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FOURIER EXPANSIONS FOR APOSTOL-BERNOULLI, APOSTOL-EULER AND APOSTOL-GENOCCHI POLYNOMIALS

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Abstract. We find Fourier expansions of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. We give a very simple proof of them.

1. Introduction and statement of main results

Let $w \in \mathbb{C}$ and x a variable. The Apostol-Bernoulli polynomials $B_n(x;w)$, Apostol-Euler polynomials $E_n(x;w)$ and Apostol-Genocchi polynomials $G_n(x;w)$ are given by the generating functions

$$\sum_{n\geq 0} B_n(x;w)\frac{t^n}{n!} = \frac{te^{xt}}{we^t-1},\ |t+log(w)|<2\pi), \tag{1.1}$$

$$\sum_{n\geq 0} E_n(x;w)\frac{t^n}{n!} = \frac{2e^{xt}}{we^t+1},\ |t+log(w)|<\pi), \tag{1.2}$$

$$\sum_{n\geq 0} G_n(x;w)\frac{t^n}{n!} = \frac{2te^{xt}}{we^t+1},\ |t+log(w)|<\pi), \tag{1.3}$$

where

$$w=|w|e^{i\theta}, -\pi\leq\theta<\pi \text{ and } \log(w)=\log(|w|)+i\theta.$$

These polynomials are a natural extension of the classical Bernoulli, Euler and Genocchi polynomials: $B_n(x)=B_n(x;1), E_n(x)=E_n(x;1), G_n(x)=G_n(x;1)$, see [3]. These polynomials have many applications in mathematics. Our main results are

Theorem 1.1. *Let $w\in\mathbb{C}\backslash\{0\}$. For $0<x<1$ if $n=1$, $0\leq x\leq 1$ if $n\geq 2$. We have*

$$B_n(x;w) = \frac{-n!}{w^x(2\pi i)^n}\sideset{}{^*}\sum_{k\in\mathbb{Z}}\frac{e^{2\pi ikx}}{\left(k-\frac{log(w)}{2\pi i}\right)^n}, \tag{1.4}$$

where $\sideset{}{^*}\sum_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}\backslash\{0\}}$ *if* $w=1$ *and* $\sideset{}{^*}\sum_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}}$ *if* $w\neq 1$.

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Theorem 1.2. *Let $w \in \mathbb{C}\backslash\{0\}$. For $0 < x < 1$ if $n = 0$, $0 \leq x \leq 1$ if $n \geq 1$. We have*

$$(1.5) \qquad E_n(x;w) = \frac{2(n!)}{w^x(2\pi i)^{n+1}} \sum_{k\in\mathbb{Z}}^{**} \frac{e^{2\pi i\left(k-\frac{1}{2}\right)x}}{\left(k - \frac{1}{2} - \frac{log(w)}{2\pi i}\right)^{n+1}},$$

where $\sum\limits_{k\in\mathbb{Z}}^{**} = \sum\limits_{k\in\mathbb{Z}\backslash\{0\}}$ *if* $w = -1$ *and* $\sum\limits_{k\in\mathbb{Z}}^{**} = \sum\limits_{k\in\mathbb{Z}}$ *if* $w \neq -1$.

Theorem 1.3. *Let $w \in \mathbb{C}\backslash\{0\}$. For $0 < x < 1$ if $n = 0$, $0 \leq x \leq 1$ if $n \geq 1$. We have*

$$(1.6) \qquad G_n(x;w) = \frac{2(n!)}{w^x(2\pi i)^{n}} \sum_{k\in\mathbb{Z}}^{**} \frac{e^{2\pi i\left(k-\frac{1}{2}\right)x}}{\left(k - \frac{1}{2} - \frac{log(w)}{2\pi i}\right)^{n}}.$$

Remark 1.4. Luo's proof [4], for Theorems 1.1 and 1.2, uses the Lipschitz summation formula [2] which is not easy to understand. In this paper we propose a very simple proof. On the other hand, Theorem 1.3 is new.

2. Proofs of main results

Proof of Theorem 1.1. We consider $\int_C f_n(t)\, dt$ with $f_n(t) = \frac{t^{-n}e^{xt}}{we^t-1}$, the contour C being a circle with radius $(2N+\epsilon)\pi$ (ϵ fixed real number such that $\epsilon\pi i \pm log(w) \neq 0 \pmod{2\pi i}$), centered at the origin. If $w \neq 1$, the poles of the integrand are $t_k = 2\pi ik - log(w), k \in \mathbb{Z}$ and $t_\infty = 0$. The residues of the functions $f_n(t)$ for $k \in \mathbb{Z}$ are easily found to be $w^{-x}(2\pi ik - log(w))^{-n}e^{2\pi ikx}$, and from Theorem 1.1 the residue at $z_\infty = 0$ is seen to be $\frac{B_n(x;w)}{n!}$. The integral around the circle C tends to zero as $N \to \infty$ provided $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$, and by the theorem of residues we obtain

$$B_n(x;w) = \frac{-n!}{w^x(2\pi i)^n} \sum_{k\in\mathbb{Z}} \frac{e^{2\pi ikx}}{\left(k - \frac{log(w)}{2\pi i}\right)^n}.$$

If $w = 1$, the poles of the integrand are $t_k = 2\pi ik, k \in \mathbb{Z}$. The residues of the functions $f_n(t)$ for $k \in \mathbb{Z}\backslash\{0\}$ are easily found to be $(2\pi ik)^{-n}e^{2\pi ikx}$, and from Theorem 1.1 the residue at $z_0 = 0$ is seen to be $\frac{B_n(x;w)}{n!}$. The integral around the circle C tends to zero as $N \to \infty$ provided $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$, and by the theorem of residues we obtain

$$B_n(x;w=1) = \frac{-n!}{(2\pi i)^n} \sum_{k\in\mathbb{Z}} \frac{e^{2\pi ikx}}{k^n}.$$

This yields the theorem. □

Proof of Theorem 1.2. We apply the same method to the function $g_n(t) = \frac{t^{-(n+1)}e^{xt}}{we^t+1}$ the contour C' being a circle with radius $(2N+1+\epsilon)\pi$ (ϵ fixed real number such that $\epsilon\pi i \pm log(w) \neq 0 \pmod{\pi i}$), centered at the origin. We omit the details. □

Proof of Theorem 1.3. We have $G_{n+1}(x;w) = (n+1)E_n(x;w)$. Thus we get Theorem 1.3 from Theorem 1.2. □

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