UPPER BOUNDS FOR RESIDUES OF DEDEKIND ZETA FUNCTIONS AND CLASS NUMBERS OF CUBIC AND QUARTIC NUMBER FIELDS

STÉPHANE R. LOUBOUTIN

Abstract. Let $K$ be an algebraic number field. Assume that $\zeta_K(s)/\zeta(s)$ is entire. We give an explicit upper bound for the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ of $K$. We deduce explicit upper bounds on class numbers of cubic and quartic number fields.

1. Introduction

Let $K$ be an algebraic number field of degree $m = r_1 + 2r_2 > 1$, where $r_1$ is the number of real places of $K$ and $r_2$ is the number of complex places of $K$. Let $\kappa_K$ be the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ of $K$. Let $d_K$ be the absolute value of the discriminant of $K$. Let $h_K$ be its class number. Then (see [Lan, Chapter XIII, Section 3, Theorem 2]):

$$h_K = \frac{w_K \sqrt{d_K}}{2^{r_1} (2\pi)^{r_2} \text{Reg}_K} \kappa_K,$$

where $w_K \geq 2$ is the number of complex roots of unity in $K$ and $\text{Reg}_K$ is the regulator of $K$. To get upper bounds on $h_K$ we need lower bounds on $\text{Reg}_K$ (e.g., see [Sil]) and upper bounds on $\kappa_K$ (e.g., see [Lou00]). If $K$ is a real quadratic number field, then

$$h_K \leq \frac{1}{2} \sqrt{d_K}$$

([Le] and [Ram, Corollary 2]); if $K$ is a real cyclic cubic number field, then

$$h_K \leq \frac{2}{3} \sqrt{d_K}$$

(see [MP], and use [Lou93] instead of [MP, Lemme 3.2] to obtain that this bound is valid for real cyclic cubic number fields of not necessarily prime discriminants). With $e = \exp(1)$, it is known that

$$\kappa_K \leq \left( \frac{e \log d_K}{2(m-1)} \right)^{m-1}$$

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If $K$ is abelian, we have a better bound:

$$\kappa_K \leq \left( \frac{\log d_K + m\lambda_K}{2(m-1)} \right)^{m-1},$$

where $\lambda_K = 0$ if $K$ is real and $\lambda_K = 5/2 - \log 6$ if $K$ is imaginary (use [Ram] Corollary 1 and notice that if $K$ is imaginary, then $m/2$ of the $m$ characters in the group of primitive Dirichlet characters associated with $K$ are odd). For some totally real number fields, an improvement on (6) is known (see [Lou01] Theorem 1), if $K$ ranges over a family of totally real number fields of a given degree $m > 1$ for which $\zeta_K(s)/\zeta(s)$ is entire, there exists $C_m$ (computable) such that $d_K \geq C_m$ implies

$$\kappa_K \leq \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!} \leq \frac{1}{\sqrt{2\pi(m-1)}} \left( \frac{e \log d_K}{2(m-1)} \right)^{m-1}.$$  

It is known that $\zeta_K(s)/\zeta(s)$ is entire if $K$ is normal (see [MM, Chapter 2, Theorem 3]), or if the Galois group of its normal closure is solvable (see [Uch, vdW] and [MM, Chapter 2, Corollary 4.2]), e.g., for any cubic or quartic number field. This paper generalizes (6) to not necessarily totally real number fields:

**Theorem 1.** Let $r_1$ and $r_2$ be given, with $r_1 + 2r_2 \geq 3$. There exists $d_{r_1,r_2}$ effectively computable such that for any number field $K$ of degree $m = r_1 + 2r_2$ with $r_1$ real places and $r_2$ complex places, we have

$$\kappa_K \leq \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!},$$

provided that (i) $d_K \geq d_{r_1,r_2}$ and (ii) that $\zeta_K(s)/\zeta(s)$ is entire.

For given $r_1$ and $r_2$, we will explain how to use any mathematical software, we use Maple, to compute such a $d_{r_1,r_2}$. It appears that for the small values of $r_1 + 2r_2 = m$, say for $3 \leq m \leq 6$, this bound (8) holds true with no restriction on the size of $d_K$ (in fact, we have an even better bound, see Theorem 3), the reason being that these computed $d_{r_1,r_2}$’s are less than or equal to the least discriminants of number fields of degree $m = r_1 + 2r_2 \leq 6$ with $r_1$ real places and $r_2$ complex places. However, even in the simplest situation where we assume that $K$ is totally real, we could not in [Lou05] obtain beforehand a $C > 0$ such that (8) holds true for $K$’s of root-discriminants $\rho_K = d_K^{1/m}$ greater than $C$.

Set

$$\gamma = \lim_{m \to \infty} \left( \sum_{k=1}^{m} \frac{1}{k} - \log m \right) = 0.57721 \cdots$$

(Euler’s constant) and

$$\lambda_{r_2,m} = 2 + r_2 \log 4 - (m-1)(\log(4\pi) - \gamma).$$

Since $\lambda_{r_2,m} < 0$ for $m \geq 3$, Theorem 1 follows from the bound

$$\kappa_K \leq \frac{(\log d_K + \lambda_{r_2,m})^{m-1}}{2^{m-1}(m-1)!} + O_{r_2,m}(\log^{m-3} d_K),$$

where $\lambda_K = 0$ if $K$ is real and $\lambda_K = 5/2 - \log 6$ if $K$ is imaginary (use [Ram] Corollary 1 and notice that if $K$ is imaginary, then $m/2$ of the $m$ characters in the group of primitive Dirichlet characters associated with $K$ are odd).
where the implied constants are effective and depend on \( r_2 \) and \( m \) only. To prove \( \text{[S]} \), we generalize the method introduced in \text{[Lou96]}. Set
\[
\gamma(1) = \lim_{m \to \infty} \left( \sum_{k=1}^{m} \log \frac{k}{k} - \frac{1}{2} \log^2 m \right) = -0.07281 \ldots
\]
and
\[
\mu_{r_2,m} = 3 + r_2 \frac{\pi^2}{12} - (m - 1) \left( \frac{\pi^2}{8} - \gamma^2 - 2\gamma(1) \right).
\]
The error term in \( \text{[S]} \) is less than or equal to zero if \( \mu_{r_2,m} > 0 \) and \( d_K \) is large enough. Now, \( \mu_{r_2,m} > 0 \) if and only if we are in one of the following cases:

<table>
<thead>
<tr>
<th>( m )</th>
<th>( r_2 )</th>
<th>( \lambda_{r_2,m} )</th>
<th>( \mu_{r_2,m} )</th>
<th>( d_K \geq )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>( 2 + \gamma - \log(4\pi) = 0.04619 \ldots )</td>
<td>1.95384 \ldots</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( 2 + \gamma - \log \pi = 1.43248 \ldots )</td>
<td>2.77631 \ldots</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>( 2 + 2\gamma - 2\log(4\pi) = -1.90761 \ldots )</td>
<td>0.90769 \ldots</td>
<td>146</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( 2 + 2\gamma - 2\log(2\pi) = -0.52132 \ldots )</td>
<td>1.73015 \ldots</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>( 2 + 3\gamma - \log(16\pi^3) = -2.47513 \ldots )</td>
<td>0.68400 \ldots</td>
<td>75100</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>( 2 + 3\gamma - \log(4\pi^3) = -1.08883 \ldots )</td>
<td>1.50647 \ldots</td>
<td>35</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>( 2 + 4\gamma - 4\log(2\pi) = -3.04264 \ldots )</td>
<td>0.46031 \ldots</td>
<td>21 \cdot 10^{10}</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>( 2 + 5\gamma - \log(16\pi^5) = -3.61015 \ldots )</td>
<td>0.23662 \ldots</td>
<td>21 \cdot 10^{31}</td>
</tr>
</tbody>
</table>

It will follow that we have a pleasingly explicit bound:

**Theorem 2.** Assume that we are in one of the eight cases of Table 1. Then,
\[
\kappa_K \leq \frac{(\log d_K + \lambda_{r_2,m})^{m-1}}{2^{m-1}(m-1)!},
\]
provided that \( d_K \) is large enough, as given in the last column of Table 1.

The results in \text{[Lou93]} and \text{[Lou96]} are the case \( m = 2 \) of Theorem 2 above. (However, in the quadratic case we have an even better bound (see \text{[Ram]}).) Finally, by taking constants slightly less than these \( \lambda_{r_2,m} \), we have a a fully explicit following result where we do not have any restriction on \( d_K \) (compare with Theorem 1):

**Theorem 3.** Let \( K \) be a number field of degree \( m \in \{2, 3, 4, 5, 6\} \) for which \( \zeta_K(s)/\zeta(s) \) is entire. Then,
\[
\kappa_K \leq \frac{(\log d_K + \lambda)^{m-1}}{2^{m-1}(m-1)!},
\]
where \( \lambda \) is as in Table 2.
Table 2

<table>
<thead>
<tr>
<th>$m$</th>
<th>$r_2 = 0$</th>
<th>$r_2 = 1$</th>
<th>$r_2 = 2$</th>
<th>$r_2 = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.04620</td>
<td>1.43249</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-1.74865</td>
<td>-0.52132</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-2.94863</td>
<td>-2.07896</td>
<td>-1.08883</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-4.21779</td>
<td>-3.29415</td>
<td>-2.41877</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-5.49315</td>
<td>-4.55901</td>
<td>-3.64104</td>
<td>-2.76490</td>
</tr>
</tbody>
</table>

Corollary 4. If $K$ is a totally real cubic number field, then

\[ h_K \leq \frac{1}{2} \sqrt{d_K}. \]  

If $K$ is a totally real quartic number field which contains no quadratic subfield, then

\[ h_K \leq \frac{5\sqrt{10}}{24} \sqrt{d_K}. \]

We refer to [Dai] for examples of number fields with very large class numbers.

2. Proof of the bound \[(8)\]

We adapt [Lou00, Proof of Theorem 7]. Let $K$ be a number field of degree $m = r_1 + 2r_2 > 1$. Assume that $\zeta_K(s)/\zeta(s)$ is entire. Set $A_{K/Q} = \sqrt{d_K/4\pi^m},$

\[ \Gamma_{K/Q}(s) = \Gamma^{r_1-1}(s/2)\Gamma^{r_2}(s) = \frac{2^{r_2(s-1)}}{\pi^{r_2/2}}\Gamma^{r_1+r_2-1}(s/2)\Gamma^{r_2}((s+1)/2) \]

(notice that $r_1 + r_2 - 1 \geq 0$ and $r_2 \geq 0$) and

\[ F_{K/Q}(s) = A_{K/Q}^s \Gamma_{K/Q}(s)(\zeta_K(s)/\zeta(s)). \]

Then, $F_{K/Q}(s)$ is entire and $F_{K/Q}(1-s) = F_{K/Q}(1-s)$. Let

\[ S_{K/Q}(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_{K/Q}(s)x^{-s}ds \quad (c > 1 \text{ and } x > 0) \]

denote the inverse Mellin transform of $F_{K/Q}(s)$. Then,

\[ S_{K/Q}(x) = \frac{1}{x} S_{K/Q}(\frac{1}{x}) \]

(notice that $F_{K/Q}(s)$ is entire, shift the vertical line of integration $\Re(s) = c > 1$ in \[(11)\] leftwards to the vertical line of integration $\Re(s) = 1 - c < 0$, then use the functional equation $F_{K/Q}(1-s) = F_{K/Q}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$). For $\Re(s) > 1$,

\[ F_{K/Q}(s) = \int_0^\infty S_{K/Q}(x)x^s \frac{dx}{x} \]

is the Mellin transform of $S_{K/Q}(x)$. Using \[(12)\], we obtain

\[ F_{K/Q}(s) = \int_1^\infty S_{K/Q}(x)(x^s + x^{1-s}) \frac{dx}{x} \]
on the whole complex plane. Now, write \( \zeta(s)/\zeta(s) = \sum_{n \geq 1} a_{K/Q}(n)n^{-s} \) and 
\( \zeta^{m-1}(s) = \sum_{n \geq 1} a_{m-1}(n)n^{-s} \) \((\Re(s) > 1)\). Then, \(|a_{K/Q}(n)| \leq a_{m-1}(n)\) (see \[Lou01\] (55))) and

\[
S_{K/Q}(x) = \sum_{n \geq 1} a_{K/Q}(n)H_{K/Q}(nx/A_{K/Q}),
\]

where

\[
H_{K/Q}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma_{K/Q}(s) x^{-s} ds.
\]

Since \( H_{K/Q}(x) > 0 \) for \( x > 0 \) (see \[Lou01\] Theorem 20)) \[\exists\], we have 
\[
|S_{K/Q}(x)| \leq \sum_{n \geq 1} a_{m-1}(n)H_{K/Q}(nx/A_{K/Q}).
\]

Plugging this into \[13\], we obtain

\[
\frac{\sqrt{d_K}}{(2\pi)^{1/2}} \kappa_K = F_{K/Q}(1) = \int_1^\infty S_{K/Q}(x)(1+1/x)dx
\]

\[
\leq \sum_{n \geq 1} a_{m-1}(n) \int_1^\infty H_{K/Q}(nx/A_{K/Q})(1+1/x)dx
\]

\[
= \sum_{n \geq 1} a_{m-1}(n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \int_1^\infty (nx/A_{K/Q})^{-s}(1+1/x)dx \right) \Gamma_{K/Q}(s) ds
\]

\[
= \sum_{n \geq 1} a_{m-1}(n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{1}{s-1} + \frac{1}{s} \right) \Gamma_{K/Q}(s)(n/A_{K/Q})^{-s} ds
\]

\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{1}{s-1} + \frac{1}{s} \right) \Gamma_{K/Q}(s) \zeta^{m-1}(s) A_{K/Q}^s ds.
\]

Therefore, we have

\[
\kappa_K \leq I_K(s) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f_K(s) ds \quad (c > 1),
\]

where

\[
f_K(s) = \tilde{\Gamma} r_2(s) \Lambda^{m-1}(s) \left( \frac{1}{s-1} + \frac{1}{s} \right) a_{K}^{(s-1)/2},
\]

\(\Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)\) and \(\tilde{\Gamma}(s) = \Gamma((s+1)/2) / (\Gamma(s/2) / \Gamma(1/2))\). Recall that \(\Lambda(s)\) has only two poles, both simple, at \(s = 1\) and \(s = 0\), and satisfies the functional equation \(\Lambda(s) = \Lambda(1-s)\). Moreover, \(1/\Gamma(s/2)\) is entire whereas \(\Gamma((s+1)/2)\) has a simple pole at each odd negative integer. It follows that \(f_K(s)\) has a pole of order \(m > 1\) at \(s = 1\), a pole of order \(m - r_2 = r_1 + r_2 \geq 1\) at \(s = 0\), and a pole of order \(r_2 \geq 0\) at each negative odd integer. Now, as in \[Lou01\] Page 1207, in the range \(\sigma_1 \leq \sigma \leq \sigma_2\) and \(|t| \geq 1\), we have \(\tilde{\Gamma}(\sigma + it) = O(\sqrt{|t|})\) and there exists \(M \geq 0\) such that \(\Lambda(\sigma + it) = O(|t|^M e^{-\pi|t|/4})\). Hence, we are allowed to shift in \[14\] the

Notice the misprints in \[Lou00\] page 273, line 1 and \[Lou01\] Theorem 20) where one should read 
\[
(M_1 * M_2)(x) = \int_0^\infty M_1(x/t)M_2(t) \frac{dt}{t}.
\]
vertical line of integration $\Re(s) = c > 1$ leftwards to the vertical line of integration $\Re(s) = 1/2$. We pick up one residue and obtain:

$$\kappa_K \leq \text{Res}_{s=1}(f_K(s)) + I_K(1/2) = \text{Res}_{s=1}(f_K(s)) + O_{r_2,m}(d_K^{-1/4}).$$

The bound (15) now follows from Lemma 5 below.

3. Computation of Some Residues

To prove Theorems 2 and 3 we need a better approximation to $I_K(s)$. By shifting in (14) the vertical line of integration $\Re(s) = c > 1$ leftwards to the vertical line of integration $\Re(s) = -2$, we pick up three residues and we obtain:

$$\kappa_K \leq \text{Res}_{s=1}(f_K(s)) + \text{Res}_{s=0}(f_K(s)) + \text{Res}_{s=-1}(f_K(s)) + I_K(-2),$$

where $\text{Res}_{s=1}(f_K(s))$ is a polynomial of degree $m - 1$ in $\log d_K$ with real coefficients, $\sqrt{d_K}\text{Res}_{s=0}(f_K(s))$ is a polynomial of degree $r_1 + r_2 - 1$ in $\log d_K$ with real coefficients, and $d_K\text{Res}_{s=-1}(f_K(s))$ is a polynomial of degree $r_2 - 1$ in $\log d_K$ with real coefficients. This section is devoted to computing these residues.

**Lemma 5.** Set

$$f_{k,l}(s) = \hat{\Gamma}(s)\Lambda(s) \left(\frac{1}{s-1} + \frac{1}{\pi}\right) e^{(s-1)X}.$$

Then, $\text{Res}_{s=1}(f_{k,l}(s))$ is a polynomial of degree $l$ in $X$ with real coefficients and

$$\text{Res}_{s=1}(f_{k,l}(s)) = \frac{(X + A_{k,l})^l}{l!} \quad (l = 1),$$

$$\text{Res}_{s=1}(f_{k,l}(s)) = \frac{(X + A_{k,l})^l}{l!} - C_{k,l} \frac{X^{l-2}}{(l-2)!} \quad (l = 2)$$

and

$$\text{Res}_{s=1}(f_{k,l}(s)) = \frac{(X + A_{k,l})^l}{l!} - C_{k,l} \frac{X^{l-2}}{(l-2)!} + O_{k,l}(X^{l-3}) \quad (l \geq 3),$$

where

$$A_{k,l} = (2 + k \log 4 - l(\log(4\pi) - \gamma))/2$$

and

$$C_{k,l} = (3 + k\pi^2/12 - l(\pi^2/8 - \gamma^2 - 2\gamma(1)))/2.$$  

**Proof.** We have $\hat{\Gamma}(s) = 1 + a(s-1) + b(s-1)^2 + O((s-1)^3)$, with $a = \log 2$ and $b = (\log^2 2 - \pi^2/12)/2$, and $(s-1)\Lambda(s) = 1 + c(s-1) + d(s-1)^2 + O((s-1)^3)$, with

$$c = -\frac{\log(4\pi) - \gamma}{2} \quad \text{and} \quad d = \frac{2(\log(4\pi) - \gamma)^2 + \pi^2 - 8\gamma^2 - 16\gamma(1)}{16}.$$

For $k \geq 0$ and $l \geq 0$, it holds that

$$(1 + az + bz^2 + O(z^3))^l (1 + cz + dz^2 + O(z^3))^1 \left(1 + z - z^2 + O(z^2)\right) = 1 + A_{k,l}z + B_{k,l}z^2 + O(z^3),$$

where $A_{k,l} = ka + lc + 1$ and $B_{k,l} = klc + k(b + \frac{e-1}{2}a^2) + l(d + \frac{e-1}{2}c^2) + ka + lc - 1$. 

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Hence, the desired results hold true with
\[ C_{k,t} = A_{k,t}^2/2 - B_{k,t} = (k(a^2 - 2b) + l(c^2 - 2d) + 3)/2. \]
We have \( a^2 - 2b = \pi^2/12 \) and \( c^2 - 2d = \gamma^2 + 2\gamma(1) - \pi^2/8. \)
\( \square \)

**Lemma 6.** Set \( r = l - k \), let \( f_{k,t}(s) \) and \( A_{k,t} \) be as in Lemma 5 and set
\[ C'_{k,t} = (3 - k\pi^2/12 - l(\pi^2/8 - \gamma^2 - 2\gamma(1)))/2. \]
If \( r = 0 \) or \( r = 1 \), then
\[ \text{Res}_{s=0}(f_{k,t}(s)) = (-1)^l(\pi/2)^k \frac{(X - A_{k,t})^r}{r!} e^{-X}. \]
If \( r = 2 \), then
\[ \text{Res}_{s=0}(f_{k,t}(s)) = (-1)^l(\pi/2)^k \left( \frac{(X - A_{k,t})^r}{r!} - C'_{k,t} \frac{X^{r-2}}{(r-2)!} \right) e^{-X}. \]
If \( r \geq 3 \), then
\[ \text{Res}_{s=0}(f_{k,t}(s)) = (-1)^l(\pi/2)^k \left( \frac{(X - A_{k,t})^r}{r!} - C'_{k,t} \frac{X^{r-2}}{(r-2)!} + O_{k,t}(X^{r-3}) \right) e^{-X}. \]

**Proof.** Here, \( \Gamma(s) = \frac{\pi s}{2} (1 - as + bs^2 + O(s^3)) \), with \( a = \log 2 \) and \( b = (\log^2 2 + \pi^2/12)/2 \), and \( s\Lambda(s) = (1 - cs + ds^2 + O(s^3)) \), with \( c \) and \( d \) as in \([17] \). \( \square \)

**Lemma 7.** Let \( f_{k,t}(s) \) be as in Lemma 5. We have
\[ \text{Res}_{s=-1}(f_{1,t}(s)) = \frac{3}{2} \frac{\pi}{6} e^{-2X}. \]

**Lemma 8.** It holds that
\[ |I_K(-2)| \leq \frac{5}{4\pi^2} \Gamma(r_2/2 + 1) \left( \frac{14}{m - 1} \right)^{r_2/2 + 1} d_K^{-3/2}. \]

**Proof.** Using
\[ |\Gamma(-2 + it)| = \left( 1 + \frac{t^2}{2} \pi \frac{t}{2} \tanh\left( \frac{\pi t}{2} \right) \right)^{1/2} \leq 2\pi |t| \]
and
\[ |\Lambda(-2 + it)| = |\Lambda(3 - it)| \leq \frac{\zeta(3)}{\pi^{3/2}} |\Gamma((3 - it)/2)| = \frac{\zeta(3)}{2\pi} \sqrt{\frac{1 + t^2}{\cosh(\pi t/2)}} \leq \frac{1}{3} e^{-\pi|t|/7}, \]
we obtain:
\[ d_K^{3/2} |I_K(-2)| \leq \frac{5}{6\pi} \int_0^\infty |\Gamma(-2 + it)|^{r_2} |\Lambda(-2 + it)|^{m-1} dt \]
\[ \leq \frac{5}{2\pi^{3/2}} \int_0^\infty (2\pi t)^{r_2/2} e^{-\pi(m-1)t/7} dt, \]
and the desired bound. \( \square \)
Table 3. Minimal discriminants

<table>
<thead>
<tr>
<th>m</th>
<th>( r_2 = 0 )</th>
<th>( r_2 = 1 )</th>
<th>( r_2 = 2 )</th>
<th>( r_2 = 3 )</th>
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<td>28037</td>
<td>9747</td>
</tr>
</tbody>
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4. Proof of Theorems \(2\) and \(3\) and contents of Tables 1 and 2

We use (16), the previous lemmas and Table 3 above (see [Odl]).

1. If \( K \) is a real quadratic field, then

\[
\kappa_K \leq \frac{\log d_K + 2 + \gamma - \log(4\pi)}{2} - \frac{\log d_K - (2 + \gamma - \log(4\pi))}{2\sqrt{d_K}} + \frac{35}{18\pi^2 d_K^{3/2}}
\]

is less than or equal to \((\log d_K + 2 + \gamma - \log(4\pi))/2\) for \( d_K \geq 3 \).

2. If \( K \) is an imaginary quadratic field, then

\[
\kappa_K \leq \frac{\log d_K + 2 + \gamma - \log \pi}{2} - \frac{\pi}{2\sqrt{d_K}} + \frac{\pi}{4d_K} + \frac{35\sqrt{14\pi}}{36\pi^2 d_K^{3/2}}
\]

is less than or equal to \((\log d_K + 2 + \gamma - \log \pi)/2\) for \( d_K \geq 3 \).

3. If \( K \) is a totally real cubic number field, then

\[
\kappa_K \leq \frac{(\log d_K + 2 + 2\gamma - 2\log(4\pi))^2}{8} - \frac{3/2 + \gamma^2 + 2\gamma(1) - \pi^2/8}{8\sqrt{d_K}} + \frac{(\log d_K - (2 + 2\gamma - 2\log(4\pi))^2}{8\sqrt{d_K}} - \frac{3/2 + \gamma^2 + 2\gamma(1) - \pi^2/8}{\sqrt{d_K}} + \frac{35}{108\pi^2 d_K^{3/2}}
\]

is less than or equal to \((\log d_K + 2 + 2\gamma - 2\log(4\pi))^2)/8\) for \( d_K \geq 146 \), and less than or equal to \((\log d_K - 1.74865)^2)/8\) for \( d_K \geq 49 \).

4. If \( K \) is a not totally real cubic number field, then

\[
\kappa_K \leq \frac{(\log d_K + 2 + 2\gamma - 2\log(2\pi))^2}{8} - \frac{3/2 + \gamma^2 + 2\gamma(1) - \pi^2/12}{8\sqrt{d_K}} + \frac{\pi}{4\sqrt{d_K}} (\log d_K - 2 - 2\gamma + 2\log(2\pi)) + \frac{\pi^2}{24d_K} + \frac{35\sqrt{7\pi}}{216\pi^2 d_K^{3/2}}
\]

is less than or equal to \((\log d_K + 2 + 2\gamma - 2\log(2\pi))^2)/8\) for \( d_K \geq 4 \).
5. The other cases are easily dealt with by using any software for symbolic computation, e.g., Maple, to compute the residues which appear in \[ (10) \].

5. **Proof of Corollary**

1. Let \( K \) be a totally real cubic field. Then, \( \text{Reg}_K \geq \frac{1}{16} \log^2 (d_K/4) \) (see [Cus] Theorem 1) or [Nak] Section 2.3). Hence,

\[
h_K = \frac{\sqrt{d_K}}{4\text{Reg}_K} \kappa_K \leq \frac{(\log d_K - 1.74865)^2}{2\log^2 (d_K/4)} \sqrt{d_K} \leq \frac{1}{2} \sqrt{d_K}.
\]

2. Let \( K \) be a totally real quartic number field which contains no real quadratic subfield. Then, \( \text{Reg}_K \geq \frac{1}{80} \log^2 d_K \) (see [Cus, Theorem 2]). By (7) (see also [Lou01] Theorem 2, point 3), we have \( \kappa_K \leq \frac{1}{48} \log d_K \). Hence, by (1), we obtain

\[
h_K = \frac{\sqrt{d_K}}{8\text{Reg}_K} \kappa_K \leq \frac{5\sqrt{10}}{24} \sqrt{d_K}.
\]

References


INSTITUT DE MATHEMATIQUES DE LUMINY, UMR 6206, 163, AVENUE DE LUMINY, CASE 907,
13288 MARSEILLE CEDEX 9, FRANCE

E-mail address: loubouti@iml.univ-mrs.fr