} - \frac{341}{122880\pi (n+1)^4} \end{aligned}$$

and the following asymptotic formula holds:

$$\begin{aligned} G_n = c_0 &+ \frac{1}{\pi} \ln (n+1) - \frac{1}{4\pi (n+1)} + \frac{5}{192\pi (n+1)^2} \\ &+ \frac{3}{128\pi (n+1)^3} - \frac{341}{122880\pi (n+1)^4} + O\left(\frac{1}{(n+1)^5} \right). \end{aligned}$$

Another direction for developing the problem of approximation of G_n was opened by Cvijović and Klinowski [4, Theorem 1] who gave some estimates in terms of the digamma function $\psi = \Gamma'/\Gamma$, namely

$$c_0 + \frac{1}{\pi} \psi \left(n + \frac{5}{4} \right) < G_n < 1.0725 + \frac{1}{\pi} \psi \left(n + \frac{5}{4} \right) \tag{1.3}$$

and

$$0.9883 + \frac{1}{\pi} \psi \left(n + \frac{3}{2} \right) < G_n < c_0 + \frac{1}{\pi} \psi \left(n + \frac{3}{2} \right) \quad (n \geq 0). \tag{1.4}$$

Motivated by the work of Cvijović and Klinowski [4], Alzer [2, Theorem 1] proved the following double inequality, for every integer $n \geq 1$,

$$c_0 + \frac{1}{\pi}\psi(n+\alpha) < G_n < c_0 + \frac{1}{\pi}\psi(n+\beta), \tag{1.5}$$

where the constants $\alpha = 5/4$ and $\beta = \psi^{-1}(\pi(1-c_0)) = 1.26621...$ are the best possible.

We establish the following double inequality that improves much of the results (1.3)–(1.5) of Cvijović, Klinowski and Alzer, and also shows that the best approximation of the form

$$G_n \approx c_0 + \frac{1}{\pi}\psi(n+b) \quad (n \to \infty)$$

is obtained for $b = 5/4$.

Theorem 3. *For every positive integer n, we have*

$$c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) + \frac{1}{64\pi n^2} - \frac{3}{128\pi n^3} < G_n$$
$$< c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) + \frac{1}{64\pi n^2} - \frac{3}{128\pi n^3} + \frac{173}{8192\pi n^4}.$$

2. The Proofs

Our new results announced in the previous section are established using an original approach. More precisely, we use an excellent result of Alzer [1, Theorem 8], who proved that for every $n \geq 0$, the functions

$$F_n(x) = \ln\Gamma(x+1) - \left(x+\frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln 2\pi - \sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \tag{2.1}$$

and

$$H_n(x) = -\ln\Gamma(x+1) + \left(x+\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

are completely monotonic (B_j is the jth Bernoulli number). In particular, $F_n > 0$ and $H_n > 0$ and, consequently,

$$s(x) < \ln\Gamma(x+1) < t(x) \tag{2.2}$$

where

$$s(x) = \ln\sqrt{2\pi} + \left(x+\frac{1}{2}\right)\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7}$$

and

$$t(x) = \ln\sqrt{2\pi} + \left(x+\frac{1}{2}\right)\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9}.$$

Then, using (2.2), we get

$$e^{u(x)} < \frac{1}{16^x}\left(\frac{\Gamma(2x+1)}{(\Gamma(x+1))^2}\right)^2 < e^{v(x)}, \tag{2.3}$$

where

$$u(x) = 2s(2x) - 4t(x) - x\ln 16\ , \quad v(x) = 2t(2x) - 4s(x) - x\ln 16.$$

We also use the inequality

$$\sum_{k=1}^{2p}\frac{(-1)^{k-1}}{k}t^k<\ln(1+t)<\sum_{k=1}^{2p+1}\frac{(-1)^{k-1}}{k}t^k\quad (t>0),$$

which easily follows from the Taylor formula.

Proof of Theorem 1. By direct verification, the required inequalities are true for $n=1,2$, so we assume $n\geq 3$. The sequence

$$a_n=G_n-c_0-\frac{1}{\pi}\ln\left(n+\frac{3}{4}\right)$$

converges to zero and it suffices to show that $(a_n)_{n\geq 1}$ is strictly decreasing. We have

$$a_n-a_{n-1}=\frac{1}{16^n}\left(\frac{\Gamma(2n+1)}{(\Gamma(n+1))^2}\right)^2-\frac{1}{\pi}\ln\left(1+\frac{1}{n-\frac{1}{4}}\right)<e^{v(n)}-\frac{1}{\pi}\sum_{k=1}^{4}\frac{(-1)^{k-1}}{k\left(n-\frac{1}{4}\right)^k},$$

so we have to prove that $f(x)<0$, where

$$f(x)=v(x)-\ln\left(\frac{1}{\pi}\sum_{k=1}^{4}\frac{(-1)^{k-1}}{k\left(x-\frac{1}{4}\right)^k}\right).$$

We have

$$f'(x)=\frac{P(x)}{33792x^{10}(4x-1)(148x-240x^2+192x^3-73)},$$

where

$$P(x)=5947392x^{11}-13381632x^{10}+\cdots-73.$$

The polynomial $P(x+3)$ has all coefficients positive, so $P>0$ on $[3,\infty)$. Thus f is strictly increasing on $[3,\infty)$, with $f(\infty)=0$, so $f(x)<0$, for every $x\in[3,\infty)$.

The sequence

$$b_n=G_n-c_0-\frac{1}{\pi}\ln\left(n+\frac{3}{4}+\frac{11}{192n}\right)$$

converges to zero and it suffices to show that $(b_n)_{n\geq 1}$ is strictly increasing. We have

$$\begin{aligned}b_n-b_{n-1}&=\frac{1}{16^n}\left(\frac{\Gamma(2n+1)}{(\Gamma(n+1))^2}\right)^2-\frac{1}{\pi}\ln\left(1+\frac{1+\frac{11}{192n}-\frac{11}{192(n-1)}}{n-\frac{1}{4}+\frac{11}{192(n-1)}}\right)\\&>e^{u(n)}-\frac{1}{\pi}\sum_{k=1}^{5}\frac{(-1)^{k-1}}{k\left(\frac{n-\frac{1}{4}+\frac{11}{192(n-1)}}{1+\frac{11}{192n}-\frac{11}{192(n-1)}}\right)^k},\end{aligned}$$

so we have to prove that $g(x)>0$, where

$$g(x)=u(x)-\ln\left(\frac{1}{\pi}\sum_{k=1}^{5}\frac{(-1)^{k-1}}{k\left(\frac{x-\frac{1}{4}+\frac{11}{192(x-1)}}{1+\frac{11}{192x}-\frac{11}{192(x-1)}}\right)^k}\right).$$

We have

$$g'(x)=-\frac{Q(x)}{33792x^{10}(192x^2-240x+59)(192x^2-192x-11)R(x)}$$

where

$$Q(x) = 392796625756697395 20x^{22} - 273610843762924191744x^{21} + \cdots + 116760686592$$

and

$$R(x) = 81\,537\,269\,760x^{12} - 448\,454\,983\,680x^{11} + 1085\,464\,903\,680x^{10} + \cdots + 175692.$$

The polynomials $Q(x+3)$ and $R(x+3)$ have all coefficients positive, so $g' < 0$ on $[3, \infty)$. Thus g is strictly decreasing on $[3, \infty)$, with $g(\infty) = 0$, so $g(x) > 0$, for every $x \in [2, \infty)$. $\square$

Proof of Theorem 2. The required inequalities are true for $n = 1, 2$, so we assume $n \geq 3$. The sequence

$$x_n = c_0 + \frac{1}{\pi}\ln(n+1) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3} - \frac{341}{122880\pi(n+1)^4} - \frac{75}{8192\pi(n+1)^5} - G_n$$

converges to zero, so it suffices to prove that $(x_n)_{n\geq 1}$ is increasing. In this sense, we have

$$x_n - x_{n-1} = \frac{1}{\pi}\ln\left(1+\frac{1}{n}\right) + \frac{1}{4\pi}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{5}{192\pi}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - \frac{3}{128\pi}\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{341}{122880\pi}\left(\frac{1}{n^4} - \frac{1}{(n+1)^4}\right) + \frac{75}{8192\pi}\left(\frac{1}{n^5} - \frac{1}{(n+1)^5}\right) - \frac{1}{16^n}\left(\frac{\Gamma(2n+1)}{(\Gamma(n+1))^2}\right)^2.$$

Using

$$\ln\left(1+\frac{1}{n}\right) > \sum_{k=1}^{10}\frac{(-1)^{k-1}}{kn^k}$$

and (2.3), we get

$$x_n - x_{n-1} > S(n) - \exp v(n), \tag{2.4}$$

where

$$S(n) = \frac{1}{\pi}\sum_{k=1}^{10}\frac{(-1)^{k-1}}{kn^k} + \frac{1}{4\pi}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{5}{192\pi}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - \frac{3}{128\pi}\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{341}{122880\pi}\left(\frac{1}{n^4} - \frac{1}{(n+1)^4}\right) + \frac{75}{8192\pi}\left(\frac{1}{n^5} - \frac{1}{(n+1)^5}\right).$$

Following (2.4), we have to prove that $h(x) > 0$, for every $x \geq 3$, where

$$h(x) = \ln S(x) - v(x).$$

We have

$$h'(x) = -\frac{W(x)}{33792x^{10}(x+1)T(x)},$$

where

$$W(x) = 2894918400x^{18} + 39555584640x^{17} + \cdots + 258048$$

and

$$T(x) = 2580480x^{14} + 12257280x^{13} + \cdots - 258048.$$

The polynomials $W(x+3)$ and $T(x+3)$ have all coefficients positive, so $W > 0$ and $T > 0$, on $[3, \infty)$. In consequence, $h' < 0$, so h is strictly decreasing on $[3, \infty)$. But $h(\infty) = 0$, thus $h(x) > 0$, for every $x \in [3, \infty)$.

The sequence

$$y_n = c_0 + \frac{1}{\pi} \ln(n+1) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3} - \frac{341}{122880\pi(n+1)^4} - G_n$$

converges to zero, so it suffices to prove that $(y_n)_{n \geq 1}$ is strictly decreasing. In this sense, we have

$$y_n - y_{n-1} = \frac{1}{\pi} \ln\left(1 + \frac{1}{n}\right) + \frac{1}{4\pi}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{5}{192\pi}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - \frac{3}{128\pi}\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{341}{122880\pi}\left(\frac{1}{n^4} - \frac{1}{(n+1)^4}\right) - \frac{1}{16^n}\left(\frac{\Gamma(2n+1)}{(\Gamma(n+1))^2}\right)^2.$$

Using

$$\ln\left(1 + \frac{1}{n}\right) < \sum_{k=1}^{9} \frac{(-1)^{k-1}}{kn^k}$$

and (2.3), we get

$$y_n - y_{n-1} < Z(n) - \exp u(n), \tag{2.5}$$

where

$$Z(n) = \frac{1}{\pi} \sum_{k=1}^{9} \frac{(-1)^{k-1}}{kn^k} + \frac{1}{4\pi}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{5}{192\pi}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - \frac{3}{128\pi}\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + \frac{341}{122880\pi}\left(\frac{1}{n^4} - \frac{1}{(n+1)^4}\right).$$

Following (2.5), we have to prove that $j(x) < 0$, for every $x > 0$, where

$$j(x) = \ln Z(x) - u(x).$$

We have

$$j'(x) = \frac{M(x)}{33792x^{10}(x+1)N(x)},$$

where

$$M(x) = 19958400000x^{17} + 26044761600x^{16} + \cdots - 293601280$$

and

$$N(x) = 2580480x^{12} + 9676800x^{11} + \cdots + 286720.$$

As all the coefficients of the polynomials $M(x+3)$ and $N(x+3)$ are positive, it results that $M > 0$ and $N > 0$ on $[3, \infty)$. In consequence, $j' > 0$. Finally, $j < 0$, on $[3, \infty)$, since j is strictly increasing, with $j(\infty) = 0$. $\square$

Proof of Theorem 3. The required inequalities are true for $n = 1, 2$ and we assume that $n \geq 3$. We prove that the sequence

$$t_n = G_n - c_0 - \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) - \frac{1}{64\pi n^2} + \frac{3}{128\pi n^3}$$

is strictly decreasing. In this sense,

$$\begin{aligned} t_n - t_{n-1} = {} & \frac{1}{16^n}\frac{(\Gamma(2n+1))^2}{(\Gamma(n+1))^4} - \frac{1}{\pi\left(n+\frac{1}{4}\right)} \\ & - \frac{1}{64\pi n^2} + \frac{3}{128\pi n^3} + \frac{1}{64\pi(n-1)^2} - \frac{3}{128\pi(n-1)^3} \end{aligned}$$

and we have to prove that $m < 0$, where

$$m(x) = v(x) - \ln\left(\frac{1}{\pi\left(x+\frac{1}{4}\right)} + \frac{1}{64\pi x^2} - \frac{3}{128\pi x^3} - \frac{1}{64\pi(x-1)^2} + \frac{3}{128\pi(x-1)^3}\right).$$

We have

$$m'(x) = \frac{L(x)}{33792x^{10}(x-1)(4x+1)(x - 29x^2 - 456x^3 + 1520x^4 - 1536x^5 + 512x^6 + 3)},$$

where

$$L(x) = 23384064x^{13} - 25681920x^{12} + \cdots + 3.$$

As the polynomials involved in the expression of m' have all coefficients positive when we replace x by $x + 3$, it follows that $m' > 0$ on $[3, \infty)$. In consequence, m is strictly increasing with $m(\infty) = 0$, so $m < 0$.

Now we prove that the sequence

$$z_n = G_n - c_0 - \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) - \frac{1}{64\pi n^2} + \frac{3}{128\pi n^3} - \frac{173}{8192\pi n^4}$$

is strictly increasing. In this sense,

$$\begin{aligned} z_n - z_{n-1} = {} & \frac{1}{16^n}\frac{(\Gamma(2n+1))^2}{(\Gamma(n+1))^4} - \frac{1}{\pi\left(n+\frac{1}{4}\right)} \\ & - \frac{1}{64\pi n^2} + \frac{3}{128\pi n^3} - \frac{173}{8192\pi n^4} + \frac{1}{64\pi(n-1)^2} \\ & - \frac{3}{128\pi(n-1)^3} + \frac{173}{8192\pi(n-1)^4}, \end{aligned}$$

and we have to prove that $n(x) > 0$, where

$$\begin{aligned} n(x) = u(x) - \ln\Bigg(& \frac{1}{\pi\left(x+\frac{1}{4}\right)} + \frac{1}{64\pi x^2} - \frac{3}{128\pi x^3} + \frac{173}{8192\pi x^4} \\ & - \frac{1}{64\pi(x-1)^2} + \frac{3}{128\pi(x-1)^3} - \frac{173}{8192\pi(x-1)^4}\Bigg). \end{aligned}$$

We have

$$n'(x) = -\frac{K(x)}{33792x^{10}(x-1)(4x+1)J(x)},$$

where

$$K(x) = 1175961600x^{14} - 845070336x^{13} + \cdots + 177152$$

and

$$J(x) = 5380x^3 - 1602x^2 - 192x + 24560x^4 - 126464x^5 + 195584x^6 - 131072x^7 + 32768x^8 + 173.$$

But the polynomials $K(x+3)$ and $J(x+3)$ have all coefficients positive, so $n' < 0$ on $[3,\infty)$. Finally, $n(x) > 0$, for every $x \in [3,\infty)$, since it is strictly decreasing with $n(\infty) = 0$. □

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