

DOUGALL'S BILATERAL ${}_2H_2$ -SERIES AND RAMANUJAN-LIKE π -FORMULAE

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ABSTRACT. The modified Abel lemma on summation by parts is employed to investigate the partial sum of Dougall's bilateral ${}_2H_2$ -series. Several unusual transformations into fast convergent series are established. They lead surprisingly to numerous infinite series expressions for π , including several formulae discovered by Ramanujan (1914) and recently by Guillera (2008).

Roughly speaking, hypergeometric series is defined to be a series $\sum C_n$ with term ratio C_{1+n}/C_n a rational function of n . In general, it can be explicitly written as

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n z^n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!}$$

where the rising shifted factorial is given by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\cdots(x+n-1) \quad \text{for } n = 1, 2, \dots.$$

The Γ -function is defined by the Euler integral

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du \quad \text{with } \Re(x) > 0$$

which admits the well-known reciprocal formulae

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{and} \quad \Gamma\left(\frac{1}{2} + x\right)\Gamma\left(\frac{1}{2} - x\right) = \frac{\pi}{\cos \pi x}.$$

By means of Cauchy's residue theorem, Dougall [21, 1907] discovered the following bilateral series identity

$${}_2H_2 \left[\begin{matrix} a, b \\ c, d \end{matrix} \middle| 1 \right] = \sum_{k=-\infty}^{+\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} = \Gamma \left[\begin{matrix} 1-a, 1-b, c, d, c+d-a-b-1 \\ c-a, d-a, c-b, d-b \end{matrix} \right]$$

Received by the editor May 13, 2010 and, in revised form, July 31, 2010.

2010 *Mathematics Subject Classification*. Primary 33C20, Secondary 40A25, 65B10.

Key words and phrases. Abel's lemma on summation by parts, Classical hypergeometric series, The Gauss summation theorem, Dougall's bilateral ${}_2H_2$ -series identity, Ramanujan's π -formulae.

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where $\Re(c+d-a-b) > 1$ for convergence. Different proofs can be found in Chu [19] and Slater [33, §6.1]. When $d = 1$, the last formula reduces to the well-known Gauss summation theorem (cf. Bailey [5, §1.3] and Slater [33, §1.7])

$${}_2F_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{where } \Re(c-a-b) > 0.$$

Throughout this paper, we shall adopt the notations of Slater [33] to denote the product and quotient forms of the shifted factorial and the Γ -function as follows:

$$\begin{aligned} [A, B, \dots, C]_n &= (A)_n (B)_n \cdots (C)_n, & \Gamma[A, B, \dots, C] &= \Gamma(A)\Gamma(B)\cdots\Gamma(C); \\ \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right]_n &= \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}, & \Gamma \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right] &= \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}. \end{aligned}$$

By utilizing the modified Abel lemma on summation by parts, the author [18, 19] has recently reviewed several identities of classical hypergeometric series. The purpose of the present work is to explore this approach further to investigate acceleration of the partial sum of Dougall's ${}_2H_2$ -series into fast convergent series. Five fundamental recurrence relations will be established. They will then be employed to establish several unusual transformation formulae which express the partial sum of Dougall's ${}_2H_2$ -series in terms of fast convergent ones. Surprisingly, these transformations will further lead to numerous infinite series expressions for π , including three typical known ones due to Ramanujan [31, 1914]:

$$\begin{aligned} (1) \quad \frac{4}{\pi} &= \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \right]_k \frac{1+6k}{4^k}, \\ (2) \quad \frac{8}{\pi} &= \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ 1, 1, 1 \end{matrix} \right]_k \frac{3+20k}{(-4)^k}, \\ (3) \quad \frac{16}{\pi} &= \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \right]_k \frac{5+42k}{64^k}. \end{aligned}$$

They are among the 17 formulae recorded by Ramanujan [31, 1914] (see Berndt [10, Chapter 29] also). However, the first rigorous mathematical proofs of Ramanujan's identities and generalizations of them were given by the Borwein brothers [11, §5] and the Chudnovsky brothers [20]. For the historical notes and introductory information on the formulae for π -series, there are three excellent survey papers by Bailey–Borwein [3], Baruah–Berndt–Chan [8] and Guillera [28]. Different proofs of Ramanujan's identities and further formulae for $1/\pi$ and $1/\pi^2$ can be found in Andrews–Berndt [2, §15.6], Baruah–Berndt [6, 7], Borweins [12, 13], Chan *et al.* [15]–[17], Guillera [24]–[29] Rogers [32] and Zudilin [35].

Two more spectacular examples from the iteration pattern [5,5,10,10] read as

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left\{ \frac{(10n)!}{(5n)!(4+5n)!} \right\}^3 \frac{5}{2^{40+60n}} \quad \text{and} \quad \frac{\pi}{\sqrt{3}} = 3^7 5^2 \sum_{n=0}^{\infty} \frac{(15n)!(15+15n)!}{(30+30n)!}$$

$$\times \left\{ \begin{array}{l} 967639691668545 \\ +70481743821538938n \\ +1679601725341815240n^2 \\ +20983953947697829200n^3 \\ +162432411074663046000n^4 \\ +844125894212126940000n^5 \\ +3073307854429525600000n^6 \\ +8009128408111512000000n^7 \\ +15034125109741980000000n^8 \\ +20170957811328600000000n^9 \\ +1887075478626000000000n^{10} \\ +1169212052916000000000n^{11} \\ +431087457860000000000n^{12} \\ +71582788200000000000n^{13} \end{array} \right\} \times \left\{ \begin{array}{l} 6729160970477568 \\ +275897240208641792n \\ +4937369471226401216n^2 \\ +51577093209056406400n^3 \\ +353721530774616375400n^4 \\ +1693516148807887876500n^5 \\ +5861504657068749183750n^6 \\ +14949937005553937671875n^7 \\ +28315966186734060515625n^8 \\ +39730036847597675859375n^9 \\ +40734779556935112890625n^{10} \\ +29647774656743291015625n^{11} \\ +14503229510183701171875n^{12} \\ +4274336559486181640625n^{13} \\ +573308927466064453125n^{14} \end{array} \right\}.$$

The rest of the paper will be organized as follows. Applying the modified Abel lemma on summation by parts, five recurrence relations for the partial sum of Dougall's ${}_2H_2$ -series will be established in the next section. They will be employed in the second section to prove 60 infinite series identities for π and $1/\pi$, including several known formulae discovered by Ramanujan [31] and Guillera [29]. Finally, the more general iteration pattern will be examined in the third section, where, further, 40 Ramanujan-like formulae will be displayed. In order to assure the accuracy, all the formulae are checked by an appropriately devised *Mathematica* package.

1. FIVE RECURRENCE RELATIONS FOR THE Ω -SUM

For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ , respectively, by

$$\nabla \tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta \tau_k = \tau_k - \tau_{k+1}$$

where Δ differs from the usual operator Δ only in the minus sign. Then **Abel's lemma** on summation by parts may be reformulated as

$$(4) \quad \sum_{k=0}^{\infty} B_k \nabla A_k = [AB]_+ - A_{-1} B_0 + \sum_{k=0}^{\infty} A_k \Delta B_k$$

provided that the limit $[AB]_+ := \lim_{n \rightarrow \infty} A_n B_{n+1}$ exists and one of both series just displayed is convergent.

Proof. According to the definition of the backward difference, we have

$$\sum_{k=0}^m B_k \nabla A_k = \sum_{k=0}^m B_k \{A_k - A_{k-1}\} = \sum_{k=0}^m A_k B_k - \sum_{k=0}^m A_{k-1} B_k.$$

Replacing k by $k + 1$ for the last sum, we get the following expression:

$$\begin{aligned} \sum_{k=0}^m B_k \nabla A_k &= A_m B_{m+1} - A_{-1} B_0 + \sum_{k=0}^m A_k \{B_k - B_{k+1}\} \\ &= A_m B_{m+1} - A_{-1} B_0 + \sum_{k=0}^m A_k \Delta B_k. \end{aligned}$$

Letting $m \rightarrow \infty$ gives rise to the formula displayed in equation (4). \square

According to Dougall's ${}_2H_2$ -series, define its partial Ω -sum by

$$\Omega(a, b, c, d) := \sum_{k=0}^{\infty} \begin{bmatrix} a, b \\ c, d \end{bmatrix}_k \quad \text{where } \Re(c + d - a - b) > 1.$$

For the four nonnegative integers p, q, r, s , a linear relation expressing $\Omega(a, b, c, d)$ in terms of $\Omega(a+p, b+q, c+r, d+s)$ will be called *iteration pattern* [**pqrs**]. Now we are ready to apply the modified Abel lemma on summation by parts to derive five recurrence relations for $\Omega(a, b, c, d)$, that will be employed to demonstrate infinite series identities for π and $1/\pi$ in the next section.

§1.1. [**1000**]. For the two sequences A_k and B_k given respectively by

$$A_k = \begin{bmatrix} 1 + a, c + d - a - 1 \\ c, d \end{bmatrix}_k \quad \text{and} \quad B_k = \frac{(b)_k}{(c + d - a - 2)_k},$$

it is almost trivial to compute the differences

$$\begin{aligned} \nabla A_k &= \begin{bmatrix} a, c + d - a - 2 \\ c, d \end{bmatrix}_k \frac{(1 + a - c)(1 + a - d)}{a(2 + a - c - d)}, \\ \Delta B_k &= \frac{(b)_k}{(c + d - a - 1)_k} \frac{2 + a + b - c - d}{2 + a - c - d}, \end{aligned}$$

as well as two limiting relations

$$[AB]_+ = 0 \quad \text{and} \quad A_{-1} B_0 = \frac{(c-1)(d-1)}{a(c+d-a-2)} \quad \text{with } \Re(c+d-a-b) > 2.$$

By means of the modified Abel lemma on summation by parts, we can manipulate the Ω -series as follows:

$$\begin{aligned} \Omega(a, b, c, d) &= \frac{a(2 + a - c - d)}{(1 + a - c)(1 + a - d)} \sum_k B_k \nabla A_k \\ &= \frac{a(2 + a - c - d)}{(1 + a - c)(1 + a - d)} \left\{ [AB]_+ - A_{-1} B_0 + \sum_k A_k \Delta B_k \right\} \\ &= \frac{(c-1)(d-1)}{(1 + a - c)(1 + a - d)} + \frac{a(2 + a + b - c - d)}{(1 + a - c)(1 + a - d)} \sum_{k=0}^{\infty} \begin{bmatrix} 1 + a, b \\ c, d \end{bmatrix}_k. \end{aligned}$$

This may be restated as the recurrence relation.

Lemma 1.1 (Recurrence relation [**1000**]).

$$\Omega(a, b, c, d) = \Omega(1 + a, b, c, d) \times \frac{a(2 + a + b - c - d)}{(1 + a - c)(1 + a - d)} + \frac{(c-1)(d-1)}{(1 + a - c)(1 + a - d)}.$$

§1.2. [0001]. Let A_k and B_k be the two sequences defined respectively by

$$A_k = \frac{(1+a+b-d)_k}{(c)_k} \quad \text{and} \quad B_k = \left[\begin{matrix} a, b \\ a+b-d, d \end{matrix} \right]_k.$$

We have no difficulty showing the differences

$$\begin{aligned} \nabla A_k &= \frac{(a+b-d)_k}{(c)_k} \frac{1+a+b-c-d}{a+b-d}, \\ \Delta B_k &= \left[\begin{matrix} a, b \\ 1+a+b-d, 1+d \end{matrix} \right]_k \frac{(d-a)(d-b)}{d(d-a-b)}, \end{aligned}$$

as well as the two limiting relations

$$[AB]_+ = 0 \quad \text{and} \quad A_{-1}B_0 = \frac{c-1}{a+b-d} \quad \text{with} \quad \Re(c+d-a-b) > 1.$$

According to the modified Abel lemma on summation by parts, the Ω -series can be reformulated as

$$\begin{aligned} \Omega(a, b, c, d) &= \frac{a+b-d}{1+a+b-c-d} \sum_k B_k \nabla A_k \\ &= \frac{a+b-d}{1+a+b-c-d} \left\{ [AB]_+ - A_{-1}B_0 + \sum_k A_k \Delta B_k \right\} \\ &= \frac{c-1}{c+d-a-b-1} + \frac{(d-a)(d-b)}{d(c+d-a-b-1)} \sum_{k=0}^{\infty} \left[\begin{matrix} a, b \\ c, 1+d \end{matrix} \right]_k \end{aligned}$$

which yields the following recurrence relation.

Lemma 1.2 (Recurrence relation [0001]).

$$\Omega(a, b, c, d) = \Omega(a, b, c, 1+d) \times \frac{(d-a)(d-b)}{d(c+d-a-b-1)} + \frac{c-1}{c+d-a-b-1}.$$

§1.3. [1001]. For the two sequences A_k and B_k given respectively by

$$A_k = \frac{(1+a)_k}{(c)_k} \quad \text{and} \quad B_k = \frac{(b)_k}{(d)_k}$$

it is not hard to verify the differences

$$\begin{aligned} \nabla A_k &= \frac{(a)_k}{(c)_k} \frac{1+a-c}{a}, \\ \Delta B_k &= \frac{(b)_k}{(1+d)_k} \frac{d-b}{d}, \end{aligned}$$

as well as two limiting relations

$$[AB]_+ = 0 \quad \text{and} \quad A_{-1}B_0 = \frac{c-1}{a} \quad \text{with} \quad \Re(c+d-a-b) > 1.$$

In view of the modified Abel lemma on summation by parts, we can manipulate the Ω -series as follows:

$$\begin{aligned} \Omega(a, b, c, d) &= \frac{a}{1+a-c} \sum_k B_k \nabla A_k \\ &= \frac{a}{1+a-c} \left\{ [AB]_+ - A_{-1}B_0 + \sum_k A_k \Delta B_k \right\} \\ &= \frac{1-c}{1+a-c} + \frac{a(d-b)}{d(1+a-c)} \sum_{k=0}^{\infty} \left[\begin{matrix} 1+a, & b \\ c, & 1+d \end{matrix} \right]_k. \end{aligned}$$

This leads us to the following recurrence relation, which can also be derived by combining Lemma 1.1 with Lemma 1.2.

Lemma 1.3 (Recurrence relation [1001]).

$$\Omega(a, b, c, d) = \Omega(1+a, b, c, 1+d) \times \frac{a(d-b)}{d(1+a-c)} + \frac{1-c}{1+a-c}.$$

Iterating m -times the equation displayed in Lemma 1.3, we get the relation

$$(5a) \quad \Omega(a, b, c, d) = \Omega(a+m, b, c, d+m) \left[\begin{matrix} a, d-b \\ d, 1+a-c \end{matrix} \right]_m$$

$$(5b) \quad + \frac{1-c}{1+a-c} \sum_{k=0}^{m-1} \left[\begin{matrix} a, d-b \\ d, 2+a-c \end{matrix} \right]_k.$$

When $\Re(1+b-c) > 0$, the limit of the last sum exists as $m \rightarrow \infty$. In order to evaluate the limit of (5a), recall the Kummer–Thomae–Whipple transformation (cf. Bailey [5, §3.2])

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] &= {}_3F_2 \left[\begin{matrix} b-a, d-a, b+d-a-c-e \\ b+d-a-c, b+d-a-e \end{matrix} \middle| 1 \right] \\ &\quad \times \Gamma \left[\begin{matrix} b, & d, & b+d-a-c-e \\ a, & b+d-a-c, & b+d-a-e \end{matrix} \right]. \end{aligned}$$

By rewriting $\Omega(a+m, b, c, d+m)$ in terms of the ${}_3F_2$ -series, we may transform it into a convergent series

$$\begin{aligned} \Omega(a+m, b, c, d+m) &= {}_3F_2 \left[\begin{matrix} 1, & b, & a+m \\ c, & d+m \end{matrix} \middle| 1 \right] \\ &= {}_3F_2 \left[\begin{matrix} c-1, c+d-a-b-1, d-1+m \\ c+d-a-1, c+d-b-1+m \end{matrix} \middle| 1 \right] \\ &\quad \times \Gamma \left[\begin{matrix} c, c+d-a-b-1, d+m \\ c+d-a-1, c+d-b-1+m \end{matrix} \right]. \end{aligned}$$

Hence we can express the limit of (5a) as follows:

$$\begin{aligned} \lim_{m \rightarrow \infty} \Omega(a+m, b, c, d+m) \left[\begin{matrix} a, d-b \\ d, 1+a-c \end{matrix} \right]_m &= \Gamma \left[\begin{matrix} c, c+d-a-b-1, d \\ c+d-a-1, c+d-b-1 \end{matrix} \right] \\ \times \lim_{m \rightarrow \infty} {}_3F_2 \left[\begin{matrix} c-1, c+d-a-b-1, d-1+m \\ c+d-a-1, c+d-b-1+m \end{matrix} \middle| 1 \right] &\times \left[\begin{matrix} a, & d-b \\ 1+a-c, & c+d-b-1 \end{matrix} \right]_m. \end{aligned}$$

First, it is not hard to check that

$$\lim_{m \rightarrow \infty} \left[\begin{matrix} a, & d-b \\ 1+a-c, & c+d-b-1 \end{matrix} \right]_m = \Gamma \left[\begin{matrix} 1+a-c, & c+d-b-1 \\ a, & d-b \end{matrix} \right].$$

Then the limit of the last ${}_3F_2$ -series can be evaluated, under $\Re(1+b-c) > 0$, by the Gauss summation theorem

$$\begin{aligned} \lim_{m \rightarrow \infty} {}_3F_2 \left[\begin{matrix} c-1, c+d-a-b-1, d-1+m \\ c+d-a-1, c+d-b-1+m \end{matrix} \middle| 1 \right] &= {}_2F_1 \left[\begin{matrix} c-1, c+d-a-b-1 \\ c+d-a-1 \end{matrix} \middle| 1 \right] \\ &= \Gamma \left[\begin{matrix} c+d-a-1, 1+b-c \\ b, & d-a \end{matrix} \right]. \end{aligned}$$

Summing up, we have proved the following limiting relation:

$$\begin{aligned} \lim_{m \rightarrow \infty} \Omega(a+m, b, c, d+m) &\left[\begin{matrix} a, d-b \\ d, 1+a-c \end{matrix} \right]_m \\ &= \Gamma \left[\begin{matrix} c, d, 1+a-c, 1+b-c, c+d-a-b-1 \\ a, & b, & d-a, & d-b \end{matrix} \right]. \end{aligned}$$

Finally, letting $m \rightarrow \infty$ in (5a-5b) and then relabeling the parameters by $a \rightarrow a+c$, $c \rightarrow c+1$, $d \rightarrow b+d$, we may reformulate the resulting equation as the following reciprocal relation.

Theorem 1.4 (Reciprocal relation).

$$\sum_{k=0}^{\infty} \frac{(a+c)_k}{(b+d)_k} \left\{ \frac{(b)_k}{(c)_{k+1}} + \frac{(d)_k}{(a)_{k+1}} \right\} = \Gamma \left[\begin{matrix} a, c, b+d, b-c, d-a \\ a+c, b, d, b+d-a-c \end{matrix} \right].$$

This can be considered as a four-parameter extension of the Gauss summation theorem, because the latter can be deduced from it through multiplying across by c and then letting $c \rightarrow 0$. Furthermore, letting $a \rightarrow \frac{1}{2}-x$, $b \rightarrow \frac{1}{2}$, $c \rightarrow x$, $d \rightarrow \frac{1}{2}+x$ in Theorem 1.4, we recover the following interesting identity discovered by Guillera through the *WZ*-method.

Proposition 1.5 (Guillera [29, Identity 1: Equation 2]).

$$2x \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ 1+x, & 1+x \end{matrix} \right]_k = \frac{4^x \sqrt{\pi}}{\cos^2(\pi x)} \frac{\Gamma^3(1+x)}{\Gamma^3(\frac{1}{2}+x)} + \frac{4x^2}{2x-1} \sum_{n=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}+x \\ 1+x, & \frac{3}{2}-x \end{matrix} \right]_n.$$

§1.4. [1101]. Define the two sequences A_k and B_k , respectively, by

$$A_k = \left[\begin{matrix} 1+a, 1+b \\ c, 2+a+b-c \end{matrix} \right]_k \quad \text{and} \quad B_k = \frac{(2+a+b-c)_k}{(d)_k}.$$

We can easily prove the differences

$$\begin{aligned} \nabla A_k &= \left[\begin{matrix} a, b \\ c, 2+a+b-c \end{matrix} \right]_k \frac{(1+a-c)(1+b-c)}{ab}, \\ \Delta B_k &= \frac{(2+a+b-c)_k}{(1+d)_k} \frac{c+d-a-b-2}{d}, \end{aligned}$$

as well as two limiting relations

$$[AB]_+ = 0 \quad \text{and} \quad A_{-1}B_0 = \frac{(c-1)(1+a+b-c)}{ab} \quad \text{with} \quad \Re(c+d-a-b) > 2.$$

Taking into account the modified Abel lemma on summation by parts, we can reformulate the Ω -series as

$$\begin{aligned} \Omega(a, b, c, d) &= \frac{ab}{(1+a-c)(1+b-c)} \sum_k B_k \nabla A_k \\ &= \frac{ab}{(1+a-c)(1+b-c)} \left\{ [AB]_+ - A_{-1}B_0 + \sum_k A_k \Delta B_k \right\} \\ &= \frac{(1-c)(1+a+b-c)}{(1+a-c)(1+b-c)} + \frac{ab(c+d-a-b-2)}{d(1+a-c)(1+b-c)} \sum_{k=0}^{\infty} \left[\begin{matrix} 1+a, & 1+b \\ & c, & 1+d \end{matrix} \right]_k. \end{aligned}$$

This reads as the following recurrence relation, which can also be derived by combining Lemma 1.1 with Lemma 1.3.

Lemma 1.6 (Recurrence relation [1101]).

$$\Omega(a, b, c, d) = \Omega(1+a, 1+b, c, 1+d) \times \frac{ab(c+d-a-b-2)}{d(1+a-c)(1+b-c)} + \frac{(1-c)(1+a+b-c)}{(1+a-c)(1+b-c)}.$$

§1.5. [1011]. For the two sequences A_k and B_k given, respectively, by

$$A_k = \frac{(1+a)_k}{(c+d-b)_k} \quad \text{and} \quad B_k = \left[\begin{matrix} b, c+d-b \\ & c, d \end{matrix} \right]_k,$$

it is not difficult to check the differences

$$\begin{aligned} \nabla A_k &= \frac{(a)_k}{(c+d-b)_k} \frac{1+a+b-c-d}{a}, \\ \Delta B_k &= \left[\begin{matrix} b, c+d-b \\ 1+c, 1+d \end{matrix} \right]_k \frac{(b-c)(b-d)}{cd}, \end{aligned}$$

as well as two limiting relations

$$[AB]_+ = 0 \quad \text{and} \quad A_{-1}B_0 = \frac{c+d-b-1}{a} \quad \text{with} \quad \Re(c+d-a-b) > 1.$$

Then the modified Abel lemma on summation by parts allows us to manipulate the Ω -series as follows:

$$\begin{aligned} \Omega(a, b, c, d) &= \frac{a}{1+a+b-c-d} \sum_k B_k \nabla A_k \\ &= \frac{a}{1+a+b-c-d} \left\{ [AB]_+ - A_{-1}B_0 + \sum_k A_k \Delta B_k \right\} \\ &= \frac{1+b-c-d}{1+a+b-c-d} + \frac{a(b-c)(b-d)}{cd(1+a+b-c-d)} \sum_{k=0}^{\infty} \left[\begin{matrix} 1+a, & b \\ 1+c, & 1+d \end{matrix} \right]_k. \end{aligned}$$

This is equivalent to the following recurrence relation, which can also be derived by combining Lemma 1.2 with Lemma 1.3.

Lemma 1.7 (Recurrence relation [1011]).

$$\Omega(a, b, c, d) = \Omega(1+a, b, 1+c, 1+d) \times \frac{a(b-c)(b-d)}{cd(1+a+b-c-d)} + \frac{1+b-c-d}{1+a+b-c-d}.$$

2. FAST CONVERGENT SERIES AND π -FORMULAE

Applying the five recurrence relations established in the last section, we shall derive five infinite series expressions for $\Omega(a, b, c, d)$ with four of them being fast convergent ones. They will lead to numerous infinite series formulae for π .

§2.1. [1011]. Iterating m -times the equation displayed in Lemma 1.7, we get

$$\begin{aligned}\Omega(a, b, c, d) &= \Omega(1 + a, b, 1 + c, 1 + d) \times \frac{a(b-c)(b-d)}{cd(1+a+b-c-d)} + \frac{1+b-c-d}{1+a+b-c-d} \\ &= (-1)^m \Omega(a+m, b, c+m, d+m) \frac{(a)_m (c-b)_m (d-b)_m}{(c)_m (d)_m (c+d-a-b-1)_m} \\ &\quad + \sum_{k=0}^{m-1} (-1)^k \frac{(a)_k (c-b)_k (d-b)_k}{(c)_k (d)_k (c+d-a-b-1)_k} \frac{c+d-b-1+2k}{c+d-a-b-1+k}.\end{aligned}$$

By means of the Weierstrass M -test on uniformly convergent series (cf. Stromberg [34, Page 141]), the limiting case $m \rightarrow \infty$ of the last equation subject to $\Re(c+d-2a+b) > 1$ leads to the following transformation into well-poised series.

Theorem 2.1 (Transformation formula: $\Re(c+d) > 1 + \max\{\Re(a+b), \Re(2a-b)\}$).

$$\Omega(a, b, c, d) = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c-b)_k (d-b)_k}{(c)_k (d)_k (c+d-a-b)_k} \frac{c+d-b-1+2k}{c+d-a-b-1}.$$

When $d = 1$, evaluating $\Omega(a, b, c, d)$ by the Gauss summation formula and then making the parameter replacements $a \rightarrow C$, $b \rightarrow 1 - B$ and $c \rightarrow 1 + A - B$, we derive from the last theorem the following expression.

Proposition 2.2 (Infinite series for Γ -function quotient: $\Re(1 + A - 2B - 2C) > 0$).

$$\Gamma \left[\begin{matrix} 1 + A - B, 1 + A - C \\ A, 1 + A - B - C \end{matrix} \right] = \sum_{k=0}^{\infty} (-1)^k \frac{(A+2k)(A)_k (B)_k (C)_k}{k!(1+A-B)_k (1+A-C)_k}.$$

This is equivalent to the limiting case $D \rightarrow -\infty$ of Dougall's well-known formula (cf. Bailey [5, §4.4] and Slater [33, §2.4])

Proposition 2.3 (Dougall [21, 1907]: $\Re(1 + A - B - C - D) > 0$).

$$\begin{aligned}{}_5F_4 &\left[\begin{matrix} A, 1 + A/2, B, C, D \\ A/2, 1 + A - B, 1 + A - C, 1 + A - D \end{matrix} \middle| 1 \right] \\ &= \Gamma \left[\begin{matrix} 1 + A - B, 1 + A - C, 1 + A - D, 1 + A - B - C - D \\ 1 + A, 1 + A - B - C, 1 + A - B - D, 1 + A - C - D \end{matrix} \right].\end{aligned}$$

Corollary 2.4 ($A = B = C = x$ in Proposition 2.2).

$$\frac{\sin \pi x}{\pi x} = \sum_{k=0}^{\infty} (-1)^k \left[\begin{matrix} x, x, x \\ 1, 1, 1 \end{matrix} \right]_k \frac{x+2k}{x}.$$

Corollary 2.5 ($A = 0$ and $B = -C = x$ in Proposition 2.2).

$$\frac{\pi x}{\sin \pi x} = 1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^k x^2}{x^2 - k^2}.$$

Two particular identities may be worth mentioning. The first one was discovered by Bauer in 1859, which can also be found in Glaisher [22, §25] and Hardy [30, Equation 2].

Example 1 ($x = 1/2$ in Corollary 2.4: Bauer [9, §4]).

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} (-1)^k \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \right]_k \{1 + 4k\}.$$

The next one was established by Glaisher in 1905 in a less well-known paper.

Example 2 ($A = B = C = D = -1/2$ in Proposition 2.3: Glaisher [22, §33]).

$$\frac{8}{\pi^2} = \sum_{k=0}^{\infty} \left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, 1, 1, 1 \end{matrix} \right]_k \{1 - 4k\}.$$

Further series for $1/\pi$ and $1/\pi^2$ with convergence rate equal to one can be found in Glaisher [22], who made a systematic investigation on π -formulae through elliptic functions before Ramanujan, even though all of the 30 principal series for $1/\pi$ and $1/\pi^2$ collected by him can be derived from Dougall's ${}_5F_4$ -series published in 1907, two years after Glaisher's paper.

§2.2. [0011]. Applying Lemma 1.2 to $\Omega(a, b, 1 + d, c)$, we have

$$\Omega(a, b, c, 1 + d) = \Omega(a, b, 1 + c, 1 + d) \frac{(c-a)(c-b)}{c(c+d-a-b)} + \frac{d}{c+d-a-b}.$$

Substituting this into the equation displayed in Lemma 1.2

$$\Omega(a, b, c, d) = \Omega(a, b, c, 1 + d) \frac{(d-a)(d-b)}{d(c+d-a-b-1)} + \frac{c-1}{c+d-a-b-1}$$

we can simplify the result to the following recurrence relation.

Lemma 2.6 (Recurrence relation [0011]).

$$\begin{aligned} \Omega(a, b, c, d) = & \Omega(a, b, 1 + c, 1 + d) \frac{(c-a)(c-b)(d-a)(d-b)}{cd(c+d-a-b-1)_2} \\ & + \frac{c-1}{c+d-a-b-1} + \frac{(d-a)(d-b)}{(c+d-a-b-1)_2}. \end{aligned}$$

Iterating this relation m -times, we get

$$\begin{aligned} \Omega(a, b, c, d) = & \Omega(a, b, c + m, d + m) \frac{(c-a)_m (c-b)_m (d-a)_m (d-b)_m}{(c)_m (d)_m (c+d-a-b-1)_{2m}} \\ & + \sum_{k=0}^{m-1} \alpha_k(a, b, c, d) \frac{(c-a)_k (c-b)_k (d-a)_k (d-b)_k}{(c)_k (d)_k (c+d-a-b-1)_{2+2k}} \end{aligned}$$

where $\alpha_k(a, b, c, d)$ is the quadratic polynomial in k given by

$$(6) \quad \alpha_k(a, b, c, d) = (c-1+k)(c+d-a-b+2k) + (d-a+k)(d-b+k).$$

By means of the Weierstrass M -test on uniformly convergent series, the limiting case $m \rightarrow \infty$ of the last equation leads to the transformation.

Theorem 2.7 (Transformation formula [0011]).

$$\Omega(a, b, c, d) = \sum_{k=0}^{\infty} \alpha_k(a, b, c, d) \frac{(c-a)_k (c-b)_k (d-a)_k (d-b)_k}{(c)_k (d)_k (c+d-a-b-1)_{2+2k}}.$$

When $a = b = \frac{1}{2}$ and $c = d = 1 + x$, the last theorem recovers the following interesting identity discovered by Guillera through the WZ -method.

Proposition 2.8 (Guillera [29, Identity 1: Equation 1]).

$$8x \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ 1+x, & 1+x \end{matrix} \right]_k = \sum_{n=0}^{\infty} \left[\begin{matrix} \frac{1}{2} + x, & \frac{1}{2} + x, & \frac{1}{2} + x \\ 1+x, & 1+x, & 1+x \end{matrix} \right]_n \frac{1+6x+6n}{4^n}.$$

Instead, for $d = 1$, we can evaluate $\Omega(a, b, c, d)$ by the Gauss summation formula and derive from the last theorem the following expression.

Proposition 2.9 (Infinite series for Γ -function quotient).

$$\begin{aligned} \Gamma \left[\begin{matrix} 1+c, 1+c-a-b \\ 1+c-a, 1+c-b \end{matrix} \right] &= \sum_{k=0}^{\infty} \frac{[1-a, 1-b, c-a, c-b]_k}{[1, c]_k (2+c-a-b)_{2k}} \\ &\times \frac{c\{3k^2 + k(1+3c-2a-2b) + (c-a)(c-b)\}}{(c-a)(c-b)(1+c-a-b)} \\ &= \frac{c}{1+c-a-b} + \sum_{k=1}^{\infty} \frac{[1-a, 1-b, 1+c-a, 1+c-b]_k}{[1, c]_k (2+c-a-b)_{2k}} \\ &\times \frac{c\{3k^2 + k(1+3c-2a-2b) + (c-a)(c-b)\}}{(c-a+k)(c-b+k)(1+c-a-b)}. \end{aligned}$$

Three infinite series with a free variable x can be derived from this proposition.

Corollary 2.10 ($a + b = c = 1$ in Proposition 2.9).

$$\frac{\sin \pi x}{\pi} = \sum_{k=0}^{\infty} \frac{[x, x, 1-x, 1-x]_k}{[1, 1]_k (2)_{2k}} \{3k^2 + 2k + x(1-x)\}.$$

Corollary 2.11 ($a = b = x$ and $c = 1 + x$ in Proposition 2.9).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \frac{[1, 1-x, 1-x]_k}{(1+x)_k (3-x)_{2k}} \frac{1+4k+3k^2-kx}{(1-x)(2-x)}.$$

Corollary 2.12 ($a = b = c = 1 - x$ in Proposition 2.9).

$$\frac{\pi x}{\sin \pi x} = 1 + \sum_{k=1}^{\infty} \frac{[1, x, x]_k}{(1+x)_{2k} (1-x)_k} \frac{x+3k}{k}.$$

Letting $x = 1/2$ in Corollary 2.10, we recover immediately the identity (1) discovered by Ramanujan [31, Equation 28] (cf. Berndt [10, Equation 20.1, Page 352] also). This is only the tip of the iceberg, which represents a large class of infinite series with convergence rate equal to $1/4$. They are displayed as examples.

Example 3 ($x = 1/2$ in Corollary 2.11).

$$\frac{3\pi}{4} = \sum_{k=0}^{\infty} \left[\begin{matrix} 1, & \frac{1}{2} \\ \frac{5}{4}, & \frac{7}{4} \end{matrix} \right]_k \frac{2+3k}{4^k}.$$

Example 4 ($x = 3/4$ in Corollary 2.12).

$$\frac{\pi}{2\sqrt{2}} = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{4^k} \left[1, \frac{3}{4}, \frac{3}{4} \right]_k \frac{1+4k}{4k}.$$

Example 5 ($x = 1/2$ in Corollary 2.12).

$$\frac{\pi}{2} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} \left[1, \frac{1}{2} \right]_k \frac{1+6k}{2k}.$$

Example 6 ($x = 1/3$ in Corollary 2.12).

$$\frac{2\pi}{3\sqrt{3}} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} \left[1, \frac{1}{3}, \frac{1}{3} \right]_k \frac{1+9k}{3k}.$$

Example 7 ($x = 2/3$ in Corollary 2.12).

$$\frac{4\pi}{3\sqrt{3}} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} \left[1, \frac{2}{3}, \frac{2}{3} \right]_k \frac{2+9k}{3k}.$$

Example 8 ($x = 1/4$ in Corollary 2.12).

$$\frac{\pi}{2\sqrt{2}} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} \left[1, \frac{1}{4}, \frac{1}{4} \right]_k \frac{1+12k}{4k}.$$

Example 9 ($x = 1/3$ in Corollary 2.10).

$$\frac{9\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \left[\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right]_k \frac{2+18k+27k^2}{4^k}.$$

Example 10 ($x = 2/3$ in Corollary 2.11).

$$\frac{16\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left[1, \frac{1}{3}, \frac{1}{3} \right]_k \frac{3+10k+9k^2}{4^k}.$$

Example 11 ($x = 1/3$ in Corollary 2.11).

$$\frac{20\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left[1, \frac{2}{3}, \frac{2}{3} \right]_k \frac{3+11k+9k^2}{4^k}.$$

Example 12 ($x = 1/4$ in Corollary 2.10).

$$\frac{8\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \left[\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \right]_k \frac{3+32k+48k^2}{4^k}.$$

Example 13 ($x = 3/4$ in Corollary 2.11).

$$\frac{15\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \left[1, \frac{1}{4}, \frac{1}{4} \right]_k \frac{4+13k+12k^2}{4^k}.$$

Example 14 ($x = 1/4$ in Corollary 2.11).

$$\frac{21\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \left[1, \frac{3}{4}, \frac{3}{4} \right]_k \frac{4+15k+12k^2}{4^k}.$$

Example 15 ($x = 1/6$ in Corollary 2.10).

$$\frac{18}{\pi} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_k \frac{5 + 72k + 108k^2}{4^k}.$$

§2.3. [1012]. Applying Lemma 1.2 to $\Omega(1+a, b, 1+c, 1+d)$ gives

$$\Omega(1+a, b, 1+c, 1+d) = \Omega(1+a, b, 1+c, 2+d) \frac{(d-a)(1+d-b)}{(1+d)(c+d-a-b)} + \frac{c}{c+d-a-b}.$$

Then substituting this into the equation displayed in Lemma 1.7, we have

$$\begin{aligned} \Omega(a, b, c, d) &= \Omega(1+a, b, 1+c, 1+d) \frac{a(b-c)(b-d)}{cd(1+a+b-c-d)} + \frac{1+b-c-d}{1+a+b-c-d} \\ &= \Omega(1+a, b, 1+c, 2+d) \frac{a(b-c)(d-a)(d-b)_2}{c(d)_2(c+d-a-b-1)_2} \\ &\quad + \frac{c+d-b-1}{c+d-a-b-1} - \frac{a(c-b)(d-b)}{d(c+d-a-b-1)_2}. \end{aligned}$$

Iterating this relation m -times results in the equality

$$\begin{aligned} \Omega(a, b, c, d) &= (-1)^m \Omega(a+m, b, c+m, d+2m) \frac{(a)_m (c-b)_m (d-a)_m (d-b)_{2m}}{(c)_m (d)_{2m} (c+d-a-b-1)_{2m}} \\ &\quad + \sum_{k=0}^{m-1} (-1)^k \frac{(a)_k (c-b)_k (d-a)_k (d-b)_{2k}}{(c)_k (d)_{1+2k} (c+d-a-b-1)_{2+2k}} \beta_k(a, b, c, d) \end{aligned}$$

where $\beta_k(a, b, c, d)$ is the cubic polynomial of k given by

$$(7a) \quad \beta_k(a, b, c, d) = (c+d-b-1+3k)(c+d-a-b+2k)(d+2k)$$

$$(7b) \quad - (a+k)(c-b+k)(d-b+2k).$$

By means of the Weierstrass M -test on uniformly convergent series, the limiting case $m \rightarrow \infty$ of the last equation subject to $\Re(c+d-a-b) > 1$ leads to the transformation.

Theorem 2.13 (Transformation formula [1012]).

$$\Omega(a, b, c, d) = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c-b)_k (d-a)_k (d-b)_{2k}}{(c)_k (d)_{1+2k} (c+d-a-b-1)_{2+2k}} \beta_k(a, b, c, d).$$

When $a = \frac{1}{2} + x$, $b = \frac{1}{2}$, $c = 1 + x$ and $d = 1 + 2x$, the last theorem recovers the following interesting identity discovered recently by Guillera.

Proposition 2.14 (Guillera [29, Identity 4: Equation 7]).

$$16x \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, \frac{1}{2} + x \\ 1 + x, 1 + 2x \end{matrix} \right]_k = \sum_{n=0}^{\infty} \{3 + 20x + 20n\} \left[\begin{matrix} \frac{1}{2} + x, \frac{1}{4} + x, \frac{3}{4} + x \\ 1 + x, 1 + x, 1 + x \end{matrix} \right]_n \left(\frac{-1}{4} \right)^n.$$

When $c = 1$, we can evaluate $\Omega(a, b, c, d)$ by the Gauss summation formula and obtain from Theorem 2.13 the following proposition.

Proposition 2.15 (Infinite series for Γ -function quotient).

$$\Gamma \left[\begin{matrix} 1+d, 2+d-a-b \\ d-a, d-b \end{matrix} \right] = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (1-b)_k (d-a)_k (d-b)_{2k}}{k! (1+d)_{2k} (2+d-a-b)_{2k}} \beta_k(a, b, 1, d).$$

Corollary 2.16 ($a = b = x$ and $d = 1 + x$ in Proposition 2.15).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} 1, & \frac{1}{2}, & x, & 1-x \\ \frac{2+x}{2}, & \frac{3+x}{2}, & \frac{3-x}{2}, & \frac{4-x}{2} \end{matrix} \right]_k \frac{2+11k+19k^2+10k^3+kx-kx^2}{(1+x)(1-x)(2-x)}.$$

In Proposition 2.15, splitting the sum with respect to k into two according to the two terms of $\beta_k(a, b, 1, d)$, then keeping the first sum invariant and replacing the summation index by $k \rightarrow k - 1$ for the second sum, we may reformulate the resulting equation as another proposition.

Proposition 2.17 (Infinite series for Γ -function quotient).

$$\Gamma \left[\begin{matrix} d, 1+d-a-b \\ d-a, d-b \end{matrix} \right] = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(1-b)_k(d-a)_k(d-b)_{2k}}{k!(d)_{2k}(1+d-a-b)_{2k}} \times \left\{ (d-b+3k) + \frac{k(d-1+2k)(d-a-b+2k)}{(d-a-1+k)(d-b-1+2k)} \right\}.$$

Corollary 2.18 ($a + b = d = 1$ in Proposition 2.17).

$$\frac{\sin \pi x}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} x, x, -x, \frac{x}{2}, \frac{x-1}{2} \\ 1, 1, 1, \frac{1}{2}, \frac{1}{2} \end{matrix} \right]_k \frac{10k^3-3k^2+2kx-k^2x+x^2-4kx^2-x^3}{x(1-x)}.$$

Corollary 2.19 ($a = b = x$ and $d = 1 + x$ in Proposition 2.17).

$$\frac{\pi x}{\sin \pi x} = \frac{1}{1-x} + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} 1, & \frac{1}{2}, & x, & 1-x \\ \frac{1+x}{2}, & \frac{2+x}{2}, & \frac{2-x}{2}, & \frac{3-x}{2} \end{matrix} \right]_k \frac{x-x^2+4k+10k^2}{2k(1-x)}.$$

Furthermore, for $d = 1$, we can evaluate $\Omega(a, b, c, d)$ by the Gauss summation formula and derive from Theorem 2.13 the following expression.

Proposition 2.20 (Infinite series for Γ -function quotient).

$$\Gamma \left[\begin{matrix} c, 2+c-a-b \\ c-a, c-b \end{matrix} \right] = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k(1-a)_k(c-b)_k(1-b)_{2k}}{(c)_k(2)_{2k}(2+c-a-b)_{2k}} \beta_k(a, b, c, 1).$$

Corollary 2.21 ($a + b = c = 1$ in Proposition 2.20).

$$\frac{\sin \pi x}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} x, x, 1-x, \frac{x}{2}, \frac{1+x}{2} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{matrix} \right]_k \left\{ (1+2k)^2(x+3k) - (x+k)^2(x+2k) \right\}.$$

Corollary 2.22 ($a = b = x$ and $c = 1 + x$ in Proposition 2.20).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} x, 1-x, \frac{1-x}{2}, \frac{2-x}{2} \\ \frac{3}{2}, 1+x, \frac{3-x}{2}, \frac{4-x}{2} \end{matrix} \right]_k \frac{2+11k+19k^2+10k^3-2x-7kx-7k^2x+x^2+kx^2}{(1-x)(2-x)}.$$

When $x = 1/2$ in Corollary 2.21, we get the identity (2) found by Ramanujan [31, Equation 35] (cf. Berndt [10, Equation 21.5, Page 353] also). From the following examples, it can be seen that (2) is the simplest one among many infinite series with convergence rate equal to $-1/4$.

Example 16 ($x = 1/3$ in Corollary 2.16).

$$\frac{40\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} 1, & \frac{1}{2} \\ \frac{7}{6}, & \frac{11}{6} \end{matrix} \right]_k \{9 + 10k\}.$$

Example 17 ($x = 1/3$ in Corollary 2.19).

$$\frac{2\pi}{3\sqrt{3}} = \frac{3}{2} + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} 1, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \right]_k \frac{1+15k}{6k}.$$

Example 18 ($x = 1/2$ in Corollary 2.19).

$$\frac{\pi}{2} = 2 + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{5}{4} \end{matrix} \right]_k \frac{1+16k+40k^2}{4k}.$$

Example 19 ($x = 1/4$ in Corollary 2.19).

$$\frac{\pi}{2\sqrt{2}} = \frac{4}{3} + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{5}{8}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8} \end{matrix} \right]_k \frac{3+64k+160k^2}{24k}.$$

Example 20 ($x = 1/6$ in Corollary 2.19).

$$\frac{\pi}{3} = \frac{6}{5} + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{6}, \frac{5}{6} \\ \frac{7}{12}, \frac{11}{12}, \frac{13}{12}, \frac{17}{12} \end{matrix} \right]_k \frac{5+144k+360k^2}{60k}.$$

Example 21 ($x = 2/3$ in Corollary 2.22).

$$\frac{8\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{6} \\ \frac{3}{2}, \frac{5}{3}, \frac{7}{6} \end{matrix} \right]_k \{5+23k+30k^2\}.$$

Example 22 ($x = 1/3$ in Corollary 2.22).

$$\frac{20\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{5}{6} \\ \frac{3}{2}, \frac{4}{3}, \frac{11}{6} \end{matrix} \right]_k \{13+40k+30k^2\}.$$

Example 23 ($x = 1/2$ in Corollary 2.22).

$$\frac{3\pi}{2} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{3}{2}, \frac{5}{4}, \frac{7}{4} \end{matrix} \right]_k \{5+21k+20k^2\}.$$

This is equivalent to the series of BBP-type due to Adamchik and Wagon [1]

$$\pi = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left\{ \frac{2}{1+4k} + \frac{2}{2+4k} + \frac{1}{3+4k} \right\}.$$

Further BBP-type series for π can be found, for example, in [1, 3, 4, 14, 23, 28].

Example 24 ($x = 1/2$ in Corollary 2.18).

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{4}, -\frac{1}{2}, -\frac{1}{4} \\ 1, 1, 1 \end{matrix} \right]_k \{1-28k^2+80k^3\}.$$

Example 25 ($x = 1/3$ in Corollary 2.18).

$$\frac{3\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{6} \\ 1, 1, 1, \frac{1}{2}, \frac{1}{2} \end{matrix} \right]_k \{1+3k-45k^2+135k^3\}.$$

Example 26 ($x = 2/3$ in Corollary 2.18).

$$\frac{3\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{2}{3}, -\frac{1}{6} \\ 1, 1, 1, \frac{1}{2}, \frac{1}{2} \end{matrix} \right]_k \{4-12k-99k^2+270k^3\}.$$

Example 27 ($x = 1/3$ in Corollary 2.21).

$$\frac{27\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{6} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{matrix} \right]_k \{8 + 105k + 315k^2 + 270k^3\}.$$

Example 28 ($x = 2/3$ in Corollary 2.21).

$$\frac{27\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{matrix} \right]_k \{10 + 105k + 306k^2 + 270k^3\}.$$

Example 29 ($x = 1/4$ in Corollary 2.21).

$$\frac{32\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{matrix} \right]_k \{15 + 240k + 752k^2 + 640k^3\}.$$

Example 30 ($x = 3/4$ in Corollary 2.21).

$$\frac{32\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left[\begin{matrix} \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2} \end{matrix} \right]_k \{21 + 240k + 720k^2 + 640k^3\}.$$

§2.4. [1022]. According to Lemma 1.7, the following equality holds:

$$\begin{aligned} \Omega(a, b, 1 + c, 1 + d) &= \Omega(1 + a, b, 2 + c, 2 + d) \frac{a(1 + c - b)(1 + d - b)}{(1 + c)(1 + d)(a + b - c - d - 1)} \\ &\quad + \frac{1 + c + d - b}{1 + c + d - a - b}. \end{aligned}$$

Substituting this into the equation displayed in Lemma 2.6

$$\begin{aligned} \Omega(a, b, c, d) &= \Omega(a, b, 1 + c, 1 + d) \frac{(c - a)(c - b)(d - a)(d - b)}{cd(c + d - a - b - 1)_2} \\ &\quad + \frac{c - 1}{c + d - a - b - 1} + \frac{(d - a)(d - b)}{(c + d - a - b - 1)_2} \end{aligned}$$

and then simplifying the result, we get the relation

$$\begin{aligned} \Omega(a, b, c, d) &= \Omega(1 + a, b, 2 + c, 2 + d) \frac{a(c - a)(d - a)(c - b)_2(d - b)_2}{(c)_2(d)_2(a + b - c - d - 1)_3} \\ &\quad + \frac{(c - a)(c - b)(d - a)(d - b)(1 + c + d - b)}{cd(c + d - a - b - 1)_3} \\ &\quad + \frac{c - 1}{c + d - a - b - 1} + \frac{(d - a)(d - b)}{(c + d - a - b - 1)_2}. \end{aligned}$$

Iterating this relation m -times yields the expression

$$\begin{aligned} \Omega(a, b, c, d) &= (-1)^m \Omega(a + m, b, c + 2m, d + 2m) \frac{(a)_m(c - a)_m(d - a)_m(c - b)_{2m}(d - b)_{2m}}{(c)_{2m}(d)_{2m}(c + d - a - b - 1)_{3m}} \\ &\quad + \sum_{k=0}^{m-1} (-1)^k \frac{(a)_k(c - a)_k(d - a)_k(c - b)_{2k}(d - b)_{2k}}{(c)_{1+2k}(d)_{1+2k}(c + d - a - b - 1)_{3+3k}} \gamma_k(a, b, c, d) \end{aligned}$$

where $\gamma_k(a, b, c, d)$ is the quintic polynomial of k given by

(8a) $\gamma_k(a, b, c, d) = (c - a + k)(d - a + k)(c - b + 2k)(d - b + 2k)(1 + c + d - b + 4k)$

(8b) $\quad + (c + 2k)(c - 1 + 2k)(d + 2k)(c + d - a - b + 3k)(1 + c + d - a - b + 3k)$

(8c) $\quad + (c + 2k)(d + 2k)(d - a + k)(d - b + 2k)(1 + c + d - a - b + 3k).$

By means of the Weierstrass M -test on uniformly convergent series, the limiting case $m \rightarrow \infty$ of the last equation leads to the transformation.

Theorem 2.23 (Transformation formula [1022]).

$$\Omega(a, b, c, d) = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (c-a)_k (d-a)_k (c-b)_{2k} (d-b)_{2k}}{(c)_{1+2k} (d)_{1+2k} (c+d-a-b-1)_{3+3k}} \gamma_k(a, b, c, d).$$

When $d = 1$, we can evaluate $\Omega(a, b, c, d)$ by the Gauss summation formula and derive from the last theorem the following further relations.

Proposition 2.24 (Infinite series for Γ -function quotient).

$$\Gamma \left[\begin{matrix} 1+c, 3+c-a-b \\ c-a, c-b \end{matrix} \right] = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (1-a)_k (c-a)_k (c-b)_{2k} (1-b)_{2k}}{(2)_{2k} (1+c)_{2k} (3+c-a-b)_{3k}} \gamma_k(a, b, c, 1).$$

Corollary 2.25 ($a + b = c = 1$ in Proposition 2.24).

$$\frac{\sin \pi x}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{27} \right)^k \left[\begin{matrix} x, 1-x, 1-x, \frac{x}{2}, \frac{x}{2}, \frac{1+x}{2}, \frac{1+x}{2} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3} \end{matrix} \right]_k \gamma_k(x, 1-x, 1, 1).$$

Corollary 2.26 ($a = b = x$ and $c = 1 + x$ in Proposition 2.24).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \left(\frac{-1}{27} \right)^k \left[\begin{matrix} 1, \frac{1}{2}, x, 1-x, \frac{1-x}{2}, \frac{2-x}{2} \\ \frac{3}{2}, \frac{2+x}{2}, \frac{3+x}{2}, \frac{4-x}{3}, \frac{5-x}{3}, \frac{6-x}{3} \end{matrix} \right]_k \frac{\gamma_k(x, x, 1+x, 1)}{(1-x^2)(2-x)(3-x)}.$$

Corollary 2.27 ($a = b = c = 1 - x$ in Proposition 2.24).

$$\frac{2(1-x^2)\pi x}{(1+x^2)\sin \pi x} = 2 + \sum_{k=1}^{\infty} \left(\frac{-1}{27} \right)^k \left[\begin{matrix} 1, x, 1-x, \frac{x}{2}, \frac{1+x}{2} \\ \frac{2-x}{2}, \frac{3-x}{2}, \frac{2+x}{3}, \frac{3+x}{3}, \frac{4+x}{3} \end{matrix} \right]_k \frac{\gamma_k(1-x, 1-x, 1-x, 1)}{k^2(2k+1)(1+x^2)}.$$

Letting $c = 1$ in Theorem 2.23 and then making the replacement $k \rightarrow k + 1$ for the sum corresponding to (8b) in the factor $\gamma_k(a, b, c, d)$, we derive the following expressions.

Proposition 2.28 (Infinite series for Γ -Function quotient).

$$\Gamma \left[\begin{matrix} 2+d, 4+d-a-b \\ 1+d-a, 1+d-b \end{matrix} \right] = \sum_{k=0}^{\infty} (-1)^k \frac{(a)_k (1-a)_k (1+d-a)_k (1-b)_{2k} (1+d-b)_{2k}}{(2)_{2k} (2+d)_{2k} (4+d-a-b)_{3k}} \gamma_k^*(a, b, d)$$

where $\gamma_k^*(a, b, d)$ is the quintic polynomial of k given by

$$\begin{aligned} \gamma_k^*(a, b, d) &= (1+d+2k)(1-a+k)(1-b+2k)(2+d-b+4k)(3+d-a-b+3k) \\ &\quad - (a+k)(1-a+k)(1-b+2k)(2-b+2k)(1+d-b+2k) \\ &\quad + (1+2k)(d+2k)(1+d+2k)(2+d-a-b+3k)(3+d-a-b+3k). \end{aligned}$$

Corollary 2.29 ($a = b = d = 1 - x$ in Proposition 2.28).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \left(\frac{-1}{27} \right)^k \left[\begin{matrix} 1, x, 1-x, \frac{x}{2}, \frac{1+x}{2} \\ \frac{3-x}{2}, \frac{4-x}{2}, \frac{3+x}{3}, \frac{4+x}{3}, \frac{5+x}{3} \end{matrix} \right]_k \frac{\gamma_k^*(1-x, 1-x, 1-x)}{(2k+1)(1-x^2)(4-x^2)}.$$

By specifying x in Corollaries 2.25, 2.26, 2.27 and 2.29, we can deduce the following infinite series expressions for π and $1/\pi$ with convergence rate equal to $-1/27$.

Example 31 ($x = 1/2$ in Corollary 2.29).

$$\frac{15\pi}{8} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[1, \frac{1}{2}\right]_{\left[\frac{7}{6}, \frac{11}{6}\right]_k} \{6 + 7k\}.$$

Example 32 ($x = 1/2$ in Corollary 2.27).

$$\frac{\pi}{2} = \frac{5}{3} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \left[1, \frac{1}{2}\right]_{\left[\frac{5}{6}, \frac{7}{6}\right]_k} \frac{3 + 28k}{6k}.$$

Example 33 ($x = 1/3$ in Corollary 2.27).

$$\frac{4\pi}{3\sqrt{3}} = \frac{5}{2} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \left[1, \frac{2}{3}, \frac{2}{3}, \frac{1}{6}\right]_{\left[\frac{5}{6}, \frac{7}{9}, \frac{10}{9}, \frac{13}{9}\right]_k} \frac{2 + 30k + 63k^2}{3k}.$$

Example 34 ($x = 3/4$ in Corollary 2.27).

$$\frac{\pi}{2\sqrt{2}} = \frac{25}{21} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \left[1, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}\right]_{\left[\frac{5}{8}, \frac{9}{8}, \frac{11}{12}, \frac{19}{12}\right]_k} \frac{21 + 304k + 448k^2}{84k}.$$

Example 35 ($x = 2/3$ in Corollary 2.27).

$$\frac{2\pi}{3\sqrt{3}} = \frac{13}{10} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \left[1, \frac{1}{3}, \frac{1}{3}, \frac{5}{6}\right]_{\left[\frac{7}{6}, \frac{8}{9}, \frac{11}{9}, \frac{14}{9}\right]_k} \frac{(5 + 21k)(1 + 12k + 18k^2)}{15k}.$$

Example 36 ($x = 1/4$ in Corollary 2.27).

$$\frac{3\pi}{2\sqrt{2}} = \frac{17}{5} + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \left[1, \frac{1}{4}, \frac{1}{8}, \frac{5}{8}\right]_{\left[\frac{7}{8}, \frac{11}{8}, \frac{13}{12}, \frac{17}{12}\right]_k} \frac{15 + 324k + 1408k^2 + 1792k^3}{20k}.$$

Example 37 ($x = 1/3$ in Corollary 2.25).

$$\frac{243\sqrt{3}}{8\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{6}, \frac{1}{6} \right]_{\left[1, 1, 1, \frac{3}{2}, \frac{3}{2}\right]_k} \{17 + 279k + 864k^2 + 756k^3\}.$$

Example 38 ($x = 1/2$ in Corollary 2.25).

$$\frac{64}{\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \right]_{\left[1, 1, 1, \frac{4}{3}, \frac{5}{3}\right]_k} \{21 + 296k + 992k^2 + 896k^3\}.$$

Example 39 ($x = 2/3$ in Corollary 2.29).

$$\frac{160\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[1, \frac{1}{3}, \frac{1}{3}, \frac{5}{6}\right]_{\left[\frac{7}{6}, \frac{11}{9}, \frac{14}{9}, \frac{17}{9}\right]_k} \{33 + 227k + 438k^2 + 252k^3\}.$$

Example 40 ($x = 2/3$ in Corollary 2.25).

$$\frac{243\sqrt{3}}{4\pi} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \right]_{\left[1, 1, 1, \frac{3}{2}, \frac{3}{2}\right]_k} \{35 + 504k + 1620k^2 + 1512k^3\}.$$

Example 41 ($x = 1/6$ in Corollary 2.27).

$$70\pi = 222 + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \left[1, \frac{1}{6}, \frac{5}{6}, \frac{1}{12}, \frac{7}{12}\right]_{\left[\frac{11}{12}, \frac{17}{12}, \frac{13}{18}, \frac{19}{18}, \frac{25}{18}\right]_k} \frac{35 + 1026k + 4608k^2 + 6048k^3}{k}.$$

Example 42 ($x = 3/4$ in Corollary 2.26).

$$\frac{315\pi}{16\sqrt{2}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{\frac{11}{8}}, \frac{\frac{1}{4}}{\frac{15}{8}}, \frac{\frac{1}{8}}{\frac{13}{12}}, \frac{\frac{5}{8}}{\frac{17}{12}} \right]_k \left\{ 44 + 305k + 688k^2 + 448k^3 \right\}.$$

Example 43 ($x = 5/6$ in Corollary 2.27).

$$110\pi = 366 + \sum_{k=1}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{\frac{7}{12}}, \frac{\frac{1}{6}}{\frac{13}{12}}, \frac{\frac{5}{6}}{\frac{17}{18}}, \frac{\frac{5}{12}}{\frac{23}{19}}, \frac{\frac{11}{12}}{\frac{29}{18}} \right]_k \frac{55 + 1314k + 5760k^2 + 6048k^3}{k}.$$

Example 44 ($x = 1/3$ in Corollary 2.29).

$$\frac{280\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{\frac{11}{6}}, \frac{\frac{2}{3}}{\frac{10}{9}}, \frac{\frac{2}{3}}{\frac{13}{9}}, \frac{\frac{1}{6}}{\frac{16}{9}} \right]_k \left\{ 57 + 308k + 498k^2 + 252k^3 \right\}.$$

Example 45 ($x = 1/4$ in Corollary 2.26).

$$\frac{1155\pi}{16\sqrt{2}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left[\frac{1}{\frac{9}{8}}, \frac{\frac{3}{4}}{\frac{13}{8}}, \frac{\frac{3}{8}}{\frac{19}{12}}, \frac{\frac{7}{8}}{\frac{23}{12}} \right]_k \left\{ 164 + 697k + 976k^2 + 448k^3 \right\}.$$

§2.5. [1122]. Applying Lemma 1.7 to $\Omega(b, 1+a, 1+c, 1+d)$, we get

$$\Omega(1+a, b, 1+c, 1+d) = \Omega(1+a, 1+b, 2+c, 2+d) \frac{b(a-c)(a-d)}{(1+c)(1+d)(a+b-c-d)} + \frac{a-c-d}{a+b-c-d}.$$

Substituting this into the equation displayed in Lemma 1.7, we may simplify the resulting equation as follows:

$$\begin{aligned} \Omega(a, b, c, d) &= \Omega(1+a, b, 1+c, 1+d) \frac{a(b-c)(b-d)}{cd(1+a+b-c-d)} + \frac{1+b-c-d}{1+a+b-c-d} \\ &= \Omega(1+a, 1+b, 2+c, 2+d) \frac{ab(c-a)(c-b)(d-a)(d-b)}{(c)_2(d)_2(c+d-a-b-1)_2} \\ &\quad + \frac{a(c-b)(d-b)(a-c-d)}{cd(c+d-a-b-1)_2} + \frac{c+d-b-1}{c+d-a-b-1}. \end{aligned}$$

Iterating this relation m -times, we get

$$\begin{aligned} \Omega(a, b, c, d) &= \Omega(a+m, b+m, c+2m, d+2m) \frac{(a)_m(b)_m(c-a)_m(c-b)_m(d-a)_m(d-b)_m}{(c)_{2m}(d)_{2m}(c+d-a-b-1)_{2m}} \\ &\quad + \sum_{k=0}^{m-1} \frac{(a)_k(b)_k(c-a)_k(c-b)_k(d-a)_k(d-b)_k}{(c)_{1+2k}(d)_{1+2k}(c+d-a-b-1)_{2+2k}} \rho_k(a, b, c, d) \end{aligned}$$

where $\rho_k(a, b, c, d)$ is the quartic polynomial of k given by

$$(9a) \quad \rho_k(a, b, c, d) = (a+k)(c-b+k)(d-b+k)(a-c-d-3k)$$

$$(9b) \quad + (c+2k)(d+2k)(c+d-b-1+3k)(c+d-a-b+2k).$$

By means of the Weierstrass M -test on uniformly convergent series, the limiting case $m \rightarrow \infty$ of the last equation yields the transformation.

Theorem 2.30 (Transformation formula [1122]).

$$\Omega(a, b, c, d) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c-a)_k(c-b)_k(d-a)_k(d-b)_k}{(c)_{1+2k}(d)_{1+2k}(c+d-a-b-1)_{2+2k}} \rho_k(a, b, c, d).$$

When $a = b = \frac{1}{2} + x$ and $c = d = 1 + 2x$, the last theorem recovers the following interesting identity discovered recently by Guillera.

Proposition 2.31 (Guillera [29, Identity 2: Equation 3]).

$$32x \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2} + x, & \frac{1}{2} + x \\ 1 + 2x, & 1 + 2x \end{matrix} \right]_k = \sum_{n=0}^{\infty} \left[\begin{matrix} \frac{1}{2} + x, & \frac{1}{2} + x, & \frac{1}{2} + x \\ 1 + x, & 1 + x, & 1 + x \end{matrix} \right]_n \frac{5 + 42x + 42n}{64^n}.$$

In particular, for $d = 1$, we can evaluate $\Omega(a, b, c, d)$ by the Gauss summation formula and derive from the last theorem the following expression.

Proposition 2.32 (Infinite series for Γ -function quotient).

$$\Gamma \left[\begin{matrix} 1 + c, 2 + c - a - b \\ c - a, c - b \end{matrix} \right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (1 - a)_k (1 - b)_k (c - a)_k (c - b)_k}{(2)_{2k} (1 + c)_{2k} (2 + c - a - b)_{2k}} \rho_k(a, b, c, 1).$$

Corollary 2.33 ($a + b = c = 1$ in Proposition 2.32).

$$\frac{\sin \pi x}{\pi x} = \sum_{k=0}^{\infty} \left[\begin{matrix} x, 1 - x \\ 1, \frac{3}{2} \end{matrix} \right]_k \frac{k(1+3k)(3+9k+7k^2) + x(1-x)(1+6k+6k^2+x-x^2)}{64^k}.$$

Corollary 2.34 ($a = b = x$ and $c = 1 + x$ in Proposition 2.32).

$$\frac{\pi x}{\sin \pi x} = \sum_{k=0}^{\infty} \left[\begin{matrix} 1, x, x, 1 - x, 1 - x \\ \frac{3}{2}, \frac{2+x}{2}, \frac{3+x}{2}, \frac{3-x}{2}, \frac{4-x}{2} \end{matrix} \right]_k \frac{2+14k+39k^2+48k^3+21k^4-x+3k^2x+x^2-3k^2x^2}{64^k (1+x)(1-x)(2-x)}.$$

Corollary 2.35 ($a = b = c = x$ in Proposition 2.32).

$$\frac{\pi x}{\sin \pi x} = \frac{1 + x^3}{1 - x^2} + \sum_{k=1}^{\infty} \frac{[1, x, 1 - x]_k^2}{[2, 1 + x, 2 - x]_{2k}} \frac{(2 + 3k)(1 - x)x + k(5 + 20k + 21k^2)}{k(1 - x)}.$$

The identity (3) due to Ramanujan [31, Equation 29] (cf. Berndt [10, Equation 20.2, Page 352] also) can be shown to be the case $x = 1/2$ of Corollary 2.33. Similarly, we can derive further infinite series formulae with convergence rate equal to $1/64$.

Example 46 ($x = 1/2$ in Corollary 2.35).

$$\pi = 3 + \sum_{k=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \frac{5}{4} \end{matrix} \right]_k \frac{(1 + 6k)(2 + 7k)}{64^k \cdot k}.$$

Example 47 ($x = 1/3$ in Corollary 2.35).

$$\frac{4\pi}{\sqrt{3}} = 7 + \sum_{k=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{3}, \frac{2}{3} \\ \frac{3}{2}, \frac{5}{6}, \frac{7}{6} \end{matrix} \right]_k \frac{4 + 39k + 63k^2}{64^k \cdot k}.$$

Example 48 ($x = 1/3$ in Corollary 2.34).

$$\frac{40\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left[\begin{matrix} 1, \frac{1}{3}, \frac{2}{3} \\ \frac{3}{2}, \frac{7}{6}, \frac{11}{6} \end{matrix} \right]_k \frac{8 + 27k + 21k^2}{64^k}.$$

Example 49 ($x = 1/4$ in Corollary 2.35).

$$3\pi\sqrt{2} = 13 + \sum_{k=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ \frac{3}{2}, \frac{5}{8}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8} \end{matrix} \right]_k \frac{6 + 89k + 320k^2 + 336k^3}{64^k \cdot k}.$$

Example 50 ($x = 1/2$ in Corollary 2.34).

$$\frac{9\pi}{4} = \sum_{k=0}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4} \end{matrix} \right]_k \frac{7 + 42k + 75k^2 + 42k^3}{64^k}.$$

Example 51 ($x = 1/5$ in Corollary 2.35).

$$\frac{8\pi\sqrt{2}}{\sqrt{5}-\sqrt{5}} = 21 + \sum_{k=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5} \\ \frac{3}{2}, \frac{3}{5}, \frac{7}{5}, \frac{9}{10}, \frac{11}{10} \end{matrix} \right]_k \frac{8 + 137k + 500k^2 + 525k^3}{64^k \cdot k}.$$

Example 52 ($x = 1/6$ in Corollary 2.35).

$$10\pi = 31 + \sum_{k=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{3}{2}, \frac{7}{12}, \frac{11}{12}, \frac{13}{12}, \frac{17}{12} \end{matrix} \right]_k \frac{10 + 195k + 720k^2 + 756k^3}{64^k \cdot k}.$$

Example 53 ($x = 2/5$ in Corollary 2.35).

$$\frac{12\pi\sqrt{2}}{\sqrt{5}+\sqrt{5}} = 19 + \sum_{k=1}^{\infty} \left[\begin{matrix} 1, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5} \\ \frac{3}{2}, \frac{4}{5}, \frac{6}{5}, \frac{7}{10}, \frac{13}{10} \end{matrix} \right]_k \frac{12 + 143k + 500k^2 + 525k^3}{64^k \cdot k}.$$

Example 54 ($x = 1/8$ in Corollary 2.35).

$$\frac{14\pi}{\sqrt{2}-\sqrt{2}} = 57 + \sum_{k=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{8}, \frac{1}{8}, \frac{7}{8}, \frac{7}{8} \\ \frac{3}{2}, \frac{9}{16}, \frac{15}{16}, \frac{17}{16}, \frac{23}{16} \end{matrix} \right]_k \frac{14 + 341k + 1280k^2 + 1344k^3}{64^k \cdot k}.$$

Example 55 ($x = 3/8$ in Corollary 2.35).

$$\frac{30\pi}{\sqrt{2}+\sqrt{2}} = 49 + \sum_{k=1}^{\infty} \left[\begin{matrix} 1, \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8} \\ \frac{3}{2}, \frac{11}{16}, \frac{13}{16}, \frac{19}{16}, \frac{21}{16} \end{matrix} \right]_k \frac{30 + 365k + 1280k^2 + 1344k^3}{64^k \cdot k}.$$

Example 56 ($x = 1/3$ in Corollary 2.33).

$$\frac{81\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1, \frac{3}{2} \end{matrix} \right]_k^3 \frac{22 + 351k + 1566k^2 + 2754k^3 + 1701k^4}{64^k}.$$

Example 57 ($x = 1/4$ in Corollary 2.34).

$$\frac{105\pi}{8\sqrt{2}} = \sum_{k=0}^{\infty} \left[\begin{matrix} 1, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ \frac{3}{2}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8} \end{matrix} \right]_k \frac{29 + 224k + 633k^2 + 768k^3 + 336k^4}{64^k}.$$

Example 58 ($x = 1/4$ in Corollary 2.33).

$$\frac{128\sqrt{2}}{\pi} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1, \frac{3}{2} \end{matrix} \right]_k^3 \frac{57 + 1056k + 4896k^2 + 8704k^3 + 5376k^4}{64^k}.$$

Example 59 ($x = 1/6$ in Corollary 2.34).

$$\frac{385\pi}{18} = \sum_{k=0}^{\infty} \left[\begin{matrix} 1, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{3}{2}, \frac{13}{12}, \frac{17}{12}, \frac{19}{12}, \frac{23}{12} \end{matrix} \right]_k \frac{67 + 504k + 1419k^2 + 1728k^3 + 756k^4}{64^k}.$$

Example 60 ($x = 1/6$ in Corollary 2.33).

$$\frac{648}{\pi} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{6}, \frac{5}{6} \\ 1, \frac{3}{2} \end{matrix} \right]_k^3 \frac{205 + 4968k + 24408k^2 + 44064k^3 + 27216k^4}{64^k}.$$

3. MORE GENERAL TRANSFORMATION AND FURTHER π -FORMULAE

In general, we can even iterate p and q times of Lemma 1.1 with respect to parameters a and b , then r and s times of Lemma 1.2 with respect to parameters c and d . The resulting relation reads as follows:

$$\begin{aligned} \Omega(a, b, c, d) &= (-1)^{pm+qm} \frac{(a)_{pm}(b)_{qm}}{(c)_{rm}(d)_{sm}} \Omega(a+pm, b+qm, c+rm, d+sm) \\ &\times \frac{(c-a)_{rm-pm}(c-b)_{rm-qn}(d-a)_{sm-pm}(d-b)_{sm-qn}}{(c+d-a-b-1)_{m(r+s-p-q)}} \\ &+ \sum_{n=0}^{m-1} (-1)^{pn+qn} \frac{(a)_{pn}(b)_{qn}}{(c)_{rn}(d)_{sn}} W_n(a, b, c, d; p, q, r, s) \\ &\times \frac{(c-a)_{rn-pn}(c-b)_{rn-qn}(d-a)_{sn-pn}(d-b)_{sn-qn}}{(c+d-a-b-1)_{n(r+s-p-q)}} \end{aligned}$$

where

$$W_n(a, b, c, d; p, q, r, s) = W(a + pn, b + qn, c + rn, d + sn; p, q, r, s)$$

and

$$\begin{aligned} W(a, b, c, d; p, q, r, s) &= \frac{(c-1)(d-1)}{(1+a-c)(1+a-d)} \sum_{i=0}^{p-1} \left[\begin{matrix} a, 2+a+b-c-d \\ 2+a-c, 2+a-d \end{matrix} \right]_i \\ &+ \frac{(c-1)(d-1)}{(1+b-c)(1+b-d)} \left[\begin{matrix} a, 2+a+b-c-d \\ 1+a-c, 1+a-d \end{matrix} \right]_p \sum_{j=0}^{q-1} \left[\begin{matrix} b, 2+a+b-c-d+p \\ 2+b-c, 2+b-d \end{matrix} \right]_j \\ &+ \frac{(2+a+b-c-d)_{p+q-1}(a)_p(b)_q}{[1+a-c, 1+a-d]_p [1+b-c, 1+b-d]_q} \left\{ \sum_{k=0}^{r-1} \frac{(d-1)[c-a-p, c-b-q]_k}{(c)_k(c+d-a-b-p-q)_k} \right. \\ &\quad \left. - \frac{(c-1)[c-a-p, c-b-q]_r}{(c-1)_r(c+d-a-b-p-q)_r} \sum_{\ell=0}^{s-1} \left[\begin{matrix} d-a-p, d-b-q \\ d, c+d-a-b-p-q+r \end{matrix} \right]_{\ell} \right\}. \end{aligned}$$

By means of the Weierstrass M -test on uniformly convergent series, the limiting case $m \rightarrow \infty$ of the last equation gives rise to the transformation.

Theorem 3.1 (Transformation formula: $p + q < r + s$).

$$\begin{aligned} \Omega(a, b, c, d) &= \sum_{n=0}^{\infty} (-1)^{pn+qn} \frac{(a)_{pn}(b)_{qn}}{(c)_{rn}(d)_{sn}} W_n(a, b, c, d; p, q, r, s) \\ &\times \frac{(c-a)_{rn-pn}(c-b)_{rn-qn}(d-a)_{sn-pn}(d-b)_{sn-qn}}{(c+d-a-b-1)_{n(r+s-p-q)}}. \end{aligned}$$

This general theorem can be utilized to derive many summation formulae involving π . For instance, the two infinite series anticipated in the introduction are obtained from the iteration pattern [5,5,10,10], respectively, under the parameter settings “ $a = b = 1/2, c = d = 1$ ” and “ $a = b = 1/3, c = 4/3, d = 1$ ”. Further infinite series with different convergence rates are computed through *Mathematica*. They are exhibited as examples, where iteration patterns and parameter settings are highlighted in the headers.

Example 61 ([0022] $a = b = 1/2$ and $c = d = 1$).

$$\frac{128}{\pi} = \sum_{n=0}^{\infty} \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{matrix} \right]_n \frac{39 + 672n + 3168n^2 + 5888n^3 + 3840n^4}{16^n}.$$

Example 62 ([0022] $a = b = 1/2$, $c = 3/2$ and $d = 1$).

$$\frac{105\pi}{8} = \sum_{n=0}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8} \end{matrix} \right]_n \frac{40 + 237n + 428n^2 + 240n^3}{16^n}.$$

Example 63 ([0022] $a = b = c = 1/2$ and $d = 1$).

$$30\pi = 88 + \sum_{n=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8} \end{matrix} \right]_n \frac{15 + 272n + 992n^2 + 960n^3}{16^n \cdot n}.$$

Example 64 ([1123] $a = 1/3$, $b = 2/3$ and $c = d = 1$).

$$\frac{729\sqrt{3}}{4\pi} = \sum_{n=0}^{\infty} \left(\frac{2}{27} \right)^{2n} \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n \{100 + 1521n + 2610n^2\}.$$

Example 65 ([1123] $a = b = 1/2$, $c = 3/2$ and $d = 1$).

$$15\pi = \sum_{n=0}^{\infty} \left(\frac{2}{27} \right)^{2n} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{4}{3}, \frac{5}{3}, \frac{7}{6}, \frac{11}{6} \end{matrix} \right]_n \{47 + 286n + 517n^2 + 290n^3\}.$$

Example 66 ([1123] $a = b = c = 1/2$ and $d = 1$).

$$12\pi = 37 + \sum_{n=1}^{\infty} \left(\frac{2}{27} \right)^{2n} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{4}{3}, \frac{5}{3}, \frac{5}{6}, \frac{7}{6} \end{matrix} \right]_n \frac{36 + 415n + 1314n^2 + 1160n^3}{n}.$$

Example 67 ([0012] $a = b = 1/2$ and $c = d = 1$).

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} \left(\frac{4}{27} \right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, 1, \frac{4}{3}, \frac{5}{3} \end{matrix} \right]_n \{9 + 118n + 400n^2 + 368n^3\}.$$

Example 68 ([0012] $a = 1/3$, $b = 2/3$ and $c = d = 1$).

$$\frac{81\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \left(\frac{4}{27} \right)^n \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n \{20 + 243n + 414n^2\}.$$

Example 69 ([0012] $a = b = 1/3$, $c = 4/3$ and $d = 1$).

$$\frac{160\pi}{9\sqrt{3}} = \sum_{n=0}^{\infty} \left(\frac{4}{27} \right)^n \left[\begin{matrix} 1, \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \\ \frac{3}{2}, \frac{11}{9}, \frac{14}{9}, \frac{17}{9} \end{matrix} \right]_n \{28 + 151n + 257n^2 + 138n^3\}.$$

Example 70 ([0012] $a = b = 2/3$, $c = 5/3$ and $d = 1$).

$$\frac{56\pi}{9\sqrt{3}} = \sum_{n=0}^{\infty} \left(\frac{4}{27} \right)^n \left[\begin{matrix} 1, \frac{2}{3}, \frac{1}{6}, \frac{1}{6} \\ \frac{3}{2}, \frac{10}{9}, \frac{13}{9}, \frac{16}{9} \end{matrix} \right]_n \{11 + 74n + 181n^2 + 138n^3\}.$$

Example 71 ([0012] $a = b = c = 1/2$ and $d = 1$).

$$3\pi = 7 + \sum_{n=1}^{\infty} \left(\frac{4}{27} \right)^n \left[\begin{matrix} 1, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{6}, \frac{7}{6} \end{matrix} \right]_n \frac{6 + 67n + 200n^2 + 184n^3}{n}.$$

Example 72 ([0012] $a = b = c = 1/3$ and $d = 1$).

$$\frac{20\pi}{\sqrt{3}} = 19 + \sum_{n=1}^{\infty} \left(\frac{4}{27}\right)^n \left[\begin{matrix} 1, \frac{1}{3}, \frac{5}{6}, \frac{5}{6} \\ \frac{3}{2}, \frac{8}{9}, \frac{11}{9}, \frac{14}{9} \end{matrix} \right]_n \frac{20 + 185n + 495n^2 + 414n^3}{n}.$$

Example 73 ([0012] $a = b = c = 2/3$ and $d = 1$).

$$\frac{8\pi}{\sqrt{3}} = 13 + \sum_{n=1}^{\infty} \left(\frac{4}{27}\right)^n \left[\begin{matrix} 1, \frac{2}{3}, \frac{1}{6}, \frac{1}{6} \\ \frac{3}{2}, \frac{7}{9}, \frac{10}{9}, \frac{13}{9} \end{matrix} \right]_n \frac{8 + 119n + 405n^2 + 414n^3}{n}.$$

Example 74 ([1013] $a = 1/3, b = 2/3$ and $c = d = 1$).

$$\frac{486\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-4}{27}\right)^n \left[\begin{matrix} \frac{1}{3}, \frac{5}{6}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9} \\ 1, 1, 1, \frac{5}{3}, \frac{5}{3} \end{matrix} \right]_n \{280 + 4314n + 10908n^2 + 7533n^3\}.$$

Example 75 ([1013] $a = 2/3, b = 1/3$ and $c = d = 1$).

$$\frac{243\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \left(\frac{-4}{27}\right)^n \left[\begin{matrix} \frac{2}{3}, \frac{1}{6}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9} \\ 1, 1, 1, \frac{4}{3}, \frac{4}{3} \end{matrix} \right]_n \{80 + 1281n + 5697n^2 + 7533n^3\}.$$

Example 76 ([1013] $a = b = 1/2, c = 3/2$ and $d = 1$).

$$30\pi = \sum_{n=0}^{\infty} \left(\frac{-4}{27}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6} \\ \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{7}{6}, \frac{11}{6} \end{matrix} \right]_n \{97 + 931n + 3222n^2 + 4548n^3 + 2232n^4\}.$$

Example 77 ([1013] $a = b = 1/3, c = 4/3$ and $d = 1$).

$$\frac{320\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \left(\frac{-4}{27}\right)^n \left[\begin{matrix} \frac{1}{3}, \frac{5}{6}, \frac{2}{9}, \frac{5}{9}, \frac{8}{9} \\ \frac{4}{3}, \frac{5}{3}, \frac{11}{9}, \frac{14}{9}, \frac{17}{9} \end{matrix} \right]_n \{604 + 5080n + 14655n^2 + 17604n^3 + 7533n^4\}.$$

Example 78 ([1013] $a = b = 2/3, c = 5/3$ and $d = 1$).

$$\frac{112\pi}{\sqrt{3}} = \sum_{n=0}^{\infty} \left(\frac{-4}{27}\right)^n \left[\begin{matrix} \frac{2}{3}, \frac{1}{6}, \frac{1}{9}, \frac{4}{9}, \frac{7}{9} \\ \frac{4}{3}, \frac{5}{3}, \frac{10}{9}, \frac{13}{9}, \frac{16}{9} \end{matrix} \right]_n \{206 + 2045n + 7941n^2 + 13095n^3 + 7533n^4\}.$$

Example 79 ([1013] $a = b = 1/4, c = 5/4$ and $d = 1$).

$$462\pi\sqrt{2} = \sum_{n=0}^{\infty} \left(\frac{-4}{27}\right)^n \left[\begin{matrix} \frac{1}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{12}, \frac{11}{12} \\ \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{19}{12}, \frac{23}{12} \end{matrix} \right]_n \times \{2137 + 15405n + 39972n^2 + 44400n^3 + 17856n^4\}.$$

Example 80 ([1013] $a = b = 3/4, c = 7/4$ and $d = 1$).

$$90\pi\sqrt{2} = \sum_{n=0}^{\infty} \left(\frac{-4}{27}\right)^n \left[\begin{matrix} \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{1}{12}, \frac{5}{12} \\ \frac{4}{3}, \frac{5}{3}, \frac{7}{4}, \frac{13}{12}, \frac{17}{12} \end{matrix} \right]_n \times \{403 + 3975n + 16100n^2 + 28368n^3 + 17856n^4\}.$$

Example 81 ([1023] $a = b = 1/2$ and $c = d = 1$).

$$\frac{512}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{64}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{4}{3}, \frac{5}{3} \end{matrix} \right]_n \{165 + 2792n + 10068n^2 + 9360n^3\}.$$

Example 82 ([1023] $a = b = 1/2$, $c = 3/2$ and $d = 1$).

$$\frac{105\pi}{2} = \sum_{n=0}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8} \end{matrix} \right]_n \left(\frac{-1}{64} \right)^n \\ \times \left\{ 166 + 1814n + 7421n^2 + 14347n^3 + 13224n^4 + 4680n^5 \right\}.$$

Example 83 ([1023] $a = b = c = 1/2$ and $d = 1$).

$$60\pi = 194 + \sum_{n=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8} \end{matrix} \right]_n \left(\frac{-1}{64} \right)^n \\ \times \frac{90 + 1703n + 10737n^2 + 30160n^3 + 38856n^4 + 18720n^5}{n}.$$

Example 84 ([0013] $a = 1/3$, $b = 2/3$ and $c = d = 1$).

$$\frac{1287\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \left[\begin{matrix} \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9} \\ 1, 1, 1, \frac{3}{2}, \frac{5}{4}, \frac{7}{4} \end{matrix} \right]_n \left(\frac{27}{256} \right)^n \\ \times \left\{ 1120 + 23202n + 130464n^2 + 260253n^3 + 166941n^4 \right\}.$$

Example 85 ([0013] $a = b = 1/2$, $c = 3/2$ and $d = 1$).

$$105\pi = \sum_{n=0}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8} \end{matrix} \right]_n \left(\frac{27}{256} \right)^n \\ \times \frac{317 + 3842n + 18247n^2 + 40606n^3 + 42192n^4 + 16488n^5}{n}.$$

Example 86 ([0013] $a = b = c = 1/2$ and $d = 1$).

$$60\pi = 149 + \sum_{n=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8} \end{matrix} \right]_n \left(\frac{27}{256} \right)^n \\ \times \frac{180 + 3167n + 19156n^2 + 52844n^3 + 67896n^4 + 32976n^5}{n}.$$

Example 87 ([1034] $a = b = 1/2$ and $c = d = 1$).

$$\frac{65536}{\pi} = \sum_{n=0}^{\infty} \left(\frac{-1}{432} \right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \\ 1, 1, 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3} \end{matrix} \right]_n \\ \times \left\{ 20895 + 564400n + 4058132n^2 + 12197472n^3 + 16285440n^4 + 7981056n^5 \right\}.$$

Example 88 ([1133] $a = 1/3$, $b = 2/3$ and $c = d = 1$).

$$\frac{19683\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2}, \frac{5}{4}, \frac{7}{4} \end{matrix} \right]_n \\ \times \frac{5420 + 129105n + 737874n^2 + 1474848n^3 + 943488n^4}{27^{2n}}.$$

Example 89 ([1133] $a = b = c = 1/5$ and $d = 1$).

$$\frac{3024\pi\sqrt{2}}{\sqrt{5} - \sqrt{5}} = 8070 + \sum_{n=1}^{\infty} \left[\begin{matrix} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{9}{10}, \frac{9}{10} \\ \frac{4}{3}, \frac{5}{3}, \frac{17}{10}, \frac{11}{15}, \frac{16}{15}, \frac{19}{20}, \frac{29}{20} \end{matrix} \right]_n \\ \times \frac{1134 + 34323n + 232833n^2 + 645080n^3 + 796500n^4 + 364000n^5}{27^{2n} \cdot n}.$$

Example 90 ([1133] $a = b = c = 3/5$ and $d = 1$).

$$\frac{2688\pi\sqrt{2}}{\sqrt{5+\sqrt{5}}} = 4430 + \sum_{n=1}^{\infty} \left[\frac{1}{\frac{4}{3}}, \frac{1}{\frac{5}{3}}, \frac{1}{\frac{11}{10}}, \frac{1}{\frac{13}{15}}, \frac{1}{\frac{23}{15}}, \frac{1}{\frac{17}{20}}, \frac{1}{\frac{27}{20}} \right]_n \\ \times \frac{1008+23816n+167281n^2+509295n^3+705500n^4+364000n^5}{27^{2n} \cdot n}.$$

Example 91 ([1124] $a = b = 1/2$ and $c = d = 1$).

$$\frac{9216}{\pi} = \sum_{n=0}^{\infty} \left(\frac{3}{8} \right)^{6n} \left[\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{\frac{5}{4}}, \frac{1}{\frac{5}{4}}, \frac{1}{\frac{7}{4}}, \frac{1}{\frac{7}{4}} \right]_n \\ \times \left\{ 2925 + 71802n + 554280n^2 + 1794832n^3 + 2464912n^4 + 1195040n^5 \right\}.$$

Example 92 ([1124] $a = 1/4$, $b = 3/4$ and $c = d = 1$).

$$\frac{262144}{\pi\sqrt{2}} = \sum_{n=0}^{\infty} \left(\frac{3}{8} \right)^{6n} \left[\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{\frac{3}{2}}, \frac{1}{\frac{3}{2}}, \frac{1}{\frac{3}{2}} \right]_n \\ \times \left\{ 58905 + 1578944n + 8002656n^2 + 14939136n^3 + 9560320n^4 \right\}.$$

Example 93 ([1124] $a = b = c = 1/2$ and $d = 1$).

$$360\pi = 1119 + \sum_{n=1}^{\infty} \left(\frac{3}{8} \right)^{6n} \left[\frac{1}{\frac{3}{4}}, \frac{1}{\frac{5}{4}}, \frac{1}{\frac{5}{4}}, \frac{1}{\frac{7}{4}}, \frac{1}{\frac{7}{4}}, \frac{1}{\frac{9}{8}}, \frac{1}{\frac{11}{8}}, \frac{1}{\frac{13}{8}} \right]_n \\ \times \frac{1440+32223n+275358n^2+1162864n^3+2565376n^4+2806352n^5+1195040n^6}{n}.$$

Example 94 ([1124] $a = b = c = 3/4$ and $d = 1$).

$$4320\pi\sqrt{2} = 19119 + \sum_{n=1}^{\infty} \left(\frac{3}{8} \right)^{6n} \left[\frac{1}{\frac{3}{2}}, \frac{1}{\frac{5}{4}}, \frac{1}{\frac{7}{8}}, \frac{1}{\frac{11}{8}}, \frac{1}{\frac{13}{16}}, \frac{1}{\frac{17}{16}}, \frac{1}{\frac{21}{16}}, \frac{1}{\frac{25}{16}} \right]_n \\ \times \frac{17280+531375n+5322772n^2+25256544n^3+62390912n^4+77511424n^5+38241280n^6}{n}.$$

Example 95 ([1134] $a = 1/5$, $b = 4/5$ and $c = d = 1$).

$$\frac{5^{10}}{2\pi\sqrt{2}}\sqrt{5-\sqrt{5}} = \sum_{n=0}^{\infty} \left(\frac{27}{50000} \right)^n \left[\frac{2}{\frac{2}{5}}, \frac{3}{\frac{3}{5}}, \frac{1}{\frac{10}{10}}, \frac{9}{\frac{9}{10}}, \frac{1}{\frac{15}{15}}, \frac{4}{\frac{4}{15}}, \frac{11}{\frac{11}{15}}, \frac{14}{\frac{14}{15}} \right]_n \\ \times \left\{ 1826748 + 77513850n + 750064850n^2 + 3126564375n^3 \right. \\ \left. + 6462824375n^4 + 6516000000n^5 + 2555437500n^6 \right\}.$$

Example 96 ([1134] $a = 2/5$, $b = 3/5$ and $c = d = 1$).

$$\frac{5^{10}}{2\pi\sqrt{2}}\sqrt{5+\sqrt{5}} = \sum_{n=0}^{\infty} \left(\frac{27}{50000} \right)^n \left[\frac{1}{\frac{1}{5}}, \frac{4}{\frac{4}{5}}, \frac{3}{\frac{3}{10}}, \frac{7}{\frac{7}{10}}, \frac{2}{\frac{2}{15}}, \frac{7}{\frac{7}{15}}, \frac{8}{\frac{8}{15}}, \frac{13}{\frac{13}{15}} \right]_n \\ \times \left\{ 2954952 + 87237600n + 779867350n^2 + 3165466875n^3 \right. \\ \left. + 6481206875n^4 + 6516000000n^5 + 2555437500n^6 \right\}.$$

Example 97 ([4488] $a = b = 1/2$ and $c = d = 1$).

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left\{ \frac{(8n)!}{(4n)!(3+4n)!} \right\}^3 \frac{1}{2^{31+48n}} \times \left\{ \begin{array}{l} 147650093325+8160777962064n+139098317327424n^2 \\ +1197446420123648n^3+6134230398836736n^4 \\ +20072164579737600n^5+43115678544691200n^6 \\ +60594339629236224n^7+53671368012595200n^8 \\ +27188396308824064n^9+6004799145246720n^{10} \end{array} \right\}.$$

Example 98 ([4488] $a = b = 1/3$, $c = 4/3$ and $d = 1$).

$$\frac{\pi}{\sqrt{3}} = 2^7 3^6 \sum_{n=0}^{\infty} \frac{(12n)!(12+12n)!}{(24+24n)!} \left\{ \begin{array}{l} 25177954620+758453650825n \\ +9733209838197n^2+70930479893210n^3 \\ +328524057424620n^4+1021122326418600n^5 \\ +2183094932530176n^6+3222033941162880n^7 \\ +3226937631759360n^8+2094002610739200n^9 \\ +794122229809152n^{10}+133590654812160n^{11} \end{array} \right\}.$$

Example 99 ([3588] $a = b = 1/2$ and $c = d = 1$).

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left\{ \frac{(6n)!(10n)!}{(3n)!(5n)!(7+8n)!} \right\}^3 \frac{(1+2n)^3}{2^{22+48n}} \times \left\{ \begin{array}{l} 170923431471384375+15297585161389476750n \\ +565470661487328872100n^2+11994998601076524911880n^3 \\ +167182680558716014742208n^4+1646467252861184903405952n^5 \\ +11975534932107848739688704n^6+66202254492960380950225408n^7 \\ +283488775041181863422083584n^8+951933287952149824105513984n^9 \\ +2524298122093567646814943232n^{10}+5299439637329702566400421888n^{11} \\ +8791167003335176149318615040n^{12}+11445656080075560220793733120n^{13} \\ +11545634384559874555665776640n^{14}+8832431201134981263461580800n^{15} \\ +4948699155399925991009615872n^{16}+1913934453266746935647010816n^{17} \\ +456352821126098194819973120n^{18}+50528158441688990812209152n^{19} \end{array} \right\}.$$

Example 100 ([3588] $a = b = 1/3$, $c = 4/3$ and $d = 1$).

$$\frac{2\pi}{\sqrt{3}} = 3^9 \sum_{n=0}^{\infty} \left\{ \frac{(9n)!(15n)!}{(23+24n)!} \right\} (1+3n)(2+3n) \times \left\{ \begin{array}{l} 2382277599313920000+147373047270939033600n+4140703455713858112000n^2 \\ +70531118938161201164288n^3+819603599483270894994016n^4 \\ +6926844518177755180042304n^5+44291363309108954778685648n^6 \\ +219907697427081930735511784n^7+862799357102906273399022538n^8 \\ +2706280624453200964717025178n^9+6834413666311588515380045481n^{10} \\ +13940847104235282407764890810n^{11}+22958753272784766032704244865n^{12} \\ +30396894247604117232052600932n^{13}+32073193428099754396901877594n^{14} \\ +26581837893789029140773813276n^{15}+16914238179325358039452675692n^{16} \\ +7970364506403439240820665986n^{17}+2618623288000958406033629277n^{18} \\ +535224232070110581653587842n^{19}+51217527485675123150312889n^{20} \end{array} \right\}.$$

ACKNOWLEDGEMENT

The author is grateful to the anonymous referee for the careful reading and valuable comments.

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