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RIESZ BASES OF WAVELETS AND APPLICATIONS TO NUMERICAL SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract. We investigate Riesz bases of wavelets in Sobolev spaces and their applications to numerical solutions of the biharmonic equation and general elliptic equations of fourth-order.

First, we study bicubic splines on the unit square with homogeneous boundary conditions. The approximation properties of these cubic splines are established and applied to convergence analysis of the finite element method for the biharmonic equation. Second, we develop a fairly general theory for Riesz bases of Hilbert spaces equipped with induced norms. Under the guidance of the general theory, we are able to construct wavelet bases for Sobolev spaces on the unit square. The condition numbers of the stiffness matrices associated with the wavelet bases are relatively small and uniformly bounded. Third, we provide several numerical examples to show that the numerical schemes based on our wavelet bases are very efficient. Finally, we extend our study to general elliptic equations of fourth-order and demonstrate that our numerical schemes also have superb performance in the general case.

1. Introduction

In this paper, we investigate Riesz bases of wavelets in Sobolev spaces and their applications to numerical solutions of the biharmonic equation and general elliptic equations of fourth-order.

We use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ to denote the set of positive integers, integers, real numbers, and complex numbers, respectively. For $s \in \mathbb{N}$, we use $\mathbb{R}^s$ to denote the s-dimensional Euclidean space with the inner product of $x = (x_1, \dots, x_s)$ and $y = (y_1, \dots, y_s)$ in $\mathbb{R}^s$ given by $x \cdot y := x_1 y_1 + \cdots + x_s y_s$.

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. An element of $\mathbb{N}_0^s$ is called a **multi-index**. The length of a multi-index $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$ is given by $|\mu| := \mu_1 + \cdots + \mu_s$. For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$ and $x = (x_1, \dots, x_s) \in \mathbb{R}^s$, define $x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}$. A polynomial is a finite sum of the form $\sum_\mu c_\mu x^\mu$ with c_μ being complex numbers. The degree of a polynomial $q = \sum_\mu c_\mu x^\mu$ is defined to be $\deg q := \max\{|\mu| : c_\mu \neq 0\}$. By Π_k we denote the linear space of all polynomials of degree at most k.

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Let Ω be a (Lebesgue) measurable subset of $\mathbb{R}^s$. Suppose f is a complex-valued (Lebesgue) measurable function on Ω. For $1 \le p < \infty$, let

$$\|f\|_{p,\Omega} := \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p},$$

and let $\|f\|_{\infty,\Omega}$ denote the essential supremum of $|f|$ on Ω. When $\Omega = \mathbb{R}^s$, we omit the reference to $\mathbb{R}^s$. For $1 \le p \le \infty$, by $L_p(\Omega)$ we denote the Banach space of all measurable functions f on Ω such that $\|f\|_{p,\Omega} < \infty$. Equipped with the norm $\|\cdot\|_{p,\Omega}$, $L_p(\Omega)$ becomes a Banach space. For $p = 2$, $L_2(\Omega)$ is a Hilbert space with the inner product given by $\langle f, g\rangle := \int_\Omega f(x)\overline{g(x)}\, dx$, $f, g \in L_2(\Omega)$. Let $\mathcal{E}_\Omega$ be the extension operator that maps each function f on Ω to the function $\tilde{f}$ on $\mathbb{R}^s$ given by $\tilde{f}(x) := f(x)$ for $x \in \Omega$ and $\tilde{f}(x) := 0$ for $x \in \mathbb{R}^s \setminus \Omega$. Then $\mathcal{E}_\Omega$ is an embedding from $L_p(\Omega)$ to $L_p(\mathbb{R}^s)$. We may identify $L_p(\Omega)$ with $\mathcal{E}_\Omega(L_p(\Omega))$. Thus we may view $L_p(\Omega)$ as the subspace of $L_p(\mathbb{R}^s)$ consisting of all functions in $L_p(\mathbb{R}^s)$ that vanish outside Ω.

For a vector $y = (y_1, \dots, y_s) \in \mathbb{R}^s$, let D_y denote the differential operator given by

$$D_y f(x) := \lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}, \qquad x \in \mathbb{R}^s.$$

Moreover, we use ∇_y to denote the difference operator given by $\nabla_y f = f - f(\cdot - y)$. Let $e_1, \dots, e_s$ be the unit coordinate vectors in $\mathbb{R}^s$. For $j = 1, \dots, s$, we write D_j for D_{e_j}. For a multi-index $\mu = (\mu_1, \dots, \mu_s)$, we use D^μ to denote the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$.

Suppose Ω is a (nonempty) open subset of $\mathbb{R}^s$. Let $C(\Omega)$ be the linear space of all continuous functions on Ω. By $C_c(\Omega)$ we denote the linear space of all continuous functions on Ω with compact support contained in Ω. For an integer $r \ge 0$, we use $C^r(\Omega)$ to denote the linear space of those functions f on Ω for which $D^\alpha f \in C(\Omega)$ for all $|\alpha| \le r$. Let $C^\infty(\Omega) := \bigcap_{r=0}^\infty C^r(\Omega)$ and $C_c^\infty(\Omega) := C_c(\Omega) \cap C^\infty(\Omega)$. For $k \in \mathbb{N}_0$ and $1 \le p \le \infty$, the Sobolev space $W_p^k(\Omega)$ consists of all functions $f \in L_p(\Omega)$ such that for each multi-index α with $|\alpha| \le k$, $D^\alpha f$ exists in the distributional sense and belongs to $L_p(\Omega)$.

The Fourier transform of a function $f \in L_1(\mathbb{R}^s)$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^s,$$

where i denotes the imaginary unit. The Fourier transform can be naturally extended to functions in $L_2(\mathbb{R}^s)$. By the Plancherel theorem, we have $\|\hat{f}\|_2 = (2\pi)^{s/2}\|f\|_2$.

For $\mu \ge 0$, we denote by $H^\mu(\mathbb{R}^s)$ the Sobolev space of all functions $f \in L_2(\mathbb{R}^s)$ such that the semi-norm

$$|f|_{H^\mu(\mathbb{R}^s)} := \left(\frac{1}{(2\pi)^s}\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 (\xi_1^2 + \cdots + \xi_s^2)^\mu \, d\xi\right)^{1/2} \tag{1.1}$$

is finite. The space $H^\mu(\mathbb{R}^s)$ is a Hilbert space with the inner product given by

$$\langle f, g\rangle_{H^\mu(\mathbb{R}^s)} := \frac{1}{(2\pi)^s}\int_{\mathbb{R}^s} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\big[1 + (\xi_1^2 + \cdots + \xi_s^2)^\mu\big]\, d\xi, \quad f, g \in H^\mu(\mathbb{R}^s).$$

For $\mu > 0$, the norm in $H^\mu(\mathbb{R}^s)$ is given by $\|f\|_{H^\mu(\mathbb{R}^s)} = (\langle f, f\rangle_{H^\mu(\mathbb{R}^s)})^{1/2}$. For a (nonempty) open subset Ω of $\mathbb{R}^s$, we use $H_0^\mu(\Omega)$ to denote the closure of $C_c^\infty(\Omega)$ in $H^\mu(\mathbb{R}^s)$.

If $\mu = k \in \mathbb{N}$, then $H^k(\mathbb{R}^s) = W_2^k(\mathbb{R}^s)$. Suppose $u \in H^k(\mathbb{R}^s)$ for some $k \in \mathbb{N}$. For $|\alpha| \le k$ we have $D^\alpha u \in L_2(\mathbb{R}^s)$ and

$$(D^\alpha u)\hat{}(\xi) = (i\xi)^\alpha \hat{u}(\xi), \quad \xi \in \mathbb{R}^s.$$

Let $\Delta := D_1^2 + \cdots + D_s^2$ be the Laplace operator. For $u \in H^2(\mathbb{R}^s)$ we have

$$(\Delta u)\hat{}(\xi) = -(\xi_1^2 + \cdots + \xi_s^2)\hat{u}(\xi), \quad \xi = (\xi_1, \ldots, \xi_s) \in \mathbb{R}^s.$$

Suppose $u \in H_0^2(\Omega)$. By the Plancherel theorem we have

$$\|\Delta u\|_{L_2(\Omega)}^2 = (2\pi)^{-s} \int_{\mathbb{R}^s} (\xi_1^2 + \cdots + \xi_s^2)^2 |\hat{u}(\xi)|^2 \, d\xi, \quad u \in H_0^2(\Omega).$$

This together with (1.1) gives

$$\|\Delta u\|_{L_2(\Omega)} = |u|_{H_0^2(\Omega)} \quad \forall\, u \in H_0^2(\Omega). \tag{1.2}$$

If, in addition, Ω is bounded, then we have the Poincaré inequality (see, e.g., [12, Chap. 5]):

$$\|u\|_{H_0^2(\Omega)} \le C|u|_{H_0^2(\Omega)} \quad \forall\, u \in H_0^2(\Omega), \tag{1.3}$$

where C is a constant independent of u. But C depends on Ω.

Now suppose that Ω is a bounded and connected open subset of $\mathbb{R}^s$, and its boundary $\partial\Omega$ is Lipschitz continuous. We are interested in the biharmonic equation with the homogeneous boundary conditions:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where f is a given function in $L_2(\Omega)$ and $\frac{\partial u}{\partial n}$ denotes the derivative of u in the direction normal to the boundary $\partial\Omega$. For $u, v \in H_0^2(\Omega)$, it follows from (1.2) and (1.3) that

$$\langle \Delta u, \Delta u\rangle = \|\Delta u\|_{L_2(\Omega)}^2 = |u|_{H_0^2(\Omega)}^2 \ge \|u\|_{H_0^2(\Omega)}^2 / C.$$

Moreover, by (1.2) we have

$$|\langle \Delta u, \Delta v\rangle| \le \|\Delta u\|_{L_2(\Omega)} \|\Delta v\|_{L_2(\Omega)} = |u|_{H_0^2(\Omega)} |v|_{H_0^2(\Omega)}.$$

Hence, by the Lax-Milgram theorem, there exists a unique element $u \in H_0^2(\Omega)$ such that

$$\langle \Delta u, \Delta v\rangle = \langle f, v\rangle \quad \forall\, v \in H_0^2(\Omega). \tag{1.5}$$

This u is the weak solution to the biharmonic equation (1.4).

In order to solve the variational problem (1.5), we use finite dimensional subspaces to approximate $H_0^2(\Omega)$. Let V be a subspace of $H_0^2(\Omega)$ with $\{v_1, \ldots, v_m\}$ as its basis. We look for $y_1, \ldots, y_m \in \mathbb{C}$ such that $u := \sum_{k=1}^m y_k v_k$ satisfies the system of equations

$$\langle \Delta u, \Delta v_j\rangle = \langle f, v_j\rangle, \quad j = 1, \ldots, m.$$

This system of equations can be written as

$$\sum_{k=1}^m a_{jk} y_k = b_j, \quad j = 1, \ldots, m, \tag{1.6}$$

where $b_j := \langle f, v_j\rangle$ and $a_{jk} := \langle \Delta v_k, \Delta v_j\rangle$, $j,k \in \{1,\dots,m\}$. Often $v_1,\dots,v_m$ are chosen to be finite elements over a triangulation of Ω with mesh size $h > 0$. Under suitable conditions, the condition number $\kappa(A)$ of the stiffness matrix $A := (a_{jk})_{1\le j,k\le m}$ is estimated by $\kappa(A) = O(h^{-4})$. Consequently, A becomes ill-conditioned when h is small. Thus, without preconditioning, it would be very difficult to solve the system of linear equations in (1.6) efficiently.

The above discussion leads us to study Riesz bases in Hilbert spaces. Let H be a Hilbert space. The inner product of two elements u and v in H is denoted by $\langle u, v\rangle$. The norm of an element u in H is given by $\|u\| := \sqrt{\langle u, u\rangle}$. Let J be a countable index set. A sequence $(v_j)_{j\in J}$ in a Hilbert space H is said to be a **Bessel sequence** if there exists a constant C such that the inequality

$$\Big\|\sum_{j\in J} c_j v_j\Big\| \le C\Big(\sum_{j\in J}|c_j|^2\Big)^{1/2} \tag{1.7}$$

holds for every sequence $(c_j)_{j\in J}$ with only finitely many nonzero terms. A sequence $(v_j)_{j\in J}$ in H is said to be a **Riesz sequence** if there exist two positive constants C_1 and C_2 such that the inequalities

$$C_1\Big(\sum_{j\in J}|c_j|^2\Big)^{1/2} \le \Big\|\sum_{j\in J} c_j v_j\Big\| \le C_2\Big(\sum_{j\in J}|c_j|^2\Big)^{1/2} \tag{1.8}$$

hold for every sequence $(c_j)_{j\in J}$ with only finitely many nonzero terms. We call C_1 a **Riesz lower bound** and C_2 a **Riesz upper bound**. If $(v_j)_{j\in J}$ is a Riesz sequence in H and the linear span of $(v_j)_{j\in J}$ is dense in H, then $(v_j)_{j\in J}$ is said to be a **Riesz basis** of H. Let A denote the matrix $\big(\langle v_j, v_k\rangle\big)_{j,k\in J}$. If (1.8) is valid, then the condition number of the matrix A is no bigger than C_2^2/C_1^2. See [34, Chap. 4] for details.

Let us consider a model problem of the biharmonic equation (1.4) when

$$\Omega = (0,1)^2 := \{(x_1,x_2)\in\mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\}.$$

In this case, we first formulate a nested sequence of finite dimensional subspaces $(V_n)_{n\ge 3}$ of $H_0^2(\Omega)$ such that $V_n \subset V_{n+1}$ for all $n \ge 3$ and $\bigcup_{n=3}^\infty V_n$ is dense in $H_0^2(\Omega)$. Then we construct a suitable wavelet space W_n such that V_{n+1} is the direct sum of V_n and W_n for $n \ge 3$. Choose a Riesz basis for V_3 and, for each $n \ge 3$, choose a Riesz basis for W_n. For $n \ge 3$, let Ψ_n be the union of the Riesz bases of V_3, $W_3,\dots,W_{n-1}$. Under certain conditions we can prove that the condition number of the matrix $B_n := (\langle \Delta u, \Delta v\rangle)_{u,v\in\Psi_n}$ is uniformly bounded (independent of n). Therefore, appropriately chosen Riesz bases of wavelets will give rise to efficient algorithms for numerical solutions of the biharmonic equation (1.4). In fact, more is true. We will demonstrate that the wavelet bases constructed in this paper are also good for numerical solutions of general elliptic equations of fourth-order.

Spline wavelets will be used in this paper. In [8] Chui and Wang initiated the study of semi-orthogonal wavelets generated from cardinal splines. Biorthogonal spline wavelets were constructed by Dahmen, Kunoth and Urban in [9]. Many applications require wavelets with very short support. Spline wavelets with short support were investigated by Jia, Wang, and Zhou in [22], and by Han and Shen in [14]. Using Hermite cubic splines, Jia and Liu in [20] constructed wavelet bases on the interval $[0,1]$ and applied those wavelets to numerical solutions to the Sturm-Liouville equation with the Dirichelet boundary condition. Spline wavelets on the interval

$[0,1]$ with homogeneous boundary conditions were constructed in [18]. Riesz bases of multiple wavelets were studied by Han and Jiang in [13], and by Li and Xian in [25].

For polygonal domains Riesz bases of spline wavelets were constructed by Davydov and Stevenson [10] on quadrangulation, and by Jia and Liu [21] on arbitrary triangulations. However, numerical schemes based on these wavelet bases have yet to be implemented.

Here is an outline of the paper. In Section 2, we investigate cubic splines on the interval $[0,1]$ and the square $[0,1]^2$ with homogeneous boundary conditions. The approximation properties of these cubic splines are established and applied to convergence analysis of the finite element method for the biharmonic equation in Section 3. In order to construct wavelet bases from these cubic splines, we discuss norm equivalence induced by multilevel decompositions in Section 4, and develop a fairly general theory for Riesz bases of Hilbert spaces equipped with the induced norms in Section 5. Under the guidance of the general theory, we are able to construct wavelet bases for Sobolev spaces on the interval $(0,1)$ and the square $(0,1)^2$ in Section 6. In Section 7 we describe the general principle for the wavelet method and show that the condition numbers of the stiffness matrices associated with the wavelet bases are relatively small and uniformly bounded. In Sections 8 and 9 we give numerical examples to demonstrate that the algorithms based on our wavelet bases are very efficient. Finally, in Section 10, we extend our study to general elliptic equations of fourth-order and indicate that our numerical schemes also have superb performance in the general case.

2. Approximation by cubic splines

In this section, we investigate cubic splines on the interval $[0,1]$ and the square $[0,1]^2$ with homogeneous boundary conditions.

A spline is a piecewise polynomial. Let us give a definition of B-splines according to the book [3] of de Boor. Suppose $k \in \mathbb{N}$. Let $\mathbf{t} = (t_j)_{j\in\mathbb{Z}}$ be a sequence of real numbers such that $t_j < t_{j+k}$ for all $j \in \mathbb{Z}$. The B-splines of order k for $\mathbf{t}$ are given by

$$B_{j,k,\mathbf{t}}(x) := (t_{j+k} - t_j)[t_j, \ldots, t_{j+k}](\cdot - x)_+^{k-1}, \quad x \in \mathbb{R},$$

where $[t_j, \ldots, t_{j+k}]$ denotes the kth order divided difference at the points $t_j, \ldots, t_{j+k}$, and $a_+ := \max(a, 0)$. It is easily seen that each $B_{j,k,\mathbf{t}}$ is supported on $[t_j, t_{j+k}]$ and agrees with a polynomial of degree at most $k-1$ on each interval (t_r, t_{r+1}) as long as $t_{r+1} > t_r$. Many useful properties of B-splines can be established by using the local linear functional introduced by de Boor and Fix in [4]. Among other things, a polynomial p of degree at most $k-1$ can be represented as a B-spline series. In other words, there exist complex numbers c_j $(j \in \mathbb{Z})$ such that $p = \sum_{j\in\mathbb{Z}} c_j B_{j,k,\mathbf{t}}$. Moreover, the B-splines $B_{j,k,\mathbf{t}}$ are locally linearly independent. To be more precise, let (a,b) be a finite interval. Then only finitely many of B-splines $B_{j,k,\mathbf{t}}$ are not identically zero on (a,b) and these B-splines are linearly independent on (a,b).

Now suppose $k = 4$, $N \geq 8$, and $\mathbf{t}$ is given by

$$t_j = \begin{cases} j-1 & \text{for } j \leq 1, \\ j-2 & \text{for } 2 \leq j \leq N+2, \\ j-3 & \text{for } j \geq N+3. \end{cases}$$

Let $\phi_j := B_{j,4,\mathbf{t}}$, $j \in \mathbb{Z}$. It is easily seen that $\phi_j(x) = \phi_{N-j}(N-x)$ for $x \in \mathbb{R}$. Each ϕ_j is a cubic spline. Write ϕ for ϕ_2. Then ϕ is supported on $[0,4]$ and

$$\phi(x) = \begin{cases} \frac{1}{6}x^3 & \text{for } 0 \le x \le 1, \\ -\frac{1}{2}x^3 + 2x^2 - 2x + \frac{2}{3} & \text{for } 1 \le x \le 2, \\ \frac{1}{2}x^3 - 4x^2 + 10x - \frac{22}{3} & \text{for } 2 \le x \le 3, \\ -\frac{1}{6}x^3 + 2x^2 - 8x + \frac{32}{3} & \text{for } 3 \le x \le 4. \end{cases}$$

It is easily seen that $\phi(1) = 1/6$, $\phi(2) = 2/3$, $\phi(3) = 1/6$. Moreover,

$$\phi(x) = \frac{1}{8}\big[\phi(2x) + 4\phi(2x-1) + 6\phi(2x-2) + 4\phi(2x-3) + \phi(2x-4)\big], \quad x \in \mathbb{R}.$$

Write ϕ_b for ϕ_1. Then ϕ_b is given by

$$\phi_b(x) = \begin{cases} \frac{1}{12}(-11x^3 + 18x^2) & \text{for } 0 \le x \le 1, \\ \frac{1}{12}(7x^3 - 36x^2 + 54x - 18) & \text{for } 1 \le x \le 2, \\ \frac{1}{6}(3-x)^3 & \text{for } 2 \le x \le 3, \\ 0 & \text{for } x \in \mathbb{R} \setminus [0,3]. \end{cases}$$

It is easily seen that $\phi_b(1) = 7/12$ and $\phi_b(2) = 1/6$. Moreover,

$$\phi_b(x) = \frac{1}{4}\phi_b(2x) + \frac{11}{16}\phi(2x) + \frac{1}{2}\phi(2x-1) + \frac{1}{8}\phi(2x-2), \quad x \in \mathbb{R}.$$

We have $\phi_b'' \in L_2(\mathbb{R})$ and

$$\phi_b''(x) = \begin{cases} \frac{1}{2}(-11x + 6) & \text{for } 0 < x \le 1, \\ \frac{1}{2}(7x - 12) & \text{for } 1 < x \le 2, \\ 3 - x & \text{for } 2 < x \le 3, \\ 0 & \text{for } x \in \mathbb{R} \setminus [0,3]. \end{cases}$$

It follows that $\lim_{x\to 0-} \phi_b''(x) = 0$ and $\lim_{x\to 0+} \phi_b''(x) = 3$. Hence, $\phi_b''' = g + 3\delta$, where g is a piecewise constant function supported on $[0,3]$ and δ is the Dirac delta function. By taking the Fourier transform of both sides of the equation $\phi_b''' = g + 3\delta$ we obtain $(i\xi)^3\hat{\phi}_b(\xi) = \hat{g}(\xi) + 3$ for $\xi \in \mathbb{R}$. Consequently, for $0 < \mu < 5/2$ we have

$$\int_{|\xi|>1} |\xi|^{2\mu} |\hat{\phi}_b(\xi)|^2 \, d\xi = \int_{|\xi|>1} \frac{|\hat{g}(\xi) + 3|^2}{|\xi|^{2(3-\mu)}} \, d\xi < \infty.$$

Moreover,

$$\int_{|\xi|\le 1} |\xi|^{2\mu} |\hat{\phi}_b(\xi)|^2 \, d\xi \le \int_{|\xi|\le 1} |\hat{\phi}_b(\xi)|^2 \, d\xi < \infty.$$

This shows that $\phi_b \in H^\mu(\mathbb{R})$ for $0 < \mu < 5/2$. An analogous argument shows that $\phi \in H^\mu(\mathbb{R})$ for $0 < \mu < 7/2$.

We see that ϕ is supported on $[0,4]$. Consequently, $\phi(\cdot - k)$ vanishes on $(1,2)$ for $k \le -3$ or $k \ge 2$. Moreover, the B-splines $\phi(\cdot - k)$ $(k = -2, -1, 0, 1)$ are linearly independent on $(1,2)$. Hence, there exists a continuous function $\tilde{\varphi}$ supported on $[1,2]$ such that

$$\langle \tilde{\varphi}, \phi(\cdot - k) \rangle = \delta_{0k} \quad \forall\, k \in \mathbb{Z},$$

where δ_{jk} denotes the Kronecker symbol: $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$. We observe that

$$\phi_k = \begin{cases} \phi(\cdot - k + 1) & \text{for } k \leq -3, \\ \phi(\cdot - k + 2) & \text{for } k = 2, \ldots, N-2, \\ \phi(\cdot - k + 3) & \text{for } k \geq N+3. \end{cases}$$

Accordingly, we define $\tilde{\phi}_j$ for $j \in \mathbb{Z} \setminus ([-3,3] \cup [N-2, N+4])$ as follows:

$$\tilde{\phi}_j := \begin{cases} \tilde{\varphi}(\cdot - j + 1) & \text{for } j \leq -4, \\ \tilde{\varphi}(\cdot - j + 2) & \text{for } j = 4, \ldots, N-3, \\ \tilde{\varphi}(\cdot - j + 3) & \text{for } j \geq N+5. \end{cases}$$

It is easily verified that $\langle \tilde{\phi}_j, \phi_k \rangle = \delta_{jk}$ for all $j \in \mathbb{Z} \setminus ([-3,3] \cup [N-2, N+4])$ and $k \in \mathbb{Z}$. Suppose $-3 \leq j \leq 3$. By the local linear independence of the B-splines ϕ_k ($k \in \mathbb{Z}$), we can find a continuous function $\tilde{\phi}_j$ supported on $[j-1, j]$ such that $\langle \tilde{\phi}_j, \phi_k \rangle = \delta_{jk}$ for all $k \in \mathbb{Z}$. Similarly, for $N-2 \leq j \leq N+4$ we can find a continuous function $\tilde{\phi}_j$ supported on $[j, j+1]$ such that $\langle \tilde{\phi}_j, \phi_k \rangle = \delta_{jk}$ for all $k \in \mathbb{Z}$. We conclude that $\langle \tilde{\phi}_j, \phi_k \rangle = \delta_{jk}$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$. In light of our construction, there is a constant M independent of N such that $\|\tilde{\phi}_j\|_\infty \leq M$ for all $j \in \mathbb{Z}$. Consequently, $(\phi_j)_{j \in \mathbb{Z}}$ is a Riesz sequence in $L_2(\mathbb{R})$.

Let Q be the linear operator given by

$$Qu := \sum_{j \in \mathbb{Z}} \langle u, \tilde{\phi}_j \rangle \phi_j,$$

where u is a locally integrable function on $\mathbb{R}$. In particular, Qu is well defined for u in $L_2(\mathbb{R})$. Since $\|\tilde{\phi}_j\|_\infty \leq M$ for all $j \in \mathbb{Z}$, Q is a bounded operator on $L_2(\mathbb{R})$. Moreover, if u is supported on $[0, N]$, then $\langle u, \tilde{\phi}_j \rangle = 0$ for $j \leq 0$ or $j \geq N$. But ϕ_j is supported in $[0, N]$ for $1 \leq j \leq N-1$. Therefore, Qu is supported on $[0, N]$, provided u is supported on $[0, N]$. We have $Q\phi_j = \phi_j$ for all $j \in \mathbb{Z}$. A polynomial $p \in \Pi_3$ can be represented as $p = \sum_{j \in \mathbb{Z}} c_j \phi_j$. Hence, $Qp = p$ for all $p \in \Pi_3$.

Now choose $N = 2^n$ for some $n \geq 3$. For $j \in \mathbb{Z}$, let $\phi_{n,j}(x) := 2^{n/2} \phi_j(2^n x)$ and $\tilde{\phi}_{n,j}(x) := 2^{n/2} \tilde{\phi}_j(2^n x)$, $x \in \mathbb{R}$. Then $\langle \tilde{\phi}_{n,j}, \phi_{n,k} \rangle = \delta_{jk}$ for all $j, k \in \mathbb{Z}$. Let Q_n be the linear operator given by

$$Q_n u = \sum_{j \in \mathbb{Z}} \langle u, \tilde{\phi}_{n,j} \rangle \phi_{n,j}, \tag{2.1}$$

where u is a locally integrable function on $\mathbb{R}$. Clearly, $Q_n \phi_{n,j} = \phi_{n,j}$ for all $j \in \mathbb{Z}$. By $\|Q_n\|$ we denote the norm of Q_n as an operator on $L_2(\mathbb{R})$. Then $\|Q_n\| = \|Q\|$ for all $n \geq 3$. Since $Q_n p = p$ for all $p \in \Pi_3$, for $0 \leq m \leq 3$ we have (see [16] and [19])

$$\|Q_n u - u\|_{H^m(\mathbb{R})} \leq C(1/2^n)^{4-m} |u|_{H^4(\mathbb{R})}, \quad u \in H^4(\mathbb{R}), \tag{2.2}$$

where C is a constant independent of n and u.

For $n \geq 3$ and $j = 1, \ldots, 2^n - 1$, $\phi_{n,j}$ and $\tilde{\phi}_{n,j}$ are supported in the interval $[0, 1]$. Let $V_n(0,1)$ be the linear span of $\{\phi_{n,j} : j = 1, \ldots, 2^n - 1\}$. Since $\phi_{n,j} \in H^\mu(\mathbb{R})$ for $0 < \mu < 5/2$, $V_n(0,1)$ is a closed subspace of $H_0^\mu(0,1)$ for $0 < \mu < 5/2$ and its dimension is $2^n - 1$. Moreover, $V_n(0,1) \subset V_{n+1}(0,1)$ for $n \geq 3$. Recall that $L_2(0,1)$ is regarded as the subspace of $L_2(\mathbb{R})$ consisting of all functions in $L_2(\mathbb{R})$ that vanish outside $(0,1)$. It is easily seen that $\{\phi_{n,j} : j = 1, \ldots, 2^n - 1\}$ is a Riesz sequence in

$L_2(0,1)$ with Riesz bounds independent of n. If $u \in L_2(0,1)$, then $\langle u, \tilde{\phi}_{n,j}\rangle = 0$ for $j \notin \{1,\dots,2^n-1\}$. Hence, for $n \ge 3$, Q_n maps $L_2(0,1)$ onto $V_n(0,1)$.

For $n \ge 3$ and $j = (j_1, j_2) \in \mathbb{Z}^2$, let

$$\phi_{n,j}(x) = \phi_{n,j_1}(x_1)\phi_{n,j_2}(x_2) \quad \text{and} \quad \tilde{\phi}_{n,j}(x) = \tilde{\phi}_{n,j_1}(x_1)\tilde{\phi}_{n,j_2}(x_2), \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

Then $\langle \tilde{\phi}_{n,j}, \phi_{n,k}\rangle = \delta_{jk}$ for all $j, k \in \mathbb{Z}^2$. Let $\mathcal{Q}_n$ be the linear operator given by

$$\mathcal{Q}_n u = \sum_{j\in\mathbb{Z}^2} \langle u, \tilde{\phi}_{n,j}\rangle \phi_{n,j}, \tag{2.3}$$

where u is a locally integrable function on $\mathbb{R}^2$. Clearly, $\mathcal{Q}_n\phi_{n,j} = \phi_{n,j}$ for all $j \in \mathbb{Z}^2$. Consequently, $\mathcal{Q}_n p = p$ for every bivariate polynomial p of degree at most 3. Hence, for $0 \le m \le 3$, there exists a constant C independent of n and u such that

$$\|u - \mathcal{Q}_n u\|_{H^m(\mathbb{R}^2)} \le C(1/2^n)^{4-m}|u|_{H^4(\mathbb{R}^2)}, \quad u \in H^4(\mathbb{R}^2). \tag{2.4}$$

Suppose $v_1, v_2 \in H^\mu(\mathbb{R})$ for some $\mu > 0$. Let v be the function on $\mathbb{R}^2$ given by $v(x_1,x_2) := v_1(x_1)v_2(x_2)$, $(x_1,x_2) \in \mathbb{R}^2$. Then $\hat{v}(\xi_1,\xi_2) = \hat{v}_1(\xi_1)\hat{v}_2(\xi_2)$ for $(\xi_1,\xi_2) \in \mathbb{R}^2$. Consequently,

$$\int_{\mathbb{R}^2} |\xi_1|^{2\mu}|\hat{v}(\xi_1,\xi_2)|^2\,d\xi_1\,d\xi_2 = \int_{\mathbb{R}} |\xi_1|^{2\mu}|\hat{v}_1(\xi_1)|^2\,d\xi_1 \int_{\mathbb{R}} |\hat{v}_2(\xi_2)|^2\,d\xi_2 < \infty$$

and

$$\int_{\mathbb{R}^2} |\xi_2|^{2\mu}|\hat{v}(\xi_1,\xi_2)|^2\,d\xi_1\,d\xi_2 = \int_{\mathbb{R}} |\hat{v}_1(\xi_1)|^2\,d\xi_1 \int_{\mathbb{R}} |\xi_2|^{2\mu}|\hat{v}_2(\xi_2)|^2\,d\xi_2 < \infty.$$

This shows $v \in H^\mu(\mathbb{R}^2)$. For $j = (j_1,j_2) \in \mathbb{Z}^2$, we have $\phi_{n,j_1} \in H^\mu(\mathbb{R})$ and $\phi_{n,j_2} \in H^\mu(\mathbb{R})$ for $0 < \mu < 5/2$. Hence, $\phi_{n,j} \in H^\mu(\mathbb{R}^2)$ for $0 < \mu < 5/2$.

Let $J_n := \{(j_1,j_2) \in \mathbb{Z}^2 : 1 \le j_1, j_2 \le 2^n - 1\}$. For $n \ge 3$ and $j \in J_n$, $\phi_{n,j}$ and $\tilde{\phi}_{n,j}$ are supported in the square $[0,1]^2$. Let $V_n((0,1)^2)$ be the linear span of $\{\phi_{n,j} : j \in J_n\}$. Then $V_n((0,1)^2)$ is a closed subspace of $H_0^\mu((0,1)^2)$ for $0 < \mu < 5/2$ and its dimension is $(2^n-1)^2$. Moreover, $V_n((0,1)^2) \subset V_{n+1}((0,1)^2)$. Recall that $L_2((0,1)^2)$ is regarded as the subspace of $L_2(\mathbb{R}^2)$ consisting of all functions in $L_2(\mathbb{R}^2)$ that vanish outside $(0,1)^2$. It is easily seen that $\{\phi_{n,j} : j \in J_n\}$ is a Riesz sequence in $L_2((0,1)^2)$ with Riesz bounds independent of n. If $u \in L_2((0,1)^2)$, then $\langle u, \tilde{\phi}_{n,j}\rangle = 0$ for $n \ge 3$ and $j \notin J_n$. Hence, for $n \ge 3$, $\mathcal{Q}_n$ maps $L_2((0,1)^2)$ onto $V_n((0,1)^2)$.

3. Convergence rates

Throughout this section we assume that Ω is the unit square $(0,1)^2$. We also write V_n for $V_n((0,1)^2)$. Given $f \in L_2(\Omega)$, let $u \in H_0^2(\Omega)$ be the unique solution to the equation (1.5). Since the inner angle at each corner of the square is $\pi/2$, we have $u \in H^4(\Omega)$, by [2, Theorem 7]. Moreover, $|u|_{H^4(\Omega)} \le C\|f\|_{L_2(\Omega)}$ for some constant C independent of f. For $n \ge 3$, let u_n be the unique solution in V_n to the following equation:

$$\langle \Delta u_n, \Delta v\rangle = \langle f, v\rangle \quad \forall\, v \in V_n. \tag{3.1}$$

In this section we will prove that

$$\|u_n - u\|_{H^m(\Omega)} \le C(1/2^n)^{4-m}\|f\|_{L^2(\Omega)} \quad \forall\, n \ge 3, \tag{3.2}$$

where $m = 0, 1, 2$ and C is a constant independent of f and n. For relevant results of finite element methods, the reader is referred to the book [31] of Strang and Fix, and the book [5] of Brenner and Scott.

The function $u \in H^4(\Omega)$ is a weak solution to the biharmonic equation (1.4). By changing its values on a set of measure zero if necessary, u becomes a function in $C^2(\overline{\Omega})$, in light of the Sobolev embedding theorem (see [12, Chap. 5]). Since $u \in H_0^2(\Omega)$, there exists a sequence $(g_n)_{n=1,2,\ldots}$ in $C_c^\infty(\Omega)$ such that $\lim_{n\to\infty} \|g_n - u\|_{H^2(\Omega)} = 0$. By the embedding theorem, $\|g_n - u\|_{C(\overline{\Omega})} \le C\|g_n - u\|_{H^2(\Omega)}$, where C is a constant independent of n. Hence, $\lim_{n\to\infty} \|g_n - u\|_{C(\overline{\Omega})} = 0$. But $g_n(x) = 0$ for all $n \in \mathbb{N}$ and $x \in \partial\Omega$. Consequently, $u(x) = 0$ for all $x \in \partial\Omega$. Let y be a nonzero vector in $\mathbb{R}^2$. Then $\lim_{n\to\infty} \|D_y g_n - D_y u\|_{H^1(\Omega)} = 0$. Let γ be a line segment $\subset \partial\Omega$ that does not contain any of the four corners of Ω. By the trace theorem (see [12, p. 258]), we have

$$\|D_y g_n - D_y u\|_{L_2(\gamma)} \le C\|D_y g_n - D_y u\|_{H^1(\Omega)},$$

where C is a constant independent of n. It follows that $\lim_{n\to\infty} \|D_y g_n - D_y u\|_{L_2(\gamma)} = 0$. But $D_y g_n = 0$ on γ for every $n \in \mathbb{N}$. Therefore, $\|D_y u\|_{L_2(\gamma)} = 0$. Note that $D_y u$ is continuous on $\partial\Omega$. This shows that $D_y u(x) = 0$ for all $x \in \gamma$. Consequently, $D_y u(x) = 0$ for all $x \in \partial\Omega$. In particular, $D_1 u(x) = D_2 u(x) = 0$ for all $x \in \partial\Omega$.

In order to prove (3.2), we first establish the following result on approximation by cubic splines:

$$\inf_{v\in V_n} \|u - v\|_{H^m(\Omega)} \le C(1/2^n)^{4-m}|u|_{H^4(\Omega)}, \quad \text{for } n \ge 3 \text{ and } u \in H_0^2(\Omega) \cap H^4(\Omega), \tag{3.3}$$

where $0 \le m \le 3$ and C is a constant independent of n and u. If we extend u to $\mathbb{R}^2$ by setting $u(x) = 0$ for $x \in \mathbb{R}^2 \setminus \Omega$, then the extended function will be in $H^2(\mathbb{R}^2)$ but, in general, will not be in $H^4(\mathbb{R}^2)$. Thus, (2.4) cannot be applied directly to the current situation.

For $N \ge 8$, let $\mathbf{t} := (t_j)_{-1\le j\le N+5}$ be the knot sequence given by

$$t_j = \begin{cases} 0 & \text{for } -1 \le j \le 2, \\ j-2 & \text{for } 3 \le j \le N+1, \\ N & \text{for } N+2 \le j \le N+5. \end{cases}$$

For $j \in \{-1, 0, N, N+1\}$, let ϕ_j be the B-spline $B_{j,4,\mathbf{t}}$. For $1 \le j \le N-1$, ϕ_j is the same as in §2. By using an argument analogous to that given in §2, we can find real-valued continuous functions $\tilde{\phi}_j$ $(-1 \le j \le N+1)$ with the following properties:

(a) $\langle \tilde{\phi}_j, \phi_k \rangle = \delta_{jk}$ for all $j, k \in \{-1, 0, \ldots, N, N+1\}$;
(b) $\|\tilde{\phi}_j\|_\infty \le M$ for $-1 \le j \le N+1$, where M is a constant independent of N;
(c) $\tilde{\phi}_j$ is supported on the interval E_j, where $E_j := [0, 1]$ for $j = -1, 0$, $E_j := [j-1, j]$ for $j = 1, \ldots, N-1$, and $E_j := [N-1, N]$ for $j = N, N+1$.

Given $n \ge 3$, we choose $N := 2^n$. For $j \in \{-1, 0, \ldots, N, N+1\}$, let

$$\phi_{n,j}(x) = 2^{n/2}\phi_j(2^n x) \quad \text{and} \quad \tilde{\phi}_{n,j}(x) = 2^{n/2}\tilde{\phi}_j(2^n x), \quad x \in [0, 1].$$

For $n \ge 3$ and $j \in \tilde{J}_n := \{(j_1, j_2) : -1 \le j_1, j_2 \le 2^n + 1\}$, let

$$\phi_{n,j}(x) = \phi_{n,j_1}(x_1)\phi_{n,j_2}(x_2) \quad \text{and} \quad \tilde{\phi}_{n,j}(x) = \tilde{\phi}_{n,j_1}(x_1)\tilde{\phi}_{n,j_2}(x_2), \quad x = (x_1, x_2) \in [0, 1]^2.$$

Then $\langle \tilde{\phi}_{n,j}, \phi_{n,k}\rangle = \delta_{jk}$ for all $j,k \in \tilde{J}_n$. Let $\tilde{\mathcal{Q}}_n$ be the linear operator given by

$$\tilde{\mathcal{Q}}_n u = \sum_{j\in \tilde{J}_n} \langle u, \tilde{\phi}_{n,j}\rangle \phi_{n,j},$$

where u is a locally integrable function on $[0,1]^2$. If p is the restriction of a bivariate polynomial of degree at most 3 to $[0,1]^2$, then $\tilde{\mathcal{Q}}_n p = p$. By the results on approximation in [23] and [16], we obtain

$$\|u - \tilde{\mathcal{Q}}_n u\|_{H^m(\Omega)} \le C(1/2^n)^{4-m}|u|_{H^4(\Omega)}, \quad u \in H^4(\Omega), \tag{3.4}$$

where $0 \le m \le 3$ and C is a constant independent of n and u.

For $n \ge 3$, let $\mathcal{Q}_n$ be the linear operator given in (2.3). Recall that J_n is the set $\{(j_1,j_2) : 1 \le j_1, j_2 \le 2^n - 1\}$. If u is a locally integrable function on $[0,1]^2$, then $\langle u, \tilde{\phi}_{n,j}\rangle = 0$ for $j \in \mathbb{Z}^2 \setminus J_n$. Hence, $\mathcal{Q}_n u \in V_n$ for $n \ge 3$. Taking (3.4) into account, we see that in order to establish (3.3), it suffices to show

$$\|\tilde{\mathcal{Q}}_n u - \mathcal{Q}_n u\|_{H^m(\Omega)} \le C(1/2^n)^{4-m}|u|_{H^4(\Omega)}, \quad u \in H^4(\Omega)\cap H_0^2(\Omega), \quad 0 \le m \le 3. \tag{3.5}$$

We have

$$\tilde{\mathcal{Q}}_n u - \mathcal{Q}_n u = \sum_{j\in \tilde{J}_n \setminus J_n} \langle u, \tilde{\phi}_{n,j}\rangle \phi_{n,j}.$$

It follows that (see [17, §3])

$$\|\tilde{\mathcal{Q}}_n u - \mathcal{Q}_n u\|_{H^m(\Omega)} \le C(2^n)^m \Bigg(\sum_{j\in \tilde{J}_n \setminus J_n} \big|\langle u, \tilde{\phi}_{n,j}\rangle\big|^2 \Bigg)^{1/2}, \quad 0 \le m \le 3. \tag{3.6}$$

We observe that $(j_1,j_2) \in \tilde{J}_n \setminus J_n$ if and only if j_1 or j_2 belongs to $\{-1, 0, N, N+1\}$. Let us consider the case $j_1 = 0$. Other cases can be treated similarly. Set $h := 1/2^n$. Since $\tilde{\phi}_{n,0}$ is supported on $[0,h]$, for $j = (j_1,j_2)$ with $j_1 = 0$ we have

$$\langle u, \tilde{\phi}_{n,j}\rangle = \int_0^1 \int_0^h u(x_1,x_2)\tilde{\phi}_{n,0}(x_1)\tilde{\phi}_{n,j_2}(x_2)\,dx_1\,dx_2.$$

Let p_1, p_2, p_3, p_4 be the unique cubic polynomials on $\mathbb{R}$ such that

$$\begin{bmatrix} p_1(0) & p_1'(0) & p_1(h) & p_1'(h) \\ p_2(0) & p_2'(0) & p_2(h) & p_2'(h) \\ p_3(0) & p_3'(0) & p_3(h) & p_3'(h) \\ p_4(0) & p_4'(0) & p_4(h) & p_4'(h) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where p' denotes the derivative of p. For $u \in H^4(\Omega)\cap H_0^2(\Omega)$ and $x = (x_1,x_2) \in \overline{\Omega}$, let

$$w(x_1,x_2) := u(0,x_2)p_1(x_1) + D_1u(0,x_2)p_2(x_1) + u(h,x_2)p_3(x_1) + D_1u(h,x_2)p_4(x_1).$$

It is easily seen that $w(0,x_2) = u(0,x_2)$, $D_1w(0,x_2) = D_1u(0,x_2)$, $w(h,x_2) = u(h,x_2)$, and $D_1w(h,x_2) = D_1u(h,x_2)$. Hence, by Theorem (4.4.4) of [5],

$$\int_0^h \big|u(x_1,x_2) - w(x_1,x_2)\big|^2\,dx_1 \le C_1 h^8 \int_0^h \big|D_1^4 u(x_1,x_2)\big|^2\,dx_1, \quad 0 \le x_2 \le 1, \tag{3.7}$$

where C_1 is a constant independent of u, h, and x_2. Since $u \in H_0^2(\Omega)\cap H^4(\Omega)$, we have $u(0,x_2) = 0$ and $D_1u(0,x_2) = 0$ for $0 \le x_2 \le 1$. Moreover, on the interval

$(0,h)$, p_3 and p_4 can be expressed as linear combinations of $\phi_{n,1}$ and $\phi_{n,2}$ (see the definition of ϕ_1 and ϕ_2 given in §2). But $\langle\tilde{\phi}_{n,0},\phi_{n,k}\rangle = 0$ for $k=1,2$. Hence,

$$\int_0^h w(x_1,x_2)\tilde{\phi}_{n,0}(x_1)\,dx_1 = 0, \quad 0 \le x_2 \le 1.$$

We conclude that

$$\langle u,\tilde{\phi}_{n,(0,j_2)}\rangle = \int_0^1\int_0^h \big[u(x_1,x_2)-w(x_1,x_2)\big]\tilde{\phi}_{n,0}(x_1)\tilde{\phi}_{n,j_2}(x_2)\,dx_1\,dx_2.$$

Recall that $\tilde{\phi}_{n,j_2}$ is supported on the interval E_{j_2}. Let χ_{j_2} denote the characteristic function of E_{j_2}. We deduce from the above equality that

$$\langle u,\tilde{\phi}_{n,(0,j_2)}\rangle = \int_0^1\int_0^h \big[u(x_1,x_2)-w(x_1,x_2)\big]\tilde{\phi}_{n,0}(x_1)\tilde{\phi}_{n,j_2}(x_2)\chi_{j_2}(x_2)\,dx_1\,dx_2.$$

Since $\|\tilde{\phi}_{n,0}\|_{L_2(0,1)} \le M$ and $\|\tilde{\phi}_{n,j_2}\|_{L_2(0,1)} \le M$, by the Schwarz inequality we obtain

$$\big|\langle u,\tilde{\phi}_{n,(0,j_2)}\rangle\big|^2 \le C_2\int_0^1\int_0^h \big|u(x_1,x_2)-w(x_1,x_2)\big|^2\chi_{j_2}(x_2)\,dx_1\,dx_2.$$

Here and in what follows, C_i ($i\in\mathbb{N}$) denotes a constant independent of u and h. It follows that

$$\sum_{j_2=-1}^{N+1}\big|\langle u,\tilde{\phi}_{n,(0,j_2)}\rangle\big|^2 \le C_2\int_0^1\int_0^h \big|u(x_1,x_2)-w(x_1,x_2)\big|^2\sum_{j_2=-1}^{N+1}\chi_{j_2}(x_2)\,dx_1\,dx_2.$$

But $\sum_{j_2=-1}^{N+1}\chi_{j_2}(x_2)\le 4$ for $0\le x_2\le 1$. Thus, the above inequality together with (3.7) gives

$$\Big(\sum_{j_2=-1}^{N+1}\big|\langle u,\tilde{\phi}_{n,(0,j_2)}\rangle\big|^2\Big)^{1/2} \le C_3(2^{-n})^4|u|_{H^4(\Omega)}, \quad u\in H_0^2(\Omega)\cap H^4(\Omega).$$

By using the same argument as for the case $j=(0,j_2)$, we obtain

$$\Big(\sum_{j\in\tilde{J}_n\setminus J_n}\big|\langle u,\tilde{\phi}_{n,j}\rangle\big|^2\Big)^{1/2} \le C_4(2^{-n})^4|u|_{H^4(\Omega)}, \quad u\in H_0^2(\Omega)\cap H^4(\Omega).$$

This in connection with (3.6) gives (3.5). Finally, since $\mathcal{Q}_n u\in V_n$, the desired estimate (3.3) follows from (3.4) and (3.5).

We are in a position to establish (3.2). Our proof follows the lines in §5.9 of [5].

Theorem 3.1. *Given $f\in L_2(\Omega)$, let u be the unique solution in $H_0^2(\Omega)$ to the equation* (1.5)*. For $n\ge 3$, let u_n be the unique solution in V_n to the equation* (3.1)*. Then the estimate* (3.2) *holds for $m=0,1,2$ and a constant C independent of f and n.*

Proof. By (1.5) and (3.1) we have $\langle\Delta(u-u_n),\Delta w\rangle = 0$ for all $w\in V_n$. Suppose $v\in V_n$. Then $u_n - v\in V_n$. Hence,

$$\|\Delta(u-u_n)\|_{L_2(\Omega)}^2 = \langle\Delta(u-u_n),\Delta(u-u_n)\rangle = \langle\Delta(u-u_n),\Delta(u-v)\rangle.$$

Taking (1.2) into account, we obtain

$$\begin{aligned}\|\Delta(u-u_n)\|^2_{L_2(\Omega)} &\le \|\Delta(u-u_n)\|_{L_2(\Omega)}\|\Delta(u-v)\|_{L_2(\Omega)}\\ &= |u-u_n|_{H^2(\Omega)}|u-v|_{H^2(\Omega)} \quad \forall\, v\in V_n.\end{aligned}$$

Write $h := 2^{-n}$. By (3.3) we have

$$\|\Delta(u-u_n)\|_{L_2(\Omega)} \le \inf_{v\in V_n} |u-v|_{H^2(\Omega)} \le Ch^2\|f\|_{L_2(\Omega)}. \tag{3.8}$$

Suppose $g\in L_2(\Omega)$. Let w be the unique element in $H^2_0(\Omega)$ such that

$$\langle \Delta w, \Delta v\rangle = \langle g, v\rangle \quad \forall\, v\in H^2_0(\Omega).$$

Then $w\in H^2_0(\Omega)\cap H^4(\Omega)$ and $\Delta^2 w = g$. Hence,

$$\langle u-u_n, g\rangle = \langle u-u_n, \Delta^2 w\rangle = \langle \Delta(u-u_n), \Delta w\rangle. \tag{3.9}$$

Let w_n be the unique element in V_n such that

$$\langle \Delta w_n, \Delta v\rangle = \langle g, v\rangle \quad \forall\, v\in V_n.$$

By using the same argument as in the preceding paragraph, we have

$$\|\Delta(w-w_n)\|_{L_2(\Omega)} \le Ch^2\|g\|_{L_2(\Omega)}. \tag{3.10}$$

Moreover, it follows from (3.9) that

$$\langle u-u_n, g\rangle = \langle \Delta(u-u_n), \Delta w\rangle - \langle \Delta(u-u_n), \Delta w_n\rangle = \langle \Delta(u-u_n), \Delta(w-w_n)\rangle.$$

Consequently,

$$\langle u-u_n, g\rangle \le \|\Delta(u-u_n)\|_{L_2(\Omega)}\|\Delta(w-w_n)\|_{L_2(\Omega)}.$$

This in connection with (3.8) and (3.10) gives

$$\langle u-u_n, g\rangle \le C^2h^4\|f\|_{L_2(\Omega)}\|g\|_{L_2(\Omega)}.$$

The above inequality is valid for all $g\in L_2(\Omega)$. Therefore, by the converse to the Hölder inequality, we obtain

$$\|u-u_n\|_{L_2(\Omega)} \le C^2h^4\|f\|_{L_2(\Omega)}.$$

It follows from (3.8) that

$$|u-u_n|_{H^2(\Omega)} = \|\Delta(u-u_n)\|_{L_2(\Omega)} \le Ch^2\|f\|_{L_2(\Omega)}.$$

Finally, by the interpolation theorem we have

$$|u-u_n|_{H^1(\Omega)} \le C_1\|u-u_n\|^{1/2}_{L_2(\Omega)}|u-u_n|^{1/2}_{H^2(\Omega)} \le C_2h^3\|f\|_{L_2(\Omega)},$$

where C_1 and C_2 are constants independent of f and n. □

4. Norm equivalence

For the shift-invariant case, norm equivalence based on wavelet decompositions was investigated in [26, Chap. 6] and [11]. The following discussions on norm equivalence are pertinent to our purpose.

For $y = (y_1, \dots, y_s) \in \mathbb{R}^s$, let $|y| := |y_1| + \cdots + |y_s|$. For $x, y \in \mathbb{R}^s$, we use $[x, y]$ to denote the line segment $\{(1-t)x + ty : 0 \le t \le 1\}$. Given a nonempty open subset Ω of $\mathbb{R}^s$ and $y \in \mathbb{R}^s$, we set $\Omega_y := \{x \in \Omega : [x-y, x] \subset \Omega\}$. The **modulus of continuity** of a function f in $L_p(\Omega)$ for $1 \le p < \infty$, or $f \in C(\Omega)$ for $p = \infty$, is defined by

$$\omega(f,h)_p := \sup_{|y|\le h} \left\|\nabla_y f\right\|_{p,\Omega_y}, \quad h > 0.$$

For a positive integer m, the mth **modulus of smoothness** of f is defined by

$$\omega_m(f,h)_p := \sup_{|y|\le h} \left\|\nabla_y^m f\right\|_{p,\Omega_{my}}, \quad h > 0.$$

For $\mu > 0$ and $1 \le p, q \le \infty$, the Besov space $B^\mu_{p,q}(\Omega)$ is the collection of those functions $f \in L_p(\Omega)$ for which the following semi-norm is finite:

$$|f|_{B^\mu_{p,q}(\Omega)} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} \left[2^{k\mu}\omega_m(f,2^{-k})_p\right]^q\right)^{1/q}, & \text{for } 1 \le q < \infty, \\ \sup_{k\in\mathbb{Z}} \left\{2^{k\mu}\omega_m(f,2^{-k})_p\right\}, & \text{for } q = \infty, \end{cases}$$

where m is the least integer greater than μ. The norm for $B^\mu_{p,q}(\Omega)$ is

$$\|f\|_{B^\mu_{p,q}(\Omega)} := \|f\|_{L_p(\Omega)} + |f|_{B^\mu_{p,q}(\Omega)}.$$

If $p = q = 2$, then the Besov space $B^\mu_{2,2}(\mathbb{R}^s)$ is the same as the Sobolev space $H^\mu(\mathbb{R}^s)$, and the semi-norms $|\cdot|_{B^\mu_{2,2}}$ and $|\cdot|_{H^\mu}$ are equivalent (see [29, Chap. 5]).

A function f in $C_c^\infty(\Omega)$ can be extended to a function in $C_c^\infty(\mathbb{R}^s)$ by setting $f(x) = 0$ for $x \in \mathbb{R}^s \setminus \Omega$. In this way, $C_c^\infty(\Omega)$ can be regarded as a subspace of $C_c^\infty(\mathbb{R}^s)$. We use $H^\mu_0(\Omega)$ to denote the closure of $C_c^\infty(\Omega)$ in $H^\mu(\mathbb{R}^s)$.

For $n \ge 3$, let $V_n(0,1)$ be the linear space of splines given in §2. Then $V_n(0,1)$ is a closed subspace of $H^\mu_0(0,1)$ for $0 \le \mu < 5/2$. Let Q_n be the operator given in (2.1). For $n \ge 3$, $Q_n|_{L_2(0,1)}$ is a projection from $L_2(0,1)$ onto $V_n(0,1)$. We have the following estimate (see [16] and [19]):

$$\|f - Q_n f\|_2 \le C\omega_4(f, 1/2^n)_2 \quad \forall\, f \in L_2(0,1), \tag{4.1}$$

where C is a constant independent of n and f.

Theorem 4.1. *For $n = 3, 4, \dots$, let Q_n be the operator given in* (2.1)*. Then there exist two positive constants C_1 and C_2 such that the inequalities*

$$\begin{aligned} C_1|f|_{H^\mu_0(0,1)} &\le \left(\left[2^{3\mu}\|Q_3 f\|_2\right]^2 + \sum_{n=4}^\infty \left[2^{n\mu}\|(Q_n - Q_{n-1})f\|_2\right]^2\right)^{1/2} \\ &\le C_2|f|_{H^\mu_0(0,1)} \end{aligned} \tag{4.2}$$

hold for all $f \in H^\mu_0(0,1)$ with $0 < \mu < 5/2$.

Proof. Given $f \in L_2(0,1)$, we write $f_n := Q_n f$ for $n \ge 3$. Let $g_3 := f_3$ and $g_n := f_n - f_{n-1}$ for $n \ge 4$. Then

$$f = \sum_{n=3}^\infty g_n,$$

where the series converges in $L_2(0,1)$. In what follows we use C_i ($i \in \mathbb{N}$) to denote a positive constant independent of f and n.

For the first inequality in (4.2), we express each g_n as

$$g_n = \sum_{k=1}^{2^n-1} b_{n,k}\phi_{n,k}, \quad n = 3, 4, \ldots, \tag{4.3}$$

where $b_{n,k} \in \mathbb{C}$ ($k = 1, \ldots, 2^n - 1$). Note that $\big(2^{-n\mu}\phi_{n,k}\big)_{n=3,4,\ldots,k=1,\ldots,2^n-1}$ is a Bessel sequence in $H_0^\mu(0,1)$ (see Theorem 1.2 of [17]). Accordingly,

$$|f|_{H_0^\mu(0,1)} = \Big|\sum_{n=3}^{\infty}\sum_{k=1}^{2^n-1} 2^{n\mu} b_{n,k} 2^{-n\mu}\phi_{n,k}\Big|_{H_0^\mu(0,1)} \le C_1\Big(\sum_{n=3}^{\infty}\sum_{k=1}^{2^n-1}\big|2^{n\mu}b_{n,k}\big|^2\Big)^{1/2}.$$

Since $\{\phi_{n,k} : 1 \le k \le 2^n - 1\}$ is a Riesz basis of $V_n(0,1)$ with Riesz bounds independent of n, it follows from (4.3) that

$$\sum_{k=1}^{2^n-1} |b_{n,k}|^2 \le C_2^2 \|g_n\|_2^2.$$

Consequently,

$$|f|_{H_0^\mu(0,1)} \le C_1 C_2 \Big(\sum_{n=3}^{\infty}\big[2^{n\mu}\|g_n\|_2\big]^2\Big)^{1/2}.$$

For the second inequality in (4.2), we deduce from (4.1) that

$$\|g_n\|_2 = \|Q_n f - Q_{n-1} f\|_2 \le \|Q_n f - f\|_2 + \|Q_{n-1} f - f\|_2 \le C_3 \omega_4(f, 1/2^n)_2.$$

Consequently,

$$\Big(\sum_{n=4}^{\infty}\big[2^{n\mu}\|g_n\|_2\big]^2\Big)^{1/2} \le C_3\Big(\sum_{n=4}^{\infty}\big[2^{n\mu}\omega_4(f,1/2^n)_2\big]^2\Big)^{1/2} \le C_4|f|_{H_0^\mu(0,1)}.$$

Moreover, for $f \in H_0^\mu(0,1)$, by Poincaré's inequality we have

$$\|Q_3 f\|_2 \le \|Q_3\|\|f\|_2 \le C_5|f|_{H_0^\mu(0,1)}.$$

This completes the proof of (4.2). □

For $n \ge 3$, let $V_n((0,1)^2)$ be the linear space of splines given in §2. Then $V_n((0,1)^2)$ is a closed subspace of $H_0^\mu((0,1)^2)$ for $0 \le \mu < 5/2$. Let $\mathcal{Q}_n$ be the operator given in (2.3). For $n \ge 3$, $\mathcal{Q}_n|_{L_2((0,1)^2)}$ is a projection from $L_2((0,1)^2)$ onto $V_n((0,1)^2)$. We have the following estimate (see [16] and [19]):

$$\|f - \mathcal{Q}_n f\|_2 \le C\omega_4(f, 1/2^n)_2 \quad \forall\, f \in L_2((0,1)^2), \tag{4.4}$$

where C is a constant independent of n and f. By using the same argument as in the proof of the above theorem, and taking (4.4) into account, we obtain the following theorem on norm equivalence for the space $H_0^\mu((0,1)^2)$.

Theorem 4.2. *For $n = 3, 4, \ldots$, let $\mathcal{Q}_n$ be the operator given in* (2.3)*. Then there exist two positive constants C_1 and C_2 such that the inequalities*

$$C_1|f|_{H_0^\mu((0,1)^2)} \le \Big(\big[2^{3\mu}\|\mathcal{Q}_3 f\|_2\big]^2 + \sum_{n=4}^{\infty}\big[2^{n\mu}\|(\mathcal{Q}_n - \mathcal{Q}_{n-1})f\|_2\big]^2\Big)^{1/2} \le C_2|f|_{H_0^\mu((0,1)^2)}$$

hold for all $f \in H_0^\mu((0,1)^2)$ with $0 < \mu < 5/2$.

As a consequence of Theorems 4.1 and 4.2, we see that $|\cdot|_{H_0^\mu(0,1)}$ and $|\cdot|_{H_0^\mu((0,1)^2)}$ are norms.

5. Wavelet bases in Hilbert spaces

In this section, we investigate Riesz bases of Hilbert spaces equipped with some induced norms. The reader is referred to [33], [30], and [21] for related work.

Let J be a countable set. For a complex-valued sequence $u = (u_j)_{j\in J}$, let

$$\|u\|_p := \left(\sum_{j\in J} |u_j|^p\right)^{1/p} \qquad \text{for } \ 1 \le p < \infty,$$

and let $\|u\|_\infty$ denote the supremum of $\{|u_j| : j \in J\}$. For $1 \le p \le \infty$, by $\ell_p(J)$ we denote the Banach space of all sequences u on J such that $\|u\|_p < \infty$. In particular, $\ell_2(J)$ is a Hilbert space with the inner product given by $\langle u, v\rangle := \sum_{j\in J} u_j\overline{v_j}$ for $u, v \in \ell_2(J)$.

Let $A = (a_{jk})_{j,k\in J}$ be a square matrix with its entries being complex numbers. For a sequence $y = (y_k)_{k\in J}$, Ay is the sequence $z = (z_j)_{j\in J}$ given by $z_j := \sum_{k\in J} a_{jk}y_k$, $j \in J$, provided the above series converges absolutely for every $j \in J$. If A is a bounded operator on $\ell_p(J)$ for some p, $1 \le p \le \infty$, then the norm of A on $\ell_p(J)$ is defined by $\|A\|_p := \sup_{\|y\|_p\le 1} \|Ay\|_p$. It is easily seen that

$$\|A\|_1 = \sup_{k\in J}\sum_{j\in J} |a_{jk}|, \quad \|A\|_\infty = \sup_{j\in J}\sum_{k\in J} |a_{jk}| \quad \text{and} \quad \|A\|_2 \le \|A\|_1^{1/2}\|A\|_\infty^{1/2}. \tag{5.1}$$

Let H be a Hilbert space. Suppose that $V_0 = \{0\}$ and $(V_n)_{n=1,2,\dots}$ is a nested family of closed subspaces of H: $V_n \subset V_{n+1}$ for all $n \in \mathbb{N}$. Assume that $\bigcup_{n=1}^\infty V_n$ is dense in H. Then every element $f \in H$ can be represented as a convergent series $\sum_{n=1}^\infty f_n$ in H with $f_n \in V_n$ for each $n \in \mathbb{N}$.

Fix $\mu > 0$ and let H_μ be a linear subspace of H. Suppose that H_μ itself is a normed linear space. We assume that there are two positive constants A_1 and A_2 such that the following two statements are valid:

(a) If $f \in H_\mu$ has a decomposition $f = \sum_{n=1}^\infty f_n$ with $f_n \in V_n$, then

$$\|f\|_{H_\mu} \le A_1\left(\sum_{n=1}^\infty \left[2^{n\mu}\|f_n\|\right]^2\right)^{1/2}. \tag{5.2}$$

(b) For each $f \in H_\mu$, there exists a decomposition $f = \sum_{n=1}^\infty f_n$ with $f_n \in V_n$ such that

$$\left(\sum_{n=1}^\infty \left[2^{n\mu}\|f_n\|\right]^2\right)^{1/2} \le A_2\|f\|_{H_\mu}. \tag{5.3}$$

By Theorems 4.1 and 4.2, for $0 < \mu < 5/2$, the space $H_0^\mu(0,1)$ relative to $L_2(0,1)$, and the space $H_0^\mu((0,1)^2)$ relative to $L_2((0,1)^2)$ satisfy the above conditions.

Suppose $f \in V_n$. We claim that there exists a decomposition $f = \sum_{m=1}^n g_m$ with $g_m \in V_m$ $(m = 1, \dots, n)$ such that

$$\left(\sum_{m=1}^n \left[2^{m\mu}\|g_m\|\right]^2\right)^{1/2} \le A_3\|f\|_{H_\mu}, \tag{5.4}$$

where A_3 is a constant independent of f and n. Indeed, there exists a decomposition $f = \sum_{n=1}^{\infty} f_n$ with $f_n \in V_n$ ($n \in \mathbb{N}$) such that (5.3) is valid. Let $g_m := f_m \in V_m$ for $m = 1, \ldots, n-1$ and $g_n := f - \sum_{m=1}^{n-1} f_m$. Clearly, $g_n \in V_n$. We have

$$\|g_n\| = \Big\|\sum_{m=n}^{\infty} f_m\Big\| \le \sum_{m=n}^{\infty} \|f_m\| \le \Big(\sum_{m=n}^{\infty} \big[2^{-m\mu}\big]^2\Big)^{1/2} \Big(\sum_{m=n}^{\infty} \big[2^{m\mu}\|f_m\|\big]^2\Big)^{1/2},$$

where the Schwarz inequality has been used to derive the last inequality. Consequently,

$$\sum_{m=1}^{n} \big[2^{m\mu}\|g_m\|\big]^2 = \sum_{m=1}^{n-1} \big[2^{m\mu}\|g_m\|\big]^2 + \big[2^{n\mu}\|g_n\|\big]^2 \le \frac{1}{1-2^{-2\mu}} \sum_{m=1}^{\infty} \big[2^{m\mu}\|f_m\|\big]^2.$$

This together with (5.3) establishes the desired inequality (5.4) with $A_3 := A_2/\sqrt{1-2^{-2\mu}}$.

For $n \in \mathbb{N}_0$, let P_n be a linear projection from V_{n+1} onto V_n, and let W_n be the kernel space of P_n. Then V_{n+1} is the direct sum of V_n and W_n. Let $\{\psi_{n,k} : k \in K_n\}$ be a Riesz basis of W_n with Riesz bounds independent of n.

Theorem 5.1. *For $\mu > 0$, $(2^{-n\mu}\psi_{n,k})_{n\in\mathbb{N}_0, k\in K_n}$ is a Bessel sequence in H_μ.*

Proof. Suppose $f = \sum_{n=0}^{\infty} g_n$, where

$$g_n = \sum_{k\in K_n} b_{n,k} 2^{-n\mu}\psi_{n,k} \in V_{n+1}.$$

Since $\{\psi_{n,k} : k \in K_n\}$ is a Riesz basis of W_n with Riesz bounds independent of n, there exists a constant C_1 independent of n such that

$$\Big\|\sum_{k\in K_n} b_{n,k}\psi_{n,k}\Big\|^2 \le C_1^2 \sum_{k\in K_n} |b_{n,k}|^2.$$

This in connection with (5.2) yields

$$\|f\|_{H_\mu}^2 \le A_1^2 \sum_{n=0}^{\infty} \big[2^{(n+1)\mu}\|g_n\|\big]^2 = 2^{2\mu} A_1^2 \sum_{n=0}^{\infty} \Big\|\sum_{k\in K_n} b_{n,k}\psi_{n,k}\Big\|^2 \le C_2^2 \sum_{n=0}^{\infty} \sum_{k\in K_n} |b_{n,k}|^2,$$

where $C_2 := 2^\mu A_1 C_1$. Hence, $(2^{-n\mu}\psi_{n,k})_{n\in\mathbb{N}_0, k\in K_n}$ is a Bessel sequence in H_μ. $\square$

Fix $n \in \mathbb{N}$ for the time being. For each m with $0 \le m \le n$, let T_m be a linear projection from V_n onto V_m. In particular, $T_0 = 0$, and T_n is the identity operator on V_n.

Theorem 5.2. *Suppose that for $j \in [m,n] \cap \mathbb{N}_0$, $\|T_m f\| \le B 2^{\nu(j-m)}\|f\|$ for all $f \in V_j$, where $0 < \nu < \mu$ and B is a constant independent of m, j, and f. Then*

$$\Big(\sum_{m=1}^{n} \big[2^{\mu m}\|(T_m - T_{m-1})f\|\big]^2\Big)^{1/2} \le C\|f\|_{H_\mu} \quad \forall\, f \in V_n, \tag{5.5}$$

where C is a constant depending on μ, ν, and B only.

Proof. Let $f \in V_n$. Then there exists a decomposition $f = \sum_{m=1}^{n} g_m$ with $g_m \in V_m$ for each m such that (5.4) is valid. For $j < m$, we have $(T_m - T_{m-1})g_j = 0$, and hence

$$(T_m - T_{m-1})f = \sum_{j=m}^{n} (T_m - T_{m-1})g_j.$$

For $m \le j \le n$, we have

$$\|(T_m - T_{m-1})g_j\| \le \|T_m g_j\| + \|T_{m-1} g_j\| \le B(1+2^\nu)2^{\nu(j-m)}\|g_j\|.$$

It follows that

$$2^{\mu m}\|(T_m - T_{m-1})f\| \le B(1+2^\nu)\sum_{j=m}^{n} 2^{-(\mu-\nu)(j-m)}\big[2^{j\mu}\|g_j\|\big].$$

Write $y_m := 2^{\mu m}\|(T_m - T_{m-1})f\|$ for $m = 1, \dots, n$ and $z_j := B(1+2^\nu)[2^{j\mu}\|g_j\|]$ for $j = 1, \dots, n$. Then $y_m \le \sum_{j=1}^n a_{mj} z_j$ for $m = 1, \dots, n$, where $a_{mj} := 2^{-(\mu-\nu)(j-m)}$ for $m \le j \le n$ and 0 otherwise. Let A denote the matrix $(a_{mj})_{1\le m,j\le n}$. It is easily seen that $\|A\|_1 \le 1/(1-2^{-(\mu-\nu)})$ and $\|A\|_\infty \le 1/(1-2^{-(\mu-\nu)})$. Hence, by (5.1) we obtain $\|A\|_2 \le 1/(1-2^{-(\mu-\nu)})$. Consequently,

$$\left(\sum_{m=1}^{n}\big[2^{\mu m}\|(T_m - T_{m-1})f\|\big]^2\right)^{1/2} \le \frac{B(1+2^\nu)}{1-2^{-(\mu-\nu)}}\left(\sum_{j=1}^{n}\big[2^{j\mu}\|g_j\|\big]^2\right)^{1/2}.$$

This in connection with (5.4) yields the desired inequality (5.5). □

Theorem 5.3. *Suppose that $0 < \nu < \mu$ and there exists a positive constant B such that*

$$\|P_m \cdots P_{n-1}\| \le B2^{\nu(n-m)}$$

for all $m, n \in \mathbb{N}$ with $m < n$. Then $\{2^{-m\mu}\psi_{m,k} : m \in \mathbb{N}_0, k \in K_m\}$ is a Riesz basis of H_μ.

Proof. By Theorem 5.1, $\{2^{-m\mu}\psi_{m,k} : m \in \mathbb{N}_0, k \in K_m\}$ is a Bessel sequence in H_μ. In order to prove that it is a Riesz basis of H_μ, let f be an element in H_μ given by

$$f = \sum_{j=0}^{n-1}\sum_{k\in K_j} b_{j,k}2^{-j\mu}\psi_{j,k}. \tag{5.6}$$

For $m = 0, 1, \dots, n-1$, let $T_m := P_m \cdots P_{n-1}$, and let T_n be the identity operator on V_n. If $m < j \le n$ and $f \in V_j$, then $\|T_m f\| = \|P_m \cdots P_{j-1} f\| \le B2^{\nu(j-m)}\|f\|$, by the hypothesis of the theorem. Since $T_m\psi_{j,k} = 0$ for $m \le j \le n-1$, it follows from (5.6) that

$$T_m f = \sum_{j=0}^{m-1}\sum_{k\in K_j} b_{j,k}2^{-j\mu}\psi_{j,k}.$$

Hence,

$$T_{m+1}f - T_m f = \sum_{k\in K_m} b_{m,k}2^{-m\mu}\psi_{m,k}.$$

Since $\{\psi_{m,k} : k \in K_m\}$ is a Riesz basis of W_m with Riesz bounds independent of m, there exists a positive constant C_1 independent of m such that

$$\left(\sum_{k\in K_m}\big|2^{-m\mu}b_{m,k}\big|^2\right)^{1/2} \le C_1\|(T_{m+1} - T_m)f\|.$$

Consequently, by Theorem 5.2 we obtain

$$\left(\sum_{m=0}^{n-1}\sum_{k\in K_m}\big|b_{m,k}\big|^2\right)^{1/2} \le C_1\left(\sum_{m=0}^{n-1}\big[2^{m\mu}\|(T_{m+1} - T_m)f\|\big]^2\right)^{1/2} \le C_2\|f\|_{H_\mu},$$

where C_2 is a constant independent of f. This shows that $\{2^{-m\mu}\psi_{m,k} : m \in \mathbb{N}_0, k \in K_m\}$ is a Riesz sequence in H_μ.

It remains to show that $\bigcup_{n=1}^\infty V_n$ is dense in H_μ. Suppose $f \in H_\mu$. Then f has a decomposition $f = \sum_{n=1}^\infty f_n$ with $f_n \in V_n$ such that (5.3) is valid. Let $g_n := \sum_{m=1}^n f_m$ for $n \in \mathbb{N}$. Clearly, $g_n \in V_n$. It follows from (5.2) that

$$\|f - g_n\|_{H_\mu} \le A_1 \Bigg(\sum_{m=n+1}^\infty \big[2^{m\mu}\|f_m\|\big]^2 \Bigg)^{1/2}.$$

By (5.3), the series $\sum_{n=1}^\infty [2^{n\mu}\|f_n\|]^2$ converges. Therefore, $\lim_{n\to\infty} \|f - g_n\|_{H_\mu} = 0$, as desired. □

6. Wavelet bases in $H^\mu(0,1)$ and $H^\mu((0,1)^2)$

In this section we use the cubic splines studied in §2 to construct wavelet bases of $H_0^\mu(0,1)$ and $H_0^\mu((0,1)^2)$.

For $j \in \mathbb{Z}$, let ϕ_j be the B-splines given in §2. Recall that $\phi_{n,j}(x) = 2^{n/2}\phi_j(2^n x)$, $x \in \mathbb{R}$. For $n \ge 3$, $\{\phi_{n,j} : j = 1, \dots, 2^n - 1\}$ is a Riesz basis of $V_n(0,1)$ with Riesz bounds independent of n. Suppose $f_n = \sum_{j=1}^{2^n-1} a_{n,j}\phi_{n,j} \in V_n(0,1)$. Then

$$B_1^2 \sum_{j=1}^{2^n-1} |a_{n,j}|^2 \le \Bigg\| \sum_{j=1}^{2^n-1} a_{n,j}\phi_{n,j} \Bigg\|_2^2 \le B_2^2 \sum_{j=1}^{2^n-1} |a_{n,j}|^2, \tag{6.1}$$

where B_1 and B_2 are positive constants independent of n. We have

$$\sum_{j=1}^{2^n-1} a_{n,j}\phi_{n,j}(k/2^n) = f_n(k/2^n) \quad \text{for} \quad k = 1, \dots, 2^n - 1.$$

Note that $\phi_{n,j}(k/2^n) = 2^{n/2}\phi_j(k)$. Let F_n denote the matrix $\big(\phi_j(k)\big)_{1\le j,k\le 2^n-1}$. Then

$$F_n = \begin{bmatrix} 7/12 & 1/6 & 0 & \cdots & 0 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/6 & 2/3 & 1/6 \\ 0 & 0 & 0 & \cdots & 0 & 1/6 & 7/12 \end{bmatrix}.$$

Let $\|F_n\|$ denote the norm of the matrix F_n on the ℓ_2 space. Clearly, $\|F_n\| \le 1$. Moreover, F_n is diagonally dominant. It is easily verified that $\|F_n^{-1}\| \le 3$ for all $n \ge 3$. Consequently,

$$C_1^2 \sum_{k=1}^{2^n-1} \big|f_n(k/2^n)\big|^2 \le \sum_{j=1}^{2^n-1} \big|2^{n/2}a_{n,j}\big|^2 \le C_2^2 \sum_{k=1}^{2^n-1} \big|f_n(k/2^n)\big|^2, \tag{6.2}$$

where C_1 and C_2 are positive constants independent of n.

Let P_n be the linear projection from V_{n+1} onto V_n given as follows: For $f_{n+1} \in V_{n+1}$, $f_n := P_n f_{n+1}$ is the unique element in $V_n(0,1)$ determined by the interpolation condition

$$f_n(k/2^n) = f_{n+1}(k/2^n) \quad \forall\, k = 1, \dots, 2^n - 1.$$

Suppose $f_n = \sum_{j=1}^{2^n-1} a_{n,j}\phi_{n,j} \in V_n$. For $m \ge 3$, let $f_m := P_m \cdots P_{n-1} f_n$. The function f_m can be represented as $f_m = \sum_{j=1}^{2^m-1} a_{m,j}\phi_{m,j}$. We have $f_m(k/2^m) =$

$f_n(k/2^m)$ for $k = 1, \dots, 2^m - 1$. It follows from (6.1) and (6.2) that

$$\begin{aligned}\|f_m\|_2^2 &\le B_2^2 \sum_{j=1}^{2^m-1} |a_{m,j}|^2 \le B_2^2 C_2^2 2^{-m} \sum_{k=1}^{2^m-1} \left|f_m(k/2^m)\right|^2 \\ &\le B_2^2 C_2^2 2^{-m} \sum_{k=1}^{2^n-1} \left|f_n(k/2^n)\right|^2 \le B_2^2 C_2^2 C_1^{-2} 2^{-m} \sum_{j=1}^{2^n-1} \left|2^{n/2} a_{n,j}\right|^2 \\ &\le B_2^2 C_2^2 C_1^{-2} B_1^{-2} 2^{n-m} \|f_n\|_2^2.\end{aligned}$$

Consequently,

$$\|P_m \cdots P_{n-1}\| \le C 2^{(n-m)/2},$$

where $C := B_2 C_2 C_1^{-1} B_1^{-1}$.

The wavelets are constructed as follows. Let

$$\psi(x) := -\frac{1}{4}\,\phi(2x) + \phi(2x-1) - \frac{1}{4}\,\phi(2x-2), \quad x \in \mathbb{R}.$$

Then ψ is supported on $[0,3]$. Moreover,

$$\psi(1) = -\frac{1}{4}\,\phi(2) + \phi(1) = 0 \quad \text{and} \quad \psi(2) = \phi(3) - \frac{1}{4}\,\phi(2) = 0.$$

Hence, $\psi(j) = 0$ for all $j \in \mathbb{Z}$. Moreover, let

$$\psi_b(x) := \phi_b(2x) - \frac{1}{4}\,\phi(2x), \quad x \in \mathbb{R}.$$

Then ψ_b is supported on $[0,2]$, and $\psi_b(j) = 0$ for all $j \in \mathbb{Z}$. For $n \ge 3$ and $x \in \mathbb{R}$, let

$$\psi_{n,j}(x) := \begin{cases} 2^{n/2}\psi_b(2^n x), & j = 1, \\ 2^{n/2}\psi(2^n x - j + 2), & j = 2, \dots, 2^n - 1, \\ 2^{n/2}\psi_b(2^n(1-x)), & j = 2^n. \end{cases}$$

We have $\psi_{n,j} \in V_{n+1}(0,1)$ and $P_n \psi_{n,j} = 0$ for $j = 1, \dots, 2^n$. Let W_n be the linear span of $\{\psi_{n,j} : j = 1, \dots, 2^n\}$. Then $\dim(W_n) = 2^n$. Moreover, $\{\psi_{n,j} : j = 1, \dots, 2^n\}$ is a Riesz basis of W_n with Riesz bounds independent of n. Thus, W_n is the kernel space of P_n, and hence V_{n+1} is the direct sum of V_n and W_n. An application of Theorem 5.3 with $\nu = 1/2$ gives the following result.

Theorem 6.1. *For $1/2 < \mu < 5/2$, the set*

$$\{2^{-3\mu}\phi_{3,j} : j = 1, \dots, 7\} \cup \{2^{-n\mu}\psi_{n,k} : n \ge 3,\ k = 1, \dots, 2^n\}$$

forms a Riesz basis of $H_0^\mu(0,1)$.

We are in a position to investigate wavelet bases in $H_0^\mu((0,1)^2)$. Recall that, for $j = (j_1, j_2) \in \mathbb{Z}^2$, $\phi_{n,j}(x) = \phi_{n,j_1}(x_1)\phi_{n,j_2}(x_2)$, $x = (x_1, x_2) \in \mathbb{R}^2$. For $n \ge 3$, $V_n((0,1)^2)$ is the linear span of $\{\phi_{n,j} : j \in J_n\}$, where $J_n = \{(j_1, j_2) \in \mathbb{Z}^2 : 1 \le j_1, j_2 \le 2^n - 1\}$. Clearly, $V_n((0,1)^2)$ is a subspace of $H_0^\mu((0,1)^2)$ for $0 \le \mu < 5/2$. Moreover, $\bigcup_{n=3}^\infty V_n((0,1)^2)$ is dense in $L_2((0,1)^2)$.

Suppose $f_n = \sum_{j \in J_n} a_{n,j}\phi_{n,j} \in V_n((0,1)^2)$. Then

$$B_1^2 \sum_{j \in J_n} |a_{n,j}|^2 \le \left\| \sum_{j \in J_n} a_{n,j}\phi_{n,j} \right\|_2^2 \le B_2^2 \sum_{j \in J_n} |a_{n,j}|^2,$$

where B_1 and B_2 are positive constants independent of n. Moreover,

$$\sum_{j\in J_n} a_{n,j}\phi_{n,j}(k/2^n) = f_n(k/2^n) \quad \text{for} \quad k \in J_n.$$

Note that $\phi_{n,j}(k/2^n) = 2^n\phi_{j_1}(k_1)\phi_{j_2}(k_2)$ for $j = (j_1, j_2) \in J_n$ and $k = (k_1, k_2) \in J_n$. The matrix $(\phi_{j_1}(k_1)\phi_{j_2}(k_2))_{1\le j_1,j_2\le 2^n-1, 1\le k_1,k_2\le 2^n-1}$ can be viewed as the Kronecker product $F_n \otimes F_n$, where F_n is the matrix $(\phi_{j_1}(k_1))_{1\le j_1,k_1\le 2^n-1}$. (See [15, Chap. 4] for the definition and properties of the Kronecker product.) We have

$$\|F_n \otimes F_n\| \le 1 \quad \text{and} \quad \|(F_n \otimes F_n)^{-1}\| = \|F_n^{-1} \otimes F_n^{-1}\| \le 9.$$

Consequently,

$$C_1^2 \sum_{k\in J_n} \left|f_n(k/2^n)\right|^2 \le \sum_{j\in J_n} \left|2^n a_{n,j}\right|^2 \le C_2^2 \sum_{k\in J_n} \left|f_n(k/2^n)\right|^2,$$

where C_1 and C_2 are positive constants independent of n. Let P_n be the linear projection from $V_{n+1}((0,1)^2)$ onto $V_n((0,1)^2)$ given as follows: For $f_{n+1} \in V_{n+1}((0,1)^2)$, $f_n := P_n f_{n+1}$ is the unique element in $V_n((0,1)^2)$ determined by the interpolation condition

$$f_n(k/2^n) = f_{n+1}(k/2^n) \quad \forall\, k \in J_n.$$

By using the same argument as above we obtain

$$\|P_m \cdots P_{n-1}\| \le C2^{n-m},$$

where C is a constant independent of m and n. Let W_n be the kernel space of P_n. Then

$$\dim(W_n) = \dim(V_{n+1}((0,1)^2)) - \dim(V_n((0,1)^2)) = (2^n-1)2^n + 2^n(2^n-1) + 2^{2n}.$$

For two functions v and w defined on $[0,1]$, we use $v \otimes w$ to denote the function on $[0,1]^2$ given by $v \otimes w(x_1, x_2) := v(x_1)w(x_2)$, $0 \le x_1, x_2 \le 1$. Let

$$\begin{aligned}
\Gamma_n' &:= \{\phi_{n,j_1} \otimes \psi_{n,k_2} : 1 \le j_1 \le 2^n - 1, 1 \le k_2 \le 2^n\},\\
\Gamma_n'' &:= \{\psi_{n,k_1} \otimes \phi_{n,j_2} : 1 \le j_2 \le 2^n - 1, 1 \le k_1 \le 2^n\},\\
\Gamma_n''' &:= \{\psi_{n,k_1} \otimes \psi_{n,k_2} : 1 \le k_1, k_2 \le 2^n\}.
\end{aligned}$$

It is easily seen that $\Gamma_n := \Gamma_n' \cup \Gamma_n'' \cup \Gamma_n'''$ is a Riesz basis of W_n with Riesz bounds independent of n.

Now an application of Theorem 5.3 with $\nu = 1$ gives the following result.

Theorem 6.2. *For $1 < \mu < 5/2$, the set*

$$\{2^{-3\mu}\phi_{3,j} : j \in J_3\} \cup \bigcup_{n=3}^{\infty} \{2^{-n\mu} w : w \in \Gamma_n\}$$

forms a Riesz basis of $H_0^\mu((0,1)^2)$.

7. The wavelet method for the biharmonic equation

In this section we apply the wavelet bases constructed in the previous section to numerical solutions of the biharmonic equation (1.4) for $\Omega = (0,1)^2$. In what follows, we write V_n for $V_n((0,1)^2)$ and always assume $n \geq 3$. Also, all the functions are real-valued.

In order to solve the variational problem (1.5) on the unit square $\Omega = (0,1)^2$, we use V_n to approximate $H_0^2(\Omega)$. Recall that $\Phi_n := \{\phi_{n,j} : j \in J_n\}$ is a basis of V_n. We look for $u_n \in V_n$ such that

$$\langle \Delta u_n, \Delta v\rangle = \langle f, v\rangle \quad \forall\, v \in V_n. \tag{7.1}$$

Suppose $u_n = \sum_{\phi\in\Phi_n} y_\phi \phi$. Let A_n be the matrix $(\langle \Delta\sigma, \Delta\phi\rangle)_{\sigma,\phi\in\Phi_n}$, and let ξ_n be the column vector $(\langle f, \phi\rangle)_{\phi\in\Phi_n}$. Then the column vector $y_n = (y_\phi)_{\phi\in\Phi_n}$ is the solution of the system of linear equations

$$A_n y_n = \xi_n. \tag{7.2}$$

However, the condition number $\kappa(A_n)$ of the matrix A_n is of the size $O(2^{4n})$. Hence, without preconditioning it would be very difficult to solve the system of equations (7.2).

Now we employ the wavelet bases constructed in the previous section to solve the variational problem (7.1). Let B_n be the matrix $(\langle \Delta\chi, \Delta\psi\rangle)_{\chi,\psi\in\Psi_n}$, where

$$\Psi_n := \{2^{-6}\phi_{3,j} : j \in J_3\} \cup \bigcup_{k=3}^{n-1} \{2^{-2k} w : w \in \Gamma_k\}. \tag{7.3}$$

We use $\lambda_{\max}(B_n)$ and $\lambda_{\min}(B_n)$ to denote the maximal and minimal eigenvalue of B_n, respectively. Then $\kappa(B_n) := \lambda_{\max}(B_n)/\lambda_{\min}(B_n)$ gives the condition number of B_n in the 2-norm. We claim that the condition number $\kappa(B_n)$ is uniformly bounded (independent of n). To justify our claim, let λ be an eigenvalue of the matrix B_n. Then there exists a nonzero column vector $z \in \mathbb{R}^{\Psi_n}$ such that $B_n z = \lambda z$. It follows that $z^T B_n z = \lambda z^T z$, where z^T denotes the transpose of the column vector z. Suppose $z = (a_\psi)_{\psi\in\Psi_n}$. It is easily seen that

$$z^T B_n z = \Big\langle \sum_{\psi\in\Psi_n} a_\psi \Delta\psi, \sum_{\psi\in\Psi_n} a_\psi \Delta\psi \Big\rangle = \Big\| \Delta\Big(\sum_{\psi\in\Psi_n} a_\psi \psi\Big)\Big\|^2_{L_2(\Omega)} = \Big|\sum_{\psi\in\Psi_n} a_\psi \psi\Big|^2_{H_0^2(\Omega)}.$$

By Theorem 6.2, there exist two positive constants C_1 and C_2 (independent of n) such that

$$C_1^2 \sum_{\psi\in\Psi_n} |a_\psi|^2 \leq \Big|\sum_{\psi\in\Psi_n} a_\psi \psi\Big|^2_{H_0^2(\Omega)} \leq C_2^2 \sum_{\psi\in\Psi_n} |a_\psi|^2. \tag{7.4}$$

On the other hand, $z^T B_n z = \lambda z^T z = \lambda \sum_{\psi\in\Psi_n} |a_\psi|^2$. Consequently,

$$C_1^2 \sum_{\psi\in\Psi_n} |a_\psi|^2 \leq \lambda \sum_{\psi\in\Psi_n} |a_\psi|^2 \leq C_2^2 \sum_{\psi\in\Psi_n} |a_\psi|^2.$$

Hence, $C_1^2 \leq \lambda \leq C_2^2$. This shows that $\kappa(B_n) \leq C_2^2/C_1^2$ for all $n \geq 3$.

The condition numbers $\kappa(B_n)$ are computed for $4 \leq n \leq 9$ and listed in Table 7.1. The numerical computation confirms our assertion that $\kappa(B_n)$ is uniformly bounded (independent of n).

TABLE 7.1. Condition number of the preconditioned matrix

Level n	Size of B_n	$\lambda_{\max}(B_n)$	$\lambda_{\min}(B_n)$	$\kappa(B_n)$
4	225×225	2.7160	0.07996	33.97
5	961×961	2.7883	0.07995	34.88
6	3969×3969	2.8082	0.07994	35.13
7	16129×16129	2.8259	0.07994	35.35
8	65025×65025	2.8434	0.07994	35.57
9	261121×261121	2.8489	0.07994	35.64

The biharmonic equation (1.4) is often decoupled as

$$-\Delta u = w \quad \text{and} \quad -\Delta w = f \quad \text{in } \Omega. \tag{7.5}$$

For the decoupled biharmonic equation (7.5) with the homogeneous boundary condition, a multigrid preconditioner was proposed in [28]. It is clear from Tables 1, 2, and 3 in [28] that the condition number of the preconditioned matrix grows like $O(h^{-1})$, where h is the mesh size. In fact, for piecewise linear approximation on the 48×48 grid, the condition number already exceeds 80. In comparison, the condition number of $\kappa(B_n)$ is uniformly bounded. For the 512×512 grid ($n = 9$), $\kappa(B_9) < 36$.

Suppose $u_n = \sum_{\psi \in \Psi_n} z_\psi \psi$. Let η_n be the column vector $\eta_n = (\langle f, \psi \rangle)_{\psi \in \Psi_n}$, and let z_n be the column vector $(z_\psi)_{\psi \in \Psi_n}$. Then u_n is the solution to the variational problem (7.1) if and only if z_n is the solution of the following system of linear equations:

$$B_n z_n = \eta_n. \tag{7.6}$$

Suppose $u_n^* = \sum_{\psi \in \Psi_n} z_\psi^* \psi$, where $z_n^* = (z_\psi^*)_{\psi \in \Psi_n}$ is the vector in $\mathbb{R}^{\Psi_n}$ such that $B_n z_n^* = \eta_n$, i.e., z_n^* is the *exact* solution to the equation (7.6). Let u be the *exact* solution to the biharmonic equation (1.4), and let $e_n^* := u_n^* - u$. Then $\|\Delta e_n^*\|_{L_2(\Omega)}$ represents the discretization error in the energy norm, and $\|e_n^*\|_{L_2(\Omega)}$ represents the discretization error in the L_2 norm.

Suppose $u_n = \sum_{\psi \in \Psi_n} z_\psi \psi$, where $z_n = (z_\psi)_{\psi \in \Psi_n}$ is an *approximate* solution to the equation (7.6). Let $e_n := u_n - u$. It was proved in Theorem 3.1 that

$$\|\Delta e_n^*\|_{L_2(\Omega)} = \big\|\Delta(u - u_n^*)\big\|_{L_2(\Omega)} \le \big\|\Delta(u - u_n)\big\|_{L_2(\Omega)} = \|\Delta e_n\|_{L_2(\Omega)}. \tag{7.7}$$

Thus, for any approximate solution u_n, the error $\|\Delta e_n\|_{L_2(\Omega)}$ is no less than the discretization error $\|\Delta e_n^*\|_{L_2(\Omega)}$ in the energy norm. If $\|\Delta e_n\|_{L_2(\Omega)} \le K \|\Delta e_n^*\|_{L_2(\Omega)}$, where K is a constant close to 1, then we say that the error of an approximate solution u_n achieves the level of discretization error in the energy norm. Similarly, if $\|e_n\|_{L_2(\Omega)} \le K \|e_n^*\|_{L_2(\Omega)}$, then we say that the error of an approximate solution u_n achieves the level of discretization error in the L_2 norm.

We observe that each $\psi \in \Psi_n$ can be uniquely expressed as $\psi = \sum_{\phi \in \Phi_n} s_{\psi\phi} \phi$. Let S_n denote the matrix $(s_{\psi\phi})_{\psi \in \Psi_n, \phi \in \Phi_n}$, which represents the wavelet transform.

Then $B_n = S_n A_n S_n^T$ and $\eta_n = S_n \xi_n$. Hence, the equation (7.6) is equivalent to (7.2) if we set $y_n = S_n^T z_n$. Actually, we will use the PCG (Preconditioned Conjugate Gradient) algorithm (see, e.g., [24, pp. 94–95]) to solve the system of linear equations

$$S_n A_n y_n = S_n \xi_n. \tag{7.8}$$

We observe that A_n is an $N^2 \times N^2$ matrix and y_n is an N^2 column vector, where $N = 2^n - 1$. Each iteration of the PCG algorithm requires a multiplication of the matrix A_n with a vector and a multiplication of the matrix S_n with a vector. A multiplication of A_n with a vector requires $O(N^2)$ work. A multiplication of the matrix S_n with a vector can be performed by using the corresponding wavelet transform. Hence, it also requires $O(N^2)$ work. We will use an approximate solution to the equation $S_n A_n y_n = S_n \xi_n$ as an initial guess for the equation $S_{n+1} A_{n+1} y_{n+1} = S_{n+1} \xi_{n+1}$ at the next level. Since $\kappa(B_n)$ is uniformly bounded, the number of iterations needed from level n to level $n+1$ is bounded (independent of n). Thus, the total work required for the solution of the equation (7.8) to the level of discretization error is $O(N^2)$. In other words, our algorithm is optimal.

8. Numerical examples: error estimates in the energy norm

The biharmonic equation arises in many applications. In fluid mechanics, the solution u of (1.4) represents the stream function, and $-\Delta u$ represents the vorticity of the fluid. In linear elasticity, u is used to describe the airy stress function. Then $(D_j D_k u)_{1 \le j,k \le 2}$ gives the Cauchy stress tensor. Thus, estimation of the error $\|\Delta e_n\|_{L_2(\Omega)}$ in the energy norm has its own significance in physics. In this section, we focus on error estimates in the energy norm. Our goal is to find efficient algorithms for computing approximate solutions that achieve the level of discretization error.

For a finite nonempty subset J, $\mathbb{R}^J$ can be viewed as a vector space. Recall that the ℓ_2 norm of a vector $z = (z_j)_{j \in J} \in \mathbb{R}^J$ is given by

$$\|z\|_2 = \left(\sum_{j \in J} |z_j|^2 \right)^{1/2}.$$

For $n \ge 3$, let Ψ_n be the set defined in (7.3). We use $\mathcal{P}_n$ to denote the mapping from $\mathbb{R}^{\Psi_n}$ to V_n that sends $(a_\psi)_{\psi \in \Psi_n}$ to $\sum_{\psi \in \Psi_n} a_\psi \psi$. For $z_n \in \mathbb{R}^{\Psi_n}$, it follows from (7.4) that

$$C_1 \|z_n\|_2 \le \|\Delta(\mathcal{P}_n z_n)\|_{L_2(\Omega)} \le C_2 \|z_n\|_2. \tag{8.1}$$

Recall that B_n is the matrix $(\langle \Delta\chi, \Delta\psi \rangle)_{\chi,\psi \in \Psi_n}$. We wish to solve the linear system of equations $B_n z_n = \eta_n$ with $\eta_n := (\langle f, \psi \rangle)_{\psi \in \Psi_n}$, where f is the function on the right-hand side of the biharmonic equation (1.4). For $z_n \in \mathbb{R}^{\Psi_n}$, $r_n := \eta_n - B_n z_n$ represents the corresponding residue. In particular, for $z_n = 0$, $r_n^0 := \eta_n$ is the initial residue. We have $B_n(z_n - z_n^*) = B_n z_n - \eta_n = -r_n$. Suppose that C_1 and C_2 are two positive constants such that (7.4) is valid. Then $\lambda_{\max}(B_n) \le C_2^2$ and $\lambda_{\min}(B_n) \ge C_1^2$. Hence,

$$C_1^2 \|z_n - z_n^*\|_2 \le \|r_n\|_2 \le C_2^2 \|z_n - z_n^*\|_2.$$

This in connection with (8.1) gives

$$\frac{C_1}{C_2^2}\,\|r_n\|_2 \le \|\Delta(u_n - u_n^*)\|_{L_2(\Omega)} \le \frac{C_2}{C_1^2}\,\|r_n\|_2,$$

where $u_n = \mathcal{P}_n z_n$ and $u_n^* = \mathcal{P}_n z_n^*$. Recall that $e_n = u_n - u$ and $e_n^* = u^* - u$, where u is the exact solution to the biharmonic equation (1.4). Consequently, $e_n - e_n^* = u_n - u_n^*$ and

$$\|\Delta(e_n - e_n^*)\|_{L_2(\Omega)} = \|\Delta(u_n - u_n^*)\|_{L_2(\Omega)} \le C\|r_n\|_2,$$

where $C = C_2/C_1^2$. This together with (7.7) yields

$$\|\Delta e_n^*\|_{L_2(\Omega)} \le \|\Delta e_n\|_{L_2(\Omega)} \le \|\Delta e_n^*\|_{L_2(\Omega)} + C\|r_n\|_2.$$

By Theorem 3.1, $\|\Delta e_n^*\|_{L_2(\Omega)} \le M_1 2^{-2n}$ for some positive constant M_1 independent of n. Therefore, if the residue r_n is made so small that $\|r_n\|_2 \le M_2 2^{-2n}$, we will have $\|\Delta e_n\|_{L_2(\Omega)} \le M 2^{-2n}$ with $M := M_1 + CM_2$.

For $k > 3$, let $\mathcal{E}_k$ be the linear mapping from $\mathbb{R}^{\Psi_{k-1}}$ to $\mathbb{R}^{\Psi_k}$ that sends $(a_\psi)_{\psi\in\Psi_{k-1}}$ to $(b_\psi)_{\psi\in\Psi_k}$, where

$$b_\psi := \begin{cases} a_\psi & \text{for } \psi \in \Psi_{k-1}, \\ 0 & \text{for } \psi \in \Psi_k \setminus \Psi_{k-1}. \end{cases}$$

Note that B_3 is a matrix of size 49×49. We first solve the equation $B_3 z_3 = \eta_3$ exactly and get the solution z_3. Then we use $\mathcal{E}_4 z_3$ as the initial vector and perform the PCG iterations for the equation $B_4 z_4 = \eta_4$ to get an approximate solution z_4. Let $u_3 := \mathcal{P}_3 z_3$ and $u_4 := \mathcal{P}_4 u_4$. Then $\tilde{e}_3 := u_3 - u_4$ represents the error in the energy norm at level 3. For $4 \le k \le n$, the above discussion motivates us to choose the threshold $\varepsilon_{n,k}$ as follows:

$$\varepsilon_{n,k} := \frac{k}{n}\frac{\|\Delta\tilde{e}_3\|_{L_2(\Omega)}}{2^{2n-5}}.$$

To solve the linear system of equations $B_n z_n = \eta_n$ for $n \ge 4$, we will use the following multilevel algorithm based on wavelets. First, let $z_{n,3} := z_3$. Second, for $4 \le k \le n$, use $z_{n,k}^0 := \mathcal{E}_k z_{n,k-1}$ as the initial vector to perform the PCG iterations for the equation $B_k z_{n,k} = r_k^0$ sufficiently many times such that the residue $r_{n,k} := r_k^0 - B_k z_{n,k}$ satisfies $\|r_{n,k}\|_2 \le \varepsilon_{n,k}$. Finally, set $z_n := z_{n,n}$ and $r_n := r_{n,n}$. Then r_n represents the residue when the algorithm stops, and z_n is the desired approximate solution.

In the above algorithm, suppose that m_k $(4 \le k \le n)$ iterations are performed for the equation $B_k z_{n,k} = r_k^0$. Note that m_k iterations at level k are equivalent to $m_k/4^{n-k}$ iterations at level n. Thus, the total number of equivalent iterations at level n will be

$$N_{it} = \sum_{k=4}^{n} \frac{m_k}{4^{n-k}}. \tag{8.2}$$

We are in a position to give numerical examples to show that the above algorithm is efficient. The following computation is conducted on a Lenovo desktop with 2 GB memory and an Intel Core 2 CPU 6400 at 2.13 GHz. We implemented our algorithm in C and used gcc to compile it.

Example 8.1. Consider the biharmonic equation (1.4) on Ω with f given by

$$f(x_1, x_2) = t\pi^4\big[4\cos(2\pi x_1)\cos(2\pi x_2) - \cos(2\pi x_1) - \cos(2\pi x_2)\big], \quad (x_1, x_2) \in \Omega,$$

where $t > 0$ is chosen so that $\|f\|_2 = 1$. The exact solution of the equation is

$$u(x_1, x_2) = t\big[1 - \cos(2\pi x_1)\big]\big[1 - \cos(2\pi x_2)\big]/16, \quad (x_1, x_2) \in \Omega = (0,1)^2. \tag{8.3}$$

TABLE 8.1. Numerical results of Example 8.1

Level n	Grid $2^n \times 2^n$	N_{it}	$\|r_n^0\|_2$	$\|r_n\|_2$	$\|\Delta e_n\|_2$	$\|\Delta e_n^*\|_2$	Time (s)
5	32×32	1.75	5.64e-3	1.01e-5	2.23e-5	2.00e-5	0.001
6	64×64	1.81	5.64e-3	1.72e-6	5.21e-6	4.99e-6	0.002
7	128×128	1.88	5.64e-3	3.02e-7	1.28e-6	1.25e-6	0.007
8	256×256	1.80	5.64e-3	7.25e-8	3.22e-7	3.11e-7	0.029
9	512×512	1.81	5.64e-3	1.32e-8	8.02e-8	7.79e-8	0.133
10	1024×1024	1.50	5.64e-3	9.31e-9	2.12e-8	1.95e-8	0.531

In Table 8.1, the third column gives the total number N_{it} of equivalent iterations at level n. For instance, for $n = 10$ and $4 \le k \le n$, m_k iterations are required for the equation $B_k z_{n,k} = r_k^0$, where $m_4 = 15$, $m_5 = 13$, $m_6 = 8$, $m_7 = 5$, $m_8 = 2$, $m_9 = 1$, and $m_{10} = 1$. By (8.2) we obtain

$$N_{it} = \sum_{k=4}^{n} \frac{m_k}{4^{n-k}} \approx 1.50.$$

The fourth column of the above table gives the initial residue, and the fifth column gives the residue when the algorithm terminates. Note that $\|r_n^0\|_2$ *depends* on n. But the first three digits of $\|r_n^0\|_2$ are the same for $n \ge 5$.

The sixth column gives the error $\|\Delta e_n\|_2 = \|\Delta e_n\|_{L_2(\Omega)}$ of the approximate solution in the energy norm. For the purpose of comparison, in the seventh column we also list the discretization error $\|\Delta e_n^*\|_2 = \|\Delta e_n^*\|_{L_2(\Omega)}$ in the energy norm. Recall that $e_n^* = u_n^* - u$ and $u_n^* = \mathcal{P}_n z_n^*$, where z_n^* is the exact solution to the equation (7.6), which is obtained by sufficiently many iterations. We find

$$\|\Delta e_n\|_2 \le 1.12 \|\Delta e_n^*\|_2 \quad \text{for } 4 \le n \le 10.$$

This demonstrates that the approximate solution obtained by our algorithm achieves the level of discretization error. Moreover, we see that $\|\Delta e_{n+1}^*\|_2 / \|\Delta e_n^*\|_2 < 0.2506 \approx 1/4$ for $4 \le n \le 9$. Thus, for the energy norm, the computation indicates that the rate of convergence is of order 2, confirming the assertion made in Theorem 3.1.

The last column of the above table gives the CPU time in seconds for solving the linear system of equations $B_n z_n = \eta_n$. At level $n = 10$, the matrix B_{10} has size 1046529×1046529. Our algorithm takes only 0.531 of a second to solve the equation $B_{10} z_{10} = \eta_{10}$.

Example 8.2. For $(x_1, x_2) \in \mathbb{R}^2$, let $z := (x_1 - 1/2)^2 + (x_2 - 1/2)^2 - 1/4$. Consider the biharmonic equation (1.4) on Ω with f given by

$$f(x_1, x_2) = \begin{cases} t\big[(4z+1)^2 \sin z + 16(4z+1)(1-\cos z) + 32(z - \sin z)\big] & \text{if } z < 0, \\ 0 & \text{if } z \geq 0, \end{cases}$$

where $t > 0$ is so chosen that $\|f\|_2 = 1$. The exact solution of the equation is given by $u(x_1, x_2) = t[\sin z - z + z^3/3]$ for $(x_1, x_2) \in \Omega$.

TABLE 8.2. Numerical results of Example 8.2

Level n	Grid $2^n \times 2^n$	N_{it}	$\|r_n^0\|_2$	$\|r_n\|_2$	$\|\Delta e_n\|_2$	$\|\Delta e_n^*\|_2$	Time (s)
5	32×32	2.00	5.19e-3	9.35e-6	2.89e-5	2.77e-5	0.001
6	64×64	1.81	5.19e-3	2.32e-6	7.47e-6	6.78e-6	0.002
7	128×128	1.92	5.19e-3	3.72e-7	1.76e-6	1.69e-6	0.007
8	256×256	1.60	5.19e-3	1.35e-7	4.50e-7	4.21e-7	0.026
9	512×512	1.51	5.19e-3	2.95e-8	1.16e-7	1.05e-7	0.122
10	1024×1024	1.43	5.19e-3	9.61e-9	2.84e-8	2.63e-8	0.523

In this example, we have $\|\Delta e_n\|_2 \leq 1.11\|\Delta e_n^*\|_2$ for $4 \leq n \leq 10$.

Example 8.3. Consider the biharmonic equation (1.4) with f given by

$$f(x_1, x_2) = te^{(3x_1 - x_2)^2}, \quad (x_1, x_2) \in \Omega = (0,1)^2,$$

where $t > 0$ is chosen so that $\|f\|_2 = 1$. In this case, the exact solution is unknown. Let $\tilde{e}_n := u_n - u_{n+1}$ and $\tilde{e}_n^* := u_n^* - u_{n+1}^*$.

TABLE 8.3. Numerical results of Example 8.3

Level n	Grid $2^n \times 2^n$	N_{it}	$\|r_n^0\|_2$	$\|r_n\|_2$	$\|\Delta \tilde{e}_n\|_2$	$\|\Delta \tilde{e}_n^*\|_2$	Time (s)
5	32×32	2.75	1.15e-3	7.02e-6	3.58e-5	3.58e-5	0.001
6	64×64	3.13	1.15e-3	1.83e-6	8.89e-6	8.87e-6	0.002
7	128×128	3.25	1.15e-3	4.20e-7	2.20e-6	2.20e-6	0.009
8	256×256	3.31	1.15e-3	8.23e-8	5.51e-7	5.51e-7	0.048
9	512×512	2.27	1.15e-3	4.98e-8	1.38e-7	1.38e-7	0.157

Example 8.4. Let $(c_{i_1,i_2})_{0\le i_1,i_2<2^{10}}$ be a random array of real numbers between 0 and 1. Consider the biharmonic equation (1.4) with f being a piecewise constant function given by

$$f(x_1,x_2) := tc_{i_1,i_2} \quad \text{for} \quad \frac{i_1}{2^{10}} < x_1 < \frac{i_1+1}{2^{10}} \quad \text{and} \quad \frac{i_2}{2^{10}} < x_2 < \frac{i_2+1}{2^{10}}$$

where $t > 0$ is chosen so that $\|f\|_2 = 1$.

TABLE 8.4. Numerical results of Example 8.4

Level n	Grid $2^n \times 2^n$	N_{it}	$\|r_n^0\|_2$	$\|r_n\|_2$	$\|\Delta\tilde{e}_n\|_2$	$\|\Delta\tilde{e}_n^*\|_2$	Time (s)
5	32×32	2.75	5.55e-3	8.78e-6	3.31e-5	3.31e-5	0.001
6	64×64	3.06	5.55e-3	2.01e-6	8.29e-6	8.29e-6	0.002
7	128×128	1.98	5.55e-3	9.55e-7	2.08e-6	2.08e-6	0.006
8	256×256	1.99	5.55e-3	2.04e-7	5.35e-7	5.35e-7	0.032
9	512×512	1.93	5.55e-3	6.20e-8	1.42e-7	1.42e-7	0.139

The biharmonic equation has been extensively studied in the literature. For numerical solutions of the biharmonic equation, the finite difference method was used in [7], [6], and [1], and the finite element method was employed in [32], [27], and [28]. Chang, Wong, Fu in [7], and Chang and Huang in [6] used difference schemes of second order, while Altas, Dym, Gupta and Manohar in [1] introduced a vector difference scheme of fourth-order in their solutions of the biharmonic equation. Sun in [32] and Oswald in [27] considered preconditioning techniques for the biharmonic equation discretized by quadratic and cubic splines, respectively. Silvester and Mihajlović in [28] proposed a multigrid preconditioning operator for the decoupled equation (7.5).

All the papers mentioned above, except [1], focused on residue reduction for the preconditioned matrices. For relative residue reduction in the ℓ_2 norm, the above numerical examples show that our algorithm requires considerably fewer iterations than those reported in [32] and [27]. Let us discuss relative residue reduction in the ℓ_∞ norm given by the quantity $\|r_n\|_\infty/\|r_n^0\|_\infty$. In Table 8.5, for $\varepsilon = 10^{-4}$, 10^{-5}, and 10^{-6}, we list the average number of iterations needed for $\tau_n := \|r_n\|_\infty/\|r_n^0\|_\infty < \varepsilon$ in the above four examples:

TABLE 8.5. Relative residue reduction

Level n	Grid $2^n \times 2^n$	$\tau_n < 10^{-4}$	$\tau_n < 10^{-5}$	$\tau_n < 10^{-6}$
8	256×256	1.8	3.0	5.0
9	512×512	1.4	2.0	2.9
10	1024×1024	1.3	1.5	1.9

The relative residue reduction in the ℓ_∞ norm was discussed in [28] for the 258×258 grid discretized by piecewise linear elements. It required 4 BICGSTAB iterations to get $\tau_8 < 6.2\times10^{-4}$ and 20 BICGSTAB iterations to get $\tau_8 < 1.2\times10^{-6}$. Further, 3 multigrid $V(1,1)$ cycles *per itearation* were performed for preconditioning (see Table 4 iii) in [28]). The algebraic multigrid method was used [7] and [6]. It was reported in Table 10 of [6] that more than 40 iterations were needed for the relative residue reduction in the ℓ_∞ norm to be less than 10^{-6}.

We remark that residue reductions are not fully comparable, because the corresponding matrices are different in different contexts. We think that it is more appropriate to compare the efficiency of numerical algorithms to achieve the level of discretization error. The wavelet method we propose has the advantage that the number of iterations needed to achieve the level of discretization error will not increase as the mesh size decreases. Thus, the wavelet method is suitable for large-scale computation. In comparison, in most of the aforementioned papers, the number of iterations would increase as the mesh size decreases.

9. Numerical examples: error estimates in the L_2 and L_∞ norms

In this section we investigate numerical solutions of the biharmonic equation and estimate errors of approximate solutions in the L_2 and L_∞ norms. We simply perform more iterations to achieve the level of discretization error. For the examples considered in this section, 7 equivalent PCG iterations based on our wavelets will be sufficient.

The following example was considered in [1]. We define

$$\|e_n\|_\infty := \max\big\{\big|e_n(i_1/2^n, i_2/2^n)\big| : 0 \le i_1, i_2 \le 2^n\big\}.$$

This definition agrees with the one given in [1].

Example 9.1. Consider the biharmonic equation (1.4) on $\Omega = (0,1)^2$ with f given by

$$f(x_1,x_2) = 16\pi^4\big[4\cos(2\pi x_1)\cos(2\pi x_2) - \cos(2\pi x_1) - \cos(2\pi x_2)\big], \quad (x_1,x_2) \in \Omega.$$

The exact solution of the equation is

$$u(x_1,x_2) = \big[1-\cos(2\pi x_1)\big]\big[1-\cos(2\pi x_2)\big], \quad (x_1,x_2) \in \Omega.$$

The numerical results are listed in Table 9.1.

Table 9.1. Error estimates in the maximum norm

Level n	Grid $2^n \times 2^n$	$\|e_n^D\|_\infty$	Time (s)	$\|e_n^{LD}\|_\infty$	Time (s)	$\|e_n^*\|_\infty$
5	32×32	6.28e-6	0.003	6.28e-6	0.004	6.28e-6
6	64×64	3.90e-7	0.007	3.90e-7	0.013	3.90e-7
7	128×128	2.44e-8	0.026	2.44e-8	0.047	2.44e-8
8	256×256	4.19e-9	0.110	1.52e-9	0.198	1.52e-9
9	512×512	4.27e-8	0.507	8.34e-11	0.885	8.31e-11

In Table 9.1, the third column gives the error $\|e_n^D\|_\infty$ by using the *double precision* arithmetic, and the fourth column gives the corresponding CPU time in seconds. Moreover, the fifth column gives the error $\|e_n^{LD}\|_\infty$ by using the *long double precision* arithmetic, and the sixth column gives the corresponding CPU time in seconds. Finally, the last column gives the discretization error $\|e_n^*\|_\infty$ in the ℓ_∞ norm.

We observe that, starting from level 8, the accuracy of the approximate solutions is affected by the roundoff errors if the double precision arithmetic is used. But the long double precision gives the desired accuracy at levels 8 and 9. We also observe that $\|e_n^*\|_\infty/\|e_{n+1}^*\|_\infty < 0.626 \approx 2^{-4}$ for $n = 5, 6, 7, 8$. Hence, the rate of convergence is of order 4.

A vector difference scheme of order 4 was used in [1]. The matrix obtained from discretization using their scheme has size $3(2^n-1)^2 \times 3(2^n-1)^2$ at level n. In comparison, the matrix B_n has size $(2^n-1)^2 \times (2^n-1)^2$. But our discretization error $\|e_n^*\|_\infty$ is smaller. For instance, for $n = 7$ we have $\|e_7^*\|_\infty \approx 2.44 \times 10^{-8}$, while the corresponding discretization error in [1] is 4.2×10^{-8}. It was reported in [1] that 3 FMG (Full Mutligrid) $W(3,2)$-cycles were used to achieve the level of discretization error. We estimate that a multiplication of their matrix with a vector costs twice as much as a multiplication of our matrix (B_n) with a vector (see the above comparison of the matrix size). Consequently, we estimate that a multigrid $V(3,2)$-cycle costs as much as 5 PCG iterations of our scheme. The computational cost of a FMG $W(3,2)$-cycle is about twice the cost of a simple $V(3,2)$-cycle (see [1]). Thus, the computational cost of 3 FMG $W(3,2)$-cycles is about the cost of 30 PCG iterations of our scheme.

Example 9.2. Consider the biharmonic equation (1.4) on Ω with f given by

$$f(x_1, x_2) = e^{(3x_1 - x_2)^2}, \quad (x_1, x_2) \in \Omega.$$

In this case, the exact solution is unknown. Recall that $\tilde{e}_n = u_n - u_{n+1}$ and $\tilde{e}_n^* = u_n^* - u_{n+1}^*$. In Table 9.2 we list numerical results of the approximate solutions that achieve the level of discretization error in the L_2 norm. The numerical computation clearly shows that the rate of convergence is of order 4, confirming the conclusion of Theorem 3.1.

TABLE 9.2. Error estimates in the L_2 norm

Level n	Grid $2^n \times 2^n$	$\|\tilde{e}_n^D\|_2$	Time (s)	$\|\tilde{e}_n^{LD}\|_2$	Time (s)	$\|\tilde{e}_n^*\|_2$
5	32×32	7.18e-7	0.003	7.18e-7	0.004	7.18e-7
6	64×64	4.18e-8	0.007	4.18e-8	0.013	4.18e-8
7	128×128	2.54e-9	0.026	2.54e-9	0.047	2.54e-9
8	256×256	1.67e-10	0.110	1.59e-10	0.198	1.58e-10
9	512×512	1.20e-9	0.507	1.00e-11	0.885	9.92e-12

10. Fourth-order elliptic equations

In this section we extend our study to general elliptic equations of fourth-order. If the Dirichlet form of a fourth-order elliptic operator is strictly coercive, then the wavelet bases constructed in §6 are still applicable to numerical solutions of the corresponding fourth-order elliptic equation with the homogeneous boundary conditions. For simplicity, let us consider the following elliptic equation:

$$\begin{cases} \Delta(a(x)\Delta u)(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = \frac{\partial u}{\partial n}(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{10.1}$$

where $a(x) = a(x_1, x_2)$ is a continuous function on Ω and there exist two positive constants K_1 and K_2 such that $K_1 \le a(x_1, x_2) \le K_2$ for all $(x_1, x_2) \in \Omega$. Consequently, there exist two positive constants C_1 and C_2 such that

$$C_1|u|^2_{H_0^2(\Omega)} \le \langle a\Delta u, \Delta u\rangle \le C_2|u|^2_{H_0^2(\Omega)} \quad \forall\, u \in H_0^2(\Omega).$$

The variational form corresponding to the preceding elliptic equation is

$$\langle a\Delta u, \Delta v\rangle = \langle f, v\rangle \quad \forall\, v \in H_0^2(\Omega).$$

Thus, our wavelet bases can be used to discretize the above equation. We give two examples of numerical computation as follows. The long double precision arithmetic will be used in the computation.

Example 10.1. Let $a(x_1, x_2) := (1+x_1)(1+x_2)$ for $(x_1, x_2) \in \Omega = (0, 1)^2$. Suppose that $f(x) = \Delta(a(x)\Delta u)(x)$ for $x \in \Omega$, where u is given by

$$u(x_1, x_2) = \big[1 - \cos(2\pi x_1)\big]\big[1 - \cos(2\pi x_2)\big]/4, \quad (x_1, x_2) \in \Omega. \tag{10.2}$$

Then u is the exact solution of (1.4). The numerical results are listed in Table 10.1.

Table 10.1. Numerical results of Example 10.1

Level n	Grid $2^n \times 2^n$	$\|e_n\|_2$	$\|e_n^*\|_2$	Time (s)
5	32×32	7.38e-7	7.38e-7	0.017
6	64×64	4.57e-8	4.57e-8	0.053
7	128×128	2.85e-9	2.85e-9	0.196
8	256×256	1.78e-10	1.78e-10	0.806
9	512×512	1.10e-11	1.10e-11	3.524

Example 10.2. Let $a(x_1, x_2) := 1+0.5\sin[10.8(x_1-x_2)]$ for $(x_1, x_2) \in \Omega = (0, 1)^2$. Let $f(x) = \Delta(a(x)\Delta u)(x)$ for $x \in \Omega$, where u is given by (10.2). Then u is the exact solution of (10.1). The numerical results are listed in Table 10.2.

TABLE 10.2. Numerical results of Example 10.2

Level n	Grid $2^n \times 2^n$	$\|e_n\|_2$	$\|e_n^*\|_2$	Time (s)
5	32×32	8.25e-7	8.25e-7	0.017
6	64×64	5.11e-8	5.11e-8	0.053
7	128×128	3.21e-9	3.18e-9	0.196
8	256×256	2.04e-10	1.99e-10	0.806
9	512×512	1.26e-11	1.21e-11	3.524

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