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hp-DISCONTINUOUS GALERKIN METHODS FOR THE HELMHOLTZ EQUATION WITH LARGE WAVE NUMBER

XIAOBING FENG AND HAIJUN WU

Abstract. In this paper we develop and analyze some interior penalty hp-discontinuous Galerkin (hp-DG) methods for the Helmholtz equation with first order absorbing boundary condition in two and three dimensions. The proposed hp-DG methods are defined using a sesquilinear form which is not only mesh-dependent (or h-dependent) but also degree-dependent (or p-dependent). In addition, the sesquilinear form contains penalty terms which not only penalize the jumps of the function values across the element edges but also the jumps of the first order tangential derivatives as well as jumps of all normal derivatives up to order p. Furthermore, to ensure the stability, the penalty parameters are taken as complex numbers with positive imaginary parts, so essentially and practically no constraint is imposed on the penalty parameters. It is proved that the proposed hp-discontinuous Galerkin methods are stable (hence, well-posed) without any mesh constraint. For each fixed wave number k, sub-optimal order (with respect to h and p) error estimates in the broken H^1-norm and the L^2-norm are derived without any mesh constraint. The error estimates as well as the stability estimates are improved to optimal order under the mesh condition $k^3h^2p^{-2} \le C_0$ by utilizing these stability and error estimates and using a stability-error iterative procedure, where C_0 is some constant independent of k, h, p, and the penalty parameters. To overcome the difficulty caused by strong indefiniteness (and non-Hermitian nature) of the Helmholtz problems in the stability analysis for numerical solutions, our main ideas for stability analysis are to make use of a local version of the Rellich identity (for the Laplacian) and to mimic the stability analysis for the PDE solutions given in [19, 20, 33], which enable us to derive stability estimates and error bounds with explicit dependence on the mesh size h, the polynomial degree p, the wave number k, as well as all the penalty parameters for the numerical solutions.

1. Introduction

This is the second installment in a series (cf. [27]) which is devoted to developing and analyzing novel interior penalty discontinuous Galerkin (IPDG) methods for

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the following Helmholtz problem with large wave number:

$$-\Delta u - k^2 u = f \qquad \text{in } \Omega := \Omega_1 \setminus D, \tag{1.1}$$

$$\frac{\partial u}{\partial n_\Omega} + \mathbf{i} k u = g \qquad \text{on } \Gamma_R := \partial\Omega_1, \tag{1.2}$$

$$u = 0 \qquad \text{on } \Gamma_D := \partial D, \tag{1.3}$$

where $k \in \mathbf{R}$, called the wave number, is a (large) positive number, $D \subset \Omega_1 \subset \mathbf{R}^d$, $d = 2, 3$, D is known as a scatterer and is assumed to be a bounded Lipschitz domain, Ω_1, which is assumed to be a polygonal/polyhedral domain and often taken as a d-rectangle in applications, defines the size of the computational domain. Note that $\partial\Omega = \Gamma_R \cup \Gamma_D$. n_Ω denotes the unit outward normal to Ω. $\mathbf{i} := \sqrt{-1}$ denotes the imaginary unit. Condition (1.2) with $g = 0$ is known as the first order absorbing boundary condition (cf. [23]), which is used to minimize the reflection at the boundary Γ_R and to limit the computation of the original scattering problem just on the finite domain Ω. Boundary condition (1.3) implies that the scatterer is sound-soft. We note that the case $D = \emptyset$ also arises in applications either as a consequence of frequency domain treatment of waves or when time-harmonic solutions of the scalar wave equation are sought (cf. [22]).

In [27] we proposed and analyzed some IPDG methods for problem (1.1)–(1.3) using piecewise linear polynomial trial and test functions. It was proved that the proposed methods are unconditionally (with respect to mesh size h) stable and well-posed for all wave numbers $k > 0$. Optimal order error estimates were established showing explicit dependence of the error bounds on h, k and all penalty parameters. However, due to the existence of a pollution term, the (broken) H^1-norm error bound deteriorates as the wave number k increases under the practical "rule of thumb" mesh constraint that kh is bounded. To improve the accuracy and efficiency of those IPDG methods, it is necessary to use (piecewise) high order polynomial trial and test functions partly because of the rigidity and low approximability of linear functions and partly because of the very oscillatory nature of high frequency waves. However, simply replacing the linear element by high order elements in the IPDG methods of [27] does not reduce the pollution very much, in particular, the theoretical error bounds do not change much because the analysis of [27] indeed strongly depends on the properties of linear functions.

Motivated by the above challenge and observation, the primary goal of this paper is to develop some new hp-interior penalty discontinuous Galerkin (hp-IPDG) methods which retain the advantages of the IPDG methods of [27] but improve their accuracy and stability by exploiting the efficiency and flexibility of piecewise high order polynomial functions. To this end, our key idea is to construct a sesquilinear form (as a discretization of the Laplacian) which is not only mesh-dependent (or h-dependent) but also degree-dependent (or p-dependent) by introducing penalty terms which not only penalize the jumps of the function values across the element edges but also the jumps of the first order tangential derivatives as well as jumps of all normal derivatives up to order p. In addition, as in [27], to ensure the stability, all penalty parameters are taken as complex numbers with positive imaginary parts. Since the Helmholtz equation with large wave number is non-Hermitian and strongly indefinite, as expected, stability estimates (or a priori estimates) for numerical solutions under practical mesh constraints is a difficult task to carry out regardless which discretization method is used. To overcome the difficulty, as in [27], the crux of our analysis is to establish and to make use of a local version of the Rellich identity (for the Laplacian) and to mimic the stability analysis for the PDE solutions given in [19, 20, 33]. The key idea here is to use the special test function $\nabla u_h \cdot (x - x_\Omega)$ (defined elementwise) with u_h denoting the *hp*-IPDG solution, such a test function is valid for any DG method. We remark that the same technique was successfully employed by Shen and Wang in [40] to establish the stability and error analysis for the spectral Galerkin approximation of the Helmholtz problem. We also note that although similar techniques to those in [27, 40] are utilized in this paper to carry out the stability analysis, the analysis of this paper is more involved because the special sesquilinear form of this paper, which contains jumps of high order normal derivatives, is a lot more complicated to deal with, even though they are similar conceptually.

Since the Helmholtz equation appears, in one way or another, directly or indirectly, in almost all wave-related problems arisen from many science, engineering, and industry applications, solving the Helmholtz equation, in one form or another, has always been and remains at the center of wave computation. We refer the reader to ([1, 2, 5, 6, 10, 13, 14, 18, 19, 22, 24, 29, 32, 36, 39, 41, 48] and the references therein) for some recent developments on numerical methods, in particular, Galerkin type methods, for the Helmholtz equation. We also refer the reader to [27] for a brief review about some theoretical issues for finite element approximations (and other types of Galerkin approximations) of the Helmholtz equation.

The *hp*-finite element method (*hp*-FEM) is a modern version of the finite element method, capable of achieving exceptionally fast (exponential) convergence. It combines the flexibility of the standard finite element method and the high order accuracy of the spectral method. Consequently, the *hp*-FEM can often attain more accurate results than the standard finite element method does while using less CPU time and resources. The *hp*-FEM has undergone intensive developments both on theory and implementation in the past twenty-five years. We refer the reader to the survey paper [7] and two recent monographs [43, 44] for a detailed exposition on the basic theory and advanced topics of the *hp*-FEM. We would like to mention that, recently, the *hp*-finite element approximations of the Helmholtz scattering problems with Dirichlet-to-Neumann boundary conditions in $\mathbf{R}^d$ $(d = 1, 2, 3)$ are considered in [37], and some first order error estimates (with respect to $\frac{h}{p}$ in H^1- and L^2-norms) are derived under the condition that both $\frac{kh}{p}$ and $\frac{\ln k}{p}$ are small enough or that $kh + k(kh)^p$ is small enough. The results are extended to the Helmholtz equation with Robin boundary conditions on smooth bounded domains or on convex polygons in [38].

Discontinuous Galerkin (DG) methods were first proposed in the 1970s, they were not popular then because they produce larger algebraic systems than standard finite element methods do. However, due to the emergence of high performance computers and fast solvers since the early 1990s, especially, massively parallel computers and parallel solvers such as multilevel and domain decomposition methods, which together with advantages of DG methods has quickly attracted renewed interests in DG methods. They have been heavily developed and tested in the past fifteen years, we refer the reader to [4] and the references therein for a review of recent developments. As is well known now, DG methods have several advantages over other types of numerical methods. For example, the trial and test spaces are easy to construct, they can naturally handle inhomogeneous boundary conditions and curved boundaries; they also allow the use of highly nonuniform and unstructured meshes, and have built-in parallelism which permits coarse-grain parallelism. In addition, the fact that the mass matrices are block diagonal is an attractive feature in the context of time-dependent problems, especially if explicit time discretizations are used. Moreover, as proved in [27], DG methods are also effective and have advantages over finite element methods for the strongly indefinite Helmholtz equation, which has not been well understood before. We refer the reader to [3, 4, 9, 16, 17, 21, 26, 42, 47] and the references therein for a detailed account on DG methods for coercive elliptic and parabolic problems, and to [45, 25, 28, 35, 34] and the references therein for recent developments on *hp*-discontinuous Galerkin (*hp*-DG) methods.

The remainder of this paper is organized as follows. In Section 2, we first introduce notation and gather some preliminaries, and then formulate our *hp*-IPDG methods. Both symmetric and nonsymmetric methods are constructed and various possible variants are also discussed. Section 3 is devoted to the stability analysis for the *hp*-IPDG methods proposed in Section 2. It is proved that the proposed *hp*-IPDG methods are stable (hence well-posed) without any mesh constraint. In Section 4, using the stability results of Section 3 we prove that for each fixed wave number k, sub-optimal order (with respect to h and p) error estimates in the broken H^1-norm and the L^2-norm are derived without any mesh constraint. Finally, using the stability estimate of Section 3, the error estimates of Section 4 and a stability-error iterative procedure we obtain some much improved (optimal order) stability and error estimates for the *hp*-IPDG solutions under the mesh condition $k^3h^2p^{-2} \le C_0$ in Section 5, where C_0 is some constant independent of k, h, p, and the penalty parameters.

2. Formulation of *hp*-interior penalty discontinuous Galerkin methods

2.1. **Notation and preliminaries.** The space, norm and inner product notation used in this paper all are standard, we refer to [11, 15, 9] for their precise definitions. On the other hand, we note that all functions in this paper are complex-valued, so the familiar terminologies such "symmetric/nonsymmetric" and "bilinear" are replaced respectively by terms "Hermitian/non-Hermitian" and "sesquilinear". For a complex number $a = a_r + \mathbf{i}a_i$ (a_r and a_i are real numbers), $\overline{a} := a_r - \mathbf{i}a_i$ denotes the complex conjugate of a. $(\cdot,\cdot)_Q$ and $\langle\cdot,\cdot\rangle_\Sigma$ for $\Sigma \subset \partial Q$ denote the complex L^2-inner product on $L^2(Q)$ and $L^2(\Sigma)$ spaces, respectively. $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\partial\Omega}$. We also define

$$H^1_{\Gamma_D}(\Omega) := \left\{u \in H^1(\Omega);\, u = 0 \text{ on } \Gamma_D\right\}.$$

Throughout the paper, C is used to denote a generic positive constant which is independent of k, h, p, and the penalty parameters. We also use the shorthand notation $A \lesssim B$ and $B \gtrsim A$ for the inequality $A \le CB$ and $B \ge CA$. $A \simeq B$ is for the statement $A \lesssim B$ and $B \lesssim A$.

We now give the definition of star-shaped domains.

Definition 2.1. $Q \subset \mathbf{R}^d$ is said to be a *star-shaped* domain with respect to $x_Q \in Q$ if there exists a nonnegative constant c_Q such that

$$(x - x_Q)\cdot n_Q \ge c_Q \qquad \forall x \in \partial Q. \tag{2.1}$$

$Q \subset \mathbf{R}^d$ is said to be *strictly star-shaped* if c_Q is positive. Here n_Q is the unit outward normal to the boundary of Q.

In this paper, we assume that Ω_1 is a strictly star-shaped domain. Recall that Ω_1 is often taken as a d-rectangle in practice. We also assume that the scatterer D is a star-shaped domain, without loss of the generality, with respect to the same point x_{Ω_1} as Ω_1. This implies that $x_{\Omega_1} \in D \subset \Omega_1$. Under these assumptions, the following stability estimates hold for problem (1.1)–(1.3).

Theorem 2.1. *Suppose $\Omega_1 \subset \mathbf{R}^d$ is a strictly star-shaped domain and $D \subset \Omega_1$ is a star-shaped domain. Then the solution u to problem* (1.1)–(1.3) *satisfies*

$$\|u\|_{H^j(\Omega)} \lesssim \Big(\frac{1}{k} + k^{j-1}\Big)\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_R)}\big) \tag{2.2}$$

for $j = 0, 1$ if $u \in H^{3/2+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. (2.2) *also holds for $j = 2$ if $u \in H^2(\Omega)$. Furthermore, it holds that*

$$\|u\|_{H^j_{loc}(\Omega)} \lesssim \Big(\frac{1}{k} + k^{j-1}\Big)\Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_R)} + \sum_{\ell=1}^{j-2} \|f\|_{H^\ell_{loc}(\Omega)}\Big) \tag{2.3}$$

for $j = 3, 4, \cdots, q$ if $u \in H^2(\Omega) \cap H^q_{loc}(\Omega)$ for some positive integer $q \geq 3$. Here $H^j_{loc}(\Omega) = \big\{v : \ v \in H^j(B) \ \text{ for any compact set } B \subset \ \text{interior } \Omega\big\}$.

Proof. Inequality (2.2) for $j = 0, 1, 2$ was proved in [19, 20, 33]. Inequality (2.3) follows from (2.2) and an application of the standard cutoff function technique together with an induction argument. We leave the derivation to the interested reader. □

2.2. **Formulation of *hp*-IPDG methods.** To formulate our *hp*-IPDG methods, we need to introduce some notation, most of them have already appeared in [27]. Let $\mathcal{T}_h$ be a family of partitions of the domain $\Omega := \Omega_1 \setminus D$ parameterized by $h \in (0, h_0)$. For any $K \in \mathcal{T}_h$, we define $h_K := \operatorname{diam}(K)$. Similarly, for each edge/face e of $K \in \mathcal{T}_h$, $h_e := \operatorname{diam}(e)$. We impose the following mild restrictions on the partition $\mathcal{T}_h$:

(i) The elements of $\mathcal{T}_h$ satisfy the shape-regular condition,
(ii) $\mathcal{T}_h$ is locally quasi-uniform, that is, if two elements K and K' are adjacent (i.e., $\operatorname{meas}(\partial K \cap \partial K') > 0$), then $h_K \simeq h_{K'}$. Where $\operatorname{meas}(e)$ stands for $(d-1)$-dimensional Lebesgue measure of e.

For convenience, we assume $\operatorname{diam}(\Omega) \simeq 1$, hence $h_e, h_K \lesssim 1$.

For any two elements K, $K' \in \mathcal{T}_h$, we call $e = \partial K \cap \partial K'$ an *interior edge/face* of $\mathcal{T}_h$ if $\operatorname{meas}(e) > 0$. Note that e could be a portion of a side/face of the element K or K' in the case of geometrically nonconforming partition. Also, for any element $K \in \mathcal{T}_h$, we call $e = \partial K \cap \partial\Omega$ a *boundary edge/face* if $\operatorname{meas}(e) > 0$. Then we define

$$\begin{aligned}
\mathcal{E}_h^I &:= \text{ set of all interior edges/faces of } \mathcal{T}_h,\\
\mathcal{E}_h^R &:= \text{ set of all boundary edges/faces of } \mathcal{T}_h \text{ on } \Gamma_R,\\
\mathcal{E}_h^D &:= \text{ set of all boundary edges/faces of } \mathcal{T}_h \text{ on } \Gamma_D,\\
\mathcal{E}_h^{RD} &:= \mathcal{E}_h^R \cup \mathcal{E}_h^D = \text{ set of all boundary edges/faces of } \mathcal{T}_h,\\
\mathcal{E}_h^{ID} &:= \mathcal{E}_h^I \cup \mathcal{E}_h^D = \text{ set of all edges/faces of } \mathcal{T}_h \text{ except those on } \Gamma_R,\\
\mathcal{E}_h &:= \mathcal{E}_h^I \cup \mathcal{E}_h^{RD} = \text{ set of all edges/faces of } \mathcal{T}_h.
\end{aligned}$$

We also define the jump $[v]$ of v on an interior edge/face $e = \partial K \cap \partial K'$ as

$$[v]|_e := \begin{cases} v|_K - v|_{K'}, & \text{if the global label of } K \text{ is larger,} \\ v|_{K'} - v|_K, & \text{if the global label of } K' \text{ is larger.} \end{cases}$$

If $e \in \mathcal{E}_h^D$, set $[v]|_e = v|_e$. The following convention is adopted in this paper:

$$\{v\}|_e := \frac{1}{2}\big(v|_K + v|_{K'}\big) \qquad \text{if } e = \partial K \cap \partial K'.$$

If $e \in \mathcal{E}_h^{RD}$, set $\{v\}|_e = v|_e$. For every $e = \partial K \cap \partial K' \in \mathcal{E}_h^I$, let n_e be the unit outward normal to edge/face e of the element K if the global label of K is bigger and of the element K' if it is the other way around. For every $e \in \mathcal{E}_h^{RD}$, let $n_e = n_\Omega$ be the unit outward normal to $\partial\Omega$.

Let $p \geq 1$ be a fixed integer, which will be used to denote the degree of the hp-IPDG methods in this paper. For each integer $0 \leq q \leq p$, we define the "energy" space

$$E^q := \prod_{K \in \mathcal{T}_h} H^{q+1}(K),$$

and the sesquilinear form $a_h^q(\cdot,\cdot)$ on $E^q \times E^q$

$$a_h^q(u,v) := b_h(u,v) + \mathbf{i}\Big(L_1(u,v) + \sum_{j=0}^{q} J_j(u,v)\Big) \qquad \forall u, v \in E^q, \tag{2.4}$$

$$b_h(u,v) := \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e \in \mathcal{E}_h^{ID}} \left(\left\langle \left\{ \frac{\partial u}{\partial n_e} \right\}, [v] \right\rangle_e + \sigma \left\langle [u], \left\{ \frac{\partial v}{\partial n_e} \right\} \right\rangle_e \right), \tag{2.5}$$

$$L_1(u,v) := \sum_{e \in \mathcal{E}_h^{ID}} \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}\, p}{h_e} \left\langle \left[\frac{\partial u}{\partial \tau_e^\ell} \right], \left[\frac{\partial v}{\partial \tau_e^\ell} \right] \right\rangle_e, \tag{2.6}$$

$$J_0(u,v) := \sum_{e \in \mathcal{E}_h^{ID}} \frac{\gamma_{0,e}\, p}{h_e} \langle [u], [v] \rangle_e, \tag{2.7}$$

$$J_j(u,v) := \sum_{e \in \mathcal{E}_h^I} \gamma_{j,e} \left(\frac{h_e}{p} \right)^{2j-1} \left\langle \left[\frac{\partial^j u}{\partial n_e^j} \right], \left[\frac{\partial^j v}{\partial n_e^j} \right] \right\rangle_e, \qquad j = 1, 2, \cdots, q, \tag{2.8}$$

and σ is a real number. $\gamma_{0,e}, \cdots, \gamma_{q,e} > 0$ and $\beta_{1,e} \geq 0$ are numbers to be specified later. $\{\tau_e^\ell\}_{\ell=1}^{d-1}$ denote an orthogonal coordinate frame on the edge/face $e \in \mathcal{E}_h$, $\frac{\partial u}{\partial \tau_e^\ell} := \nabla u \cdot \tau_e^\ell$ stands for the tangential derivative of u in the direction τ_e^ℓ, and $\frac{\partial^j u}{\partial n_e^j}$ denotes the jth order normal derivative of u on e.

It is easy to check that $(-\Delta u, v) = a_h^q(u,v)$ for all $u \in H^{q+1}(\Omega)$ and $v \in E^q$. Hence, $a_h^q(\cdot,\cdot)$ is a consistent discretization for $-\Delta$. When $\sigma = 1$, $a_h^q(\cdot,\cdot)$ is symmetric, that is, $a_h^q(u,v) = a_h^q(\overline{v},\overline{u})$. On the other hand, when $\sigma \neq 1$, $a_h^q(\cdot,\cdot)$ is nonsymmetric. In particular, $\sigma = -1$ would correspond to the nonsymmetric IPDG method studied in [42] for coercive elliptic problems. In this paper, for the ease of presentation, we only consider the case $\sigma = 1$. The penalty constants in $\mathbf{i}\big(L_1(u,v) + J_0(u,v) + \cdots + J_q(u,v)\big)$ are $\mathbf{i}\beta_{1,e}, \mathbf{i}\gamma_{0,e}, \cdots, \mathbf{i}\gamma_{q,e}$, respectively. So they are pure imaginary numbers with positive imaginary parts. It turns out that if any of them is replaced by a complex number with positive imaginary part, the ideas of the paper still apply. Here we set their real parts to be zero because the terms from real parts do not help much (and do not cause any problem either) in our analysis.

Next, we introduce the following semi-norms on the space E^q:

$$|v|_{1,h} := \Big(\sum_{K\in\mathcal{T}_h} \|\nabla v\|^2_{L^2(K)}\Big)^{\frac12}, \tag{2.9}$$

$$\|v\|_{1,h,q} := \Bigg(|v|^2_{1,h} + \sum_{e\in\mathcal{E}_h^{ID}} \bigg(\frac{\gamma_{0,e}\,p}{h_e}\|[v]\|^2_{L^2(e)} + \sum_{\ell=1}^{d-1}\frac{\beta_{1,e}p}{h_e}\bigg\|\bigg[\frac{\partial v}{\partial \tau_e^\ell}\bigg]\bigg\|^2_{L^2(e)}\bigg) + \sum_{j=1}^{q}\sum_{e\in\mathcal{E}_h^I}\gamma_{j,e}\Big(\frac{h_e}{p}\Big)^{2j-1}\bigg\|\bigg[\frac{\partial^j v}{\partial n_e^j}\bigg]\bigg\|^2_{L^2(e)}\Bigg)^{\frac12}, \tag{2.10}$$

$$|||v|||_{1,h,q} := \Bigg(\|v\|^2_{1,h,q} + \sum_{e\in\mathcal{E}_h^{ID}}\frac{h_e}{\gamma_{0,e}\,p}\bigg\|\bigg\{\frac{\partial v}{\partial n_e}\bigg\}\bigg\|^2_{L^2(e)}\Bigg)^{\frac12}. \tag{2.11}$$

Clearly, $\|\cdot\|_{1,h,q}$ and $|||\cdot|||_{1,h,q}$ are norms on E^q if $\partial D \neq \emptyset$ but only semi-norms if $\partial D = \emptyset$.

It is easy to check that the sesquilinear form $a_h^q(\cdot,\cdot)$ satisfies: For any $v\in E^q$,

$$\operatorname{Re} a_h^q(v,v) = |v|^2_{1,h} - 2\operatorname{Re}\sum_{e\in\mathcal{E}_h^{ID}}\bigg\langle\bigg\{\frac{\partial v}{\partial n_e}\bigg\}, [v]\bigg\rangle_e, \tag{2.12}$$

$$\operatorname{Im} a_h^q(v,v) = L_1(v,v) + J_0(v,v) + \cdots + J_q(v,v). \tag{2.13}$$

Using the sesquilinear form $a_h^q(\cdot,\cdot)$ we now introduce the following weak formulation for (1.1)–(1.2): Find $u\in E^q\cap H^1_{\Gamma_D}(\Omega)\cap H^2_{\mathrm{loc}}(\Omega)$ such that

$$a_h^q(u,v) - k^2(u,v) + \mathbf{i}k\langle u,v\rangle_{\Gamma_R} = (f,v) + \langle g,v\rangle_{\Gamma_R} \tag{2.14}$$

for any $v\in E^q\cap H^1_{\Gamma_D}(\Omega)\cap H^2_{\mathrm{loc}}(\Omega)$. The above formulation is consistent with the boundary value problem (1.1)–(1.2) because $a_h^q(\cdot,\cdot)$ is consistent with $-\Delta$.

For any $K\in\mathcal{T}_h$, let $\mathcal{P}_p(K)$ denote the set of all polynomials whose degrees do not exceed p. We define our *hp*-IPDG approximation space V_h^p as

$$V_h^p := \prod_{K\in\mathcal{T}_h}\mathcal{P}_p(K).$$

Clearly, $V_h^p\subset E^q\subset L^2(\Omega)$; but $V_h^p\not\subset H^1(\Omega)$. We are now ready to define our *hp*-IPDG methods based on the weak formulation (2.14): For each $0\le q\le p$, find $u_h^q\in V_h^p$ such that

$$a_h^q(u_h^q,v_h) - k^2(u_h^q,v_h) + \mathbf{i}k\langle u_h^q,v_h\rangle_{\Gamma_R} = (f,v_h) + \langle g,v_h\rangle_{\Gamma_R} \qquad \forall v_h\in V_h^p. \tag{2.15}$$

Remark 2.1. (a) When $p=q=1$, the above method (2.15) is exactly the scheme proposed in [27]. The L_1 term, which penalizes the jumps of the first order tangential derivatives, plays an important role for getting a better (theoretical) stability estimate in [27]. However, our analysis, to be given in the next section, suggests that the L_1 term plays a less pivotal role for high order IPDG methods.

(b) In fact, (2.15) defines $p+1$ different IPDG methods for $q=0,1,\cdots,p$. $q=1$ would correspond to using high order elements in the IPDG formulation proposed in [27].

(c) The idea of penalizing the jumps of normal derivatives (i.e., the J_1 term above) for second order PDEs was used earlier by Douglas and Dupont [21] in the context of C^0 finite element methods, by Baker [9] (with a different weighting, also see [26]) for fourth order PDEs. The idea of using multipenalties $J_0, J_1, \cdots, J_p$ with positive penalty parameters was first used by Arnold in [3] for coercive elliptic and parabolic PDEs. The use of the L_1 term was first introduced in [27].

In the next two sections, we shall study the stability and error analysis for the *hp*-IPDG method (2.15). We are especially interested in knowing how the stability constants and error constants depend on the wave number k (and mesh size h and element degree p, of course) and on the penalty parameters, and what is the "optimal" relationship between mesh size h and the wave number k.

3. Stability estimates

The goal of this section is to derive stability estimates (or a priori estimates) for schemes (2.15). To this end, momentarily, we assume that the solution u_h^q to (2.15) exists and will revisit the existence and uniqueness issues later at the end of the section. We would like to note that because of its strong indefiniteness, unlike in the case of coercive elliptic and parabolic problems (cf. [3, 4, 9, 21, 26, 42, 47]), the well-posedness of scheme (2.15) is difficult to prove under practical mesh constraints.

To derive stability estimates for scheme (2.15), our approach is to mimic the stability analysis for the Helmholtz problem (1.1)–(1.2) given in [19, 20, 33]. The key ingredients of our analysis are to use a special test function $v_h = \alpha \cdot \nabla u_h^q$ (defined elementwise) with $\alpha(x) := x - x_{\Omega_1}$ in (2.15) and to use the Rellich identity (cf. [20] and below) on each element. Due to existence of multiple penalty terms in $a_h^p(\cdot,\cdot)$, which do not appear in [19, 20, 33], the analysis to be given below is much more delicate and complicated than those of [19, 20, 33], although they are similar conceptually. Since most proofs of this section are along the same lines as those of the proofs in Section 4 of [27], we shall omit some details if they are already given in [27], but shall provide them if there are meaningful differences.

We first cite the following lemma which establishes three integral identities and plays a crucial role in our analysis. A proof of the lemma can be found in [27, Lemma 4.1].

Lemma 3.1. *Let $\alpha(x) := x - x_{\Omega_1}$, $v \in E^1$, $K, K' \in \mathcal{T}_h$ and $e \in \mathcal{E}_h^{ID}$. Then it holds that*

$$d\,\|v\|_{L^2(K)}^2 + 2\,\mathrm{Re}(v, \alpha \cdot \nabla v)_K = \int_{\partial K} \alpha \cdot n_K |v|^2, \tag{3.1}$$

$$(d-2)\,\|\nabla v\|_{L^2(K)}^2 + 2\,\mathrm{Re}\left(\nabla v, \nabla(\alpha \cdot \nabla v)\right)_K = \int_{\partial K} \alpha \cdot n_K |\nabla v|^2, \tag{3.2}$$

$$\begin{aligned}\left\langle \left\{ \frac{\partial v}{\partial n_e} \right\}, [\alpha \cdot \nabla v] \right\rangle_e &- \langle \alpha \cdot n_e \{\nabla v\}, [\nabla v] \rangle_e \\ &= \sum_{\ell=1}^{d-1} \int_e \left(\alpha \cdot \tau_e^\ell \left\{ \frac{\partial v}{\partial n_e} \right\} - \alpha \cdot n_e \left\{ \frac{\partial v}{\partial \tau_e^\ell} \right\} \right) \left[\frac{\partial \overline{v}}{\partial \tau_e^\ell} \right],\end{aligned} \tag{3.3}$$

where x_{Ω_1} denotes the point in the star-shaped domain definition for Ω_1 (see Definition 2.1).

Remark 3.1. The identity (3.2) can be viewed as a local version of the Rellich identity for the Laplacian Δ (cf. [19, 20]). Since $V_h^p \subset E^1$, hence, (3.1)–(3.3) hold for any function $v = v_h \in V_h^p$.

We also need the following trace and inverse inequalities (cf. [43, 46, 12]).

Lemma 3.2. *For any $K \in \mathcal{T}_h$ and $z \in \mathcal{P}_p(K)$,*

$$\|z\|_{L^2(\partial K)} \lesssim p\, h^{-\frac{1}{2}} \|z\|_{L^2(K)}, \qquad \|\nabla z\|_{L^2(K)} \lesssim p^2\, h^{-1} \|z\|_{L^2(K)}.$$

Now, taking $v_h = u_h^q$ in (2.15) yields

$$a_h^q(u_h^q, u_h^q) - k^2 \|u_h^q\|_{L^2(\Omega)}^2 + \mathbf{i}k \|u_h^q\|_{L^2(\Gamma_R)}^2 = (f, u_h^q) + \langle g, u_h^q\rangle_{\Gamma_R}. \tag{3.4}$$

Therefore, taking the real part and the imaginary part of the above equation and using (2.12) and (2.13) we get the following lemma.

Lemma 3.3. *Let $u_h^q \in V_h^p$ solve (2.15). Then*

$$\left|u_h^q\right|_{1,h}^2 - 2\,\mathrm{Re} \sum_{e\in\mathcal{E}_h^{ID}} \left\langle \left\{\frac{\partial u_h^q}{\partial n_e}\right\}, [u_h^q] \right\rangle_e - k^2 \|u_h^q\|_{L^2(\Omega)}^2 \le \left|(f, u_h^q) + \langle g, u_h^q\rangle_{\Gamma_R}\right|, \tag{3.5}$$

$$\sum_{e\in\mathcal{E}_h^{ID}} \left(\frac{\gamma_{0,e}p}{h_e} \|[u_h^q]\|_{L^2(e)}^2 + \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}p}{h_e} \left\| \left[\frac{\partial u_h}{\partial \tau_e^\ell}\right] \right\|_{L^2(e)}^2 \right) + k \|u_h^q\|_{L^2(\Gamma_R)}^2 + \sum_{j=1}^{q} \sum_{e\in\mathcal{E}_h^I} \gamma_{j,e} \left(\frac{h_e}{p}\right)^{2j-1} \left\| \left[\frac{\partial^j u_h^q}{\partial n_e^j}\right] \right\|_{L^2(e)}^2 \le \left|(f, u_h^q) + \langle g, u_h^q\rangle_{\Gamma_R}\right|. \tag{3.6}$$

From (3.5) and (3.6) we can bound $\left|u_h^q\right|_{1,h}$ and the jumps in terms of $\|u_h^q\|_{L^2(\Omega)}^2$. In order to get the desired a priori estimates, we need to derive a reverse inequality whose coefficients can be controlled. Such a reverse inequality, which is often difficult to get under practical mesh constraints, and stability estimates for u_h^q will be derived next.

Theorem 3.1. *Let $u_h^q \in V_h^p$ solve (2.15) and suppose $\beta_{1,e} \ge 0, \gamma_{0,e}, \cdots, \gamma_{q,e} > 0$. Then*

$$\|u_h^q\|_{L^2(\Omega)} + \frac{1}{k}\|u_h^q\|_{1,h,q} + \|u_h^q\|_{L^2(\Gamma_R)} + \frac{1}{k}\Big(c_{\Omega_1} \sum_{e\in\mathcal{E}_h^R} \|\nabla u_h^q\|_{L^2(e)}^2\Big)^{\frac{1}{2}} + \frac{1}{k}\Big(\sum_{e\in\mathcal{E}_h^D} c_D \Big(k^2 \|u_h^q\|_{L^2(e)}^2 + \|\nabla u_h^q\|_{L^2(e)}^2\Big)\Big)^{\frac{1}{2}} \lesssim C_{\mathrm{sta},q}\, M(f,g), \tag{3.7}$$

where

$$M(f,g) := \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_R)}, \tag{3.8}$$

$$
\begin{aligned}
(3.9)\quad C_{\mathrm{sta},q} := \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^2}\max_{e\in\mathcal{E}_h^D}\Big(\frac{\gamma_{0,e}\,p}{h_e} + \frac{p^5}{\gamma_{0,e}h_e^2} + \frac{\beta_{1,e}\,p^5}{h_e^3} + \frac{p^2}{h_e}\Big)\\
+\begin{cases}
\dfrac{1}{k^2}\max\limits_{e\in\mathcal{E}_h^I}\Big(\dfrac{p\,k^2h_e^2+p^5}{\gamma_{0,e}\,h_e^2} + \dfrac{p}{h_e}\max\limits_{0\le j\le q-1}\sqrt{\dfrac{\gamma_{j,e}}{\gamma_{j+1,e}}} + \dfrac{p^2}{h_e} \\
\qquad\qquad + \dfrac{p^3}{h_e^2}\sqrt{\dfrac{\beta_{1,e}}{\gamma_{1,e}}} + \dfrac{\gamma_{q,e}\,p^{2q+3}}{h_e^2}\Big), \qquad \textit{if } q<p,\\
\dfrac{1}{k^2}\max\limits_{e\in\mathcal{E}_h^I}\Big(\dfrac{p\,k^2h_e^2+p^5}{\gamma_{0,e}\,h_e^2} + \dfrac{p}{h_e}\max\limits_{0\le j\le q-1}\sqrt{\dfrac{\gamma_{j,e}}{\gamma_{j+1,e}}} + \dfrac{p^2}{h_e} \\
\qquad\qquad + \dfrac{p^3}{h_e^2}\sqrt{\dfrac{\beta_{1,e}}{\gamma_{1,e}}}\Big), \qquad \textit{if } q=p.
\end{cases}
\end{aligned}
$$

Proof. We divide the proof into three steps.

Step 1: Derivation of a representation identity for $\|u_h^q\|_{L^2(\Omega)}$. Define $v_h \in E$ by $v_h|_K = \alpha\cdot\nabla u_h^q|_K$ for every $K\in\mathcal{T}_h$. Since $v_h|_K$ is a polynomial of degree no more than p on K, hence, $v_h\in V_h^p$. Using this v_h as a test function in (2.15) and taking the real part of the resulted equation we get

$$
(3.10)\qquad -k^2\,\mathrm{Re}(u_h^q, v_h) = \mathrm{Re}\big((f,v_h) + \langle g, v_h\rangle_{\Gamma_R} - a_h^p(u_h^q, v_h) - \mathbf{i}k\,\langle u_h^q, v_h\rangle_{\Gamma_R}\big).
$$

It follows from (3.1), (3.4), and (3.10) that (compare with (4.11) of [27])

$$
\begin{aligned}
(3.11)\quad 2k^2\,\|u_h^q\|_{L^2(\Omega)}^2 &= k^2\sum_{K\in\mathcal{T}_h}\int_{\partial K}\alpha\cdot n_K\big|u_h^q\big|^2 + (d-2)\,\mathrm{Re}\big((f,u_h^q) + \langle g, u_h^q\rangle_{\Gamma_R}\\
&\quad - a_h^p(u_h^q,u_h^q)\big) + 2\,\mathrm{Re}\big((f,v_h) + \langle g,v_h\rangle_{\Gamma_R} - a_h^p(u_h^q,v_h) - \mathbf{i}k\,\langle u_h^q, v_h\rangle_{\Gamma_R}\big)\\
&= k^2\sum_{K\in\mathcal{T}_h}\int_{\partial K}\alpha\cdot n_K\big|u_h^q\big|^2 + (d-2)\,\mathrm{Re}\big((f,u_h^q) + \langle g,u_h^q\rangle_{\Gamma_R}\big)\\
&\quad + 2\,\mathrm{Re}\big((f,v_h) + \langle g,v_h\rangle_{\Gamma_R}\big) + 2k\,\mathrm{Im}\,\langle u_h^q, v_h\rangle_{\Gamma_R}\\
&\quad - \sum_{K\in\mathcal{T}_h}\Big((d-2)\,\|\nabla u_h^q\|_{L^2(K)}^2 + 2\,\mathrm{Re}(\nabla u_h^q, \nabla v_h)_K\Big)\\
&\quad + 2\sum_{e\in\mathcal{E}_h^{ID}}\bigg((d-2)\,\mathrm{Re}\Big\langle\Big\{\frac{\partial u_h^q}{\partial n_e}\Big\}, [u_h^q]\Big\rangle_e + \mathrm{Re}\Big\langle\Big\{\frac{\partial u_h^q}{\partial n_e}\Big\}, [v_h]\Big\rangle_e\\
&\quad + \mathrm{Re}\Big\langle [u_h^q], \Big\{\frac{\partial v_h}{\partial n_e}\Big\}\Big\rangle_e\bigg) + 2\,\mathrm{Im}\Big(L_1(u_h^q, v_h) + \sum_{j=0}^{q} J_j(u_h^q, v_h)\Big).
\end{aligned}
$$

Using the identity $\big|a\big|^2 - \big|b\big|^2 = \mathrm{Re}(a+b)(\bar a - \bar b)$ we have

$$
(3.12)\qquad \sum_{K\in\mathcal{T}_h}\int_{\partial K}\alpha\cdot n_K\big|u_h^q\big|^2 = 2\sum_{e\in\mathcal{E}_h^I}\mathrm{Re}\,\langle\alpha\cdot n_e\,\{u_h^q\}, [u_h^q]\rangle_e + \big\langle\alpha\cdot n_\Omega, |u_h^q|^2\big\rangle_{\partial\Omega}.
$$

Using the identity again followed by the Rellich identity (3.2) we get (compare with (4.13) of [27])

$$\begin{aligned}
(3.13)\quad \sum_{K\in\mathcal{T}_h}\Big((d-2)\,\|\nabla u_h^q\|_{L^2(K)}^2 + 2\operatorname{Re}(\nabla u_h^q,\nabla v_h)_K\Big) &= \sum_{K\in\mathcal{T}_h}\int_{\partial K}\alpha\cdot n_K\big|\nabla u_h^q\big|^2\\
&= 2\sum_{e\in\mathcal{E}_h^I}\operatorname{Re}\left\langle\alpha\cdot n_e\left\{\nabla u_h^q\right\},[\nabla u_h^q]\right\rangle_e + \sum_{e\in\mathcal{E}_h^{RD}}\left\langle\alpha\cdot n_e,|\nabla u_h^q|^2\right\rangle_e\\
&= 2\sum_{e\in\mathcal{E}_h^{ID}}\operatorname{Re}\left\langle\alpha\cdot n_e\left\{\nabla u_h^q\right\},[\nabla u_h^q]\right\rangle_e + \sum_{e\in\mathcal{E}_h^{R}}\left\langle\alpha\cdot n_e,|\nabla u_h^q|^2\right\rangle_e\\
&\qquad - \sum_{e\in\mathcal{E}_h^{D}}\left\langle\alpha\cdot n_e,|\nabla u_h^q|^2\right\rangle_e.
\end{aligned}$$

Plugging (3.12) and (3.13) into (3.11) gives (compare with (4.15) of [27])

$$\begin{aligned}
(3.14)\quad & 2k^2\,\|u_h^q\|_{L^2(\Omega)}^2\\
&= (d-2)\operatorname{Re}\big((f,u_h^q)+\langle g,u_h^q\rangle_{\Gamma_R}\big) + 2\operatorname{Re}\big((f,v_h)+\langle g,v_h\rangle_{\Gamma_R}\big)\\
&\quad + 2k^2\sum_{e\in\mathcal{E}_h^I}\operatorname{Re}\left\langle\alpha\cdot n_e\left\{u_h^q\right\},[u_h^q]\right\rangle_e + k^2\left\langle\alpha\cdot n_\Omega,|u_h^q|^2\right\rangle_{\partial\Omega}\\
&\quad + 2k\operatorname{Im}\left\langle u_h^q,v_h\right\rangle_{\Gamma_R} - \sum_{e\in\mathcal{E}_h^R}\left\langle\alpha\cdot n_e,|\nabla u_h^q|^2\right\rangle_e + \sum_{e\in\mathcal{E}_h^D}\left\langle\alpha\cdot n_e,|\nabla u_h^q|^2\right\rangle_e\\
&\quad - 2\sum_{e\in\mathcal{E}_h^{ID}}\operatorname{Re}\left\langle\left\{\frac{\partial u_h^q}{\partial n_e}\right\},[u_h^q]\right\rangle_e + 2(d-1)\sum_{e\in\mathcal{E}_h^{ID}}\operatorname{Re}\left\langle\left\{\frac{\partial u_h^q}{\partial n_e}\right\},[u_h^q]\right\rangle_e\\
&\quad + 2\sum_{e\in\mathcal{E}_h^{ID}}\operatorname{Re}\left(-\left\langle\alpha\cdot n_e\left\{\nabla u_h^q\right\},[\nabla u_h^q]\right\rangle_e + \left\langle\left\{\frac{\partial u_h^q}{\partial n_e}\right\},[v_h]\right\rangle_e\right)\\
&\quad + 2\sum_{e\in\mathcal{E}_h^{ID}}\operatorname{Re}\left\langle[u_h^q],\left\{\frac{\partial v_h}{\partial n_e}\right\}\right\rangle_e + 2\operatorname{Im}\Big(L_1(u_h^q,v_h)+\sum_{j=0}^{q}J_j(u_h^q,v_h)\Big).
\end{aligned}$$

Step 2: Derivation of a reverse inequality. Our task now is to estimate each term on the right-hand side of (3.14). Since the terms on the first four lines can be bounded in the exact same way as in [27], we omit their derivations and only give the final results here for the reader's convenience.

$$\begin{aligned}
(3.15)\qquad & 2\operatorname{Re}\big((f,v_h)+\langle g,v_h\rangle_{\Gamma_R}\big)\\
&\qquad\le CM(f,g)^2 + \frac{1}{8}\big|u_h^q\big|_{1,h}^2 + \frac{c_{\Omega_1}}{4}\sum_{e\in\mathcal{E}_h^R}\|\nabla u_h^q\|_{L^2(e)}^2.
\end{aligned}$$

$$\begin{aligned}
(3.16)\qquad & 2k^2\sum_{e\in\mathcal{E}_h^I}\operatorname{Re}\left\langle\alpha\cdot n_e\left\{u_h^q\right\},[u_h^q]\right\rangle_e\\
&\qquad\le \frac{k^2}{3}\,\|u_h^q\|_{L^2(\Omega)}^2 + C\sum_{e\in\mathcal{E}_h^I}\frac{p\,k^2}{\gamma_{0,e}}\frac{\gamma_{0,e}p}{h_e}\,\|[u_h^q]\|_{L^2(e)}^2.
\end{aligned}$$

$$(3.17)\qquad k^2\left\langle\alpha\cdot n_\Omega,|u_h^q|^2\right\rangle_{\partial\Omega} \le Ck^2\,\|u_h^q\|_{L^2(\Gamma_R)}^2 + \sum_{e\in\mathcal{E}_h^D}k^2\left\langle\alpha\cdot n_e,|u_h^q|^2\right\rangle_e.$$

$$2k\,\mathrm{Im}\,\langle u_h^q, v_h\rangle_{\Gamma_R} - \sum_{e\in\mathcal{E}_h^R}\left\langle \alpha\cdot n_e, |\nabla u_h^q|^2\right\rangle_e \le Ck^2\,\|u_h^q\|_{L^2(\Gamma_R)}^2 - \frac{c_{\Omega_1}}{2}\sum_{e\in\mathcal{E}_h^R}\|\nabla u_h^q\|_{L^2(e)}^2. \tag{3.18}$$

$$2(d-1)\sum_{e\in\mathcal{E}_h^{ID}}\mathrm{Re}\left\langle \left\{\frac{\partial u_h^q}{\partial n_e}\right\}, [u_h^q]\right\rangle_e \le \frac{1}{8}\left|u_h^q\right|_{1,h}^2 + C\sum_{e\in\mathcal{E}_h^{ID}}\frac{p}{\gamma_{0,e}}\frac{\gamma_{0,e}\,p}{h_e}\,\|[u_h^q]\|_{L^2(e)}^2. \tag{3.19}$$

The extra work here is to estimate the terms on the last two lines in (3.14). For an edge/face $e\in\mathcal{E}_h^{ID}$, let Ω_e denote the set of element(s) in $\mathcal{T}_h$ containing e as one edge/face. By (3.3) we obtain

$$\begin{aligned}
&2\sum_{e\in\mathcal{E}_h^{ID}}\mathrm{Re}\left(-\left\langle \alpha\cdot n_e\left\{\nabla u_h^q\right\}, [\nabla u_h^q]\right\rangle_e + \left\langle\left\{\frac{\partial u_h^q}{\partial n_e}\right\}, [v_h]\right\rangle_e\right)\\
&=2\sum_{e\in\mathcal{E}_h^{ID}}\sum_{\ell=1}^{d-1}\mathrm{Re}\int_e\left(\alpha\cdot\tau_e^\ell\left\{\frac{\partial u_h^q}{\partial n_e}\right\} - \alpha\cdot n_e\left\{\frac{\partial u_h^q}{\partial \tau_e^\ell}\right\}\right)\left[\frac{\partial \overline{u_h^q}}{\partial \tau_e^\ell}\right]\\
&\lesssim \sum_{e\in\mathcal{E}_h^{ID}} p^3 h_e^{-\frac{3}{2}}\sum_{K\in\Omega_e}\|\nabla u_h^q\|_{L^2(K)}\,\|[u_h^q]\|_{L^2(e)}\\
&\le\frac{1}{8}\left|u_h^q\right|_{1,h}^2 + C\sum_{e\in\mathcal{E}_h^{ID}}\frac{p^5}{\gamma_{0,e}h_e^2}\frac{\gamma_{0,e}\,p}{h_e}\,\|[u_h^q]\|_{L^2(e)}^2.
\end{aligned}\tag{3.20}$$

The first term on line six of (3.14) is bounded as follows (compare with (4.20) of [27]):

$$\begin{aligned}
2\sum_{e\in\mathcal{E}_h^{ID}}\mathrm{Re}\left\langle [u_h^q], \left\{\frac{\partial v_h}{\partial n_e}\right\}\right\rangle_e &\lesssim \sum_{e\in\mathcal{E}_h^{ID}} p\,h_e^{-\frac{1}{2}}\,\|[u_h^q]\|_{L^2(e)}\sum_{K\in\Omega_e}\|\nabla v_h\|_{L^2(K)}\\
&\lesssim \sum_{e\in\mathcal{E}_h^{ID}} p^3 h_e^{-\frac{3}{2}}\,\|[u_h^q]\|_{L^2(e)}\sum_{K\in\Omega_e}\|\nabla u_h^q\|_{L^2(K)}\\
&\le\frac{1}{8}\left|u_h^q\right|_{1,h}^2 + C\sum_{e\in\mathcal{E}_h^{ID}}\frac{p^5}{\gamma_{0,e}h_e^2}\frac{\gamma_{0,e}\,p}{h_e}\,\|[u_h^q]\|_{L^2(e)}^2.
\end{aligned}\tag{3.21}$$

The penalty term $L_1(\cdot,\cdot)$ is estimated as follows. Recall that $v_h|_K = \alpha\cdot\nabla u_h^q|_K$ with $\alpha = x - x_{\Omega_1}$ for each $K\in\mathcal{T}_h$. Noting that

$$\begin{aligned}
\frac{\partial v_h}{\partial \tau_e^\ell} &= \frac{\partial u_h^q}{\partial \tau_e^\ell} + \alpha\cdot\nabla\Big(\frac{\partial u_h^q}{\partial \tau_e^\ell}\Big)\\
&= \frac{\partial u_h^q}{\partial \tau_e^\ell} + \alpha\cdot n_e\frac{\partial}{\partial \tau_e^\ell}\Big(\frac{\partial u_h^q}{\partial n_e}\Big) + \sum_{m=1}^{d-1}\alpha\cdot\tau_e^m\frac{\partial}{\partial \tau_e^m}\Big(\frac{\partial u_h^q}{\partial \tau_e^\ell}\Big), \quad 1\le\ell\le d-1,
\end{aligned}\tag{3.22}$$

by the definition of $L_1(\cdot,\cdot)$ and Lemma 3.2 we get

$$
\begin{aligned}
2\,\mathrm{Im}\, L_1(u_h^q, v_h) &= 2\,\mathrm{Im} \sum_{e\in\mathcal{E}_h^{ID}} \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}p}{h_e} \Bigg(\left\langle \left[\frac{\partial u_h^q}{\partial \tau_e^\ell}\right], \left[\alpha\cdot n_e \frac{\partial}{\partial \tau_e^\ell}\Big(\frac{\partial u_h^q}{\partial n_e}\Big)\right]\right\rangle_e \\
&\qquad + \sum_{m=1}^{d-1} \left\langle \left[\frac{\partial u_h^q}{\partial \tau_e^\ell}\right], \left[\alpha\cdot \tau_e^m \frac{\partial}{\partial \tau_e^m}\left(\frac{\partial u_h^q}{\partial \tau_e^\ell}\right)\right]\right\rangle_e \Bigg) \\
&\le 2\,\mathrm{Im} \sum_{e\in\mathcal{E}_h^{D}} \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}p}{h_e} \left\langle \alpha\cdot n_e \frac{\partial u_h^q}{\partial \tau_e^\ell}, \frac{\partial}{\partial \tau_e^\ell}\Big(\frac{\partial u_h^q}{\partial n_e}\Big)\right\rangle_e \\
&\quad + C \sum_{e\in\mathcal{E}_h^{I}} \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}p^3}{h_e^2} \left\| \left[\frac{\partial u_h^q}{\partial \tau_e^\ell}\right]\right\|_{L^2(e)} \left\| \left[\frac{\partial u_h^q}{\partial n_e}\right]\right\|_{L^2(e)} \\
&\quad + C \sum_{e\in\mathcal{E}_h^{ID}} \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}p^3}{h_e^2} \left\| \left[\frac{\partial u_h^q}{\partial \tau_e^\ell}\right]\right\|_{L^2(e)} \left\| \left[\frac{\partial u_h^q}{\partial \tau_e^\ell}\right]\right\|_{L^2(e)} \\
&\le 2\,\mathrm{Im} \sum_{e\in\mathcal{E}_h^{D}} \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}p}{h_e} \left\langle \alpha\cdot n_e \frac{\partial u_h^q}{\partial \tau_e^\ell}, \frac{\partial}{\partial \tau_e^\ell}\Big(\frac{\partial u_h^q}{\partial n_e}\Big)\right\rangle_e \\
&\quad + C \sum_{e\in\mathcal{E}_h^{I}} \sum_{\ell=1}^{d-1} \frac{p^3}{h_e^2}\sqrt{\frac{\beta_{1,e}}{\gamma_{1,e}}} \left(\frac{\gamma_{1,e}h_e}{p} \left\| \left[\frac{\partial u_h^q}{\partial n_e}\right]\right\|_{L^2(e)}^2 + \frac{\beta_{1,e}p}{h_e} \left\| \left[\frac{\partial u_h^q}{\partial \tau_e^\ell}\right]\right\|_{L^2(e)}^2 \right) \\
&\quad + C \sum_{e\in\mathcal{E}_h^{ID}} \sum_{\ell=1}^{d-1} \frac{p^2}{h_e}\frac{\beta_{1,e}p}{h_e} \left\| \left[\frac{\partial u_h^q}{\partial \tau_e^\ell}\right]\right\|_{L^2(e)}^2 .
\end{aligned} \tag{3.23}
$$

We remark that $\mathrm{Im}\, L_1(u_h^q, v_h) = 0$ when $p = q = 1$.

Next we estimate the penalty terms $J_j(u_h^q, v_h)$. Since u_h^q and v_h are piecewise polynomials of degree p in general and those terms contain jumps of high order normal derivatives, it is quite delicate to control those terms as shown below.

By direct calculations we get that on each edge/face e of $K \in \mathcal{T}_h$,

$$
\begin{aligned}
\frac{\partial^j v_h}{\partial n_e^j} &= j\frac{\partial^j u_h^q}{\partial n_e^j} + \alpha\cdot\nabla\Big(\frac{\partial^j u_h^q}{\partial n_e^j}\Big) \\
&= j\frac{\partial^j u_h^q}{\partial n_e^j} + \alpha\cdot n_e \frac{\partial^{j+1} u_h^q}{\partial n_e^{j+1}} + \sum_{m=1}^{d-1} \alpha\cdot\tau_e^m \frac{\partial}{\partial \tau_e^m}\Big(\frac{\partial^j u_h^q}{\partial n_e^j}\Big), \quad 1\le j\le p-1,
\end{aligned} \tag{3.24}
$$

$$
\frac{\partial^p v_h}{\partial n_e^p} = p\frac{\partial^p u_h^q}{\partial n_e^p} + \alpha\cdot\nabla\Big(\frac{\partial^p u_h^q}{\partial n_e^p}\Big) = p\frac{\partial^p u_h^q}{\partial n_e^p}. \tag{3.25}
$$

Here we have used the fact that $(p+1)$th order derivatives of u_h^q are zero because u_h^q is a polynomial of degree at most p.

For $j = 1, 2, \cdots, q-1$, by (3.24) we have

$$
\begin{aligned}
(3.26)\quad 2\,\mathrm{Im}\, J_j(u_h^q, v_h) &= 2\,\mathrm{Im} \sum_{e\in\mathcal{E}_h^I} \gamma_{j,e}\left(\frac{h_e}{p}\right)^{2j-1} \Bigg(\alpha\cdot n_e \left\langle \left[\frac{\partial^j u_h^q}{\partial n_e^j}\right], \left[\frac{\partial^{j+1} u_h^q}{\partial n_e^{j+1}}\right]\right\rangle_e \\
&\qquad + \sum_{\ell=1}^{d-1} \alpha\cdot\tau_e^\ell \left\langle \left[\frac{\partial^j u_h^q}{\partial n_e^j}\right], \left[\frac{\partial}{\partial \tau_e^\ell}\Big(\frac{\partial^j u_h^q}{\partial n_e^j}\Big)\right]\right\rangle_e \Bigg) \\
&\lesssim \sum_{e\in\mathcal{E}_h^I} \gamma_{j,e}\left(\frac{h_e}{p}\right)^{2j-1} \left\|\left[\frac{\partial^j u_h^q}{\partial n_e^j}\right]\right\|_{L^2(e)} \\
&\qquad \times \left(\left\|\left[\frac{\partial^{j+1} u_h^q}{\partial n_e^{j+1}}\right]\right\|_{L^2(e)} + \frac{p^2}{h_e}\left\|\left[\frac{\partial^j u_h^q}{\partial n_e^j}\right]\right\|_{L^2(e)}\right) \\
&\lesssim \sum_{e\in\mathcal{E}_h^I} \frac{p}{h_e}\sqrt{\frac{\gamma_{j,e}}{\gamma_{j+1,e}}} \left(\gamma_{j,e}\left(\frac{h_e}{p}\right)^{2j-1} \left\|\left[\frac{\partial^j u_h^q}{\partial n_e^j}\right]\right\|_{L^2(e)}^2\right. \\
&\qquad \left. +\gamma_{j+1,e}\left(\frac{h_e}{p}\right)^{2j+1} \left\|\left[\frac{\partial^{j+1} u_h^q}{\partial n_e^{j+1}}\right]\right\|_{L^2(e)}^2\right) \\
&\qquad + \sum_{e\in\mathcal{E}_h^I} \frac{p^2}{h_e}\gamma_{j,e}\left(\frac{h_e}{p}\right)^{2j-1} \left\|\left[\frac{\partial^j u_h^q}{\partial n_e^j}\right]\right\|_{L^2(e)}^2.
\end{aligned}
$$

If $q < p$, then, from Lemma 3.2 and the inequality $\left\|\frac{\partial^q \varphi}{\partial n_e^q}\right\|_{L^2(\partial K)} \lesssim p h_e^{-\frac{1}{2}} |\varphi|_{H^q(K)}$, we have

$$
\begin{aligned}
(3.27)\quad \mathrm{Im}\, 2J_q(u_h^q, v_h) &\lesssim \sum_{e\in\mathcal{E}_h^I} \gamma_{q,e}\left(\frac{h_e}{p}\right)^{2q-1} \left\|\frac{\partial^q u_h^q}{\partial n_e^q}\right\|_{L^2(e)} p\, h_e^{-\frac{1}{2}} \sum_{K\in\Omega_e} |v_h|_{H^q(K)} \\
&\lesssim \sum_{e\in\mathcal{E}_h^I} \gamma_{q,e}\left(\frac{h_e}{p}\right)^{2q-1} \frac{p^{2q+1}}{h_e^{q+\frac{1}{2}}} \left\|\frac{\partial^q u_h^q}{\partial n_e^q}\right\|_{L^2(e)} \sum_{K\in\Omega_e} \|\nabla u_h^q\|_{L^2(K)} \\
&\le \frac{1}{8}|u_h^q|_{1,h}^2 + C\sum_{e\in\mathcal{E}_h^I} \frac{\gamma_{q,e}\, p^{2q+3}}{h_e^2} \gamma_{q,e}\left(\frac{h_e}{p}\right)^{2q-1} \left\|\frac{\partial^q u_h^q}{\partial n_e^q}\right\|_{L^2(e)}^2.
\end{aligned}
$$

If $q = p$, (3.25) and the definition of $J_p(\cdot,\cdot)$ immediately imply that (compare with (4.14) of [27])

$$
(3.28)\qquad 2\,\mathrm{Im}\, J_p(u_h^q, v_h) = 2\,\mathrm{Im}\, J_p(u_h^q, u_h^q) = 0.
$$

The estimate for $\mathrm{Im}\, J_0(u_h^q, v_h)$ is similar to (3.26), so we get

$$
(3.29)\quad 2\,\mathrm{Im}\, J_0(u_h^q, v_h) = 2\,\mathrm{Im} \sum_{e\in\mathcal{E}_h^{ID}} \frac{\gamma_{0,e}\, p}{h_e} \left\langle [u_h^q], \left[\alpha\cdot n_e \frac{\partial u_h^q}{\partial n_e} + \sum_{j=1}^{d-1} \alpha\cdot\tau_e^j \frac{\partial u_h^q}{\partial \tau_e^j}\right]\right\rangle_e
$$

$$
\begin{aligned}
&\le 2\,\mathrm{Im} \sum_{e\in\mathcal{E}_h^D} \frac{\gamma_{0,e}\, p}{h_e} \left\langle \alpha\cdot n_e u_h^q, \frac{\partial u_h^q}{\partial n_e} \right\rangle_e \\
&\quad + C \sum_{e\in\mathcal{E}_h^I} \frac{\gamma_{0,e}\, p}{h_e} \|[u_h^q]\|_{L^2(e)} \left\| \left[\frac{\partial u_h^q}{\partial n_e} \right] \right\|_{L^2(e)} \\
&\quad + C \sum_{e\in\mathcal{E}_h^{ID}} \frac{\gamma_{0,e}\, p}{h_e} \|[u_h^q]\|_{L^2(e)} \frac{p^2}{h_e} \|[u_h^q]\|_{L^2(e)} \\
&\le 2\,\mathrm{Im} \sum_{e\in\mathcal{E}_h^D} \frac{\gamma_{0,e}\, p}{h_e} \left\langle \alpha\cdot n_e u_h^q, \frac{\partial u_h^q}{\partial n_e} \right\rangle_e + C \sum_{e\in\mathcal{E}_h^{ID}} \frac{p^2}{h_e} \frac{\gamma_{0,e}\, p}{h_e} \|[u_h^q]\|_{L^2(e)}^2 \\
&\quad + C \sum_{e\in\mathcal{E}_h^I} \frac{p}{h_e} \sqrt{\frac{\gamma_{0,e}}{\gamma_{1,e}}} \left(\frac{\gamma_{0,e}\, p}{h_e} \|[u_h^q]\|_{L^2(e)}^2 + \frac{\gamma_{1,e} h_e}{p} \left\| \left[\frac{\partial u_h^q}{\partial n_e} \right] \right\|_{L^2(e)}^2 \right).
\end{aligned}
$$

We also need the following estimate (compare with (4.22) of [27])

$$
\begin{aligned}
(3.30)\quad & \sum_{e\in\mathcal{E}_h^D} \Bigg(k^2 \left\langle \alpha\cdot n_e, |u_h^q|^2 \right\rangle_e + \left\langle \alpha\cdot n_e, |\nabla u_h^q|^2 \right\rangle_e \\
&\quad + \sum_{\ell=1}^{d-1} \frac{2\beta_{1,e} p}{h_e} \mathrm{Im} \left\langle \alpha\cdot n_e \frac{\partial u_h^q}{\partial \tau_e^\ell}, \frac{\partial}{\partial \tau_e^\ell}\Big(\frac{\partial u_h^q}{\partial n_e}\Big) \right\rangle_e \\
&\quad + \frac{2\gamma_{0,e}\, p}{h_e} \mathrm{Im} \left\langle \alpha\cdot n_e u_h^q, \frac{\partial u_h^q}{\partial n_e} \right\rangle_e \Bigg) \\
&\le - \sum_{e\in\mathcal{E}_h^D} \Bigg\langle \alpha\cdot n_D, k^2 |u_h^q|^2 + |\nabla u_h^q|^2 - 2 \sum_{\ell=1}^{d-1} \frac{\beta_{1,e} p}{h_e} \left| \frac{\partial u_h^q}{\partial \tau_e^\ell} \right| \left| \frac{\partial}{\partial \tau_e^\ell}\Big(\frac{\partial u_h^q}{\partial n_e}\Big) \right| \\
&\quad - 2\frac{\gamma_{0,e}\, p}{h_e} |u_h^q| |\nabla u_h^q| \Bigg\rangle_e \\
&\le -c_D \sum_{e\in\mathcal{E}_h^D} \Big(k^2 \|u_h^q\|_{L^2(e)}^2 + \frac{1}{2} \|\nabla u_h^q\|_{L^2(e)}^2 \Big) \\
&\quad + C \sum_{e\in\mathcal{E}_h^D} \frac{\beta_{1,e}\, p^5}{h_e^3} \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}\, p}{h_e} \left\| \frac{\partial u_h^q}{\partial \tau_e^\ell} \right\|_{L^2(e)}^2 + C \sum_{e\in\mathcal{E}_h^D} \frac{\gamma_{0,e}\, p}{h_e} \frac{\gamma_{0,e}\, p}{h_e} \|u_h^q\|_{L^2(e)}^2,
\end{aligned}
$$

where we have used the inverse inequality and the assumption that D is star-shaped to derive the last inequality.

Step 3*: Finishing up.* We only prove the case of $q = p$ since the proof for $q < p$ is the same except using (3.27) instead of (3.28). Substituting (3.26), (3.28), (3.29) (with $q = p$) and (3.15)–(3.19) into (3.14), and using (3.30) we obtain

$$
\begin{aligned}
&2k^2 \|u_h^p\|_{L^2(\Omega)}^2 \\
&\le (d-2)\,\mathrm{Re}\big((f, u_h^p) + \langle g, u_h^p\rangle_{\Gamma_R}\big) + CM(f,g)^2 + \frac{k^2}{3} \|u_h^p\|_{L^2(\Omega)}^2 \\
&\quad - \frac{3}{8} |u_h^p|_{1,h}^2 + |u_h^p|_{1,h}^2 - 2 \sum_{e\in\mathcal{E}_h^{ID}} \mathrm{Re} \left\langle \left\{ \frac{\partial u_h^p}{\partial n_e} \right\}, [u_h^p] \right\rangle_e
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{c_{\Omega_1}}{4} \sum_{e\in\mathcal{E}_h^R} \|\nabla u_h^p\|_{L^2(e)}^2 + Ck^2 \|u_h^p\|_{L^2(\Gamma_R)}^2 \\
&- c_D \sum_{e\in\mathcal{E}_h^D} \Big(k^2 \|u_h^p\|_{L^2(e)}^2 + \frac{1}{2}\|\nabla u_h^p\|_{L^2(e)}^2 \Big) \\
&+ C \sum_{e\in\mathcal{E}_h^I} \left(\frac{p\,(k^2+1)}{\gamma_{0,e}} + \frac{p^5}{\gamma_{0,e} h_e^2} + \frac{p^2}{h_e} \right) \frac{\gamma_{0,e}\, p}{h_e} \|[u_h^p]\|_{L^2(e)}^2 \\
&+ C \sum_{e\in\mathcal{E}_h^D} \left(\frac{\gamma_{0,e}\, p}{h_e} + \frac{p^5}{\gamma_{0,e} h_e^2} + \frac{p^2}{h_e} \right) \frac{\gamma_{0,e}\, p}{h_e} \|[u_h^p]\|_{L^2(e)}^2 \\
&+ C \sum_{e\in\mathcal{E}_h^D} \frac{\beta_{1,e}\, p^5}{h_e^3} \sum_{\ell=1}^{d-1} \frac{\beta_{1,e}\, p}{h_e} \left\| \frac{\partial u_h^p}{\partial \tau_e^\ell} \right\|_{L^2(e)}^2 \\
&+ C \sum_{e\in\mathcal{E}_h^I} \sum_{\ell=1}^{d-1} \frac{p^3}{h_e^2} \sqrt{\frac{\beta_{1,e}}{\gamma_{1,e}}} \left(\frac{\gamma_{1,e} h_e}{p} \left\| \left[\frac{\partial u_h^p}{\partial n_e} \right] \right\|_{L^2(e)}^2 + \frac{\beta_{1,e} p}{h_e} \left\| \left[\frac{\partial u_h^p}{\partial \tau_e^\ell} \right] \right\|_{L^2(e)}^2 \right) \\
&+ C \sum_{e\in\mathcal{E}_h^{ID}} \sum_{\ell=1}^{d-1} \frac{p^2}{h_e} \frac{\beta_{1,e} p}{h_e} \left\| \left[\frac{\partial u_h^p}{\partial \tau_e^\ell} \right] \right\|_{L^2(e)}^2 \\
&+ C \sum_{j=1}^{p-1} \sum_{e\in\mathcal{E}_h^I} \frac{p^2}{h_e} \gamma_{j,e} \left(\frac{h_e}{p} \right)^{2j-1} \left\| \left[\frac{\partial^j u_h^p}{\partial n_e^j} \right] \right\|_{L^2(e)}^2 \\
&+ C \sum_{j=0}^{p-1} \sum_{e\in\mathcal{E}_h^I} \frac{p}{h_e} \sqrt{\frac{\gamma_{j,e}}{\gamma_{j+1,e}}} \left(\gamma_{j,e} \left(\frac{h_e}{p} \right)^{2j-1} \left\| \left[\frac{\partial^j u_h^p}{\partial n_e^j} \right] \right\|_{L^2(e)}^2 \right. \\
&\qquad\qquad\qquad \left. + \gamma_{j+1,e} \left(\frac{h_e}{p} \right)^{2j+1} \left\| \left[\frac{\partial^{j+1} u_h^p}{\partial n_e^{j+1}} \right] \right\|_{L^2(e)}^2 \right).
\end{aligned}
$$

Therefore, it follows from Lemma 3.3 and (3.9) that

$$
\begin{aligned}
&2k^2 \|u_h^p\|_{L^2(\Omega)}^2 + \frac{3}{8} |u_h^p|_{1,h}^2 + \frac{c_{\Omega_1}}{4} \sum_{e\in\mathcal{E}_h^R} \|\nabla u_h^p\|_{L^2(e)}^2 \\
&\qquad + c_D \sum_{e\in\mathcal{E}_h^D} \Big(k^2 \|u_h^p\|_{L^2(e)}^2 + \frac{1}{2}\|\nabla u_h^p\|_{L^2(e)}^2 \Big) \\
&\le CM(f,g)^2 + \frac{4k^2}{3} \|u_h^p\|_{L^2(\Omega)}^2 + Ck^2 C_{\mathrm{sta},p} \big| (f, u_h^p) + \langle g, u_h^p \rangle_{\Gamma_R} \big| \\
&\le Ck^2 C_{\mathrm{sta},p}^2 M(f,g)^2 + \frac{5k^2}{3} \|u_h^p\|_{L^2(\Omega)}^2 ,
\end{aligned}
$$

where we have used the following inequality, which is a consequence of (3.6),

$$
k^2 \|u_h^p\|_{L^2(\Gamma_R)}^2 \le k^2 \|u_h^p\|_{L^2(\Omega)}^2 + M(f,g)^2
$$

to derive the last inequality. $M(f,g)$ and $C_{\mathrm{sta},p}$ are defined by (3.8) and (3.9), respectively. Hence,

$$\begin{aligned}\|u_h^p\|_{L^2(\Omega)} + \frac{1}{k}|u_h^p|_{1,h} + \frac{1}{k}\Big(c_{\Omega_1}\sum_{e\in\mathcal{E}_h^R}\|\nabla u_h^p\|^2_{L^2(e)}\Big)^{\frac12}\\ + \frac{1}{k}\Big(\sum_{e\in\mathcal{E}_h^D} c_D\Big(k^2\|u_h^p\|^2_{L^2(e)} + \|\nabla u_h^p\|^2_{L^2(e)}\Big)\Big)^{\frac12} \lesssim C_{\mathrm{sta},p}M(f,g),\end{aligned}$$

which together with (3.6) gives (3.7). The proof is completed. □

As (2.15) can be written as a linear system, an immediate consequence of the above stability estimate is the following well-posedness theorem for (2.15).

Theorem 3.2. *For $k>0$ and $h>0$, the hp-IPDG method* (2.15) *has a unique solution u_h^q provided that $\gamma_{0,e},\gamma_{1,e},\cdots,\gamma_{q,e}>0$ and $\beta_{1,e}\ge 0$.*

Next we consider the case of quasi-uniform meshes. Note that large penalty parameters ($\gamma_j, j\ge 1$) for jumps of normal derivatives may cause a large interpolation error in the norm $\|\cdot\|_{1,h,q}$, and hence, may pollute the error estimates of the IPDG solution (see Section 4). It is interesting to minimize the stability constant $C_{\mathrm{sta},q}$ under the constraints of $\beta_{1,e}\ge 0$ and $\frac{p}{\gamma_{0,e}}+\sum_{j=1}^q p^{2j-1}\gamma_{j,e}\lesssim 1$. We have the following consequence of Theorem 3.1. The proof is straightforward and is omitted.

Theorem 3.3. *Let $h=\max h_e$. Suppose the mesh $\mathcal{T}_h$ is quasi-uniform, that is, $h_e\simeq h$. Suppose $k\gtrsim 1$ and $kh\lesssim 1$. Assume that $\gamma_{j,e}\simeq\gamma_j, j=0,1,\cdots,q$, $\sum_{j=1}^q p^{2j-1}\gamma_j\lesssim 1$, $0\le\beta_{1,e}\lesssim\frac{h^2}{p^4}\gamma_0$, and that*

$$\begin{cases}\gamma_0\simeq p^{\frac73}h^{-\frac23},\quad \gamma_j\gtrsim p^{-\frac{10}{3}}h^{\frac23}\,\gamma_{j-1} & \text{if } q=p,\\ \gamma_0\simeq\min\Big\{p^{\frac{3q+1}{q+1}}h^{-\frac{q}{q+1}},p^{\frac73}h^{-\frac23}\Big\},\quad \gamma_j\gtrsim\Big(\frac{\gamma_0\,h}{p^4}\Big)^2\gamma_{j-1},\quad \gamma_q\lesssim\frac{1}{\gamma_0p^{2q-2}} & \text{if } q<p.\end{cases}$$

Suppose $D=\emptyset$ if $q\ge 2$ or $q=p=1$. Then

$$\|u_h^q\|_{L^2(\Omega)}+\frac1k\|u_h^q\|_{1,h,q}\lesssim C_{\mathrm{sta},q}M(f,g),$$

where

$$C_{\mathrm{sta},q}\lesssim\begin{cases}\dfrac{p^{\frac83}}{k^2h^{\frac43}} & \text{if } q=p,\\ \max\Big\{\dfrac{p^{\frac83}}{k^2h^{\frac43}},\dfrac{p^{\frac{2q+4}{q+1}}}{k^2h^{\frac{q+2}{q+1}}}\Big\} & \text{if } q<p.\end{cases}$$

It is clear that, in the above theorem, $\gamma_0\simeq p^{\frac73}h^{-\frac23}$ if $q=p$ or $q\ge 2$, $\gamma_0\simeq p^{\frac{3q+1}{q+1}}h^{-\frac{q}{q+1}}$ otherwise. We conclude this section with several remarks.

Remark 3.2. (a) The *hp*-IPDG method (2.15) is well-posed for all $h,k>0$ provided that all penalty parameters are positive. As a comparison, we recall that the standard finite element method is well-posed only if mesh size h satisfies a constraint $h=O(k^{-\rho})$ for some $\rho\ge 1$, hence, the existence is only guaranteed for very small mesh size h when wave number k is large.

(b) It is well known that [3, 4, 9, 26, 42, 47] symmetric IPDG methods for coercive elliptic and parabolic PDEs often require the penalty parameter $\gamma_{0,e}$ is sufficiently large to guarantee the well-posedness of numerical solutions, and the low bound for

$\gamma_{0,e}$ is theoretically hard to determine and is also problem-dependent. However, this is no issue for scheme (2.15), which solves the (indefinite) Helmholtz equation, because they are well-posed for all $\gamma_{0,e} > 0$.

(c) In the linear element case of $q = p = 1$, a better estimate is obtained in [27] with the help of the penalty term L_1. Unfortunately the penalty term L_1 does not help very much for higher order elements.

(d) The stability estimates will be improved greatly when $k^3h^2p^{-2} \le C_0$, where C_0 is some constant independent of k, h, p, and the penalty parameters (see Theorem 5.1). The above estimates are only interesting when $k\,h \lesssim 1$ and $k^3h^2p^{-2} \gtrsim 1$.

(e) We can see that choosing $q = p$ is helpful for the stability analysis (cf. (3.26)–(3.28) in the proof of Theorem 3.1 and Theorem 3.3). On the other hand, noting that if the jump of the hp-IPDG solution u_h^p (a piecewise polynomial of degree p or less) and the jumps of its j-th normal derivatives, $j = 1, \cdots, p$, are all zeros, then u_h^p is a polynomial of degree p or less on the whole domain Ω. Therefore, the hp-IPDG method with $q = p$ can be viewed as an "interior penalty discontinuous spectral method". Numerical tests for the linear element case in [27] shows that penalizing the jump of normal derivative (i.e. $p = q = 1$) improves the stability for $k^3h^2 \gtrsim 1$ and gives better numerical solutions. Numerical tests for the higher order element case will be given in a separate work.

4. Error estimates

In this section, we derive the error estimates of the solutions of scheme (2.15). This will be done in two steps. First, we introduce elliptic projections of the PDE solution u and derive error estimates for the projections. We note that such a result also has an independent interest. Second, we bound the error between the projections and the IPDG solutions by making use of the stability results obtained in Section 3. In this and the next section, we assume that the mesh $\mathcal{T}_h$ is quasi-uniform, that $\gamma_{j,e} \simeq \gamma_j > 0$ for $j = 0, 1, \cdots, q$, and that $\beta_{1,e} \simeq \beta_1 \ge 0$. Let $h = \max h_e$. For simplicity, we also assume that the mesh $\mathcal{T}_h$ is conforming, that is, $\mathcal{T}_h$ contains no hanging nodes, since the parallel results for nonconforming meshes can be derived in a similar way.

4.1. **Elliptic projection and its error estimates.**

For any $w \in E^q \cap H^1_{\Gamma_D}(\Omega) \cap H^2_{\mathrm{loc}}(\Omega)$, we define its elliptic projection $\tilde{w}_h^q \in V_h^p$ by

$$a_h^q(\tilde{w}_h^q, v_h) + \mathbf{i}k \left\langle \tilde{w}_h^q, v_h \right\rangle_{\Gamma_R} = a_h^q(w, v_h) + \mathbf{i}k \left\langle w, v_h \right\rangle_{\Gamma_R} \qquad \forall v_h \in V_h^p. \tag{4.1}$$

In other words, $\tilde{w}_h^q$ is an IPDG approximation to the solution w of the following (complex-valued) Poisson problem:

$$\begin{aligned} -\Delta w &= F && \text{in } \Omega, \\ \frac{\partial w}{\partial n_{\Gamma_R}} + \mathbf{i}kw &= \psi && \text{on } \Gamma_R, \\ w &= 0 && \text{on } \Gamma_D \end{aligned}$$

for some given functions F and ψ which are determined by w.

Before estimating the projection error, we state the following continuity and coercivity properties for the sesquilinear form $a_h^q(\cdot,\cdot)$. Since they follow easily from (2.4)–(2.13), so we omit their proofs to save space.

Lemma 4.1. *For any $v \in E^q$ and $w \in E^q \cap H^1_{\Gamma_D}$, the mesh-dependent sesquilinear form $a_h^q(\cdot,\cdot)$ satisfies*

$$|a_h^q(v,w)|, |a_h^q(w,v)| \lesssim \|v\|_{1,h,q} \, |||w|||_{1,h,q}. \tag{4.2}$$

In addition, for any $0<\varepsilon<1$, there exists a positive constant c_ε independent of k, h, p, and the penalty parameters such that

$$\operatorname{Re} a_h^q(v_h,v_h) + \Big(1-\varepsilon+\frac{c_\varepsilon\, p}{\gamma_0}\Big)\operatorname{Im} a_h^q(v_h,v_h) \ge (1-\varepsilon)\,\|v_h\|^2_{1,h,q} \qquad \forall v_h \in V_h^p. \tag{4.3}$$

To estimate the projection error, we also need the following approximation properties of the space $V_h^p \cap H^1_{\Gamma_D}(\Omega)$.

Lemma 4.2.

(i) *Let $\mu = \min\{p+1, s\}$ and $q<\mu$. Suppose $u \in H^s(\Omega)\cap H^1_{\Gamma_D}(\Omega)$. Then there exists $\hat u_h \in V_h^p \cap H^1_{\Gamma_D}(\Omega)$ such that*

$$\|u-\hat u_h\|_{L^2(\Gamma_R)} \lesssim \frac{h^{\mu-\frac12}}{p^{s-\frac12}}\,\|u\|_{H^s(\Omega)}, \tag{4.4}$$

$$|||u-\hat u_h|||_{1,h,q} \lesssim \Big(1+\frac{p}{\gamma_0}+\sum_{j=1}^q p^{2j-1}\gamma_j\Big)^{\frac12}\,\frac{h^{\mu-1}}{p^{s-1}}\,\|u\|_{H^s(\Omega)}. \tag{4.5}$$

(ii) *Suppose $u \in H^2(\Omega)\cap H^1_{\Gamma_D}(\Omega)$. Then there exists $\hat u_h \in V_h^p\cap H^1_{\Gamma_D}(\Omega)$ such that (4.4) holds with $s=2$ and*

$$\begin{aligned}\Big(|||u-\hat u_h|||^2_{1,h,0} + \sum_{j=1}^q J_j(\hat u_h,\hat u_h)\Big)^{\frac12} \\ \lesssim \Big(1+\frac{p}{\gamma_0}+p\,\gamma_1+\sum_{j=2}^q p^{2j-2}\gamma_j\Big)^{\frac12}\,\frac{h}{p}\,\|u\|_{H^2(\Omega)},\end{aligned} \tag{4.6}$$

where $\gamma_1=0$ if $q=0$. Note that the left-hand side of (4.6) equals $|||u-\hat u_h|||_{1,h,q}$ if $u\in H^{\max\{q+1,2\}}(\Omega)\cap H^1_{\Gamma_D}(\Omega)$.

Proof. The following hp approximation properties are well-known for the hp finite element functions (cf. [8, 30, 31]):

- There exists $\check u_h \in V_h^p$ such that, for $j=0,1,\cdots,s$,

$$\|u-\check u_h\|_{H^j(\mathcal{T}_h)} := \Big(\sum_{K\in\mathcal{T}_h}\|u-\check u_h\|^2_{H^j(K)}\Big)^{\frac12} \lesssim \frac{h^{\mu-j}}{p^{s-j}}\,\|u\|_{H^s(\Omega)}. \tag{4.7}$$

- There exists $\hat u_h \in V_h^p \cap H^1_{\Gamma_D}(\Omega)$ such that

$$\|u-\hat u_h\|_{H^j(\Omega)} \lesssim \frac{h^{\mu-j}}{p^{s-j}}\,\|u\|_{H^s(\Omega)}, \quad j=0,1. \tag{4.8}$$

Here the invisible constants in the two inequalities above depend on s but are independent of k, h, p, and the penalty parameters. Then (4.4) follows from (4.8) and the trace inequality.

It follows from the inverse inequalities in Lemma 3.2 that, for $1 \le j \le q+1$,

$$\begin{aligned} \|u - \hat{u}_h\|_{H^j(\mathcal{T}_h)} &\le \|u - \check{u}_h\|_{H^j(\mathcal{T}_h)} + \|\check{u}_h - \hat{u}_h\|_{H^j(\mathcal{T}_h)} \\ &\lesssim \|u - \check{u}_h\|_{H^j(\mathcal{T}_h)} + \frac{p^{2(j-1)}}{h^{j-1}} \|\check{u}_h - \hat{u}_h\|_{H^1(\Omega)} \\ &\lesssim \frac{h^{\mu-j}}{p^{s-2j+1}} \|u\|_{H^s(\Omega)} . \end{aligned} \tag{4.9}$$

Therefore, by the following local trace inequality,

$$\|v\|^2_{L^2(\partial K)} \lesssim h_K^{-1} \|v\|^2_{L^2(K)} + \|v\|_{L^2(K)} \|\nabla v\|_{L^2(K)} ,$$

we have

$$\begin{aligned} &\sum_{K\in\mathcal{T}_h} \sum_{e\subset\partial K} \left\| \frac{\partial^j (u-\hat{u}_h)}{\partial n_e^j} \right\|^2_{L^2(e)} \\ &\lesssim h^{-1} \|u-\hat{u}_h\|^2_{H^j(\mathcal{T}_h)} + \|u-\hat{u}_h\|_{H^j(\mathcal{T}_h)} \|u-\hat{u}_h\|_{H^{j+1}(\mathcal{T}_h)} \\ &\lesssim \frac{h^{2\mu-2j-1}}{p^{2s-4j}} \|u\|^2_{H^s(\Omega)} . \end{aligned} \tag{4.10}$$

Noting that $\hat{u}_h$ is continuous, we have from (4.8) and (4.10),

$$\begin{aligned} |||u-\hat{u}_h|||^2_{1,h,q} = &\left|u-\hat{u}_h\right|^2_{1,h} + \sum_{j=1}^{q} \sum_{e\in\mathcal{E}_h^I} \gamma_{j,e} \left(\frac{h_e}{p}\right)^{2j-1} \left\| \left[\frac{\partial^j (u-\hat{u}_h)}{\partial n_e^j} \right] \right\|^2_{L^2(e)} \\ &+ \sum_{e\in\mathcal{E}_h^{ID}} \frac{h_e}{\gamma_{0,e}\, p} \left\| \left\{ \frac{\partial (u-\hat{u}_h)}{\partial n_e} \right\} \right\|^2_{L^2(e)} \\ \lesssim &\left(1 + \frac{p}{\gamma_0} + \sum_{j=1}^{q} p^{2j-1}\gamma_j\right) \frac{h^{2\mu-2}}{p^{2s-2}} \|u\|^2_{H^s(\Omega)} . \end{aligned} \tag{4.11}$$

That is, (4.5) holds.

(4.6) can be proved similarly as above. It is clear that (4.8) and (4.9) hold with $s = 2$ and (4.10) holds with $s = 2$ and $j = 1$, that is,

$$\|u-\hat{u}_h\|_{H^j(\Omega)} \lesssim \frac{h^{2-j}}{p^{2-j}} \|u\|_{H^2(\Omega)} , \quad j = 0, 1, \tag{4.12}$$

$$\|u-\hat{u}_h\|_{H^2(\mathcal{T}_h)} \lesssim p \|u\|_{H^2(\Omega)} , \tag{4.13}$$

$$\sum_{e\in\mathcal{E}_h^I} \left\| \left[\frac{\partial \hat{u}_h}{\partial n_e} \right] \right\|^2_{L^2(e)} \lesssim \sum_{K\in\mathcal{T}_h} \sum_{e\subset\partial K} \left\| \frac{\partial (u-\hat{u}_h)}{\partial n_e} \right\|^2_{L^2(e)} \lesssim h \|u\|^2_{H^2(\Omega)} . \tag{4.14}$$

We have from Lemma 3.2 and (4.13) that, for $2 \le j \le q$,

$$\begin{aligned} \sum_{e\in\mathcal{E}_h^I} \left\| \left[\frac{\partial^j \hat{u}_h}{\partial n_e^j} \right] \right\|^2_{L^2(e)} &\lesssim \sum_{K\in\mathcal{T}_h} \sum_{e\subset\partial K} \left\| \frac{\partial^j \hat{u}_h}{\partial n_e^j} \right\|^2_{L^2(e)} \lesssim \frac{p^2}{h} \|\hat{u}_h\|^2_{H^j(\mathcal{T}_h)} \\ &\lesssim \left(\frac{p^2}{h}\right)^{2j-3} \|\hat{u}_h\|^2_{H^2(\mathcal{T}_h)} \lesssim \left(\frac{p}{h}\right)^{2j-3} p^{2j-2} \|u\|^2_{H^2(\mathcal{T}_h)} . \end{aligned} \tag{4.15}$$

Now (4.6) follows by combining (4.12)–(4.15). This completes the proof of the lemma. $\square$

Let u be the solution of problem (1.1)–(1.3) and $\tilde{u}_h^q$ its elliptic projection defined as above. Define $\eta := u - \tilde{u}_h^q$. Then (4.1) immediately implies the following Galerkin orthogonality:

$$a_h^q(\eta, v_h) + \mathbf{i}k \langle \eta, v_h \rangle_{\Gamma_R} = 0 \qquad \forall v_h \in V_h^p. \tag{4.16}$$

Lemma 4.3. *Suppose problem (1.1)–(1.3) is $H^{\max\{q+1,2\}}$-regular. Then the following error estimate holds:*

$$\begin{aligned} &\|u - \tilde{u}_h^q\|_{1,h,q} + \sqrt{\lambda k}\, \|u - \tilde{u}_h^q\|_{L^2(\Gamma_R)} \\ &\qquad \lesssim \inf_{z_h \in V_h^p \cap H^1_{\Gamma_D}(\Omega)} \Big(\lambda |||u - z_h|||_{1,h,q} + \sqrt{\lambda k}\, \|u - z_h\|_{L^2(\Gamma_R)} \Big), \end{aligned} \tag{4.17}$$

Moreover, suppose that the following Poisson problem is H^2-regular in the sense that for any $F \in L^2(\Omega)$ there is a unique $w \in H^2(\Omega)$ such that

$$-\Delta w = F \qquad \text{in } \Omega, \tag{4.18}$$

$$\frac{\partial w}{\partial n} - \mathbf{i}kw = 0 \qquad \text{on } \Gamma_R, \tag{4.19}$$

$$w = 0 \qquad \text{on } \Gamma_D, \tag{4.20}$$

and

$$|w|_{H^2(\Omega)} \lesssim \|F\|_{L^2(\Omega)}. \tag{4.21}$$

Then there also holds the estimate

$$\begin{aligned} \|u - \tilde{u}_h^q\|_{L^2(\Omega)} \lesssim & \frac{h}{p} \Big(1 + \frac{p}{\gamma_0} + p\,\gamma_1 + \sum_{j=2}^{q} p^{2j-2}\gamma_j + \frac{kh}{\lambda p} \Big)^{\frac12} \\ & \times \inf_{z_h \in V_h^p \cap H^1_{\Gamma_D}(\Omega)} \Big(\lambda |||u - z_h|||_{1,h,q} + \sqrt{\lambda k}\, \|u - z_h\|_{L^2(\Gamma_R)} \Big), \end{aligned} \tag{4.22}$$

where $\lambda := 1 + \frac{p}{\gamma_0}$ and $\gamma_1 = 0$ if $q = 0$.

Proof. For any $z_h \in V_h^p \cap H^1_{\Gamma_D}(\Omega)$, let $\eta_h = \tilde{u}_h^q - z_h$. From $\eta_h + \eta = u - z_h$ and (4.16), we have

$$a_h^q(\eta_h, \eta_h) + \mathbf{i}k \langle \eta_h, \eta_h \rangle_{\Gamma_R} = a_h^q(u - z_h, \eta_h) + \mathbf{i}k \langle u - z_h, \eta_h \rangle_{\Gamma_R}. \tag{4.23}$$

Take $\varepsilon = \frac12$ in (4.3) and assume, without loss of generality, that $c_{\frac12} > \frac12$. It follows from (4.3) and (4.23) that

$$\begin{aligned} \frac12 \|\eta_h\|_{1,h,q}^2 \le & \operatorname{Re} a_h^q(\eta_h, \eta_h) + \Big(\frac12 + \frac{c_{\frac12}\, p}{\gamma_0} \Big) \operatorname{Im} a_h^q(\eta_h, \eta_h) \\ = & \operatorname{Re}\Big(a_h^q(u - z_h, \eta_h) + \mathbf{i}k \langle u - z_h, \eta_h \rangle_{\Gamma_R} \Big) - \Big(\frac12 + \frac{c_{\frac12}\, p}{\gamma_0} \Big) k \langle \eta_h, \eta_h \rangle_{\Gamma_R} \\ & + \Big(\frac12 + \frac{c_{\frac12}\, p}{\gamma_0} \Big) \operatorname{Im} \Big(a_h^q(u - z_h, \eta_h) + \mathbf{i}k \langle u - z_h, \eta_h \rangle_{\Gamma_R} \Big) \\ \le & C\lambda \Big(\|\eta_h\|_{1,h,q} |||u - z_h|||_{1,h,q} + k \|u - z_h\|_{L^2(\Gamma_R)}^2 \Big) - \frac{\lambda k}{4} \|\eta_h\|_{L^2(\Gamma_R)}^2. \end{aligned}$$

Therefore,

$$\|\eta_h\|_{1,h,q}^2 + \lambda k \|\eta_h\|_{L^2(\Gamma_R)}^2 \lesssim \lambda^2 |||u - z_h|||_{1,h,q}^2 + \lambda k \|u - z_h\|_{L^2(\Gamma_R)}^2 \tag{4.24}$$

which together with $\eta = u - z_h - \eta_h$ yields (4.17).

To show (4.22), we use the standard Nitsche's duality argument (cf. [11, 15]). Let w be the solution of (4.18)–(4.20) with $F := \eta$ and let $\hat{w}_h$ be defined in Lemma 4.2(ii) (with u replaced by w). From (4.16),

$$a_h^0(\eta, \hat{w}_h) + \mathbf{i}k \left\langle \eta, \hat{w}_h \right\rangle_{\Gamma_R} = -\mathbf{i} \sum_{j=1}^{q} J_j(\eta, \hat{w}_h).$$

Testing the conjugated (4.18) with $F = \eta$ by η and using the above equality and Lemma 4.2(ii) we get

$$\begin{aligned}
&\|\eta\|_{L^2(\Omega)}^2 = a_h^0(\eta, w) + \mathbf{i}k \left\langle \eta, w \right\rangle_{\Gamma_R} \\
&= a_h^0(\eta, w - \hat{w}_h) + \mathbf{i}k \left\langle \eta, w - \hat{w}_h \right\rangle_{\Gamma_R} - \mathbf{i} \sum_{j=1}^{q} J_j(\eta, \hat{w}_h) \\
&\lesssim \|\eta\|_{1,h,0} \, |||w - \hat{w}_h|||_{1,h,0} + k \|\eta\|_{L^2(\Gamma_R)} \|w - \hat{w}_h\|_{L^2(\Gamma_R)} \\
&\quad + \sum_{j=1}^{q} J_j(\eta, \eta)^{\frac12} J_j(\hat{w}_h, \hat{w}_h)^{\frac12} \\
&\lesssim \|\eta\|_{1,h,q} \Big(|||w - \hat{w}_h|||_{1,h,0}^2 + \sum_{j=1}^{q} J_j(\hat{w}_h, \hat{w}_h) \Big)^{\frac12} + k \|\eta\|_{L^2(\Gamma_R)} \|w - \hat{w}_h\|_{L^2(\Gamma_R)} \\
&\lesssim \|\eta\|_{1,h,q} \Big(1 + \frac{p}{\gamma_0} + p\,\gamma_1 + \sum_{j=2}^{q} p^{2j-2} \gamma_j \Big)^{\frac12} \frac{h}{p} |w|_{H^2(\Omega)} + k \|\eta\|_{L^2(\Gamma_R)} \, \frac{h^{\frac32}}{p^{\frac32}} |w|_{H^2(\Omega)},
\end{aligned}$$

which together with (4.17) and (4.21) gives (4.22). The proof is completed. □

Remark 4.1. The regularity assumption on problem (4.18)–(4.20) generally imposes a restriction on the domain Ω (hence on the scatterer D). If Γ_R and Γ_D are smooth or $D = \emptyset$ and Ω is a convex polygon, then the assumption is known to be true (cf. [11]). But for general domain Ω, one only gets $H^{1+\alpha}$ regularity for some $\alpha \in (0, 1]$, as a result, (4.22) then has to be replaced by a suboptimal estimate. We also note that the invisible constant in the estimate (4.21) depends on the domain Ω, the dependence is complicated unless Ω has a very simple geometry.

By combining Lemma 4.3 and Lemma 4.2(i) we have the following estimates for the projection error.

Lemma 4.4. *Let $\mu = \min\{p+1, s\}$ and $q < \mu$. Suppose problem* (1.1)–(1.3) *is H^s-regular and* (4.18)–(4.20) *is H^2-regular. Then the following estimates hold:*

$$\|u - \tilde{u}_h^q\|_{1,h,q} + \sqrt{\lambda k}\, \|u - \tilde{u}_h^q\|_{L^2(\Gamma_R)} \lesssim C_{\mathrm{err},q} \, \frac{h^{\mu-1}}{p^{s-1}} \|u\|_{H^s(\Omega)}, \tag{4.25}$$

$$\|u - \tilde{u}_h^q\|_{L^2(\Omega)} \lesssim \widehat{C}_{\mathrm{err},q} \, \frac{h^{\mu}}{p^{s}} \|u\|_{H^s(\Omega)}, \tag{4.26}$$

where

$$C_{\mathrm{err},q} := \lambda\Big(1 + \frac{p}{\gamma_0} + \sum_{j=1}^{q} p^{2j-1}\gamma_j + \frac{kh}{\lambda p}\Big)^{\frac12},$$

$$\widehat{C}_{\mathrm{err},q} := \Big(1 + \frac{p}{\gamma_0} + p\,\gamma_1 + \sum_{j=2}^{q} p^{2j-2}\gamma_j + \frac{kh}{\lambda p}\Big)^{\frac12} C_{\mathrm{err},q}, \qquad \lambda := 1 + \frac{p}{\gamma_0}.$$

Remark 4.2. The requirement $q < s$ in the lemma is clear since the projection $\tilde{u}_h^q$ is not defined for $q > s$. However, for $q < s$, $\|u - \tilde{u}_h^q\|_{1,h,q}$ can be bounded without using full regularity of u, and such a bound is also useful (see Lemma 5.1).

4.2. **Error estimates for** $u - u_h^q$**.** In this subsection we shall derive error estimates for scheme (2.15). This will be done by exploiting the linearity of the Helmholtz equation and making use of the stability estimates derived in Theorem 3.1 and the projection error estimates established in Lemma 4.4.

Let u and u_h^q denote the solutions of (1.1)–(1.3) and (2.15), respectively. Assume that $u \in H^s(\Omega)$ with $s \ge q+1$, then (2.14) holds for $v_h \in V_h^p$. Define the error function $e_h := u - u_h^q$. Subtracting (2.15) from (2.14) yields the following error equation:

$$a_h^q(e_h, v_h) - k^2(e_h, v_h) + \mathbf{i}k\langle e_h, v_h\rangle_{\Gamma_R} = 0 \qquad \forall v_h \in V_h^p. \tag{4.27}$$

Let $\tilde{u}_h^q$ be the elliptic projection of u as defined in the previous subsection. Write $e_h = \eta - \xi$ with $\eta := u - \tilde{u}_h^q, \xi := u_h^q - \tilde{u}_h^q$. From (4.27) and (4.16) we get

$$\begin{aligned} a_h^q(\xi, v_h) - k^2(\xi, v_h) + \mathbf{i}k\langle \xi, v_h\rangle_{\Gamma_R} &= a_h^q(\eta, v_h) - k^2(\eta, v_h) + \mathbf{i}k\langle \eta, v_h\rangle_{\Gamma_R} \\ &= -k^2(\eta, v_h) \qquad \forall v_h \in V_h^p. \end{aligned} \tag{4.28}$$

The above equation implies that $\xi \in V_h^p$ is the solution of scheme (2.15) with source terms $f = -k^2\eta$ and $g \equiv 0$. Then an application of Theorem 3.1 and Lemma 4.4 immediately gives

Lemma 4.5. $\xi = u_h^q - \tilde{u}_h^q$ *satisfies the following estimate:*

$$\|\xi\|_{L^2(\Omega)} + \frac{1}{k}\,\|\xi\|_{1,h,q} \lesssim C_{\mathrm{sta},q}\, k^2\, \widehat{C}_{\mathrm{err},q}\, \frac{h^\mu}{p^s}\, \|u\|_{H^s(\Omega)}. \tag{4.29}$$

We are ready to state our error estimate results for scheme (2.15), which follows from Lemma 4.5, Lemma 4.4 and an application of the triangle inequality.

Theorem 4.1. *Let* u *and* u_h^q *denote the solutions of* (1.1)–(1.3) *and* (2.15)*, respectively. Suppose* $u \in H^s(\Omega) \cap H^1_{\Gamma_D}(\Omega)$*. Let* $\mu = \min\{p+1, s\}$ *and* $q < \mu$*. Then under the assumptions of Lemma 4.4 we have*

$$\|u - u_h^q\|_{1,h,q} \lesssim \Big(C_{\mathrm{err},q} + \frac{k^3 h}{p}\, C_{\mathrm{sta},q}\widehat{C}_{\mathrm{err},q}\Big)\, \frac{h^{\mu-1}}{p^{s-1}}\, \|u\|_{H^s(\Omega)}, \tag{4.30}$$

$$\|u - u_h^q\|_{L^2(\Omega)} \lesssim \widehat{C}_{\mathrm{err},q}\, \Big(1 + k^2 C_{\mathrm{sta},q}\Big)\, \frac{h^\mu}{p^s}\, \|u\|_{H^s(\Omega)}. \tag{4.31}$$

Remark 4.3. $q < s$ is required in the theorem because $\|u - u_h^q\|_{1,h,q}$ is not defined for $q > s$. However, we note that the *hp*-IPDG solution u_h^q is always well-defined regardless the regularity of underlying PDE solution u. For $q < s$, $\|u - u_h^q\|_{1,h,q}$ can be bounded without using full regularity of u, and such a bound is also useful (see Lemma 5.2).

By combining Theorem 4.1 and Theorem 3.3 we have the following theorem that gives the best convergence order so far which we can obtain theoretically for the method (2.15) under the mesh condition $k^3h^2p^{-2} \gtrsim 1$ (cf. Theorem 5.1).

Theorem 4.2. *Under the assumptions of Theorem* 3.3 *and* 4.1*, we have*

$$\|u-u_h^q\|_{1,h,q} + k\,\|u-u_h^q\|_{L^2(\Omega)} \lesssim \begin{cases} k\,p^{\frac{5}{3}}h^{-\frac{1}{3}}\,\frac{h^{\mu-1}}{p^{s-1}}\,\|u\|_{H^s(\Omega)} & \text{if } q=p,\\ k\,\max\left\{p^{\frac{5}{3}}h^{-\frac{1}{3}}, p^{\frac{q+3}{q+1}}h^{-\frac{1}{q+1}}\right\}\,\frac{h^{\mu-1}}{p^{s-1}}\,\|u\|_{H^s(\Omega)} & \text{if } q<p. \end{cases} \tag{4.32}$$

Proof. The proof is obvious since $C_{\mathrm{err},q} \simeq \widehat{C}_{\mathrm{err},q} \simeq 1$. □

Remark 4.4. (a) Estimates (4.30)–(4.32) are so-called preasymptotic error estimates which are suboptimal in h and k. They can be improved to optimal order when $k^3h^2p^{-2} \le C_0$, where C_0 is some constant independent of k, h, p, and the penalty parameters (see Theorem 5.1). The second term on the right-hand side of (4.30) is called a pollution term for $\|u-u_h^q\|_{1,h,q}$.

(b) Suppose $\|u\|_{H^s(\Omega)} \lesssim k^{s-1}$. Then Theorem 4.2 shows that $\|u-u_h^q\|_{1,h,q} + k\,\|u-u_h^q\|_{L^2(\Omega)} \to 0$ if $q=p$ and $p^{\frac{8}{3}-s}k^sh^{\mu-\frac{4}{3}} \to 0$, or if $q=1<p$ and $p^{3-s}k^sh^{\mu-\frac{3}{2}} \to 0$.

5. Stability-error iterative improvement

In this section we derive some improved optimal order stability and error estimates for the hp-IPDG solution under the mesh condition that $k^3h^2p^{-2} \le C_0$ by using a stability-error iterative procedure, where C_0 is some constant independent of k, h, p, and the penalty parameters (see Theorem 5.1).

By combining Lemma 4.3 and Lemma 4.2(ii), we have the following estimates for the projection error when only the H^2-norm of the solution u is allowed in the error bound.

Lemma 5.1. *Suppose problem* (1.1)–(1.3) *is* $H^{\max\{q+1,2\}}$*-regular and* (4.18)–(4.20) *is* H^2*-regular. Then the following estimates hold:*

$$\|u-\tilde{u}_h^q\|_{1,h,q} + \sqrt{\lambda k}\,\|u-\tilde{u}_h^q\|_{L^2(\Gamma_R)} \lesssim C_{\mathrm{err},2,q}\,M(f,g)\,\frac{k\,h}{p}, \tag{5.1}$$

$$\|u-\tilde{u}_h^q\|_{L^2(\Omega)} \lesssim \widehat{C}_{\mathrm{err},2,q}\,M(f,g)\,\frac{k\,h^2}{p^2}, \tag{5.2}$$

where

$$C_{\mathrm{err},2,q} := \lambda\Big(1+\frac{p}{\gamma_0}+p\,\gamma_1+\sum_{j=2}^{q}p^{2j-2}\gamma_j+\frac{kh}{\lambda p}\Big)^{\frac{1}{2}},$$

$$\widehat{C}_{\mathrm{err},2,q} := \lambda\Big(1+\frac{p}{\gamma_0}+p\,\gamma_1+\sum_{j=2}^{q}p^{2j-2}\gamma_j+\frac{kh}{\lambda p}\Big), \qquad \lambda := 1+\frac{p}{\gamma_0}.$$

By a similar argument to that used to prove Theorem 4.1, we have the following error bounds which only involves $M(f,g)$.

Lemma 5.2. *Let u and u_h^q denote the solutions of (1.1)–(1.3) and (2.15), respectively. Suppose $u \in H^{\max\{q+1,2\}}(\Omega) \cap H^1_{\Gamma_D}(\Omega)$ and (4.18)–(4.20) is H^2-regular. Then*

$$\|u - u_h^q\|_{1,h,q} \lesssim \Big(C_{\mathrm{err},2,q} + \frac{k^3 h}{p}\, C_{\mathrm{sta},q}\widehat{C}_{\mathrm{err},2,q}\Big)\, M(f,g)\, \frac{k\,h}{p}, \tag{5.3}$$

$$\|u - u_h^q\|_{L^2(\Omega)} \lesssim \widehat{C}_{\mathrm{err},2,q}\, \Big(1 + k^2 C_{\mathrm{sta},q}\Big)\, M(f,g)\, \frac{k\,h^2}{p^2}. \tag{5.4}$$

We are now ready to state our final main theorem of this paper.

Theorem 5.1. *Let u and u_h^q denote the solutions of (1.1)–(1.3) and (2.15), respectively. Suppose $u \in H^{\max\{q+1,2\}}(\Omega) \cap H^1_{\Gamma_D}(\Omega)$ and (4.18)–(4.20) is H^2-regular. Assume that $k \gtrsim 1$, $k\,h \lesssim 1$, and that $p\gamma_0{}^{-1} + \sum_{j=1}^q p^{2j-1}\gamma_j \lesssim 1$. Then there exists a constant $C_0 > 0$, which is independent of k, h, p, and the penalty parameters, such that if $k^3h^2p^{-2} \le C_0$, then the following stability estimates hold:*

$$\|u_h^q\|_{1,h,q} \lesssim M(f,g), \tag{5.5}$$

$$\|u_h^q\|_{L^2(\Omega)} \lesssim \frac{1}{k} M(f,g). \tag{5.6}$$

Moreover, if $u \in H^s(\Omega) \cap H^1_{\Gamma_D}(\Omega)$, then the following error estimates hold:

$$\|u - u_h^q\|_{1,h,q} \lesssim \Big(1 + \frac{k^2 h}{p}\Big)\, \frac{h^{\mu-1}}{p^{s-1}}\, \|u\|_{H^s(\Omega)}\,, \tag{5.7}$$

$$\|u - u_h^q\|_{L^2(\Omega)} \lesssim k\, \frac{h^{\mu}}{p^s}\, \|u\|_{H^s(\Omega)}\,. \tag{5.8}$$

Proof. We only prove (5.5) since (5.6) can be proved similarly and (5.7)–(5.8) follow from the improved stability estimates and the argument used in the proof of Theorem 4.1.

From Theorem 3.1 we have

$$\|u_h^q\|_{1,h,q} \lesssim k\, C_{\mathrm{sta},q}\, M(f,g), \tag{5.9}$$

where $C_{\mathrm{sta},q}$ is defined in (3.9). From Lemma 5.2 we have

$$\|u - u_h^q\|_{1,h,q} \lesssim \Big(C_{\mathrm{err},2,q} + \frac{k^2 h}{p}\, k\, C_{\mathrm{sta},q}\widehat{C}_{\mathrm{err},2,q}\Big) \frac{k\,h}{p}\, M(f,g). \tag{5.10}$$

Now it follows from Theorem 2.1 and the triangle inequality that

$$\begin{aligned} \|u_h^q\|_{1,h,q} &\le \|u\|_{1,h,q} + \|u - u_h^q\|_{1,h,q} = |u|_{1,h} + \|u - u_h^q\|_{1,h,q} \\ &\lesssim \Big(1 + C_{\mathrm{err},2,q}\, \frac{k\,h}{p} + \frac{k^3 h^2}{p^2} \widehat{C}_{\mathrm{err},2,q}\, k\, C_{\mathrm{sta},q}\Big) M(f,g). \end{aligned} \tag{5.11}$$

Repeating the above process yields that there exists a constant C_1 independent of k, h, p, and the penalty parameters, and a sequence of positive numbers Λ_j such that

$$\|u_h^q\|_{1,h,q} \le \Lambda_j M(f,g), \tag{5.12}$$

with

$$\Lambda_0 \simeq k\, C_{\mathrm{sta},q}, \quad \Lambda_j = C_1\Big(1 + C_{\mathrm{err},2,q}\, \frac{k\,h}{p}\Big) + C_1\, \widehat{C}_{\mathrm{err},2,q}\, \frac{k^3 h^2}{p^2}\, \Lambda_{j-1}, \quad j = 1, 2, \cdots.$$

A simple calculation yields that if $C_1\,\widehat{C}_{\mathrm{err},2,q}\,k^3h^2 < \theta\,p^2$ for some positive constant $\theta < 1$, then

$$\lim_{j\to\infty} \Lambda_j = \frac{C_1 p\,(p + C_{\mathrm{err},2,q}\,k\,h)}{p^2 - C_1\,\widehat{C}_{\mathrm{err},2,q}\,k^3h^2},$$

which implies (5.5) by noting that $C_{\mathrm{err},2,q}, \widehat{C}_{\mathrm{err},2,q} \simeq 1$ and that $C_{\mathrm{err},2,q}\,k\,h \lesssim \big(C^2_{\mathrm{err},2,q}k^3h^2\big)^{\frac12} \lesssim \big(\widehat{C}_{\mathrm{err},2,q}k^3h^2\big)^{\frac12} \lesssim p$. □

Note that the stability estimates in (5.5) and (5.6) are of the same order as the PDE stability estimates given in Theorem 2.1.

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Department of Mathematics, The University of Tennessee, Knoxville, Tennessee 37996

E-mail address: xfeng@math.utk.edu

Department of Mathematics, Nanjing University, Nanjing, Jiangsu 210093, People's Republic of China

E-mail address: hjw@nju.edu.cn