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IMPROVEMENTS TO TURING'S METHOD

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Abstract. This article improves the estimate of the size of the definite integral of $S(t)$, the argument of the Riemann zeta-function. The primary application of this improvement is Turing's Method for the Riemann zeta-function. Analogous improvements are given for the arguments of Dirichlet L-functions and of Dedekind zeta-functions.

1. Introduction

In determining the number of non-trivial zeroes of the Riemann zeta-function $\zeta(s)$ in a given range, one proceeds in two stages. First, one can compute a number of zeroes along the critical line using Gram's Law[1] or Rosser's Rule[2] (see, e.g. [5, Chs VI-VII]), which gives one a lower bound on the total number of zeroes in the critical strip in that range. To conclude that one has found the *precise* number of zeroes in this range, one needs an additional argument.

The earliest method employed was due to Backlund [1] and relies on showing that $\Re\zeta(s) \neq 0$ along the lines connecting $2, 2+iT, \frac{1}{2}+iT$. This is very labour intensive; nevertheless, Backlund was able to perform these procedures for $T = 200$ and later Hutchinson extended this to $T = 300.468$ (see [5, *loc. cit.*] for more details). In both cases the zeroes of $\zeta(s)$ located via Gram's Law were verified to be the only zeroes in the given ranges. Titchmarsh [16] continued to use this method to show that the Riemann hypothesis is valid for $|t| \leq 1468$.

Apart from its computational intricacies, this method of Backlund is bound to fail for sufficiently large T. To see this, it is convenient to introduce the function $S(T)$ defined as

$$S(T) = \pi^{-1} \arg \zeta(\tfrac{1}{2} + iT), \tag{1.1}$$

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[1]Briefly: The Gram points $\{g_n\}_{n\geq -1}$ are easily computed and have an average spacing equal to that of the non-trivial zeroes of $\zeta(\sigma+it)$, viz. $g_{n+1} - g_n \asymp (\log g_n)^{-1}$. Gram's Law states that for $t \in (g_n, g_{n+1}]$ there is exactly one zero of $\zeta(\frac{1}{2}+it)$. Gram's Law was shown to fail infinitely often by Titchmarsh [16], and shown to fail in a positive proportion of cases by the author [19].

[2]For some n, one defines a Gram block of length p as the interval $(g_n, g_{n+p}]$, wherein there is an even number of zeroes in each of the intervals $(g_n, g_{n+1}]$ and $(g_{n+p-1}, g_{n+p}]$, and an odd number of zeroes in each of the intervals $(g_{n+1}, g_{n+2}], \ldots, (g_{n+p-2}, g_{n+p-1}]$. Rosser's Rule then states that a Gram block of length p contains exactly p zeroes of $\zeta(\frac{1}{2}+it)$. Rosser's Rule holds more frequently than Gram's Law, but its infinite failure was first shown by Lehman [11]; see also the work of the author [*op. cit.*]

where if T is not an ordinate of a zero of $\zeta(s)$, the argument is determined by continuous variation along the lines connecting $2, 2+iT, \frac{1}{2}+iT$. If T coincides with a zero of $\zeta(s)$, then

$$S(T) = \lim_{\delta\to 0}\{S(t+\delta)+S(t-\delta)\}.$$

The interest in the properties of $S(T)$ is immediate once one considers its relation to the function $N(T)$, the number of non-trivial zeroes of $\zeta(\sigma+it)$ for $|t|\leq T$. In the equation

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O(T^{-1}) + S(T), \tag{1.2}$$

the error term is continuous in T, whence it follows that $S(T)$ increases by $+1$ whenever T passes over a zero of the zeta-function. Concerning the behaviour of $S(T)$ are the following estimates:

$$\int_0^T S(t)\,dt = O(\log T), \tag{1.3}$$

due to Littlewood (see, e.g. [17, pp. 221-222]), and

$$S(T) = \Omega_{\pm}\left(\frac{(\log T)^{\frac{1}{3}}}{(\log\log T)^{\frac{7}{3}}}\right), \tag{1.4}$$

due to Selberg [15].

Returning to Backlund's approach: if $\Re\zeta(s)\neq 0$ along the lines connecting $2, 2+iT, \frac{1}{2}+iT$, then, when varied along these same lines, $|\arg\zeta(s)|<\frac{\pi}{2}$, if one takes the principal argument. It therefore follows that $|S(T)|<\frac{1}{2}$, whence $S(T)$ is bounded, which contradicts (1.4).

1.1. **Turing's method.** A more efficient procedure in producing an upper bound on the number of zeroes in a given range was proposed by Turing [20] in 1953. This relies on a quantitative version of Littlewood's result (1.3), given below.

Theorem 1.1. *Given $t_0>0$, there are positive constants a and b such that, for $t_2>t_1>t_0$, the following estimate holds:*

$$\left|\int_{t_1}^{t_2} S(t)\,dt\right| \leq a + b\log t_2. \tag{1.5}$$

Since $S(t)$ increases by $+1$ whenever t passes over a zero (on the line or not), the existence of too many zeroes in the range $t\in(t_1,t_2)$ would cause the integral in (1.5) to be too large.

Turing's paper [20] contains several errors, which are fortunately corrected by Lehman [11]. Furthermore, Lehman also improves the constants a and b, thereby making Turing's Method more easily applicable. Here additional improvements on the constants in Turing's Method are given in §2. Rumely [14] has adapted Turing's Method to Dirichlet L-functions and this is herewith improved in §3. Finally, in §4 the analogous improvements to the argument of Dedekind zeta-functions is given, following the work of Tollis [18].

It is interesting to note the motivation of Turing as he writes in [20, p. 99]:

> The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards finding such zeros than proving that none existed.

Indeed, Turing's Method has become the standard technique used in modern verification of the Riemann hypothesis.

2. Turing's method for the Riemann zeta-function

2.1. **New results.** In general let the triple of numbers (a, b, t_0) satisfy Theorem 1.1. Turing showed that $(2.07, 0.128, 168\pi)$ satisfied (1.5) and Lehman showed that $(1.7, 0.114, 168\pi)$ does so as well. Brent [3, Thm. 2] used the result of Lehman [*op. cit.*, Thm. 4] to prove the following.

Theorem 2.1 (Lehman–Brent). *If N consecutive Gram blocks with union $[g_n, g_p)$ satisfy Rosser's Rule, where*

$$N \geq \frac{b}{6\pi}\log^2 g_p + \frac{(a - b\log 2\pi)}{6\pi}\log g_p, \tag{2.1}$$

then

$$N(g_n) \leq n + 1; \quad N(g_p) \geq p + 1.$$

Since, by assumption these N Gram blocks together contain exactly $p-n$ zeroes, this shows that up to height g_p there are at most $p+1$ zeroes; and this is precisely the upper bound one has sought. Using the constants of Lehman viz. $(a = 1.7, b = 0.114)$ it is seen that one must find at least

$$N \geq 0.0061\log^2 g_p + 0.08\log g_p \tag{2.2}$$

consecutive Gram blocks to apply Theorem 2.1. This constraint on N has been used in the modern computational search for zeroes, and appears in the early works, e.g. [3] right through to the more recent [6].

Turing makes the remark several times in his paper [20] that the constant b could be improved at the expense of the constant a. In (2.1) the first term dominates when g_p is large, and therefore for computation at a large height it is desirable to choose b to be small. Indeed, what is sought is the minimisation of

$$F(a, b, g_p) = b\log\frac{g_p}{2\pi} + a. \tag{2.3}$$

Current verification of the Riemann hypothesis has surpassed the height $T = 10^{12}$, see, e.g. [6] wherein (2.2) requires the location of at least 8 Gram blocks. In §2.3 the function $F(a, b, g_p)$ is minimised at $g_p = 2\pi\cdot 10^{12}$ which leads to

Theorem 2.2. *If $t_2 > t_1 > 168\pi$, then*

$$\left|\int_{t_1}^{t_2} S(t)\,dt\right| \leq 2.067 + 0.059\log t_2.$$

It should be noted that the constants achieved in Theorem 2.2 are valid[3] for all $t_2 > t_1 > 168\pi$, and that at $t_1 > 2\pi\cdot 10^{12}$ these constants minimise the right side of (2.3). The above theorem and Theorem 2.1 immediately lead to the following.

[3]The constant 168π which occurs in the triples of Turing and Lehman seems to be a misprint. In the proof of the rate of growth of $\zeta(\frac{1}{2}+it)$, given here in Lemma 2.5, Turing and Lehman require $t > 128\pi$ so that the error terms in the Riemann–Siegel formula are small. A computational check shows that Lemma 2.5 in fact holds for all $t > 1$. Choosing a moderately large value of t_0 ensures that the small errors accrued (i.e. the δ in Lemma 2.7 and the ϵ in Lemma 2.11) are suitably small. At no point do Turing and Lehman require the imposition of a t_0 greater than 128π. It is worthwhile to note that one could replace 168π in Theorem 2.2 by a smaller number, and although this has little application to the zeta-function, it may be useful for future applications to Dedekind zeta-functions; cf. §4.

Corollary 2.3. *If N consecutive Gram blocks with union $[g_n, g_p)$ satisfy Rosser's Rule where*

$$N \geq 0.0031 \log^2 g_p + 0.11 \log g_p,$$

then

$$N(g_n) \leq n + 1; \qquad N(g_p) \geq p + 1.$$

The above corollary shows that, in order to apply Turing's Method at height $g_p = 2\pi \cdot 10^{12}$, one needs to find only 6 Gram blocks in which the Rosser Rule is valid.

2.2. **Proof of Theorem 2.2.** This section closely follows the structure of Lehman's refinement [11] of Turing's work [20]. Some of the lemmas are identical to those in these papers, and their proofs are deferred to [11]. To begin, one rewrites the integral of the function $S(t)$ using the following.

Lemma 2.4. *If $t_2 > t_1 > 0$, then*

$$\pi \int_{t_1}^{t_2} S(t)\, dt = \Re \int_{\frac{1}{2}+it_2}^{\infty+it_2} \log \zeta(s)\, ds - \Re \int_{\frac{1}{2}+it_1}^{\infty+it_1} \log \zeta(s)\, ds. \tag{2.4}$$

Proof. This is Lemma 1 in [11], and the proof is based on Littlewood's theorem for analytic functions, but more detail is supplied in [10], or [5, pp. 190-192]. □

Henceforth, Lemmas 2.5–2.8 are used to bound the first integral on the right-hand side of (2.4), and Lemmas 2.9–2.11 are needed to bound the second integral.

Lemma 2.5. *If $t \geq 128\pi$, then*

$$|\zeta\left(\tfrac{1}{2} + it\right)| \leq 2.53\, t^{\frac{1}{4}}.$$

Proof. See the argument in [11] where some corrections are given to Titchmarsh's explicit calculation of the error in the Riemann–Siegel formula. □

This estimate can certainly be improved insofar as reducing the exponent of t is concerned. Currently the best bound on the growth of the zeta-function is due to Huxley [7], viz. $\zeta(\frac{1}{2} + it) \ll t^{\alpha+\epsilon}$, where $\alpha = \frac{32}{205} \approx 0.1561$. However, the methods used to attain this bound are complicated and the calculation of the implied constant would prove lengthy. The coarser, but simpler proof (see [17, Ch. V §5]) due to van der Corput yields

$$|\zeta\left(\tfrac{1}{2} + it\right)| \leq A t^{\frac{1}{6}} \log t, \tag{2.5}$$

where the calculation of the constant A is not too time consuming. Indeed following the arguments in [17, Chs IV-V] and using a result of Karatsuba [8, Lem. 1], one can take $A \leq 20$.

The logarithmic term in (2.5) is relatively innocuous since, for a given $\eta > 0$ one can then take t_0 so large that

$$\log t \leq A' t^{\eta},$$

where $A' = A'(\eta, t_0)$ can be easily computed, whence

$$|\zeta\left(\tfrac{1}{2} + it\right)| \leq A A' t^{\frac{1}{6}+\eta}.$$

Turing [20, p. 108] makes reference to the improvements made possible by these refined estimates on the growth of $\zeta(\frac{1}{2} + it)$. The following lemmas will be written with

$$|\zeta\left(\tfrac{1}{2} + it\right)| \leq K t^{\theta}, \tag{2.6}$$

so that the benefit of such a refinement as that in (2.5) can be seen clearly.

The bound on $\zeta(s)$ on the line $\sigma = \frac{1}{2}$ can be combined with that on the line $\sigma = c > 1$, whence the Phragmén–Lindelöf theorem can be applied throughout the strip $\frac{1}{2} \le \sigma \le c$. The papers of Turing and Lehman use the value $c = \frac{5}{4}$ and some improvement will be given later by choosing an optimal value of c at the end of the proof. The following result is needed.

Lemma 2.6. *Let a, b, Q and k be real numbers, and let $f(s)$ be regular analytic in the strip $-Q \le a \le \sigma \le b$ and satisfy the growth condition*

$$|f(s)| < C \exp\left\{e^{k|t|}\right\},$$

for a certain $C > 0$ and for $0 < k < \pi/(b-a)$. Also assume that

$$|f(s)| \le \begin{cases} A|Q+s|^{\alpha} & \text{for } \Re(s) = a, \\ B|Q+s|^{\beta} & \text{for } \Re(s) = b \end{cases}$$

with $\alpha \ge \beta$. Then throughout the strip $a \le \sigma \le b$ the following holds:

$$|f(s)| \le A^{(b-\sigma)/(b-a)} B^{(\sigma-a)/(b-a)} |Q+s|^{\alpha(b-\sigma)/(b-a)+\beta(\sigma-a)/(b-a)}.$$

Proof. See [13, pp. 66-67]. □

Take $Q = 0$; $a = \frac{1}{2}$; $b = c$; $f(s) = (s-1)\zeta(s)$, whence all the conditions of Lemma 2.6 are satisfied. Then on the line $\sigma = \frac{1}{2}$ it follows that

$$|f(s)| \le Kt^{\theta}|s-1| \le K|s|^{\theta+1},$$

by virtue of (2.6). On the line $\sigma = c$,

$$|f(s)| \le |s-1|\zeta(c) \le \zeta(c)|s|,$$

since $c > 1$. So one can take $A = K$; $\alpha = \theta + 1$; $B = \zeta(c)$; $\beta = 1$ and then apply Lemma 2.6 to obtain

$$|(s-1)\zeta(s)| \le \left[K^{c-\sigma}\{\zeta(c)\}^{\sigma-\frac{1}{2}}|s|^{\theta(c-\sigma)+c-\frac{1}{2}}\right]^{1/(c-\frac{1}{2})}. \tag{2.7}$$

For sufficiently large t, let C_1 and C_2 be numbers satisfying

$$|s-1| \ge C_1|s|; \qquad |s| \le C_2|t|.$$

When $t > 168\pi$ one can take $C_1^{-1} \ge 1+\delta$ and $C_2 \le 1+\delta$, where $\delta = 2\cdot 10^{-6}$. This gives an estimate on the growth of $\zeta(s)$ in terms of t only, and, together with (2.7) proves

Lemma 2.7. *Let K, θ and t_0 satisfy the relation that $|\zeta(\frac{1}{2}+it)| \le Kt^{\theta}$ whenever $t > t_0 > 168\pi$. Also, let $\delta = 2\cdot 10^{-6}$ and let c be a parameter satisfying $1 < c \le \frac{5}{4}$. Then throughout the region $\frac{1}{2} \le \sigma \le c$ the following estimate holds:*

$$|\zeta(s)| \le (1+\delta)\left\{K^{c-\sigma}\{\zeta(c)\}^{\sigma-\frac{1}{2}}((1+\delta)\,t)^{\theta(c-\sigma)}\right\}^{1/(c-\frac{1}{2})}.$$

Now, in the integral

$$\int_{\frac{1}{2}+it}^{\infty+it} \log|\zeta(s)|\,ds$$

one seeks to apply the convexity bound of Lemma 2.7 over the range $\frac{1}{2} \leq \sigma \leq c$, and to trivially estimate $\zeta(s)$ for $\sigma > c$. To this end, write

$$\int_{\frac{1}{2}+it}^{\infty+it} \log|\zeta(s)|\, ds = \int_{\frac{1}{2}+it}^{c+it} \log|\zeta(s)|\, ds + m(c),$$

where

$$m(c) := \int_{c+it}^{\infty+it} \log|\zeta(s)|\, ds \leq \int_{c}^{\infty} \log|\zeta(\sigma)|\, d\sigma,$$

since $c > 1$. The application of Lemma 2.7 proves

Lemma 2.8. *Under the same assumptions as Lemma 2.7, the following estimate holds:*

$$\Re \int_{\frac{1}{2}+it}^{\infty+it} \log\zeta(s)\, ds < a_1 + b_1 \log t,$$

where

$$a_1 = \int_{c}^{\infty} \log|\zeta(\sigma)|\, d\sigma + \frac{1}{2}\left(c - \tfrac{1}{2}\right) \log\{K\zeta(c)\} + \delta,$$

and

$$b_1 = \frac{\theta}{2}(c - \tfrac{1}{2}).$$

The improvements in the following lemmas come from writing $\zeta(s+d)$ in place of $\zeta(s+1)$ which is used in the methods of Turing and Lehman. One then seeks the optimal value of $d \leq 1$ at the end of the proof. Write

$$\begin{aligned}\Re \int_{\frac{1}{2}+it}^{\infty+it} \log\zeta(s)\, ds &= \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it} \log\left|\frac{\zeta(s)}{\zeta(s+d)}\right| ds \\ &\quad + \int_{\frac{1}{2}+d+it}^{\infty+it} \log|\zeta(s)|\, ds + \int_{\frac{1}{2}+d+it}^{\frac{1}{2}+2d+it} \log|\zeta(s)|\, ds,\end{aligned} \tag{2.8}$$

where $\frac{1}{2} < d \leq 1$. Since $d > \frac{1}{2}$, then $\Re(s) > 1$ in the second and third integrals on the right side of the above equation. Thus, $\zeta(s) \geq \zeta(2\sigma)/\zeta(\sigma)$, so that, after suitable changes of variables, (2.8) becomes

$$\Re \int_{\frac{1}{2}+it}^{\infty+it} \log\zeta(s)\, ds \geq \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it} \log\left|\frac{\zeta(s)}{\zeta(s+d)}\right| ds + I(d), \tag{2.9}$$

where

$$\begin{aligned}I(d) &= \frac{1}{2}\int_{1+2d}^{\infty} \log\zeta(\sigma)\, d\sigma - \int_{\frac{1}{2}+d}^{\infty} \log\zeta(\sigma)\, d\sigma \\ &\quad + \frac{1}{2}\int_{1+2d}^{1+4d} \log\zeta(\sigma)\, d\sigma - \int_{\frac{1}{2}+d}^{\frac{1}{2}+2d} \log\zeta(\sigma)\, d\sigma,\end{aligned} \tag{2.10}$$

and these integrals, all convergent, will be evaluated at the end of the proof. The integrand the right side of (2.9) can be rewritten[4] using the Weierstrass product formula (cf. [4, pp. 82-83])

$$\zeta(s) = \frac{e^{bs}}{2(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \tag{2.11}$$

[4]This method of approach is slightly easier than that given in Turing's paper, as noted by Lehman [11, p. 310].

where the product is taken over zeroes ρ and b is a constant such that

$$b = \frac{1}{2}\log\pi - \sum_{\rho}\frac{1}{\rho},$$

when the sum converges if each zero is paired with its conjugate. Thus

$$\begin{aligned}\log\left|\frac{\zeta(s)}{\zeta(s+d)}\right| &= \log\left|\frac{s+d-1}{s-1}\right| - \log\left|\frac{\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{s}{2}+1+\frac{d}{2})}\right| \\ &\quad + \sum_{\rho}\log\left|\frac{s-\rho}{s+d-\rho}\right| - \frac{d}{2}\log\pi,\end{aligned}$$

and so it follows that

$$\begin{aligned}\Re\int_{\frac{1}{2}+it}^{\infty+it}\log\zeta(s)\,ds &\geq \sum_{\rho}\int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it}\log\left|\frac{s-\rho}{s+d-\rho}\right|ds \\ &\quad - \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it}\log\left|\frac{\Gamma(\frac{s}{2}+1)}{\Gamma(\frac{s+d}{2}+1)}\right|ds \\ &\quad + \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it}\log\left|\frac{s+d-1}{s-1}\right|ds + I(d) - \frac{d^2}{2}\log\pi \\ &= I_1 - I_2 + I_3 + I(d) - \frac{d^2}{2}\log\pi.\end{aligned} \tag{2.12}$$

The following lemmas are needed for evaluation of I_1 and I_2. Since $\frac{1}{2} < d \leq 1$, it is easily seen that $I_3 \geq 0$ but since the argument of the logarithm tends to one as $t \to \infty$, no further improvements are possible. To estimate the integral I_2 the following result is required, which is a quantitative version of the classical estimate

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z + O\left(\frac{1}{z}\right);$$

see, e.g. [21, Ch. XII].

Lemma 2.9. *Define the symbol Θ in the following way: $f(x) = \Theta\{g(x)\}$ means that $|f(x)| \leq g(x)$. If $\Re z > 0$, then*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + \Theta\left(\frac{2}{\pi^2|(\Im z)^2 - (\Re z)^2|}\right).$$

Proof. See [11, Lem. 8]. □

Using the mean-value theorem for integrals, I_2 can be written as

$$I_2 = -\frac{1}{2}\int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it}\left\{\int_0^d \Re\frac{\Gamma'\left(1+\frac{s+\xi}{2}\right)}{\Gamma\left(1+\frac{s+\xi}{2}\right)}\,d\xi\right\}ds = -\frac{d^2}{2}\Re\frac{\Gamma'\left(\sigma+\frac{it}{2}\right)}{\Gamma\left(\sigma+\frac{it}{2}\right)},$$

for some $\sigma : \frac{5}{4} < \sigma < d + \frac{5}{4}$, whence by Lemma 2.9

$$I_2 = -\frac{d^2}{2}\log\frac{t}{2} + \epsilon, \tag{2.13}$$

where $|\epsilon|$ is comfortably less than $9 \cdot 10^{-5}$ when $t > 168\pi$ and decreases rapidly with increasing t.

In [20], the integrand in I_1 is evaluated using an approximate solution to a differential equation. This is then summed over the zeroes ρ. Using the fact that if $\rho = \beta + i\gamma$ lies off the critical line, then so too does $1 - \overline{\rho}$, Booker [2] was able to sharpen the bound on I_1. His result is given in the $d = 1$ case of the following.

Lemma 2.10 (Booker)**.** *Given a complex number w with $|\Re(w)| \leq \frac{1}{2}$, then for $\frac{1}{2} < d \leq 1$ the following holds:*

$$\int_0^d \log\left|\frac{(x+d+w)(x+d-\overline{w})}{(x+w)(x-\overline{w})}\right| dx \leq d^2(\log 4)\Re\left(\frac{1}{d+w} + \frac{1}{d-\overline{w}}\right).$$

Proof. The proof for $d = 1$ is given as Lemma 4.4 in [2]. The adaptation to values of d such that $\frac{1}{2} < d \leq 1$ is straightforward. □

Write

$$I_1 = \sum_\rho \int_0^d \log\left|\frac{\sigma + \frac{1}{2} + it - \rho}{\sigma + d + \frac{1}{2} + it - \rho}\right| d\sigma,$$

and apply Lemma 2.10 with $w = \frac{1}{2} + it - \rho$, pairing together ρ and $1 - \overline{\rho}$, whence

$$I_1 \geq -d^2(\log 4)\sum_\rho \Re\left(\frac{1}{\frac{1}{2} + d + it - \rho}\right).$$

Here the improvement of Booker's result is seen, as Lehman [*op. cit.*] has 1.48 in the place of $\log 4 \approx 1.38$. Rather than appealing to the Mittag-Leffler series for $\zeta(\frac{1}{2} + it)$ as in [11], here one proceeds directly by rewriting the sum over the zeroes using the Weierstrass product (2.11). By logarithmically differentiating (2.11) and taking real parts, it is seen that

$$\begin{aligned} -I_1 \leq d^2(\log 4)\Bigg\{&\Re\frac{\zeta'\left(\frac{1}{2} + d + it\right)}{\zeta\left(\frac{1}{2} + d + it\right)} + \frac{d - \frac{1}{2}}{(d - \frac{1}{2})^2 + t^2} \\ &+ \frac{1}{2}\Re\frac{\Gamma'\left(\frac{1}{2} + \frac{\frac{1}{2}+d+it}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\frac{1}{2}+d+it}{2}\right)} - \frac{1}{2}\log\pi\Bigg\}, \end{aligned}$$

and thus, using Lemma 2.9 with $t > 168\pi$ one has

$$-I_1 \leq d^2(\log 4)\left\{\Re\frac{\zeta'\left(\frac{1}{2} + d + it\right)}{\zeta\left(\frac{1}{2} + d + it\right)} + \frac{1}{2}\log\frac{t}{2} - \frac{1}{2}\log\pi + \epsilon'\right\},$$

where $|\epsilon'| \leq 10^{-4}$. Finally, since $d > \frac{1}{2}$, then

$$\Re\frac{\zeta'\left(\frac{1}{2} + d + it\right)}{\zeta\left(\frac{1}{2} + d + it\right)} \leq \left|\frac{\zeta'\left(\frac{1}{2} + d + it\right)}{\zeta\left(\frac{1}{2} + d + it\right)}\right| \leq -\frac{\zeta'\left(\frac{1}{2} + d\right)}{\zeta\left(\frac{1}{2} + d\right)},$$

and so

$$-I_1 \leq d^2(\log 4)\left\{-\frac{\zeta'\left(\frac{1}{2} + d\right)}{\zeta\left(\frac{1}{2} + d\right)} + \frac{1}{2}\log t - \frac{1}{2}\log 2\pi + \epsilon'\right\}. \tag{2.14}$$

The results for I_1 and I_2 contained in equations (2.14) and (2.13), respectively, can be used in (2.12) to give

$$-\Re\int_{\frac{1}{2}+it}^{\infty+it}\log\zeta(s)\,ds \leq d^2(\log 4)\left\{-\frac{\zeta'\left(\frac{1}{2}+d\right)}{\zeta\left(\frac{1}{2}+d\right)}+\frac{1}{2}\log t-\frac{1}{2}\log 2\pi\right\}$$
$$-\frac{d^2}{2}\log\frac{t}{2}-I(d)+\frac{d^2}{2}\log\pi+3\epsilon',$$

where $I(d)$ is defined by equation (2.10). This then proves

Lemma 2.11. *For $t > 168\pi$, d satisfying $\frac{1}{2} < d \leq 1$, and $\epsilon' = 10^{-4}$, the following estimate holds:*

$$-\Re\int_{\frac{1}{2}+it}^{\infty+it}\log\zeta(s)\,ds \leq a_2+b_2\log t,$$

where

$$\begin{aligned} a = {} & d^2(\log 4)\left\{-\frac{\zeta'(\frac{1}{2}+d)}{\zeta(\frac{1}{2}+d)}-\frac{1}{2}\log 2\pi+\frac{1}{4}\right\}+\frac{d^2}{2}\log\pi \\ & -\frac{1}{2}\int_{1+2d}^{\infty}\log\zeta(\sigma)\,d\sigma+\int_{\frac{1}{2}+d}^{\infty}\log\zeta(\sigma)\,d\sigma \\ & -\frac{1}{2}\int_{1+2d}^{1+4d}\log\zeta(\sigma)\,d\sigma+\int_{\frac{1}{2}+d}^{\frac{1}{2}+2d}\log\zeta(\sigma)\,d\sigma+3\epsilon', \end{aligned}$$

and

$$b=\frac{d^2}{2}\left(\log 4-1\right).$$

Lemmas 2.4, 2.7 and 2.11 prove at once

Theorem 2.12. *Let $t_2 > t_1 > t_0 > 168\pi$ and let the pair of numbers K, θ satisfy the relation that $\zeta(\frac{1}{2}+it) \leq Kt^{\theta}$ for $t > t_0$. Also, let $\mu = 3\cdot 10^{-6}$. If the parameters c and d are chosen such that $1 < c \leq \frac{5}{4}$ and $\frac{1}{2} < d \leq 1$, then*

$$\left|\int_{t_1}^{t_2}S(t)\,dt\right| \leq a+b\log t_2,$$

where

$$\begin{aligned} \pi a = {} & d^2(\log 4)\left\{-\frac{\zeta'(\frac{1}{2}+d)}{\zeta(\frac{1}{2}+d)}-\frac{1}{2}\log 2\pi+\frac{1}{4}\right\}+\frac{d^2}{2}\log\pi \\ & -\frac{1}{2}\int_{1+2d}^{\infty}\log\zeta(\sigma)\,d\sigma+\int_{\frac{1}{2}+d}^{\infty}\log\zeta(\sigma)\,d\sigma-\frac{1}{2}\int_{1+2d}^{1+4d}\log\zeta(\sigma)\,d\sigma \\ & +\int_{\frac{1}{2}+d}^{\frac{1}{2}+2d}\log\zeta(\sigma)\,d\sigma+\frac{1}{2}(c-\tfrac{1}{2})\log\{K\zeta(c)\}+\int_{c}^{\infty}\log\zeta(\sigma)\,d\sigma+\mu, \end{aligned} \tag{2.15}$$

and

$$2\pi b = \theta(c-\tfrac{1}{2})+d^2(\log 4-1). \tag{2.16}$$

2.3. **Calculations.** Taking the parameters $c = \frac{5}{4}$ and $d = 1$, $\theta = \frac{1}{4}$ and $K = 2.53$ one has, from the pair of equations (2.15) and (2.16) that $a = 1.61$ and $b = 0.0914$. These can be compared with the constants of Lehman, viz. ($a = 1.7, b = 0.114$). It can also be seen that the minimal value of b attainable by this method is 0.0353.

Since the application of Turing's Method involves Gram blocks one wishes to minimise the bound given in (2.1). That is, one wishes to minimise the quantity $F(a, b, g_p)$ given in (2.3). Here, the values of a and b have been chosen to be optimal, for the application to Gram blocks, at height $g_p = 2\pi \cdot 10^{12}$. Since it has been shown above that a and b are themselves functions of c and d, write $F(c, d)$ for $F(a, b, 2\pi \cdot 10^{12})$.

Since there are no terms in (2.15) and (2.16) which involve both c and d, one can write $F(c, d) = F_c(c) + F_d(d)$ and optimise each of the functions F_c and F_d separately. The presence of integrals involving the zeta-function in equations (2.15) and (2.16) makes the optimisation process difficult, even for a computer programme. Therefore, small values of $F_c(c)$ and $F_d(d)$ were sought over the intervals

$$d = d(N) = 0.99 - 2N\Delta; \quad c = c(N) = 1.24 - N\Delta,$$

where $\Delta = 0.02$ and $0 \leq N \leq 12$. This showed that values of $F(c, d) \leq 3.72$ were clustered around $d = 0.71$ and $c = 1.08$. A further search for small values was conducted with

$$d = d(N) = 0.68 + N\Delta; \quad c = c(N) = 1.05 + N\Delta,$$

where, this time, $\Delta = 0.01$ and $0 \leq N \leq 20$. The smallest value found in this second search was $F(c, d) = 3.6805\ldots$, corresponding to $d = 0.74$ and $c = 1.1$. For simplicity the choice of $d = \frac{3}{4}$ and $c = \frac{11}{10}$ gives $F(c, d) = 3.6812\ldots$ and obtains the constants in Theorem 2.2, viz.

$$a(\tfrac{11}{10}, \tfrac{3}{4}) = 2.0666; \quad b(\tfrac{11}{10}, \tfrac{3}{4}) = 0.0585.$$

3. Dirichlet L-functions

3.1. **Introduction.** In the works of Rumely [14] and Tollis [18], analogues for Turing's Method are developed for Dirichlet L-functions, and for Dedekind zeta-functions, respectively. Each of these proofs is based on [11], so it is fitting to apply the above adaptations to yield better constants in these analogous cases. Since many of the details in the proofs are identical to those in §2, this section and §4 are less ponderous than the previous one.

3.2. **Analogies to the functions $Z(t)$, $\theta(t)$ and $S(t)$.** Let χ be a primitive Dirichlet character with conductor $Q > 1$, and let $L(s, \chi)$ be the Dirichlet L-series attached to χ. Furthermore, define $\delta = (1 - \chi(-1))/2$ so that δ is 0 or 1 according to whether χ is an even or odd character. Then the function

$$\xi(s, \chi) = \left(\frac{Q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s+\delta}{2}\right) L(s, \chi) \tag{3.1}$$

is entire and satisfies the functional equation

$$\xi(s, \chi) = W_\chi \xi(1 - s, \overline{\chi}),$$

where

$$W_\chi = i^{-\delta}\tau(\chi)Q^{-\frac{1}{2}}; \quad \tau(\chi) = \sum_{n=1}^{Q} \chi(n) e^{\frac{2\pi n i}{Q}}.$$

It is easily seen that $|W_\chi| = 1$ and so one may write $W_\chi = e^{i\theta_\chi}$ and, for $s = \frac{1}{2} + it$,

$$\theta(t, \chi) := \frac{t}{2} \log \frac{Q}{\pi} + \Im \log \Gamma\left(\frac{s+\delta}{2}\right) - \frac{\theta_\chi}{2}.$$

Then the functions $Z(t, \chi)$ and $\theta(t, \chi)$ are related by the equation

$$Z(t, \chi) = e^{i\theta(t,\chi)} L(s, \chi),$$

where $Z(t, \chi)$ is real. This is analogous to the equation

$$Z(t) = e^{i\theta(t)} \zeta(\frac{1}{2} + it),$$

which can be found in [17, Ch. IV, §17].

One can now show that $\theta(t, \chi)$ is ultimately monotonically increasing. This means that the *Gram points* g_n can be defined for Dirichlet L-functions as those points at which $\theta(g_n, \chi) = n\pi$.

Similarly to (1.1), define, whenever t is not an ordinate of a zero of $L(s, \chi)$, the function

$$S(t, \chi) = \frac{1}{\pi} \arg L(\tfrac{1}{2} + it, \chi), \tag{3.2}$$

where, as before, the argument is determined via continuous variation along the straight lines connecting 2, $2 + it$ and $\frac{1}{2} + it$, with a continuity condition if t coincides with a zero. It is known that

$$\int_{t_1}^{t_2} S(t, \chi)\, dt = O(\log Qt_2),$$

and Turing's Method for Dirichlet L-functions requires a quantitative version of this result.

3.3. Theorem and new results.

Theorem 3.1. *Let (a, b, t_0) denote the following triple of numbers. Given $t_0 > 0$ there are positive constants a and b such that, whenever $t_2 > t_1 > t_0$ the following estimate holds:*

$$\left| \int_{t_1}^{t_2} S(t, \chi)\, dt \right| \le a + b \log \frac{Qt_2}{2\pi}, \tag{3.3}$$

Rumely [14] has shown that $(1.8397, 0.1242, 50)$ satisfies (3.3). Analogous to Theorem 2.1 is

Theorem 3.2 (Rumely)**.** *For $t_2 > t_1 > 50$ the following estimate holds:*

$$\int_{t_1}^{t_2} \left| S(t, \chi) \frac{\theta'(t, \chi)}{\pi} \right| \le 0.1592 \log \frac{QT}{2\pi} \left(a + b \log \frac{QT}{2\pi} \right) := B(Q, t_2). \tag{3.4}$$

The constant 0.1592 comes from applying Stirling's formula to the function $\theta(t, \chi)$. It is this bound which is used in practical calculations. As in the case of the zeta-function, a and b are roughly inversely proportional, so one can choose these parameters in such a way that the quantity $B(Q, t_2)$ is minimised for a given Q and t_2.

At $Q = 100$ and $t_2 = 2500$, Rumely's constants $(a = 1.8397, b = 0.1242)$ give the value

$$B(Q, t_2) \approx 5.32;$$

however, there is a misprint in [14] and this is quoted as 4.824, which does not appear to affect his numerical calculations. The values of a and b have been optimised in (3.4) for $Q = 100$ and $t_2 = 2500$, which proves the following.

Theorem 3.3. *If $t_2 > t_1 > t_0$, then the following estimate holds:*

$$\left| \int_{t_1}^{t_2} S(t,\chi)\, dt \right| \leq 1.975 + 0.084 \log \left(\frac{Q t_2}{2\pi} \right).$$

It therefore follows that $B(100, 2500) \approx 4.82$. Further reductions in the size of $B(Q, t_2)$ are possible if the quantity Qt_2 is taken much larger, which will certainly happen in future calculations.

3.4. **Proof of Theorem 3.3.** Littlewood's lemma on the number of zeroes of an analytic function in a rectangle is used to prove

Lemma 3.4. *If $t_2 > t_1 > 0$, then*

$$\int_{t_1}^{t_2} S(t,\chi)\, dt = \frac{1}{\pi} \int_{\frac{1}{2}+it_2}^{\infty+it_2} \log |L(s,\chi)|\, d\sigma - \frac{1}{\pi} \int_{\frac{1}{2}+it_1}^{\infty+it_1} \log |L(s,\chi)|\, d\sigma. \tag{3.5}$$

Proof. The proof of this is the same as for Lemma 2.4. □

The following lemma is a convexity estimate which will be used to give an upper bound on the first integral in (3.5).

Lemma 3.5 (Rademacher)**.** *Suppose $1 < c < \frac{3}{2}$. Then, for $1 - c \leq \sigma \leq c$, for all moduli $Q > 1$, and for all primitive characters χ with modulus Q,*

$$|L(s,\chi)| \leq \left(\frac{Q|1+s|}{2\pi} \right)^{\frac{c-\sigma}{2}} \zeta(c). \tag{3.6}$$

Proof. See [12, Thm. 3]. □

Rumely chooses $c = \frac{5}{4}$, but here the value of c will be chosen optimally at the end of the argument. In preparation for taking the logarithm of both sides of (3.6) note that for $\frac{1}{2} \leq \sigma \leq c$ and $t \geq t_0$, one can find an $\epsilon > 0$ such that $\log(|1+s|/t) \leq \epsilon$. This will be used to express $|\log L(s,\chi)|$ as a function of t rather than s. Indeed, if $\sigma \leq \frac{5}{4}$ and $t > t_0$ it is easy to show that

$$\frac{|1+s|}{t} \leq 1 + \frac{81}{32 t_0^2} = 1 + \epsilon.$$

Write

$$\int_{\frac{1}{2}+it}^{\infty+it} \log |L(s,\chi)|\, ds = \int_{\frac{1}{2}}^{c} \log |L(s,\chi)|\, d\sigma + \int_{c}^{\infty} \log |L(s,\chi)|\, d\sigma, \tag{3.7}$$

where the convexity result will be applied to the first integral on the right side. To estimate the second, note that for $\sigma \geq c > 1$ one can write

$$|L(s,\chi)| = \left| \sum_{n=1}^{\infty} \chi(n) n^{-s} \right| \leq \sum_{n=1}^{\infty} n^{-\sigma} = \zeta(\sigma).$$

With this estimation and the convexity estimate of (3.6), equation (3.7) becomes

$$\int_{\frac{1}{2}+it}^{\infty+it} \log|L(s,\chi)|\,d\sigma \le \frac{1}{4}\left(c-\tfrac{1}{2}\right)^2\left\{\log\frac{Qt}{2\pi}+\epsilon\right\} + \left(c-\tfrac{1}{2}\right)\log\zeta(1+\eta) + \int_c^{\infty}\log|\zeta(\sigma)|\,d\sigma.$$

This then proves

Lemma 3.6. *If $t > t_0 > 0$ and c is a parameter satisfying $1 < c \le \frac{5}{4}$, then throughout the region $\frac{1}{2} \le \sigma \le c$ the following estimate holds:*

$$\int_{\frac{1}{2}+it}^{\infty+it} \log|L(s,\chi)|\,d\sigma \le a_1 + b_1\log\frac{Qt}{2\pi},$$

where

$$a_1 = \frac{729}{2048t_0^2} + \left(c-\tfrac{1}{2}\right)\log\zeta(c) + \int_c^{\infty}\log|\zeta(\sigma)|\,d\sigma \tag{3.8}$$

and

$$b_1 = \frac{1}{4}(c-\tfrac{1}{2})^2.$$

Rumely uses $t_0 = 50$, whence one can take the first term in (3.8) to be at most $1.5\cdot 10^{-4}$.

The improvements in the following lemmas arise from taking d to be in the range $\frac{1}{2} < d \le 1$ and choosing the value of d optimally at the end of the proof. One writes

$$\int_{\frac{1}{2}+it}^{\infty+it} \log|L(s,\chi)|\,d\sigma$$

as a sum of integrals in the style of (2.8). For $\sigma > 1$ one can write

$$\begin{aligned}\log|L(s,\chi)| &= -\sum_p \log|1-\chi(p)p^{-s}| \ge -\sum_p \log(1+p^{-\sigma})\\ &= \sum_p\left\{-\log(1-p^{-2\sigma})+\log(1-p^{-\sigma})\right\} = \log\zeta(2\sigma)-\log\zeta(\sigma),\end{aligned}$$

whence

$$\int_{\frac{1}{2}+it}^{\infty+it} \log|L(s,\chi)|\,d\sigma \ge \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it}\log\left|\frac{L(s,\chi)}{L(s+d,\chi)}\right|d\sigma + I(d),$$

where $I(d)$ is the same function defined in (2.10) in §2.2. Now the integrand on the right of the above equation can be expanded using the Weierstrass Product[5] (see, e.g. [4, pp. 84-85])

$$\left(\tfrac{Q}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\tfrac{s+\delta}{2}\right)L(s,\chi) = \xi(s,\chi) = e^{A+Bs}\prod_\rho(1-\tfrac{s}{\rho})e^{\frac{s}{\rho}}, \tag{3.9}$$

with

$$B = -\lim_{T\to\infty}\sum_{|\rho|<T}\frac{1}{\rho}. \tag{3.10}$$

[5]Note that equation (18) of [14] has $(Q/\pi)^s$, rather than $(Q/\pi)^{s/2}$.

In the same manner as Turing's Method for the zeta-function, one arrives at

$$\begin{aligned} \int_{\frac{1}{2}+it}^{\infty+it} \log|L(s,\chi)|\,d\sigma &\geq \frac{d^2}{2}\log\frac{Q}{\pi} + \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it} \log\left|\frac{\Gamma\left(\frac{s+d+\delta}{2}\right)}{\Gamma\left(\frac{s+\delta}{2}\right)}\right| d\sigma \\ &\quad + \sum_{\rho}\int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it} \log\left|\frac{s-\rho}{s+d-\rho}\right| d\sigma + I(d) \\ &= \frac{d^2}{2}\log\frac{Q}{\pi} + I_1 + I_2 + I(d). \end{aligned} \tag{3.11}$$

As before, one uses the second mean-value theorem for integrals to address I_1, whence

$$I_1 = \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it} \log\left|\frac{\Gamma\left(\frac{s+d+\delta}{2}\right)}{\Gamma\left(\frac{s+\delta}{2}\right)}\right| d\sigma = \frac{d^2}{2}\Re\frac{\Gamma'\left(\tau+\frac{it}{2}\right)}{\Gamma\left(\tau+\frac{it}{2}\right)},$$

for

$$\tau \in \left(\tfrac{1}{4}+\tfrac{\delta}{2},\, d+\tfrac{1}{4}+\tfrac{\delta}{2}\right) \subset \left(\tfrac{1}{4},\, d+\tfrac{3}{4}\right),$$

since δ is either 0 or 1. Using Lemma 2.9, one has that

$$I_1 \geq \frac{d^2}{2}\left\{\log\frac{t}{2} - \epsilon'\right\}, \tag{3.12}$$

where

$$\epsilon' = \frac{11}{t_0^2}, \tag{3.13}$$

and, since, $t_0 > 50$, it follows that $\epsilon' < 5\cdot 10^{-3}$. The application of Lemma 2.10 to I_2, with zeroes ρ paired with $1-\overline{\rho}$ gives

$$I_2 \geq -d^2(\log 4)\sum_{\rho}\Re\left(\frac{1}{d+\frac{1}{2}+it-\rho}\right).$$

Now logarithmically differentiate the Weierstrass product in (3.9), take real parts, and use (3.10), to arrive at

$$\sum_{\rho}\Re\left(\frac{1}{s-\rho}\right) = \frac{1}{2}\log\frac{Q}{\pi} + \frac{1}{2}\Re\left(\frac{\Gamma'\left(\frac{s+\delta}{2}\right)}{\Gamma\left(\frac{s+\delta}{2}\right)}\right) + \Re\left(\frac{L'(s,\chi)}{L(s,\chi)}\right). \tag{3.14}$$

For $\sigma = \Re(s) > 1$, one can write

$$\Re\left(\frac{L'(s,\chi)}{L(s,\chi)}\right) = \Re\left(\sum_{p}\frac{\chi(p)\log p}{p^s-\chi(p)}\right) \leq \sum_{p}\frac{\log p}{p^{\sigma}-1} = -\frac{\zeta'(\sigma)}{\zeta(\sigma)}. \tag{3.15}$$

Thus when $s = d+\frac{1}{2}+it$, an application of Lemma 2.9 to (3.14) together with (3.15) gives

$$I_2 \geq -d^2(\log 4)\left(\frac{1}{2}\log\frac{Qt}{2\pi} + \frac{5}{t_0^2} - \frac{\zeta'(\frac{1}{2}+d)}{\zeta(\frac{1}{2}+d)}\right). \tag{3.16}$$

The results for I_2, contained in (3.16), and for I_1, contained in (3.12) and (3.13), can be combined with (3.11) to prove

Lemma 3.7. *For $t > t_0 > 50$ and for a parameter d satisfying the condition $\frac{1}{2} < d \leq 1$, the following estimate holds:*

$$-\int_{\frac{1}{2}+it}^{\infty+it} \log|L(s,\chi)|\, d\sigma \leq a_2 + b_2 \log\frac{Qt}{2\pi},$$

where

$$a = \frac{13d^2}{t_0^2} - d^2(\log 4)\frac{\zeta'\left(\frac{1}{2}+d\right)}{\zeta\left(\frac{1}{2}+d\right)} - \frac{1}{2}\int_{2d+1}^{\infty} \log\zeta(\sigma)\, d\sigma$$
$$+ \int_{\frac{1}{2}+d}^{\infty} \log\zeta(\sigma)\, d\sigma - +\frac{1}{2}\int_{2d+1}^{4d+1} \log\zeta(\sigma)\, d\sigma + \int_{\frac{1}{2}+d}^{\frac{1}{2}+2d} \log\zeta(\sigma)\, d\sigma$$

and

$$b = \frac{d^2}{2}(\log 4 - 1).$$

Lemmas 3.4, 3.6 and 3.7 prove at once

Theorem 3.8. *If $t_2 > t_1 > t_0 > 50$ and c and d are parameters such that $1 < c \leq \frac{5}{4}$ and $\frac{1}{2} < d \leq 1$, the following estimate holds:*

$$\left|\int_{t_1}^{t_2} S(t,\chi)\, dt\right| \leq a + b\log\left(\frac{Qt_2}{2\pi}\right),$$

where

$$\begin{aligned} a\pi = {} & (c - \tfrac{1}{2})\log\zeta(c) + \int_c^{\infty} \log\zeta(\sigma)\, d\sigma - d^2(\log 4)\frac{\zeta'\left(\frac{1}{2}+d\right)}{\zeta\left(\frac{1}{2}+d\right)} \\ & - \frac{1}{2}\int_{2d+1}^{\infty} \log\zeta(\sigma)\, d\sigma + \int_{\frac{1}{2}+d}^{\infty} \log\zeta(\sigma)\, d\sigma \\ & - \frac{1}{2}\int_{2d+1}^{4d+1} \log\zeta(\sigma)\, d\sigma + \int_{\frac{1}{2}+d}^{\frac{1}{2}+2d} \log\zeta(\sigma)\, d\sigma + \frac{15d^2}{t_0^2} \end{aligned} \tag{3.17}$$

and

$$2b\pi = \frac{1}{2}\left(c - \tfrac{1}{2}\right)^2 + d^2(\log 4 - 1). \tag{3.18}$$

3.5. **Calculations and improvements.** In (3.17) and (3.18) Rumely has $c = \frac{5}{4}$ and $d = 1$ as well as 1.48 in place of $\log 4 \approx 1.38$, and thus he calculates[6]

$$a = 1.839; \quad b = 0.1212.$$

Even with the same values of c and d, the inclusion of Lemma 2.10 gives the result here that

$$a(\tfrac{5}{4}, 1) = 1.794; \quad b(\tfrac{5}{4}, 1) = 0.1063.$$

For the values of $Q = 100$, $t_2 = 2500$ the quantity $B(Q, t_2)$ — defined in Theorem 3.2 — was minimised over two intervals using a computer programme, similarly to §2.3. This yielded the optimal value for $B(Q, t_2)$ at $c = 1.17$ and $d = 0.88$, whence the constants

$$a(1.17, 0.88) = 1.9744; \quad b(1.17, 0.88) = 0.0833,$$

which appear in Theorem 3.3.

[6]The number 0.1242 quoted by Rumely in his Theorem 2 is a result of a rounding error from his Lemma 2.

4. Dedekind zeta-functions

Let K be a number field of degree N with discriminant D and with ring of integers $\mathcal{O}_K$. Let the signature of the field be (r_1, r_2), by which it is meant that K has r_1 real embeddings and r_2 pairs of complex embeddings, whence $N = r_1 + 2r_2$. Then for $\Re(s) > 1$ the Dedekind zeta-function is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a} \in \mathcal{O}_K} (N\mathfrak{a})^{-s} = \sum_{n \geq 1} a_n n^{-s},$$

where $\mathfrak{a}$ ranges over the non-zero ideals of $\mathcal{O}_K$ and a_n is the number of ideals with norm n. Like the Riemann zeta-function, the Dedekind zeta-function can be extended via analytic continuation to the entire complex plane where it is defined as a meromorphic function with a simple pole at $s = 1$. If

$$\Lambda_K(s) = \Gamma(\tfrac{s}{2})^{r_1} \Gamma(s)^{r_2} \left(\frac{\sqrt{|D_K|}}{\pi^{\frac{N}{2}} 2^{r_2}} \right)^s \zeta_K(s),$$

then the Dedekind zeta-function satisfies the functional equation

$$\Lambda_K(s) = \Lambda_K(1-s). \tag{4.1}$$

One can define (see, e.g. [18]) the functions analogous to $Z(t)$ and $\theta(t)$ by

$$Z_K(t) = e^{i\theta_K(t)} \zeta_K(\tfrac{1}{2} + it).$$

Analogous to the function $S(t)$ define,

$$S_K(t) = \frac{1}{\pi} \arg \zeta_K(\tfrac{1}{2} + it); \quad S_K^1(t) = \int_0^t S_K(u)\, du,$$

where the valuation of the argument is determined, if t is not an ordinate of a zero, by continuous variation along the line from $\infty + it$ to $\frac{1}{2} + it$ and $S(0) = 0$. The modified Turing criterion for Dedekind zeta-functions relies on the following.

Theorem 4.1. *Given $t_0 > 0$ there are positive constants a, b and g such that, whenever $t_2 > t_1 > t_0$ the following estimate holds:*

$$\left| \int_{t_1}^{t_2} S_K(t)\, dt \right| \leq a + bN + g \log \left(|D_K| \left(\frac{t_2}{2\pi} \right)^N \right). \tag{4.2}$$

If one denotes the quadruple (a, b, g, t_0) as those numbers satisfying (4.2), then the work of Tollis [18] leads to the quadruple $(0.2627, 1.8392, 0.122, 40)$. Analogous to Theorem 3.2, is the following:

Theorem 4.2 (Tollis)**.** *For $t_2 > t_1 > 40$, then*

$$\begin{aligned} \left| \int_{t_1}^{t_2} S_K(t) \frac{\theta_K'(t)}{\pi}\, dt \right| &\leq \left(\frac{b}{2\pi} N + \frac{a}{2\pi} \right) \log \left(|D_K| \left(\frac{t_2}{2\pi} \right)^N \right) \\ &\quad + \frac{g}{2\pi} \log^2 \left(|D_K| \left(\frac{t_2}{2\pi} \right)^N \right) \\ &= B(D_K, t_2, N). \end{aligned}$$

For a given D_K, t_2, N one wishes to choose the constants a, b and g so as to minimise $B(D_K, t_2, N)$. For the sample values $N = 4$, $D_K = 1000$ and $t_2 = 80$ one finds that Tollis's constants give $B(D_K, t_2, N) \approx 26.44$. As will be shown in §4.2, very little improvement can be given on the constants of Tollis. Nevertheless, the inclusion of Lemma 2.10 is enough to prove

Theorem 4.3. *Given $t_2 > t_1 > 40$ the following estimate holds:*

$$\left| \int_{t_1}^{t_2} S_K(t)\, dt \right| \le 0.264 + 1.843N + 0.105 \log \left(|D_K| \left(\frac{t_2}{2\pi} \right)^N \right). \tag{4.3}$$

The improvements to Tollis's work will most likely be of use in the search for zeroes of Dedekind zeta-functions of large discriminant or degree but at *small* height. For this reason the constant t_0 has been retained in the following equations, and appears in Theorem 4.8 from which Theorem 4.3 is derived.

4.1. **Proof of Theorem 4.3.** As before, one begins by proving

Lemma 4.4.

$$\pi \int_{t_1}^{t_2} S_K(t)\, dt = \int_{\frac{1}{2}+it_2}^{\infty+it_2} \log |\zeta_K(s)|\, ds - \int_{\frac{1}{2}+it_1}^{\infty+it_1} \log |\zeta_K(s)|\, ds.$$

Proof. The proof is the same as in Lemma 2.4. □

The convexity estimate required is

Lemma 4.5 (Rademacher)**.** *For $1 < c < \frac{3}{2}$ and $s = \sigma + it$ then throughout the range $1 - c \le \sigma \le c$, the following estimate holds:*

$$|\zeta_K(s)| \le 3 \left| \frac{1+s}{1-s} \right| \zeta(c)^N \left(|D_K| \left\{ \frac{|1+s|}{2\pi} \right\}^N \right)^{\frac{c-\sigma}{2}}. \tag{4.4}$$

Proof. See [12, Thm. 4]. □

Note that, for $\frac{1}{2} \le \sigma \le c \le \frac{5}{4}$ and for $t > t_0$ one can write

$$\log |1+s| \le \log t + \frac{81}{32t_0^2}.$$

This then enables one to place an upper bound on (4.4) in terms of t rather than s. Now write

$$\int_{\frac{1}{2}+it}^{\infty+it} \log |\zeta_K(s)|\, ds = \int_{\frac{1}{2}+it}^{c+it} \log |\zeta_K(s)|\, ds + \int_c^\infty \log |\zeta_K(\sigma)|\, d\sigma,$$

where the second integral on the right-hand side is estimated trivially by the relation

$$\log |\zeta_K(\sigma + it)| \le N \log \zeta(\sigma), \tag{4.5}$$

since $\sigma > 1$. The inequality in (4.5) can be seen by taking the prime ideal decomposition as in, e.g. [12, p. 199]. An application of the convexity estimates from Lemma 4.5 proves the following.

Lemma 4.6. *For $t > t_0 > 0$ and for a parameter c satisfying $1 < c \le \frac{5}{4}$, the following estimate holds:*

$$\int_{\frac{1}{2}+it}^{\infty+it} \log |\zeta_K(s)|\, ds \le a_1 + b_1 N + g_1 \log \left(|D_K| \left(\frac{t_2}{2\pi} \right)^N \right),$$

where

$$a_1 = \left(c - \tfrac{1}{2}\right)\left(\frac{81}{32t_0^2} + \log 3\right),$$

$$b_1 = \left(c - \tfrac{1}{2}\right)\left(\log\zeta(c) + \frac{81\left(c - \frac{1}{2}\right)}{128t_0^2}\right) + \int_c^\infty \log\zeta(\sigma)\,d\sigma,$$

and

$$g_1 = \frac{1}{4}\left(c - \tfrac{1}{2}\right)^2.$$

One writes

$$\int_{\frac{1}{2}+it}^{\infty+it} \log|\zeta_K(s)|\,ds$$

as a sum of three integrals in the style of (2.8). Thence, when $\sigma > 1$ one can use the fact that

$$\log|\zeta_K(s)| \geq N(\log|\zeta(2\sigma)| - \log|\zeta(s)|),$$

to write

$$\int_{\frac{1}{2}+it}^{\infty+it} \log|\zeta_K(s)|\,ds \geq \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it} \log\left|\frac{\zeta_K(s)}{\zeta_K(s+d)}\right| ds + NI(d),$$

where $I(d)$ is the same function defined in (2.10) in §2.2. One aims at using the functional equation to estimate the integrand on the right-hand side. Using a result of Lang [9, Ch. XIII] one can write out the Weierstrass product viz.

$$s(s-1)\Lambda_K(s) = e^{a+bs}\prod_\rho\left(1 - \frac{s}{\rho}\right)e^{s/\rho}, \tag{4.6}$$

where

$$\Re b = -\sum_\rho \Re\frac{1}{\rho}.$$

This then gives

$$\begin{aligned}\int_{\frac{1}{2}+it}^{\infty+it} \log|\zeta_K(s)|\,ds &\geq d^2\log\left(\frac{\sqrt{|D_K|}}{\pi^{\frac{N}{2}}2^{r_2}}\right) + r_1\int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it}\log\left|\frac{\Gamma(\frac{s+d}{2})}{\Gamma(\frac{s}{2})}\right| ds \\ &\quad + r_2\int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it}\log\left|\frac{\Gamma(s+d)}{\Gamma(s)}\right| ds \\ &\quad + \int_{\frac{1}{2}+it}^{\frac{1}{2}+d+it}\sum_\rho\log\left|\frac{s-\rho}{s+d-\rho}\right| ds + NI(d) \\ &\geq d^2\log\left(\frac{\sqrt{|D_K|}}{\pi^{\frac{N}{2}}2^{r_2}}\right) + I_1 + I_2 + I_3 + NI(d).\end{aligned} \tag{4.7}$$

Applying the second mean-value theorem for integrals gives

$$I_1 = \frac{r_1 d^2}{2}\Re\frac{\Gamma'\left(\frac{it}{2} + \tau_1\right)}{\Gamma\left(\frac{it}{2} + \tau_1\right)}; \quad I_2 = r_2 d^2\Re\frac{\Gamma'\left(it + \tau_2\right)}{\Gamma\left(it + \tau_2\right)},$$

where $\frac{1}{4} < \tau_1 < d + \frac{1}{4}$ and $\frac{1}{2} < \tau_2 < 2d + \frac{1}{2}$. Hence, Lemma 2.9 gives

$$I_1 \geq \frac{r_1 d^2}{2}\left(\log\frac{t}{2} - \frac{7}{2t_0^2}\right); \quad I_2 \geq r_2 d^2\left(\log t - \frac{11}{4t_0^2}\right). \tag{4.8}$$

The integral I_3 is estimated using Lemma 2.10, and logarithmic differentiation of (4.6) gives

$$\sum_{\rho} \Re\left(\frac{1}{s-\rho}\right) = \Re\left(\frac{1}{s} + \frac{1}{s-1}\right) + \frac{r_1}{2}\Re\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$
$$+ r_2\Re\frac{\Gamma'(s)}{\Gamma(s)} + \log\left|\frac{\sqrt{|D_K|}}{\pi^{\frac{N}{2}}2^{r_2}}\right| + \Re\left(\frac{\zeta_K'(s)}{\zeta_K(s)}\right).$$

Since

$$\Re\frac{\zeta_K'(s)}{\zeta_K(s)} \leq -N\frac{\zeta'(\sigma)}{\zeta(\sigma)},$$

when $\sigma > 1$, the expression for I_3 becomes

$$I_3 \geq -d^2(\log 4)\left\{\frac{2}{t_0^2} + \frac{r_1}{2}\left(\log\frac{t}{2} + \frac{2}{t_0^2}\right) + r_2\left(\log t + \frac{3}{2t_0^2}\right)\right.$$
$$\left. + \log\frac{\sqrt{|D_K|}}{\pi^{\frac{N}{2}}2^{r_2}} - N\frac{\zeta'(d+\frac{1}{2})}{\zeta(d+\frac{1}{2})}\right\}.$$

Thus the estimates for I_1 and I_2, which are contained in (4.8) and the estimate of I_3 above prove, via (4.7):

Lemma 4.7. *If $t > t_0 > 0$ and d is a parameter that satisfies $\frac{1}{2} < d \leq 1$, then the following estimate holds:*

$$-\int_{\frac{1}{2}+it}^{\infty+it} \log|\zeta_K(s)|\,ds \leq a_2 + b_2N + g_2\log\left(|D_K|\left(\frac{t}{2\pi}\right)^N\right),$$

where

$$a_2 = \frac{4d^2\log 2}{t_0^2},$$

$$b_2 = d^2(\log 2)\left\{\log 2 - \frac{1}{2} - 2\frac{\zeta'\left(\frac{1}{2}+d\right)}{\zeta\left(\frac{1}{2}+d\right)} + \frac{8}{t_0^2}\right\} - I(d)$$

and

$$g_2 = \frac{d^2}{2}(\log 4 - 1),$$

and $I(d)$ is defined by (2.10) *in* §2.2.

Lemmas 4.4, 4.6 and 4.7 prove at once

Theorem 4.8. *If $t_2 > t_1 > t_0 > 0$ and the parameters c and d satisfy $1 < c \leq \frac{5}{4}$ and $\frac{1}{2} < d \leq 1$, then the following estimate holds:*

$$\left|\int_{t_1}^{t_2} S_K(t)\,dt\right| \leq a + bN + g\log\left(|D_K|\left(\frac{t}{2\pi}\right)^N\right),$$

where

$$\pi a = \left(c - \tfrac{1}{2}\right)\left(\frac{81}{32t_0^2} + \log 3\right) + \frac{4d^2\log 2}{t_0^2},$$

$$\pi b = \left(c - \tfrac{1}{2}\right)\left(\log\zeta(c) + \frac{81\left(c-\tfrac{1}{2}\right)}{128t_0^2}\right) + \int_c^\infty \log\zeta(\sigma)\,d\sigma$$
$$+ d^2(\log 2)\left\{\log 2 - \frac{1}{2} - 2\frac{\zeta'\left(\tfrac{1}{2}+d\right)}{\zeta\left(\tfrac{1}{2}+d\right)} + \frac{8}{t_0^2}\right\} - I(d),$$

and

$$\pi g = \frac{1}{4}\left(c - \tfrac{1}{2}\right)^2 + \frac{d^2}{2}(\log 4 - 1).$$

4.2. **Calculations.** Given the values $D_K = 1000$, $N = 4$, $t_0 = 40$ and $t_2 = 100$, the quantity to be minimised is

$$F(a,b,g) = a + 4b + 18g,$$

with a, b and g defined in Theorem 4.8. Proceeding with an optimisation programme similar to that in §2.3, one finds that in fact the 'trivial estimate', viz. the values $c = \frac{5}{4}$ and $d = 1$ produce the minimum value of $F(a,b,g)$ and hence the minimum value of $B(D_K, t_2, N)$ as defined in (4.3). The optimisation argument is only better than the trivial estimate when one of the parameters D_K, t_2 or N is large, which will certainly occur in future calculations.

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