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ON A NONLINEAR SUBDIVISION SCHEME AVOIDING GIBBS OSCILLATIONS AND CONVERGING TOWARDS C^s FUNCTIONS WITH $s > 1$

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ABSTRACT. This paper presents a new nonlinear dyadic subdivision scheme eliminating the Gibbs oscillations close to discontinuities. Its convergence, stability and order of approximation are analyzed. It is proved that this scheme converges towards limit functions Hölder continuous with exponent larger than 1.299. Numerical estimates provide a Hölder exponent of 2.438. This subdivision scheme is the first one that simultaneously achieves the control of the Gibbs phenomenon and has limit functions with Hölder exponent larger than 1.

1. INTRODUCTION

Subdivision schemes are useful tools for generating smooth curves and surfaces. For convergent schemes, starting from discrete sets of control points and using basic rules of low complexity, curves or surfaces can be obtained as limits (called limit functions) of sequences of points generated by recursive application of the subdivision scheme.

A simple example of a subdivision scheme is the family of interpolatory subdivision schemes, based on local Lagrange interpolation that has been derived and analyzed in [11]. Another example is the family of spline subdivision schemes related to spline spaces [8].

The four-point interpolatory scheme [16], [15], is a convergent linear scheme of the first family, involving four-point stencils at each subdivision, for which the limit functions are at least in the space C^{2-} (see Definitions 1 and 2 in Section 3). The Chaikin algorithm [7] is an example of a spline subdivision scheme, with lower complexity than the previous example and converging towards C^{2-} functions.

For applications, for instance, to computer-aided geometric design or image processing, complexity and convergence/regularity are not the only quality criteria. On the one hand, the order of approximation, which characterizes the precision of the scheme, is crucial. On the other hand, oscillations that could occur in the limit functions in the vicinity of strongly varying data (coming from the sampling of discontinuous functions), called Gibbs oscillations, are undesirable.

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In the last decade, various attempts to improve the properties of linear subdivision schemes have lead to nonlinear subdivision schemes. For such schemes, the subdivision rules become data dependant; in addition to the previously defined criteria, a stability property should be added to ensure that the nonlinear scheme is linearly affected by perturbations of the data (for linear schemes, the stability is a direct consequence of the convergence).

For nonlinear subdivision schemes, few general results of convergence or stability are available; see for instance [5], [9], [12], [22], [10], [19] and [17].

A large family of nonlinear subdivision schemes comprising, e.g., ENO, WENO or PPH schemes [9], [4], is built from schemes constructed as a perturbation of the four-point linear interpolatory C^{2-} Lagrange scheme based on centered degree 3 polynomial interpolation. These schemes are interpolatory subdivision schemes and are constructed to avoid the Gibbs oscillations occurring classically for linear interpolatory schemes (see Figure 1 in Section 6). The schemes of this family are, unfortunately, characterized by a low regularity of the limit functions, typically C^{1-}. Moreover, the ENO scheme is unstable.

In [14], a new linear and noninterpolatory four-point subdivision scheme has been presented. Its refinement rule is based on local cubic interpolation followed by a shift of 1/4 or, in other words, an evaluation at positions 1/4 and 3/4 rather than the standard evaluation at 1/2 that leads to the interpolatory scheme. This new scheme has been shown to be convergent towards a C^2 curve.

The aim of this paper is to analyze a new scheme formulated using the same trick (shift of 1/4) for the PPH-type schemes [4] which are derived by modifying the classical four-point interpolatory subdivision scheme substituting the arithmetic mean with the harmonic mean (see formula (2) in Section 2). After the definition of the scheme in Section 2 we successively analyze its convergence (Section 3), its stability (Section 4) and its order of approximation (in Section 5). Its behavior in the presence of strongly varying data (Gibbs oscillations) is analyzed in Section 6. The last section is devoted to concluding remarks.

2. A new nonlinear subdivision scheme

As mentioned above, the starting point of our work is the construction of N. Dyn, M.S. Floater and K. Hormann in [14]. There, a new linear and noninterpolatory four-point dyadic subdivision scheme that generates C^2 curves is presented. Its refinement rule is based on the local cubic Lagrange interpolation based on the values $\{f_{n-1}, f_n, f_{n+1}, f_{n+2}\}$ at the positions $\{-1, 0, 1, 2\}$ followed by an evaluation at positions 1/4 and 3/4. For all $f \in l^\infty(\mathbb{Z})$, the scheme is then given by

$$\begin{aligned}(Sf)_{2n} &= -\frac{7}{128}f_{n-1} + \frac{105}{128}f_n + \frac{35}{128}f_{n+1} - \frac{5}{128}f_{n+2},\\ (Sf)_{2n+1} &= -\frac{5}{128}f_{n-1} + \frac{35}{128}f_n + \frac{105}{128}f_{n+1} - \frac{7}{128}f_{n+2}.\end{aligned} \tag{1}$$

Following [4] where a nonlinear scheme is derived by modifying the classical four-point interpolatory subdivision scheme substituting the harmonic mean for the arithmetic mean, we first obtain two new formulations of the scheme (1).

1:

$$
\begin{aligned}
(Sf)_{2n} &= \frac{49}{64}f_n + \frac{14}{64}f_{n+1} + \frac{1}{64}f_{n+2} - \frac{7}{64}\frac{(d^2f_n + d^2f_{n+1})}{2},\\
(Sf)_{2n+1} &= \frac{15}{64}f_n + \frac{50}{64}f_{n+1} - \frac{1}{64}f_{n+2} - \frac{5}{64}\frac{(d^2f_n + d^2f_{n+1})}{2};
\end{aligned}
$$

2:

$$
\begin{aligned}
(Sf)_{2n} &= -\frac{1}{64}f_{n-1} + \frac{50}{64}f_n + \frac{15}{64}f_{n+1} - \frac{5}{64}\frac{(d^2f_n + d^2f_{n+1})}{2},\\
(Sf)_{2n+1} &= \frac{1}{64}f_{n-1} + \frac{14}{64}f_n + \frac{49}{64}f_{n+1} - \frac{7}{64}\frac{(d^2f_n + d^2f_{n+1})}{2},
\end{aligned}
$$

where (d^2f), the second order difference, is defined by $d^2f_n = f_{n+1} - 2f_n + f_{n-1}$.

The two formulations differ essentially in the points $f_k, n-1 \le k \le n+2$ contributing to the first three components of the right-hand side of **1** and **2**.

Using the same strategy as in [4], we define the new nonlinear subdivision scheme S_{PPHA} associated to (1) by

If $|d^2f_n| \ge |d^2f_{n+1}|$**:**

$$
\begin{aligned}
(S_{\text{PPHA}}f)_{2n} &= \frac{49}{64}f_n + \frac{14}{64}f_{n+1} + \frac{1}{64}f_{n+2} - \frac{7}{64}\text{PPH}(d^2f_n, d^2f_{n+1}),\\
(S_{\text{PPHA}}f)_{2n+1} &= \frac{15}{64}f_n + \frac{50}{64}f_{n+1} - \frac{1}{64}f_{n+2} - \frac{5}{64}\text{PPH}(d^2f_n, d^2f_{n+1}),
\end{aligned}
$$

and if $|d^2f_n| < |d^2f_{n+1}|$**:**

$$
\begin{aligned}
(S_{\text{PPHA}}f)_{2n} &= -\frac{1}{64}f_{n-1} + \frac{50}{64}f_n + \frac{15}{64}f_{n+1} - \frac{5}{64}\text{PPH}(d^2f_n, d^2f_{n+1}),\\
(S_{\text{PPHA}}f)_{2n+1} &= \frac{1}{64}f_{n-1} + \frac{14}{64}f_n + \frac{49}{64}f_{n+1} - \frac{7}{64}\text{PPH}(d^2f_n, d^2f_{n+1}),
\end{aligned}
$$

where PPH stands for the harmonic mean defined by

$$
(x,y) \in \mathbb{R}^2 \mapsto \text{PPH}(x,y) := \frac{xy}{x+y}(\text{sgn}(xy)+1), \tag{2}
$$

with $\text{sgn}(x) = 1$ if $x \ge 0$ and $\text{sgn}(x) = -1$ if $x < 0$.

The motivation for the substitution of the arithmetic mean by the harmonic mean is the elimination of oscillations near strongly varying data due to the fact that

$$
|\text{PPH}(x,y)| \le 2\min(|x|,|y|) \tag{3}
$$

replaces

$$
\left|\frac{x+y}{2}\right| \le \max(|x|,|y|).
$$

Before making a detailed analysis of the properties of the new scheme S_{PPHA} we summarize the most important properties of the harmonic mean in the following proposition (properties 1 to 9 are proved in [5] and property 10 is straightforward).

Proposition 1 (Properties of the harmonic mean)**.** *For all* $(x,y) \in \mathbb{R}^2$*, the harmonic mean* $\text{PPH}(x,y)$ *satisfies:*

1. $\text{PPH}(x,y) = \text{PPH}(y,x)$.
2. $\text{PPH}(x,y) = 0 \quad \text{if } xy \le 0$.
3. $\text{PPH}(-x,-y) = -\text{PPH}(x,y)$.

4. $\mathrm{PPH}(x,y)=\frac{\mathrm{sign}(x)+\mathrm{sign}(y)}{2}\min(|x|,|y|)\left[1+\left|\frac{x-y}{x+y}\right|\right].$
5. $|\mathrm{PPH}(x,y)|\le\max(|x|,|y|)$.
6. $|\mathrm{PPH}(x,y)|\le 2\min(|x|,|y|)$.
7. *For* $x,y>0$, $\min(x,y)\le\mathrm{PPH}(x,y)\le\frac{x+y}{2}$.
8. *If* $x=O(1)$, $y=O(1)$, $|y-x|=O(h)$ *and* $xy>0$, *then*
$$|\frac{x+y}{2}-\mathrm{PPH}(x,y)|=O(h^2).$$
9. $|\mathrm{PPH}(x_1,y_1)-\mathrm{PPH}(x_2,y_2)|\le 2\max(|x_1-x_2|,|y_1-y_2|)$.
10. *For all* $(c_1,c_2)\in\mathbb{R}^2$,
$$\begin{aligned}|c_1x-c_2\mathrm{PPH}(x,y)| &\le \max(|c_1|,|c_2|)\max(|x|,|y|)\ \textit{if}\ c_1c_2\ge 0,\\ |c_1x-c_2\mathrm{PPH}(x,y)| &\le (|c_1|+|c_2|)\max(|x|,|y|)\ \textit{if}\ c_1c_2<0.\end{aligned}$$

3. Convergence and regularity

We recall the following definitions.

Definition 1 (Convergence of a subdivision scheme)**.** A dyadic subdivision scheme S is said to be uniformly convergent if
$$\forall f\in l^\infty(\mathbb{Z}),\ \exists S^\infty f\in C^0(\mathbb{R})\ \text{s.t.}\ \lim_{j\to+\infty}\sup_{n\in\mathbb{Z}}|(S^jf)_n-S^\infty f(n2^{-j})|=0.$$

Definition 2 ($C^{\alpha-}$ convergence of a subdivision scheme)**.** A convergent subdivision scheme S is said to be $C^{\alpha-}$ convergent if for all $f\in l^\infty(\mathbb{Z}), S^\infty f\in C^{\alpha-}$ where for $0<\alpha\le 1$,
$$C^{\alpha-}=\{f\ \text{continuous, bounded and verifying}\ \forall\alpha_1<\alpha,\ \exists C>0,\ \text{s.t.}\ \forall x,y\in\mathbb{R},\\ |f(x)-f(y)|\le C|x-y|^{\alpha_1}\},$$
and for $\alpha>1$, writing $\alpha=p+r>0$ with $p\in\mathbb{N}$ and $0\le r<1$,
$$C^{\alpha-}=\{f\ \text{with}\ f^{(p)}\in C^{r-}\}.$$

Definition 3 (L^∞ stability of the limit function)**.** Let S be a linear uniformly convergent subdivision scheme and let ϕ be its limit function defined by $\phi=S^\infty\delta$ with $\delta_n=0\quad\forall n\in\mathbb{N}\setminus\{0\}$ and $\delta_0=1$. The limit function ϕ is said to be L^∞ stable if
$$\exists A,B>0\ \text{s.t.}\ \forall f\in l^\infty(\mathbb{Z}), A||f||_\infty\le||\sum_{n\in\mathbb{Z}}f_n\phi(.-n)||_{L^\infty}\le B||f||_\infty,$$
where $||f||_\infty=\sup_{n\in\mathbb{Z}}\{|f_n|\}$.

In order to derive the convergence, we rewrite the nonlinear subdivision scheme S_{PPHA} as a perturbation of a classical two-point linear subdivision scheme, S_{C}, introduced by G. Chaikin in [7] and defined by
$$\begin{aligned}(S_{\mathrm{C}}f)_{2n} &= \frac{3}{4}f_n+\frac{1}{4}f_{n+1},\\ (S_{\mathrm{C}}f)_{2n+1} &= \frac{1}{4}f_n+\frac{3}{4}f_{n+1}.\end{aligned}\tag{4}$$
The scheme S_{C} is known to be convergent with a regularity C^{2-} (i.e., for any $f\in l^\infty(\mathbb{Z}), S_{\mathrm{C}}^\infty f\in C^{2-}$). Moreover, its limit function is L^∞ stable.

Writing:

If $|d^2 f_n| \geq |d^2 f_{n+1}|$**:**

$$\begin{aligned}(S_{\text{PPHA}}f)_{2n} &= \frac{3}{4}f_n + \frac{1}{4}f_{n+1} + \frac{1}{64}d^2 f_{n+1} - \frac{7}{64}\text{PPH}(d^2 f_n, d^2 f_{n+1}),\\ (S_{\text{PPHA}}f)_{2n+1} &= \frac{1}{4}f_n + \frac{3}{4}f_{n+1} - \frac{1}{64}d^2 f_{n+1} - \frac{5}{64}\text{PPH}(d^2 f_n, d^2 f_{n+1}),\end{aligned}$$

and if $|d^2 f_n| < |d^2 f_{n+1}|$**:**

$$\begin{aligned}(S_{\text{PPHA}}f)_{2n} &= \frac{3}{4}f_n + \frac{1}{4}f_{n+1} - \frac{1}{64}d^2 f_n - \frac{5}{64}\text{PPH}(d^2 f_n, d^2 f_{n+1}),\\ (S_{\text{PPHA}}f)_{2n+1} &= \frac{1}{4}f_n + \frac{3}{4}f_{n+1} + \frac{1}{64}d^2 f_n - \frac{7}{64}\text{PPH}(d^2 f_n, d^2 f_{n+1}),\end{aligned}$$

we get that S_{PPHA} can be expressed as

$$S_{\text{PPHA}}f = S_{\text{C}}f + F(d^2 f). \tag{5}$$

Introducing the function

$$R(x,y) = \begin{cases} y - \text{PPH}(x,y) & \text{if } |x| \geq |y|,\\ -x + \text{PPH}(x,y) & \text{if } |x| < |y|,\end{cases} \tag{6}$$

the expression of F reads

$$F(d^2 f)_{2n} = \frac{1}{64}R(d^2 f_n, d^2 f_{n+1}) - \frac{6}{64}\text{PPH}(d^2 f_n, d^2 f_{n+1}), \tag{7}$$

$$F(d^2 f)_{2n+1} = -\frac{1}{64}R(d^2 f_n, d^2 f_{n+1}) - \frac{6}{64}\text{PPH}(d^2 f_n, d^2 f_{n+1}). \tag{8}$$

Before analyzing the convergence and the stability of S_{PPHA}, we establish the following useful properties of the function R:

Proposition 2 (Properties of the function R)**.**

1. *For all* $(x,y) \in \mathbb{R}^2$, $|R(x,y)| \leq \max(|x|,|y|)$.
2. *For all* $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$,

$$|R(x_1,y_1) - R(x_2,y_2)| \leq \max(|x_1 - x_2|, |y_1 - y_2|).$$

Proof. Property 1 is a direct consequence of Proposition 1–10. To get property 2 we note that the function R is continuous and we prove that its first-order partial derivatives R_x and R_y exist and satisfy $||R_x|| + ||R_y|| \leq 1$ almost everywhere.

Indeed, if $x \cdot y > 0$,

$$R_x(x,y) = \begin{cases} -\frac{2y^2}{(x+y)^2} & \text{if } |x| > |y|,\\ -\frac{x^2+2xy-y^2}{(x+y)^2} & \text{if } |x| < |y|,\end{cases} \tag{9}$$

$$R_y(x,y) = \begin{cases} -\frac{y^2+2xy-x^2}{(x+y)^2} & \text{if } |x| > |y|,\\ \frac{2x^2}{(x+y)^2} & \text{if } |x| < |y|,\end{cases} \tag{10}$$

and if $x \cdot y \leq 0$,

$$R_x(x,y) = \begin{cases} 0 & \text{if } |x| > |y|,\\ -1 & \text{if } |x| < |y|,\end{cases} \tag{11}$$

$$R_y(x,y) = \begin{cases} 1 & \text{if } |x| > |y|,\\ 0 & \text{if } |x| < |y|.\end{cases} \tag{12}$$

Therefore, by direct calculation, $||R_x|| + ||R_y||$ is bounded almost everywhere by one that concludes the proof. $\square$

To analyze the convergence of S_{PPHA}, we use the following result proved in [3] and [2]:

Let S_{NL} be a subdivision scheme defined by

$$\forall f \in l^{\infty}(\mathbb{Z}), \quad \forall n \in \mathbb{Z} \qquad (S_{NL}f)_n = (Sf)_n + F(\delta f)_n, \tag{13}$$

where F is a nonlinear operator defined on $l^{\infty}(\mathbb{Z})$, δ is a linear and continuous operator on $l^{\infty}(\mathbb{Z})$ and S is a linear and convergent subdivision scheme with an L^{∞} stable limit function. Then

Theorem 1. *If F, S and δ given in* (13) *verify:*

$$\exists M > 0 \quad s.t. \quad \forall f \in l^{\infty}(\mathbb{Z}) \quad ||F(f)||_{\infty} \leq M||f||_{\infty}, \tag{14}$$

$$\exists c < 1 \quad s.t. \quad \forall f \in l^{\infty}(\mathbb{Z}) \quad ||\delta S(f) + \delta F(\delta f)||_{\infty} \leq c||\delta f||_{\infty}, \tag{15}$$

then the subdivision scheme S_{NL} is uniformly convergent. Moreover, if S is $C^{\alpha-}$ convergent, then S_{NL} is $C^{\beta-}$ convergent with $\beta = \min(\alpha, -log_2(c))$.

Using Theorem 1, we will prove the following result:

Theorem 2. *The nonlinear subdivision scheme S_{PPHA} is $C^{\beta-}$ convergent with $\beta \geq -log_2(\frac{13}{32}) > 1$.*

Proof. For the perturbation F defined by (7) and (8), it is easy to see using Proposition 1-5 and Proposition 2-1 that for all $f \in l^{\infty}(\mathbb{Z})$,

$$||F(f)||_{\infty} \leq \frac{7}{64}||f||_{\infty}, \tag{16}$$

i.e. hypothesis (14).

We now consider hypothesis (15) related, in this case, to the contraction of the second-order differences $(d^2 f)$. To simplify the notation, we call $f^1 = S_{\text{PPHA}}(f)$, thus

$$\begin{aligned}
(d^2f^1)_{2n} &= \frac{1}{64}(16(d^2f)_n - 6\text{PPH}(d^2f_{n-1}, d^2f_n) + 6\text{PPH}(d^2f_n, d^2f_{n+1}) \\
&\quad -R(d^2f_{n-1}, d^2f_n) - 3R(d^2f_n, d^2f_{n+1})), \\
(d^2f^1)_{2n+1} &= \frac{1}{64}(16(d^2f)_{n+1} + 6\text{PPH}(d^2f_n, d^2f_{n+1}) - 6\text{PPH}(d^2f_{n+1}, d^2f_{n+2}) \\
&\quad +3R(d^2f_n, d^2f_{n+1}) + R(d^2f_{n+1}, d^2f_{n+2})).
\end{aligned}$$

Using properties 5 and 10 of Proposition 1 as well as property 1 of Proposition 2 we deduce that for all $f \in l^{\infty}(\mathbb{Z})$,

$$||d^2f^1||_{\infty} \leq \frac{13}{32}||d^2f||_{\infty}. \tag{17}$$

Therefore, hypothesis (15) of Theorem 1 is satisfied and consequently, the convergence of S_{PPHA} is achieved.

For the regularity, we again use Theorem 1. According to the values $\alpha = 2$ and $c = \frac{13}{32}$ we get the regularity constant $\beta = \min\left(2, -\log_2\left(\frac{13}{32}\right)\right) \approx 1.299$. $\square$

Numerical regularity. Following [21], the regularity of a limit function can be evaluated numerically. Using S_1 and S_2 the subdivision schemes for the differences of order 1 and 2 associated to S_{PPHA} (which can be derived due to the specific definition of S_{PPHA}), the following quantities are estimated for $k = 1, 2$ and different values of j:

$$-\log_2\left(2^k \frac{||(S_k^{j+1}f)_{n+1} - (S_k^{j+1}f)_n||_\infty}{||(S_k^j f)_{n+1} - (S_k^j f)_n||_\infty}\right).$$

They provide an estimate for β_1 and β_2 such that the limit functions belong to $C^{1+\beta_1-}$ and $C^{2+\beta_2-}$. From Table 1, the numerical estimate of the regularity is $C^{2.438-}$. Recalling that the corresponding numerical estimate for the linear scheme [14] is $C^{2.67-}$, we observe that the nonlinear perturbation has a very weak influence on the regularity.

Table 1. Numerical estimates of the limit functions regularity $C^{1+\beta_1-}$ and $C^{2+\beta_2-}$ for S_{PPHA}.

j	5	6	7	8	9	10
β_1	0.9999	0.9999	1	1	1	1
β_2	0.4395	0.7738	1.2615	0.6541	0.4387	0.4388

4. Stability

We first recall the following definition.

Definition 4 (Stability of a convergent scheme)**.** A convergent subdivision scheme is stable if

$$\exists C < +\infty \text{ s.t. } \forall f^0, g^0 \in L^\infty(\mathbb{Z}) \quad ||S^\infty f - S^\infty g||_{L^\infty} \leq C||f^0 - g^0||_\infty. \tag{18}$$

As for the convergence, to derive the stability of S_{PPHA} we use the following theorem of [2].

Theorem 3. *If F, S and δ given in* (13) *verify: $\exists M > 0, c < 1$ such that $\forall f, g \in l^\infty(\mathbb{Z})$,*

$$||F(f) - F(g)||_\infty \leq M||f - g||_\infty, \tag{19}$$

$$\|\delta(S_{NL}f - S_{NL}g)\|_\infty \leq c\|\delta(f - g)\|_\infty, \tag{20}$$

then the nonlinear subdivision scheme S_{NL} is stable.

Using Theorem 3, we will prove the following result:

Theorem 4. *The scheme S_{PPHA} is stable.*

Proof. We check the hypotheses of Theorem 3.

First, we start with hypothesis (19) for F.

Using the expressions of F, (7) and (8), Proposition 1-9 and Proposition 2-2, we get for all $f, g \in l^\infty(\mathbb{Z})$,

$$||F(f) - F(g)||_\infty \quad \leq \quad \frac{1 + 7 \cdot 2}{64}||f - g||_\infty.$$

Second, we have to verify the contraction hypothesis (20).

For any couple $(f,g) \in (l^\infty(\mathbb{Z}))^2$, we study $(d^2f^1 - d^2g^1)_k$ for $k = 2n+1$ (Case 1) or $k = 2n$ (Case 2). Only Case 1 is considered since the bound expressions are similar in both cases. Using Proposition 1-9 as well as Proposition 2-2 we get

$$\begin{aligned}
64|(d^2f^1)_{2n+1} - (d^2g^1)_{2n+1}| &\leq 16|(d^2f)_{n+1} - (d^2g)_{n+1}| \\
&\quad +6|\text{PPH}(d^2f_n, d^2f_{n+1}) - \text{PPH}(d^2g_n, d^2g_{n+1})| \\
&\quad +6|\text{PPH}(d^2f_{n+1}, d^2f_{n+2}) - \text{PPH}(d^2g_{n+1}, d^2g_{n+2})| \\
&\quad +3|R(d^2f_n, d^2f_{n+1}) - R(d^2g_n, d^2g_{n+1})| \\
&\quad +|R(d^2f_{n+1}, d^2f_{n+2} - R(d^2g_{n+1}, d^2g_{n+2})| \\
&\leq (16+12+12+3+1)||(d^2f) - (d^2g)||_\infty \\
&= 44||(d^2f) - (d^2g)||_\infty.
\end{aligned}$$

Thus, the hypotheses of Theorem 3 are verified and the stability of S_{PPHA} is established. □

5. Order of approximation

In this section, we consider the reproduction of polynomials and the order of approximation of S_{PPHA}.

We recall the following definitions.

Definition 5 (Reproduction of polynomials)**.** A dyadic subdivision scheme S is said to reproduce polynomials of degree k if for any polynomial P of degree k and for any sequence f such that $\forall n \in \mathbb{Z}, f_n = P(n)$ then:

$$\exists \tilde{P} \text{ a polynomial of degree } k \text{ such that } (Sf)_n = \tilde{P}(2^{-1}n).$$

Definition 6 (Order of approximation)**.** A dyadic subdivision scheme S is said to have an order k of approximation if for any function $g \in C^k$ and any $h > 0$, $f = g(h.)$ implies that

$$|Sf - g(2^{-1}h.)| \leq Ch^k.$$

We then have the following result:

Proposition 3 (Reproduction of polynomials)**.** *S_{PPHA} reproduces the polynomials of degree 2 with the translation of $\frac{1}{4}$.*

Proof. We remark that for any P, polynomial of degree 2, and $p = (P(n))_{n\in\mathbb{Z}}$, we have

$$\text{PPH}(d^2p_n, d^2p_{n+1}) = \frac{d^2p_n + d^2p_{n+1}}{2}.$$

Therefore, for any initial sequence $p = (p_n)_{n\in\mathbb{Z}}$, $S_{\text{PPHA}}(p)$ coincides with the application to p of the linear scheme [14]. In particular, the results of N. Dyn, M.S. Floater and K. Hormann [14] can be applied and the property of Definition 5 is satisfied with $\tilde{P}(.) = P(. + 1/4)$. □

Concerning the order of approximation the following result holds.

Proposition 4 (Order of approximation)**.** *For any function $g \in C^4([0,1])$ and $h > 0$, if*

$$f = (g((n - \frac{1}{2})h))_{n\in\mathbb{Z}},$$

then if $d^2 f_n d^2 f_{n+1} > 0$ *for all* $n \in \mathbb{Z}$*, then*

$$||(S_{\text{PPHA}} f)_n - g(2^{-1}hn)||_\infty = O(h^4),$$

otherwise

$$||(S_{\text{PPHA}} f)_n - g(2^{-1}hn)||_\infty = O(h^3).$$

Proof. According to Proposition 1, if for all $n \in \mathbb{N}$, $d^2 f_n d^2 f_{n+1} > 0$, then

$$|\text{PPH}(d^2 f_n, d^2 f_{n+1}) - \frac{d^2 f_n + d^2 f_{n+1}}{2}| = O(h^4).$$

Therefore, if S stands for the linear scheme defined in [14], according to the definition of S_{PPHA},

$$||S_{\text{PPHA}} f - Sf||_\infty = O(h^4).$$

Since (see [14]) the scheme S is of order of approximation 4 we get the result when $d^2 f_n d^2 f_{n+1} > 0$. Otherwise, the reproduction of polynomials leads to

$$||(S_{\text{PPHA}} f)_n - g(2^{-1}h(n))||_\infty = O(h^3). \qquad \square$$

Remark 1. Following [21] one can also establish, using the stability of S_{PPHA}, that $||S^\infty_{\text{PPHA}} f - g||_\infty = O(h^3)$.

6. Elimination of the Gibbs phenomenon

In this section, we focus on the behavior of the scheme in the presence of strongly varying data. The reference framework deals with the sampling of a step function as shown on Figure 1. As can be seen on the left in Figure 1, high-order linear schemes suffer from an oscillating behavior called Gibbs phenomenon.

According to D. Gottlieb and C.W. Shu [18], given a punctually discontinuous function f and its sampling f^h defined by $f^h_n = f(nh)$, the Gibbs phenomenon deals with the convergence of $S^\infty(f^h)$ towards f when h goes to 0. It can be characterized by two features ([18]):

1. Away from the discontinuity the convergence is rather slow and for any point x,
$$|f(x) - S^\infty(f^h)(x)| = O(h).$$
2. There is an overshoot, close to the discontinuity, that does not diminish with the reduction of h. Thus,
$$\max |f(x) - S^\infty(f^h)(x)| \text{ does not tend to zero with } h.$$

We will now prove that the nonlinear scheme S_{PPHA} does not suffer from the Gibbs phenomenon oscillations, as can be guessed from Figure 1.

We indeed have the following:

Proposition 5 (Elimination of Gibbs oscillations)**.** *Given* $0 \leq \xi \leq h$*, let* f *be any function defined by*

$$\forall x \leq \xi, f(x) = f_-(x) \textit{ with } f_- \in C^4(]-\infty, \xi]),$$
$$\forall x \geq \xi, f(x) = f_+(x) \textit{ with } f_+ \in C^4([\xi, +\infty[),$$

with $f_-(\xi) > f_+(\xi)$*.*

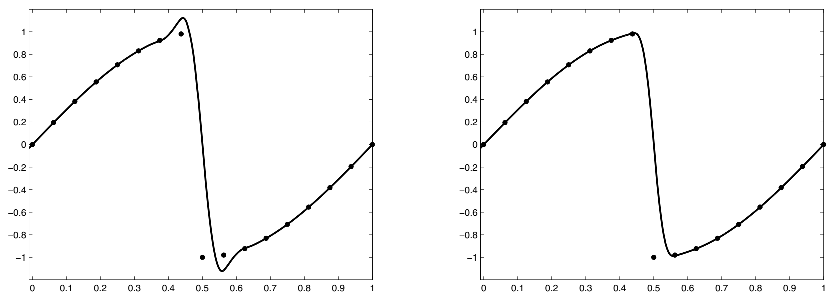

FIGURE 1. Comparison of limit functions for the same initial sequence (sampling of function (23)). Left, the linear scheme (1), right, the nonlinear scheme S_{PPHA}

If h is sufficiently small to ensure that $d^2 f_0 < 0$ and $d^2 f_1 > 0$, we have:

- *if $|x| \geq 3h$, then $|f(x+\frac{1}{2}h) - S^{\infty}_{\text{PPHA}}(f^h)(x)| = O(h^3)$,*
- *if $|x| \leq 3h$, there exists $\alpha_h = 0(h)$ such that $f_-(0) + \alpha_h \geq S^{\infty}_{\text{PPHA}}(f^h)(x) \geq f_+(h) - \alpha_h$.*

Proof. For any iteration j , there exists p_j^-, p_j^+ such that, for all $n \notin [2p_j^-, 2p_j^+]$ the evaluation of $S^{j+1}_{\text{PPHA}} f)_n$ is performed starting only from regular data. For $j = 0, p_0^- = -1, p_0^+ = 2$ and by induction, $p_j^- = -2^{j+1} - 2^j + 2, p_j^+ = 2^{j+1} + 2^j - 1$. Therefore, according to Proposition 4, for $x \geq 3h$, $|f(x+\frac{1}{2}h) - S^{\infty}_{\text{PPHA}}(f^h)(x)| = O(h^3)$.

To prove the second part of the proposition, we first consider the initial data and iterate the scheme.

We recall that, by hypothesis, for all $i \geq 0, f_{-i} = f_- + O(h)$ and $f_i = f_+ + O(h)$. Computing the second differences gives that $d^2 f_{-1} = O(h^2), d^2 f_2 = O(h^2)$ while $d^2 f_0 = f_1 - f_0 + O(h)$ and $d^2 f_1 = -(f_1 - f_0) + O(h)$. Applying the scheme S_{PPHA} provides:

$$
\begin{aligned}
(S_{\text{PPHA}} f)_{-2} &= \frac{3}{4} f_{-1} + \frac{1}{4} f_0 + O(h^2), \\
(S_{\text{PPHA}} f)_{-1} &= \frac{1}{4} f_{-1} + \frac{3}{4} f_0 + O(h^2), \\
(S_{\text{PPHA}} f)_{0} &= \frac{3}{4} f_0 + \frac{1}{4} f_1 + \frac{1}{64}(f_0 - f_1) + O(h), \\
(S_{\text{PPHA}} f)_{1} &= \frac{3}{4} f_0 + \frac{1}{4} f_{n+1} - \frac{1}{64}(f_0 - f_1) + O(h).
\end{aligned}
$$

One should notice that all of these points belong to an interval of the form $[f^+ - O(h), f^- + O(h)]$. Without loss of generality, we focuss on negative indices. A direct evaluation of second-order differences gives:

$$
\begin{aligned}
d^2 S_{\text{PPHA}} f_0 &= (S_{\text{PPHA}} f)_1 - 2(S_{\text{PPHA}} f)_0 + (S_{\text{PPHA}} f)_{-1} = \frac{13}{64}(f_1 - f_0) + O(h), \\
d^2 S_{\text{PPHA}} f_{-1} &= (S_{\text{PPHA}} f)_0 - 2(S_{\text{PPHA}} f)_{-1} + (S_{\text{PPHA}} f)_{-2} = \frac{17}{64}(f_1 - f_0) + O(h),
\end{aligned}
$$

and another application of the scheme provides:

$$\begin{array}{rcl}(S^2_{\text{PPHA}}f)_n &=& (S^2_C f)_n + O(h^2), n \in [2p_1^-, -3],\\ (S^2_{\text{PPHA}}f)_{-2} &=& \frac{1}{4}(\frac{3}{4}f_{-1} + \frac{1}{4}f_1) + \frac{3}{4}f_0 + \frac{c_1}{64}(f_0 - f_1) + O(h),\\ (S^2_{\text{PPHA}}f)_{-1} &=& \frac{1}{4}(\frac{1}{4}f_{-1} + \frac{3}{4}f_1) + \frac{3}{4}f_0 + \frac{c_2}{64}(f_0 - f_1) + O(h),\\ (S^2_{\text{PPHA}}f)_0 &=& \frac{3}{4}f_0 + \frac{1}{4}f_1 + \frac{1}{2}\frac{1}{64}(f_0 - f_1) + O(h),\end{array}$$

with $c_1 = \frac{1}{4} - \frac{13}{64} - 5\text{PPH}(\frac{13}{64}, \frac{17}{64})$ and $c_2 = \frac{3}{4} + \frac{13}{64} - 7\text{PPH}(\frac{13}{64}, \frac{17}{64})$ $(c_i < 0)$.

From this stage, we are now able to prove that Gibbs oscillations cannot appear.

Since after two iterations the second-order differences are bounded by $\frac{17}{64}(f_1 - f_0) + O(h)$, due to (17),

$$\forall j \geq 2, \forall n \in [p^-_{j-1}, 0], |d^2 S^j_{\text{PPHA}} f_n| \leq (\frac{13}{32})^{j-2}\frac{17}{64}(f_0 - f_1) + O(h).$$

According to (16) and due to the stability of S_C we get that the total perturbation is bounded for all $j \geq 2$ by

$$\frac{7}{64}\sum_{m=1}^{+\infty}(\frac{13}{32})^m \frac{17}{64}(f_0 - f_1) = \frac{7}{64}\frac{13}{19}\frac{17}{64}(f_0 - f_1). \tag{21}$$

***Lower bound*:** Due to the "corner cutting property" of the scheme S_C, for all $j \geq 2$, and all $n \in [2p^-_{j-1}, 0]$, we get that

$$S_C^{j-2}(S^2_{\text{PPHA}}f)_n \in [f_0 + 0(h), \min\{(S^2_{\text{PPHA}}f)_n, n \in [2p_1^-, 0]\}].$$

Adding the total perturbation (21) we obtain finally that for all j and all $n \in [2p^-_{j-1}, 0]$, $(S^j_{\text{PPHA}}f)_n \geq f_+ - O(h)$.

***Upper bound*:** Taking into account the regularity of f on $]-\infty, \xi[$ and the data after two iterations, a direct calculation gives that for $n \in [p^-_{j-1}, 2^j - 1]$,

$$(S^j_{\text{PPHA}}f)_n = (S^j_C f)_n + O(h^2),$$

while for $n \in [2^j, 0]$,

$$(S^j_{\text{PPHA}}f)_n \leq A_j f_{-1} + B_j f_1 + \frac{3}{4}f_0 + \frac{7}{64}\frac{13}{19}\frac{17}{64}(f_0 - f_1) + O(h). \tag{22}$$

Here, A_{j+1} and $B_{j+1}, j \geq 2$ are provided from a convex combination of A_j and B_j, therefore, according to their values for $j = 2$, $A_j, B_j \in [\frac{1}{16}, \frac{1}{4}]$ and $A_j + B_j = \frac{1}{4}$.

Rewritting the right-hand term of (22) we get that

$$(S^j_{\text{PPHA}}f)_n \leq A_j(f_{-1} - f_0) + B_j(f_1 - f_0) + f_0 + \frac{7}{64}\frac{17}{64}\frac{13}{19}(f_0 - f_1) + O(h) \leq f_0.$$

Therefore, for all $n \in [p^-_{j-1}, 0], (S^j_{\text{PPHA}}f)_n \leq f_- + O(h)$; this concludes the proof. □

Before finishing this paper, we return to Figure 1 and to the comparison between the limit functions obtained with SPPHA and the limit function obtained with linear subdivision schemes starting from the sampling f^h of the discontinuous function:

$$f(x) = \begin{cases} \sin(\pi x) & \text{for } x \in [0, 0.5[\,, \\ -\sin(\pi x) & \text{for } x \in [0.5, 1]\,. \end{cases} \tag{23}$$

It appears from Figure 1 that the limit function of the nonlinear scheme S_{PPHA} (right) behaves much better close to the discontinuity than do the limit functions associated to the linear scheme of comparable complexity (left). Moreover, from Proposition 4 we know that the limit function of the scheme S_{PPHA} is, in regular regions, of higher order than the Chaikin scheme corresponding function.

7. Conclusions

In this paper, a new nonlinear subdivision scheme has been defined. It has many desirable properties. It is convergent with a regularity proved to be at least $C^{1.299-}$ and numerically estimated at $C^{2.438-}$. By construction, it is adapted to the presence of isolated discontinuities, and the Gibbs phenomenon is eliminated. The scheme is also stable, a property that, due to nonlinearity is not a consequence of the convergence. Moreover, its order of convergence is 3. Given that it is constructed from a four-point centered stencil, all of these properties make this scheme an excellent candidate for various applications. An example is given in Figure 2 devoted to 2D curve generation.

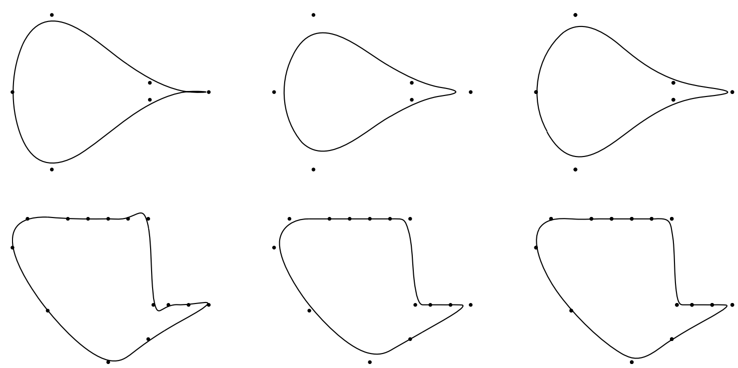

Figure 2. Application to 2D curve generation: Initial points (•); left, linear scheme (1), middle, Chaikin scheme (4), right nonlinear scheme S_{PPHA}

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