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THE IMPACT OF $\zeta(s)$ COMPLEX ZEROS ON $\pi(x)$ FOR $x < 10^{10^{13}}$

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ABSTRACT. An analysis of the local variations of the prime counting function $\pi(x)$ due to the impact of the non-trivial, complex zeros ϱ_k of $\zeta(s)$ is provided for $x < 10^{10^{13}}$ using up to 200 billion $\zeta(s)$ complex zeros. A new bound for $|\mathrm{li}(x) - \pi(x)| < x^{1/2}(\log\log\log x + e + 1)/e\log x$ is proposed consistent with the error growth rate in Littlewood's proof that $\mathrm{li}(x) - \pi(x)$ changes sign infinitely often. This bound is also consistent with all presently known cases where $\pi(x) > \mathrm{li}(x)$ including many new examples listed. This implies that Littlewood's constant $\mathrm{K} = 1/e$, the lower bound for Skewes' number is 3.17×10^{114} and the positive constant c in the Riemann Hypothesis equivalent $|\mathrm{li}(x) - \pi(x)| < c\log(x)x^{1/2}$ is less than 3×10^{-27}.

1. INTRODUCTION

Let L_j be the j^{th} iterated logarithm of x; $\mathrm{L}_j = \log(\mathrm{L}_{j-1})$; $\mathrm{L}_1 = \log(x)$; $\mathrm{L}_2 = \log\log(x)$, etc. We use $\mathrm{li}(x)$ as the standard logarithmic integral. Following convention with $\gamma =$ Euler's constant $(0.57721\ldots)$,

$$\mathrm{li}(x) = \lim_{\epsilon\to 0}\left(\int_0^{1-\varepsilon}\frac{dt}{\log\, t} + \int_{1+\varepsilon}^{x}\frac{dt}{\log\, t}\right) = \sum_{k=1}^{\infty}\frac{\mathrm{L}_1^k}{k\cdot k!} + \mathrm{L}_2 + \gamma.$$

We also use the asymptotic approximation to $\mathrm{li}(x)$ (see Appendix 1)

$$\mathrm{li}(x) \approx \frac{x}{\mathrm{L}_1}\sum_{k=0}^{\lfloor \mathrm{L}_1 - 1\rfloor}\frac{k!}{\mathrm{L}_1^k} + \mathrm{O}(\mathrm{L}_1^{-1/2}). \tag{1}$$

Bernhard Riemann's 8-page paper in 1859 provided a major breakthrough in our understanding of the prime counting function $\pi(x)$, and the influence the complex zeros of the Riemann zeta function $\zeta(s)$ have on it [21]. He showed that much of the error in using $\mathrm{li}(x)$ as an estimate for $\pi(x)$ comes from including prime powers while the rest comes from what Riemann called the "periodic" terms that trace back to oscillations due to a logarithmic summation involving the complex zeros of $\zeta(s)$. The Riemann Hypothesis (RH), that all $\zeta(s)$ complex zeros "very likely" have their real component equal 1/2 still influences today's number theory efforts. Riemann's correction for removing the prime power counts is given by (see Appendix 1)

$$\mathrm{R}(x) = \sum_{n=2}^{\mathrm{T}}\frac{-\mu(n)}{n}\mathrm{li}\left(x^{\frac{1}{n}}\right) \approx \sum_{n=2}^{\mathrm{T}} -\mu(n)\frac{x^{\frac{1}{n}}}{\mathrm{L}_1}\sum_{j=0}^{\lfloor \frac{\mathrm{L}_1}{n} - 1\rfloor} j!\left(\frac{n}{\mathrm{L}_1}\right)^j. \tag{2}$$

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Where T is the summation upper limit (see Section 2) and $\mu(n)$ is the Möbius function defined as:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ contains one or more multiple prime factors (n is not square-free),} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes.} \end{cases}$$

The common estimate for $\pi(x)$ is given by: $\pi(x) \approx \mathcal{R}(x) = \mathrm{li}(x) - R(x)$. Tables of $\pi(x)$ (e.g. Riesel [18]) show a smaller variation on average using the Riemann formula versus $\mathrm{li}(x)$. Many mathematicians assumed that $\mathrm{li}(x) > \pi(x)$ for all x (see [13, ix], [1, 1], [5, 235]) until J. E. Littlewood proved that $\mathrm{li}(x) - \pi(x) < 0$ eventually, and changes sign infinitely often. However, Littlewood made no estimate where the first "crossover" would occur [18]. The sum of the complex oscillations can peak in a sharp spike and drive $\pi(x)$ to exceed $\mathrm{li}(x)$ (local density of primes $>$ than $1/\log x$), or drive the local prime density $< 1/\log x$, and $\mathrm{li}(x)$ greatly exceeds $\pi(x)$. Littlewood also proved that just removing prime powers is, in the long run, not a much better estimate for $\pi(x)$ than is $\mathrm{li}(x)$. The first region where $\mathrm{li}(x)$ is a better estimate of $\pi(x)$ than $\mathcal{R}(x)$ is, occurs from $x = 3,445,027$ to $3,445,031$. Rubinstein and Sarnak [25] showed that over large extended regions the proportion of x such that $\mathrm{li}(x) > \pi(x)$ is about 0.99999973, later verified by Bays and Hudson [1].

The first x for which $\mathrm{li}(x) < \pi(x)$ is known as Skewes' number (referred to here as S_1). In 1933, Stanley Skewes (Littlewood's former student) proved that given the Riemann Hypothesis, S_1 would occur no later than $x = \mathtt{e\hat{}(e\hat{}(e}^{79}\mathtt{))}$ [28] which he lowered to $x = e^{e^{20}}$ in 1955 [29]. Even if the RH should prove false, Skewes showed that a crossover would still occur before $x = \mathtt{10\hat{}(10\hat{}(10}^{963}\mathtt{))}$. Upper bounds for S_1 have decreased steadily as computers were used to investigate the oscillations in $\mathrm{li}(x) - \pi(x)$. R. Lehman's theorem [17] helped reduce S_1 to 1.5926×10^{1165} in 1966; H. J. J. te Riele cut it to 6.6578×10^{370} in 1987 [31]. Bays and Hudson reduced S_1 to 1.39822×10^{316} (hinting that S_1 could be 1.398×10^{316}) [1]. Using 10^6 complex zeros they plotted $10,000$ values of $\mathrm{li}(x) - \pi(x)$ between 10^6 and 10^{400}. Chao and Plymen used 10^7 complex zeros to refine Bays and Hudson's results and confirm a crossover at 1.39801×10^{316} [4]. Saouter and Demichel refined the error terms in Lehman's theorem and, with 22 million complex zeros, found the lowest upper bound to date for S_1 at $1.3971667 \times 10^{316}$ [26]. Our analysis conducted with up to 200 billion complex zeros gives computational evidence that a crossover may exist near $1.397162914 \times 10^{316}$ (unconfirmed via Lehman's theorem) [7]. While it is possible that another region less than this may harbor a very narrow crossover, we did not find one. Similarly, the lower bound for S_1 has increased with additional computational attention over the last 50 years. Gauss claims to have confirmed $\mathrm{li}(x) > \pi(x)$ for $x < 300,000$ [10]. Rosser and Schoenfeld showed $S_1 > 10^8$ in 1962 [22], then Brent improved the lower bound to 8×10^{10} in 1975 [3] and Kotnik raised it to 2×10^{14} in 2008 [15]. We (and independently Andry Kulsha [16]) have confirmed $S_1 > 10^{18}$ while the detailed tables of $\pi(x)$ from Tomás e Silva, suggest $S_1 > 10^{20}$ [30].

2. $\zeta(s)$ complex zeros impact on $\pi(x)$

We start with the Riemann analytical form for $\pi(x)$ as given by Edwards [9]

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{J}\left(x^{\frac{1}{n}}\right) \tag{3}$$

with the analytical formula for $\mathrm{J}(x)$ defined as

$$\mathrm{J}(x) = \mathrm{li}(x) - \sum_{\rho_k} \mathrm{li}(x^{\rho_k}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} \quad \text{for } x \geq 2. \tag{4}$$

To calculate $\pi(x)$, both $\mathrm{li}(x)$ and the sum, $\sum_{\rho_k} \mathrm{li}(x^{\rho_k})$ taken over the complex zeros ρ_k of $\zeta(s)$, for x and the $1/n$ roots of x are required, (where $\sum_{\rho_k} \mathrm{li}(x^{\rho_k})$ is Riemann's "periodic term"). The last term, extremely small for $x > 20$, does contribute small corrections from the $x^{1/n}$ roots of x. Let

$$\mathrm{I}(x) = \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} - \log 2. \tag{5}$$

Since we count primes starting at 2, the Möbius summation is halted when $x < 2$ by setting $T = \lfloor L_1/\log 2 \rfloor$. We define two correction terms, $\mathrm{C}(x)$ (which is usually ignored since $|\mathrm{C}(x)| \ll 1$ for large x) and $\mathrm{Z}(x)$:

$$\mathrm{C}(x) = \sum_{n-1}^{\mathrm{T}} \frac{\mu(n)}{n} \mathrm{I}\left(x^{\frac{1}{n}}\right) \quad \text{with } |\mathrm{C}(x)| = O\left(\frac{1}{L_1}\right) \quad \text{and} \tag{6a}$$

$$\mathrm{Z}(x) = \sum_{n=1}^{\mathrm{T}} \frac{\mu(n)}{n} \sum_{\rho_k} \mathrm{li}\left\{\left[x^{\frac{1}{n}}\right]^{\rho_k}\right\} = \sum_{n=1}^{T} \frac{\mu(n)}{n} \sum_{\rho_k} \left(\mathrm{li}(x^{\rho_k/n}) + \mathrm{li}(x^{\rho_k^*/n})\right), \tag{6b}$$

where $\rho_k = \varphi + i\alpha_k$ and $\rho_k^* = \varphi - i\alpha_k$, are the k^{th} symmetric, complex root pair of $\zeta(s)$ and per the RH, we assume $\varphi = 1/2$. Recognizing that $\mathrm{Z}(x)$ and $\mathrm{C}(x)$ may be positive or negative, equation (3) now becomes

$$\pi(x) = \mathrm{li}(x) - \mathrm{R}(x) - \mathrm{Z}(x) + \mathrm{C}(x). \tag{7}$$

From Riesel and Göhl [24], (6b) can be made computationally tractable by the following:

$$\mathrm{li}(x^{\rho_k}) + \mathrm{li}(x^{\rho_k^*}) = 2\Re\{\mathrm{li}(x^{\rho_k})\} \approx 2\Re\left(\frac{\sqrt{x} \cdot e^{i\alpha_k \mathrm{L}_1}}{\left(\frac{1}{2} + i\alpha_k\right)\mathrm{L}_1}\right) = \frac{2\sqrt{x}}{|\rho_k|\mathrm{L}_1} \cdot \cos(\alpha_k \mathrm{L}_1 - \arg\rho_k).$$

This allows $\mathrm{Z}(x)$ to be expressed in a form similar to (2) for $\mathrm{R}(x)$ with $\mathrm{Z}(x) = \sum_n \mathrm{Z}_n$,

$$\mathrm{Z}_n(x) = \frac{\mu(n)}{n} \sum_{\rho_k} \mathrm{li}(x^{\rho_k/n}) \approx \sum_{\rho_k} \frac{2\mu(n)x^{1/2n}}{|\rho_k|\mathrm{L}_1} \cdot \cos\left(\frac{\alpha_k \mathrm{L}_1}{n} - \arg\rho_k\right). \tag{8}$$

Clearly, $\mathrm{Z}_n(x)$ is proportional to $x^{1/2n}$ whereas $\mathrm{R}_n(x)$ is proportional to $x^{1/n}$. Since $\mathrm{li}(x)$ and $\mathrm{R}(x)$ can be calculated precisely and $|\mathrm{C}(x)| \ll 1$, the "unpredictable" oscillatory error in $\pi(x)$ is essentially all in $\mathrm{Z}(x)$. For brevity, let $\mathrm{lierr}(x) = \mathrm{li}(x) - \pi(x)$ so the error in using $\mathrm{lierr}(x)$ to estimate $\pi(x)$ becomes

$$\mathrm{lierr}(x) = \mathrm{R}(x) + \mathrm{Z}(x) - \mathrm{C}(x).$$

Littlewood proved $\mathrm{lierr}(x)$ can be < 0; $\mathrm{R}(x)$ is strictly positive, so $-\mathrm{Z}(x)$ must eventually exceed $\mathrm{R}(x)$ in magnitude. Letting $\mathrm{V}(x) = \mathrm{Z}(x)/\mathrm{R}(x)$ be the ratio of the

$\zeta(\mathrm{s})$ (non-trivial zeta zeros correction) to $\mathrm{R}(x)$ (the prime powers correction) allows us to express $\mathrm{lierr}(x)$ (where S_1 now becomes the first occurrence of $\mathrm{V}(x) = -1$) as

$$\mathrm{lierr}(x) = \mathrm{R}(x)[1 + \mathrm{V}(x)] - \mathrm{C}(x). \tag{9}$$

This allows $\mathrm{V}(x)$ to define the envelope of the complex zeta zero impact. When $\pi(x)$ is known, $\mathrm{V}(x)$ can be computed directly, without resorting to any complex summation, as

$$V(x) = \frac{\mathrm{li}(x) - \pi(x) + \mathrm{C}(x)}{\mathrm{R}(x)} - 1. \tag{10a}$$

For large x, $\mathrm{V}(x)$ is estimated using $\mathrm{Z}_1(x)$ and $\mathrm{R}(x)$ is approximated as $\mathrm{li}(x^{1/2})/2 \approx x^{1/2}/(\mathrm{L}_1 - 2 - 4/\mathrm{L}_1)$ (see Appendix A) leading to:

(10b)

$$V(x) = \frac{\mathrm{Z}(x)}{\mathrm{R}(x)} \cong \frac{\sum_\rho \Re\{\mathrm{li}(x^\rho)\}}{\mathrm{li}(x^{1/2})/2} \approx \sum_{\rho_k} \frac{2\sqrt{x}\cos(\alpha_k \mathrm{L}_1 - \arg\rho_k)}{|\rho_k|\mathrm{L}_1} \cdot \frac{\mathrm{L}_1 - 2 - 4/\mathrm{L}_1}{\sqrt{x}}$$
$$\cong \sum_{\rho_k} \frac{2\cos(\alpha_k \mathrm{L}_1 - \arg\rho_k)}{|\rho_k|} \cdot \left(1 - \frac{2}{\mathrm{L}_1} - \frac{4}{\mathrm{L}_1^2}\right).$$

For sufficiently large x ($x > 10^{87}, \mathrm{L}_1 > 200$) we can drop the last term and estimate $\mathrm{V}(x)$ concisely as

$$\mathrm{V}(x) = \frac{\mathrm{Z}(x)}{\mathrm{R}(x)} \approx \sum_{\rho_k} \frac{2\cos(\alpha_k \mathrm{L}_1 - \arg\rho_k)}{|\rho_k|} \approx \sum_{\rho_k} \frac{2\sin(\alpha_k \mathrm{L}_1)}{\alpha_k}. \tag{10c}$$

To balance computational accuracy and efficiency, we recommend using the cosine summation for the first 10,000 zeros, before switching to the sine summation (as $\arg\rho_k \approx \pi/2$ for large ρ_k). The computational benefit of (10b) and (10c) is of course working with $\mathrm{L}_1(x)$ instead of $x^{1/2}$ for extremely large x. A thorough search for $\pm\mathrm{V}(x)$ peaks using the exact expression (10a) for $x < 10^{18}$ was made by one author (Stoll). Two approximate $\mathrm{V}(x)$ peaks for $10^{18} < x < 10^{20}$ were derived from the extensive $\pi(x)$ tables of Thomás Oliveira é Silva [30]. Equation (10b) was then used for $10^{20} < x < 10^{10^{13}}$ using a sampling approach with 10,000 zeros to identify regions of high $|\mathrm{V}(x)|$ followed by telescoping in with smaller step sizes and 10^5 to 10^7 non-trivial zeros to isolate peaks. The most significant $\mathrm{V}(x)$ peaks are listed in Tables 2 and 3 of Section 3.

We now propose a symmetric upper and lower bound for $x \geq 3$ for $\mathrm{V}(x)$, linear in $\mathrm{L}_3(x)$, where $\mathrm{V}(x)$ can be positive or negative based on (10b), of

(11)

$$|\mathrm{V}(x)| < \left(c_1\mathrm{L}_3 + c_2 + \frac{c_3}{\mathrm{L}_1^2}\right) \quad \text{with} \quad c_1 = c_2 = \frac{1}{e} = 0.3678794\ldots \quad \text{and} \quad c_3 = 4/3.$$

The slope of $1/e$ was derived empirically, but appears to hold over 10^{13} orders of magnitude of x. The third term is only required for $x \leq 10$ and is dropped for simplicity in later analysis. Equating (10b) and (11) provides a simple approximation for the summation of the cosine (or sine) terms

$$\left|\sum_{\rho_k} \frac{2\cos(\alpha_k \mathrm{L}_1 - \arg\rho_k)}{|\rho_k|}\right| < \frac{\mathrm{L}_3 + 1}{e} \cdot \left(1 + \frac{2}{\mathrm{L}_1}\right) = \mathrm{O}(\mathrm{L}_3). \tag{12}$$

It is straightforward to rearrange equation (7) into the form A. E. Ingham used in his review of Littlewood's proof that $\pi(x) - \mathrm{li}(x) > 0$, with $\mathrm{Q}(x)$ being an unknown function proportional to L_3 [13],

$$\pi(x) - \mathrm{li}(x) = \frac{x^{\frac{1}{2}}}{\mathrm{L}_1}\left(-1 + Q(x)\frac{\mathrm{L}_1}{x^{\frac{1}{2}}} - o(1)\right).$$

The -1 denotes that "normally" $\pi(x)$ would be less than $\mathrm{li}(x)$ but for the impact of $\mathrm{Q}(x)$. From (12) and recalling $\mathrm{Z}(x) = \mathrm{R}(x)\mathrm{V}(x)$, we set $\mathrm{Q}(x)$ equal to $\mathrm{Z}(x)$ leading to

$$\pi(x) - \mathrm{li}(x) \approx \frac{x^{\frac{1}{2}}}{\mathrm{L}_1}\left(-1 + \frac{x^{\frac{1}{2}}}{\mathrm{L}_1}\left(\frac{\mathrm{L}_3 + 1}{e}\right)\frac{\mathrm{L}_1}{x^{\frac{1}{2}}} - o(1)\right) = \frac{x^{\frac{1}{2}}}{\mathrm{L}_1}\left(-1 + \left(\frac{\mathrm{L}_3 + 1}{e}\right) - o(1)\right).$$

This is of the form Ingham finally gives [13, 105] for when $\pi(x)$ could exceed $\mathrm{li}(x)$ (with $0 < \varepsilon < 1$),

$$\pi(x) - \mathrm{li}(x) > \frac{x^{\frac{1}{2}}}{\mathrm{L}_1}\left(-1 + \frac{1}{2}(1 - \varepsilon)\mathrm{L}_3 - \varepsilon\right).$$

Ingham made no attempt to estimate ε, but he notes that the right-hand side of the above "is certainly positive when x is large enough, *it remains negative at any rate over the range* $10 \le x \le 10^{700}$" (italics are ours). The problem with this last equation, however, is that the assumption $x > 10^{700}$ is already built-in such that L_3 can *only* be greater than 2 for any positive ε, which is incorrect since the current upper bound for $\mathrm{L}_3(\mathrm{S}_1) \approx 1.8855888$. Since ε is of $o(1)$, we set the last term to be $k = \delta - \varepsilon$ where $0 \le \delta \le 1$. Using data available in the 1970s[1] to 1990s including Lehman's and especially te Riele's reductions of S_1, one might have estimated the slope as $1/e$, leading to $\varepsilon = 1 - 2/e \approx 0.264241$ and may even have conjectured $\delta \approx \pi/2 - 1 \approx 0.5708$ leading to an estimate for $\mathrm{L}_3 \approx 1.885$ and thus $\mathrm{S}_1 \approx 8.0 \times 10^{314}$, remarkably close to the present estimate.

3. Numerical results

Table 1 shows computed values for $\mathrm{Z}(x)$ using 1, 7 and 17 Z_n terms and 1000 or 64,000 complex zeros. Riesel and Göhl determined the complex zero impact by adding terms "horizontally"—first finding the contribution from each $x^{1/n}$, for $2 < n < (\log x/\log 2 - 1)$, for ρ_1 then repeating and summing only up to ρ_{29} for $x \le 100$ [24]. We compute "vertically"—calculating each $\mathrm{Z}_n(x)$ separately, using thousands of complex zeros then summing these results. This permits examination of higher order Z_n term contributions to $\mathrm{Z}(x)$. Table 1 supports the conjecture that using additional zeros is more important for computing $\mathrm{Z}(x)$ peaks than are the higher order terms (recall from (8) that $\mathrm{Z}_n(x) \sim x^{1/2n}$).

The $\mathrm{V}(x)$ limit curves, shown in Figure 2, bound the known extreme $\mathrm{V}(x)$ values from $5 \le x \le 10^{10^{13}}$. The implied Skewes' number from (11) occurs when $\mathrm{V}(x) = -1$ at $x = 3.168323 \times 10^{114}$, significantly less than the current S_1 upper bound. While this sets a feasible lower limit to S_1, there is no expectation at all that S_1 will be found at any significant value less than that already identified. However, we do find $\mathrm{V}(x) = -0.9204$ at 4.489×10^{41} and $\mathrm{V}(x) = +0.99896$ at 1.90988×10^{215}.

[1] Paulo Ribenboim [20, 177] notes that Bays and Hudson had a full table of primes up to 1.2×10^{12} stored on magnetic tape in 1976. Since then, computer methods have made it feasible to compute $\mathrm{li}(x) - \pi(x)$ and conjecture an envelope.

TABLE 1. Comparison of Z(x) estimates using 1000 and 64,000 zeros

x	True Z(x)	# zeros*	Z_1 only	Σ to Z_7	Σ to Z_{17}
24,137	-11.492	1000 64000	-10.134 -10.598	-10.974 -11.424	-11.333 -11.400
302,830	30.429	1000 64000	37.194 29.895	26.520 29.240	26.55 29.27
110,102,617	-446.41	1000 64000	-385.92 -439.90	-382.99 -436.87	-382.88 -436.81
3,745,619,057	-1680.85	1000 64000	-1485.65 -1620.98	-1482.12 -1617.57	-1482.08 -1617.53
110,486,344,210	9614.97	1000 64000	8165.30 9367.08	8180.48 9380.15	8180.43 9380.18
330,957,852,107	-16,339.60	1000 64000	-13,482.64 -15,963.18	-13.485.48 -15,963.19	-13.485.45 -15,963.11
36,219,717,668,608	161,158.89	1000 64000	134,326.23 157,635.26	134,328.63 157,632.98	134,328.70 157,633.00

TABLE 2. $\pi(x)$ components and V(x) at significant peaks for $x < 10^{20}$

x	$\pi(x)$	Lierr(x)	R(x)	Z(x)	V(x)
3,445,943	246,651	79.03818	163.019	-83.988	- 0.51517
30,909,673	1,910,834	169.7254	400.449	-230.727	- 0.57616
36,917,098	2,256,803	692.4810	431.385	261.093	+ 0.60524
110,102,617	6,308,959	238.6247	685.052	-446.410	- 0.65168
11,467,849,446	518,601,766	8595.216	5242.640	3352.556	+ 0.63947
110,486,344,210	4,531,230,536	24,164.270	14,549.31	9614.973	+ 0.66086
330,957,852,107	12,987,694,192	7,626.300	23,965.91	-16,339.60	- 0.68179
36,219,717,668,608	**1,199,924,248,491**	**369,491.06**	**208,332.19**	**161,158.89**	**+ 0.77357**
1,212,562,524,413,152	35,981,229,109,582	1,849,017.18	1,071,954.8	777,062.41	+ 0.72490
16,452,596,773,450,799	453,114,351,748,945	1,196,278.77	3,650,290.9	-2,454,283.53	- 0.67230
266,175,790,131,587,542	6,808,276,716,152,988	23,364,674.45	13,593,108.6	9,771,565.82	+ 0.71886
1.325 E+18	32,553,643,155,282,975	7,846,849.35	29,086,015.9	-21,239,165.97	- 0.73022
5.550 E+19	1,248,894,293,512,468,306	298,732,223.14	171,902,701.7	126,829,521.5	+ 0.73780

Table 2 lists precise values for V(x) peak values for $x < 10^{20}$. The V(x) values for $10^{20} < x < 10^{10^{13}}$ given in Table 3 are from Demichel in his detailed research of V(x) spikes [6], [7]. He used (10b) and files containing 200 billion non-trivial zeros he helped to compute and verify in prior work [11]. Plots of V(x) confirming all previously discovered crossovers and new regions with large V(x) are shown in [6].

We observe that maximal values of lierr(x) and V(x) seem to occur when x is even (prior to the next prime where V(x) decreases), corresponding to the end of a string of large prime gaps and minimal values of lierr(x) and V(x) occur at odd numbers (a prime) corresponding to the end of a region of high prime density. The most extreme exact V(x) value identified for $x < 10^{18}$ is at $36,219,717,668,608$ with $\mathrm{V}(x) = +0.7735669$. We can also estimate x such that V(x) equals a given value M with $\mathrm{L}_3(x) \cong \mathrm{M}e - 1$. Thus $|\mathrm{Z}(x)|$ could equal twice R(x) at $\mathrm{L}_3 \cong 2e - 1 = 4.4366$ or $x \sim 10^{10^{36.33}}$.

TABLE 3. Most significant $\mathrm{V}(x)$ peaks for $10^{20} < x < 10^{10^{13}}$

x	$\mathrm{L}_3(x)$	Source of x (Date)	Zeros Used	$\mathrm{V}(x)$
2.08963594805312 E+31	1.45355376	D (2008)	10^6	+0.85111265
4.489584432 E+41	1.51806506	D (2008)	10^{10}	-0.92040123
5.38952511422773 E+84	1.66269816	D (2008)	10^8	-0.93251498
1.25855876915892 E+179	1.79541545	B&H (2000)/D(2005)	10^6	-0.95469600
1.33793033022140 E+190	1.80528667	B&H (2000)/D(2005)	10^6	-0.96875484
1.9098756608800 E+215	1.82551280	D (2005)	5 x 10^9	+0.99896402
1.397162914 E+316	1.8855888859	D (2005) [2]	2 x 10^{11}	-1.000
1.397166708 E+316	1.8855888865	S&D (2009)	2.2 x 10^7	-1.002
1.398215 E+316	1.88558904	C & P (2005)/D(2005)	10^7	-1.005
1.39822803460925 E+316	1.88558904	B&H (2000)/D(2005)	B&H-10^6 /D-10^8	-1.01770266
6.65769904020884 E+370	1.90950668	R (1987)/D(2005)	R-10^4 / D-10^6	-1.04965574
2.56771067186843 E+807	2.01861127	D (2008)	5 x 10^9	+1.04307053
1.592776 E+1165	2.06618941	L (1966)	L-2.5x10^5/D-10^6	-1.042
2.76934778827004 E+4943	2.23429015	D (2008)	10^5	-1.11199752
1.303324322225356 E+651,157	2.65468731	B&H (2000)/D(2005)	10^6	-1.26421901
1.1952052716413331 E+1,748,085	2.72182581	D (2008)	10^5	-1.28300855
1.382217859037819 E+6,359,808	2.80333258	D (2005)	10^5	-1.2851
4.63188213923253 E+25,462,014	2.88405914	D (2008)	10^5	+1.30691350
6.5722 E+30,802,655	2.89464837	D (2005)	10^5	-1.332
1.49741610590218 E+140,451,912	2.97524423	D (2008)	10^5	+1.34051829
1.14937434741219 E+263,854,435	3.00691652	D (2008)	10^5	+1.36190288
3.3380935222734 E+1,048,348,162	3.07290228	D (2008)	10^5	-1.36843786
5.60458099779389 E+1,715,984,906	3.09545477	D (2008)	10^5	+1.41803522
4.66560656850816 E+3,926,214,944	3.13222689	D (2008)	10^5	+1.45919474
5.51920707045967 E+16,251,858,572	3.19234726	D (2008)	10^6	-1.50218678
7.48642968663951 E+1,741,296,446,971	3.36797434	D (2008)	10^5	+1.51320794
1.26735548862278 E+2,176,190,682,786	3.37562756	D (2008)	10^5	-1.51154647
4.9055169754782 E+3,648,403,832,739	**3.39314314**	**D (2008)**	**10^6**	**-1.55246992**
2.9778893050643 E+4,026,762,218,841	3.39645335	D (2008)	10^5	-1.51359970
2.30170547097451 E+7,424,180,200,481	3.41673589	D (2008)	10^5	-1.51177469

4. Bounds on $|\mathrm{li}(x) - \pi(x)|$ and $\pi(x)$

The prime number theorem, proven independently by Hadamard [12] and de la Vallée-Poussin [32] in 1896, stipulates that $\lim_{x\to\infty} \pi(x)/\mathrm{li}(x) = 1$. Since then, there has been significant interest in expressing bounds on both the absolute and relative error of $\mathrm{li}(x) - \pi(x)$, or alternatively, bounds on $\pi(x)$ itself. De la Vallée-Poussin also proved (assuming RH) the relative error for the $\mathrm{li}(x)$ approximation approaches zero as

$$\left|\frac{\mathrm{li}(x) - \pi(x)}{\mathrm{li}(x)}\right| < e^{-\sqrt{\mathrm{cL}_1}}.$$

H. von Koch [14] improved this in 1901 to show the relative error goes to zero faster than

$$\left|\frac{\mathrm{li}(x) - \pi(x)}{\mathrm{li}(x)}\right| < \frac{-\mathrm{cL}_1^2}{\sqrt{x}} = cx^{-\frac{1}{2}+\frac{2\mathrm{L}_2}{\mathrm{L}_1}} < cx^{-\frac{1}{2}+\varepsilon}. \tag{13}$$

Since $\lim_{x\to\infty} \mathrm{li}(x) = x/\mathrm{L}_1$, it is straightforward to show an equivalent form of the RH from (13) that for every positive real number ε and some positive constant c,

[2]Lowest conjectured estimate for Skewes' number based on deep exploration with 200 billion zeros. Sources: B&H = Bays & Hudson, C&P = Chao & Plymen, D = Demichel, L = Lehman, S&D = Saouter & Demichel, R = te Riele.

the absolute error for $\mathrm{li}(x) - \pi(x)$ is

(14)
$$|\mathrm{li}(x) - \pi(x)| = \left|\frac{\mathrm{li}(x) - \pi(x)}{\mathrm{li}(x)}\right| \cdot \mathrm{li}(x) < \frac{\mathrm{cL}_1^2}{\sqrt{x}} \cdot \frac{x}{\mathrm{L}_1} = \mathrm{cL}_1\sqrt{x} = cx^{\frac{1}{2}+\frac{\mathrm{L}_2}{\mathrm{L}_1}} = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

In 1976, L. Schoenfeld [27] provided an estimate for c (assuming RH) by proving

$$|\mathrm{li}(x) - \pi(x)| < \frac{\mathrm{L}_1}{8\pi}\sqrt{x} \text{ for } x \geq 2657. \tag{15}$$

From (9) and (11) we have for $x \geq 10$,

$$|\mathrm{lierr}(x)| < \mathrm{R}(x)[(\mathrm{L}_3 + 1)/e + 1]. \tag{16}$$

Using the approximation for $\mathrm{R}(x)$ shown in Appendix 1 we can now express new, tight bounds on $\mathrm{lierr}(x)$ for $x \geq 662$ as

$$\begin{aligned} &\frac{\sqrt{x}}{\mathrm{L}_1}\left(\frac{x^{\frac{1}{6}}}{x^{\frac{1}{6}}-1}\right)\left[\frac{\mathrm{L}_1^2 - 2\mathrm{L}_1 - 8}{\mathrm{L}_1^2 - 4\mathrm{L}_1 - 8}\right]\left[\frac{e - \mathrm{L}_3 - 1}{e}\right] < \mathrm{lierr}(x) \\ &\quad < \frac{\sqrt{x}}{\mathrm{L}_1}\left(\frac{x^{\frac{1}{6}}}{x^{\frac{1}{6}}-1}\right)\left[\frac{\mathrm{L}_1^2 - 2\mathrm{L}_1 - 8}{\mathrm{L}_1^2 - 4\mathrm{L}_1 - 8}\right]\left[\frac{\mathrm{L}_3 + e + 1}{e}\right]. \end{aligned} \tag{17}$$

Theorem 1. *If* (11) *is true, then* $|\mathrm{lierr}(x)| < \frac{3(\mathrm{L}_3+4)}{8(\mathrm{L}_1-2)}\sqrt{x}$ *for* $x \geq 32,051$.

Proof. First we note that $\frac{3(\mathrm{L}_3+4)}{8(\mathrm{L}_1-2)}\sqrt{x} > \mathrm{R}(x)[(\mathrm{L}_3 + e_1)/e]$ for $x \geq 102,468,440$. Since $e+1 < 4$, and $3/8 > 1/e$, and using $R(x) \sim \sqrt{x}(\mathrm{L}_1 - 2)$ from (A.5) the bound is true for $x \geq 102,468,440$. The validity between $32,051$ and $102,468,440$ was confirmed explicitly using *Mathematica 7.0* with 30 digits precision. So,

$$|\mathrm{lierr}(x)| < \mathrm{R}(x)\left[\frac{\mathrm{L}_3 + e + 1}{e}\right] < \frac{3(\mathrm{L}_3 + 4)}{8(\mathrm{L}_1 - 2)}\sqrt{x} \text{ for } x \geq 32,051. \tag{18}$$

We have strong computational reasons to believe we can find even tighter bounds on this error however. □

Conjecture 1. *If* (11) *is true, then* $|\mathrm{lierr}(x)| < \frac{(\mathrm{L}_3+4)}{e\mathrm{L}_1}$ *for* $x \geq 36,969,811$.

We find $\frac{(\mathrm{L}_3+4)}{e\mathrm{L}_1}\sqrt{x} > \mathrm{R}(x)(\mathrm{L}_3 + e + 1)/e$ for $x \geq 4.98 \times 10^{17}$. For x greater than this, the limit is imposed by $\mathrm{V}(x)$. For $36,969,811 < x < 5 \times 10^{17}$ the validity was confirmed against the known regions of large $\mathrm{V}(x)$ variation.

Again, if (11) is valid for all x, we can also rewrite (9) using $\mathrm{R}(x) \sim \sqrt{x}/\mathrm{L}_1$ to express the error in exponent form such that for all x greater than some sufficiently large x_0,

$$|\mathrm{lierr}(x)| < \frac{\sqrt{x}}{e\mathrm{L}_1}(\mathrm{L}_3 + e + 1) \approx x^{\frac{1}{2} - \frac{\mathrm{L}_2 - \mathrm{L}_4 + 1}{\mathrm{L}_1} + \frac{2(e+1)}{\mathrm{L}_1(2\mathrm{L}_3 + e + 1)}} = O(x^{\frac{1}{2}-\varepsilon}). \tag{19a}$$

The exponent stays $< 1/2$, and the error stays $< \sqrt{x}$ since $(\mathrm{L}_3 + e + 1)/e\mathrm{L}_1$ is < 1 for all $x > e$.

Littlewood's proof asserts that for suitably large x and a positive constant K,

$$|\mathrm{lierr}(x)| > \frac{\mathrm{K}\sqrt{\mathrm{x}}}{\mathrm{L}_1}\mathrm{L}_3. \tag{19b}$$

For both (19a) and (19b) to be true implies that K must be $1/e$. Finally, given $\mathrm{V}(x)$, the relative error becomes

$$\left|\frac{\mathrm{li}(x)-\pi(x)}{\mathrm{li}(x)}\right| < \frac{\mathrm{L}_3+4}{e\sqrt{x}} \ll \frac{\mathrm{c}\mathrm{L}_1^2}{\sqrt{x}} \text{ for } x \geq 36,917,119. \tag{20}$$

Bounds on $\pi(x)$. L. Panaitopol [19] proved the following $\pi(x)$ bounds used by Chao and Plymen in their derivation and estimate of S_1 [4]

$$\text{for } x \geq 59,\quad \frac{x}{\mathrm{L}_1 - 1 + 1/\sqrt{\mathrm{L}_1}} < \pi(x) < \frac{x}{\mathrm{L}_1 - 1 - 1/\sqrt{\mathrm{L}_1}} \text{ for } x \geq 6.$$

Pierre Dusart proved bounds on $\pi(x)$ based on a truncated, modified expression for $\mathrm{li}(x)$ for the lower bound $(1.8 < 2)$ and a different truncated, modified expression for $\mathrm{li}(x)$ $(2.51 > 2)$ for the upper bound [8]. These bounds are significantly "tighter" than Panaitopol's for large x:

$$\text{for } x \geq 32,299,\ \frac{x}{\mathrm{L}_1}\left(1+\frac{1}{\mathrm{L}_1}+\frac{1.8}{\mathrm{L}_1^2}\right) < \pi(x) < \frac{x}{\mathrm{L}_1}\left(1+\frac{1}{\mathrm{L}_1}+\frac{2.51}{\mathrm{L}_1^2}\right) \quad \text{for } x \geq 355,991.$$

From (11), we find the following bounds for $\pi(x)$ which are even "tighter" than Dusart's at large x:

$$\text{for } x \geq 11,\ \mathrm{li}(x) - \frac{\mathrm{R}(x)}{e}(\mathrm{L}_3+1+e) < \pi(x) < \mathrm{li}(x) + \frac{\mathrm{R}(x)}{e}(\mathrm{L}_3+1-e). \tag{21}$$

This equation may be of use in improving Lehman's theorem (see Introduction).

Theorem 2. *The Riemann Hypothesis is true if for some positive constants c and* δ*, where* $0 < \delta < 2$*, we find* $|\mathrm{V}(x)| < c(\mathrm{L}_1^{2-\delta} - 2\mathrm{L}_1^{1-\delta}) - 1$ *for all* $x > x_0$.

Proof. We start with (9), assume large x, and compare it to the RH condition (18).

$$\begin{aligned}|\mathrm{lierr}(x)| &= \mathrm{R}(x)(1+\mathrm{V}(x)) \approx \frac{\sqrt{x}}{\mathrm{L}_1}\left(1+\frac{2}{\mathrm{L}_1}\right)\{1 + c(\mathrm{L}_1^{2-\delta} - 2\mathrm{L}_1^{1-\delta}) - 1\} \\ &= \mathrm{c}\sqrt{x}(\mathrm{L}_1^{1-\delta} - 4/\mathrm{L}_1^{1+\delta}) < \mathrm{c}\mathrm{L}_1\sqrt{x}.\end{aligned}$$

The constant δ must be > 0 to confirm RH and it must be < 2 for $\mathrm{lierr}(x)$ to be greater than $\sqrt{x}/\mathrm{L}_1$ which we know to be true. This emphasizes the true focus of the RH condition; it is the $\mathrm{Z}(x)$ term that must be limited to $< \mathrm{c}\mathrm{L}_1\sqrt{x}$. If (11) is not valid for all x, $\mathrm{lierr}(x)$ could eventually exceed $\sqrt{x}$, and yet not violate the RH. Assume for some $x > x_0$, that $|\mathrm{V}(x)| < \mathrm{c}\mathrm{L}_1 - 1$, meaning $\mathrm{Z}(x)$ grows such that $\mathrm{Z}(x) \sim \mathrm{c}x^{1/2}$, then

$$|\mathrm{lierr}(x)| = \frac{\sqrt{x}}{\mathrm{L}_1}\left(1+\frac{2}{\mathrm{L}_1}+\cdots\right)(1+\mathrm{c}\mathrm{L}_1-1) = c\sqrt{x}\left(1+\frac{2}{\mathrm{L}_1}+\cdots\right) < \mathrm{c}\mathrm{L}_1\sqrt{x}.$$

□

5. Relevant figures

This section contains figures relevant to the previous sections. Figure 1 shows the known $\mathrm{V}(x)$ peaks contained within the proposed $\pm\mathrm{V}(x)$ limit curves for $x < 10^{10^{13}}$. The x-axis is labelled with both L_3 values and the approximate x-values for each L_3 tick mark. Figure 2 shows $\mathrm{V}(x)$ values for increasing numbers of complex zeros that conform the region of $\pi(x) > \mathrm{li}(x)$ found by Bays and Hudson (cited in section 1) along with an estimate of the width of the crossover region. Figure 3 shows the maximum $|\mathrm{V}(x)|$ found to date with $\mathrm{V}(x) = -1.55247 \cdot \mathrm{R}(x)$ discovered at $x = 4.9055169 \times 10^{10^{12.562}}$.

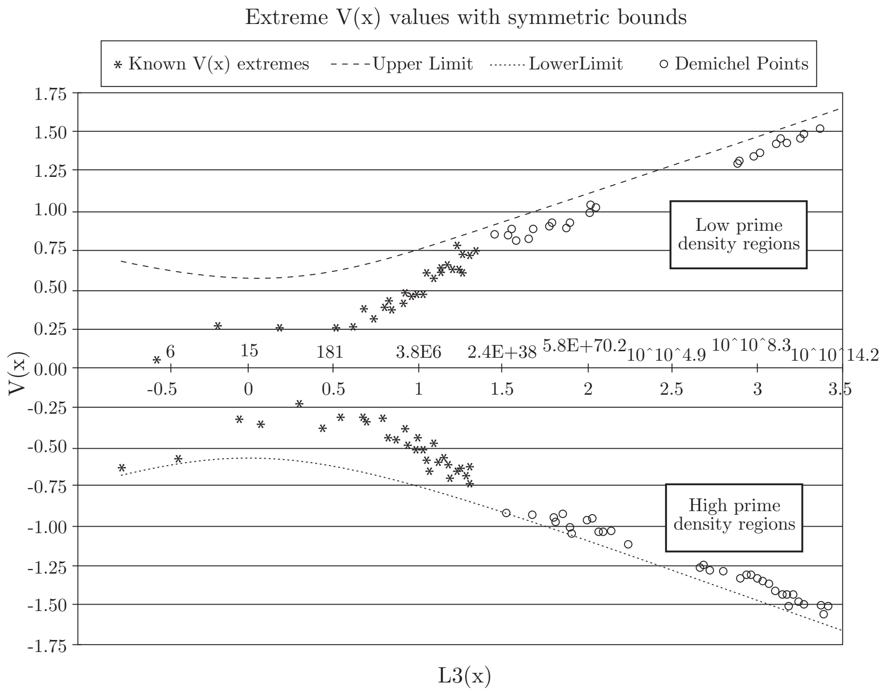

FIGURE 1. V(x) values with limit curves for $5 < x < 10^{10^{13}}$

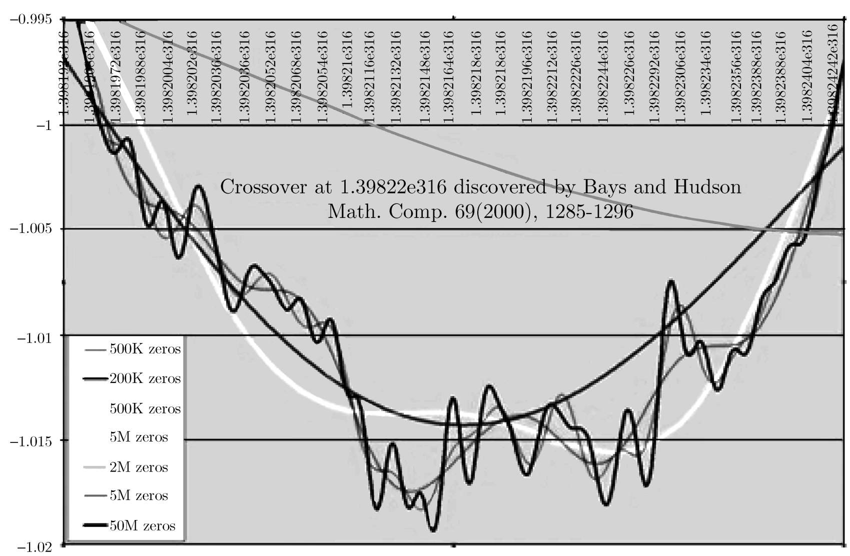

FIGURE 2. Confirmation of the Bays and Hudson crossover of width $\sim 5.6 \times 10^{311}$

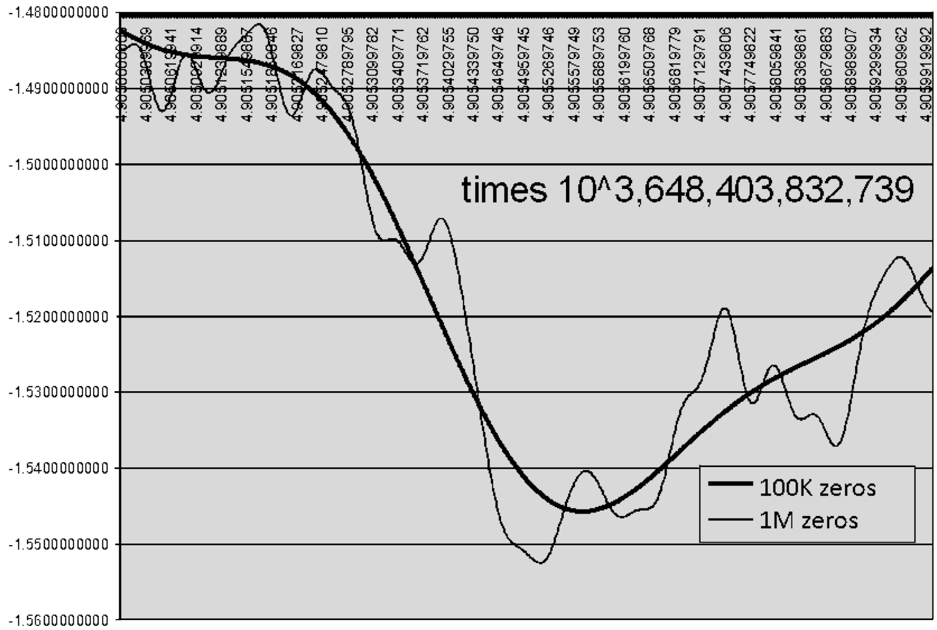

FIGURE 3. Largest $|\mathrm{V}(x)|$ found at $\mathrm{L}_3 = 3.39314314$

6. Conclusions

We have shown computational evidence that the non-trivial zeta zero contribution to $\pi(x)$ grows no faster than $(\mathrm{L}_3 + 1) \cdot \mathrm{R}(x)/e$ for $x < 10^{10^{13}}$. This implies the Riemann zeta function complex zeros are distributed with sufficient orderliness that the complex summation cancels to an extremely high degree which in turn leads to the tight bounds on estimates of $\pi(x)$ and $|\mathrm{lierr}(x)|$. We also conjecture that the limits for $\mathrm{Z}(x)$ and $\mathrm{V}(x)$ are:

$$
\begin{aligned}
&|\mathrm{Z}(x)| < \frac{\sqrt{x}}{e\mathrm{L}_1}\left[1 + \frac{2}{\mathrm{L}_1} + \frac{8}{\mathrm{L}_1^2}\right](\mathrm{L}_3 + 1) = O\left(\frac{\mathrm{L}_3\sqrt{x}}{\mathrm{L}_1}\right) \quad \text{and} \\
&|\mathrm{V}(x)| = \left|\frac{\mathrm{Z}(x)}{\mathrm{R}(x)}\right| \approx \sum_{\alpha_k} \sin(\alpha_k \mathrm{L}_1)/\alpha_k = O(\mathrm{L}_3).
\end{aligned} \tag{22}
$$

If $\mathrm{V}(x)$ from (11) can be shown to be a valid limit for $\mathrm{Z}(x)/\mathrm{R}(x)$, then the statements below follow:

$$
\begin{aligned}
|\mathrm{li}(x) - \pi(x)| &< \frac{\sqrt{x}}{e\mathrm{L}_1}\left(1 + \frac{2}{\mathrm{L}_1} + \frac{\alpha}{\mathrm{L}_1^2}\right)(\mathrm{L}_3 + e + 1) \quad \text{where } \alpha > 8 \\
&\quad \text{for } \alpha = 24, \text{ this holds for } x \geq 3; \text{ for } \alpha = 9, \text{ it holds for } x \geq 463{,}189 \\
&< \sqrt{\mathrm{li}(x)} + \frac{1}{5} \text{ for } x \geq 3, \text{(the 1/5 may be dropped for integer values of } x) \\
&< \sqrt{\frac{x}{\mathrm{L}_1 - x}} + \frac{2}{5} \text{ for } x \geq 3, \text{(the 2/5 may be dropped for integer values of } x) \\
&< \sqrt{x}.
\end{aligned}
$$

Then from (13) or (14) the Riemann Hypothesis would be confirmed.

Table 4. Limits on c

cL_n^r	c<
$\mathrm{cL}_3^{3/2}$	0.17
cL_3^2	0.07
cL_2	0.03
$\mathrm{cL}_1^{1/2}$	2×10^{-7}
cL_1	5×10^{-14}
cL_1^2	3×10^{-27}

From the numerical evidence gathered so far, we can place limits on alternative forms for $\mathrm{V}(x)$. Assume that $\mathrm{V}^*(x) = \mathrm{V}(x) + \mathrm{cL}_n^r$. Table 4 shows the upper bound on c such that the impact of the 2nd term would not have been detected so far in our search for larger $\mathrm{V}(x)$. The limit is found by assuming the proposed $\mathrm{V}(x)$ limit is violated by 50% at the maximum L_3 explored (3.41767). The bottom row shows that the positive constant c in the RH limit $x^{1/2}/\mathrm{L}_1 \cdot [\mathrm{L}_3 + 1)/e + \mathrm{cL}_1^2] \approx \mathrm{cL}_1 x^{1/2}$, would have to be less than 3×10^{-27} to have not been detected by $10^{10^{13}}$.

More investigation is needed on the statistical distribution of the zeta function's complex zeros to determine if there is a provable limit for $|\mathrm{V}(x)|$ as a function of L_3. Recent work by Bogomolny and others [2] shows the statistical distribution of zeta zero spacing is well modeled by a circular unitary ensemble of $N \times N$ random unitary matrices with deviations described by unitary matrices of finite dimension.

Appendix A

li(x) and R(x) approximations

The equation for $\mathrm{li}(x)$ given below in (A.1) is well known [8, p. 87]. It is easily obtained through repeated integration by parts and is often given without any upper limit. However, like Stirling's approximation for $\Gamma(x)$, it is fast and useful in other analysis. As an asymptotic expansion expressed as a divergent sequence, it quickly converges to nearly the correct value up to a certain k (here depending on $\log x$), then diverges after that. This summation begins to diverge at $k = \lfloor \mathrm{L}_1 \rfloor$ so we stop at $k = \lfloor \mathrm{L}_1 - 1 \rfloor$. The error is almost logarithmically linear between integer changes in L_1, but the envelope of the error is not symmetric:

$$\mathrm{li}(x) \cong \frac{x}{\mathrm{L}_1} \sum_{k=0}^{\lfloor \mathrm{L}_1 - 1 \rfloor} \frac{k!}{\mathrm{L}_1^k} \begin{cases} +1.2/\sqrt{\mathrm{L}_1}, \\ -0.6/\sqrt{\mathrm{L}_1}. \end{cases} \tag{A.1}$$

For significantly higher accuracy, $\mathrm{li}(x)$ can be expressed as a finite sum with smaller, symmetric errors as

$$\begin{aligned} \mathrm{li}(x) \cong & \frac{x}{\mathrm{L}_1} \left(\sum_{k=0}^{\lfloor \mathrm{L}_1 - 1 \rfloor} \frac{k!}{\mathrm{L}_1^k} + \frac{(\mathrm{L}_1 - \lfloor \mathrm{L}_1 \rfloor) \lfloor \mathrm{L}_1 \rfloor !}{\mathrm{L}_1^{\lfloor \mathrm{L}_1 \rfloor}} \right) \\ & - \left(\frac{300 \mathrm{L}_1^{3/2}}{359 \mathrm{L}_1^2 + \mathrm{L}_2 \mathrm{L}_1 - 3\mathrm{L}_3 + 3} \right) - \frac{\sin(2\pi \mathrm{L}_1)}{7/3 \mathrm{L}_1^2 + 61(\mathrm{L}_1 - 2)}. \end{aligned} \tag{A.2}$$

The remaining error from (A.2) is estimated to be $< 23\mathrm{L}_2^2/(\mathrm{L}_1^2 + 48\mathrm{L}_1)^2$ for $x > 113$.

For the Möbius terms we use the following asymptotic expansion easily derived from (A.1):

$$\frac{1}{n}\mathrm{li}\left(x^{\frac{1}{n}}\right) \approx \left(\frac{x^{\frac{1}{n}}}{\mathrm{L}_1}\right) \sum_{j=0}^{\lfloor (L_1/n)-1 \rfloor} j!\left(\frac{n}{\mathrm{L}_1}\right)^j. \tag{A.3}$$

With this, the Riemann correction term $\mathrm{R}(x)$ becomes

(A.4)

$$\begin{aligned}
\mathrm{R}(x) &= \sum_{n=2}^{\infty} \frac{-\mu(n)}{n}\mathrm{li}\left(x^{\frac{1}{n}}\right) = \frac{-\mu(n)x^{\frac{1}{n}}}{\mathrm{L}_1} \sum_{n=2}^{\lfloor \frac{L_1}{\ln(2)} \rfloor} \sum_{j=0}^{\lfloor \frac{L_1}{n}-1 \rfloor} j!\,(n\mathrm{L}_1)^j \\
&\cong \frac{x^{1/2}}{\mathrm{L}_1}\left\{1+\frac{2}{\mathrm{L}_1}+\frac{8}{\mathrm{L}_1^2}+\frac{48}{\mathrm{L}_1^3}+\frac{384}{\mathrm{L}_1^4}+\cdots\right\}+\frac{x^{1/3}}{\mathrm{L}_1}\left\{1+\frac{3}{\mathrm{L}_1}+\frac{18}{\mathrm{L}_1^2}+\frac{162}{\mathrm{L}_1^3}+\cdots\right\} \\
&\quad + \frac{x^{1/5}}{\mathrm{L}_1}\left\{1+\frac{5}{\mathrm{L}_1}+\frac{50}{\mathrm{L}_1^2}+\cdots\right\} - \frac{x^{1/6}}{\mathrm{L}_1}....
\end{aligned}$$

Finally, $\mathrm{R}(x)$ can be asymptotically approximated (in decreasing order of accuracy) as:

(A.5)

$$\begin{aligned}
\mathrm{R}(x) &\cong \left(\frac{x^{\frac{1}{2}}+x^{\frac{1}{3}}+x^{\frac{1}{5}}}{\mathrm{L}_1}\right)\left[\frac{\mathrm{L}_1^2-2\mathrm{L}_1-8}{\mathrm{L}_1^2-4\mathrm{L}_1-8}\right] \cong \frac{\sqrt{x}}{\mathrm{L}_1}\left(\frac{x^{\frac{1}{6}}}{x^{\frac{1}{6}}-1}\right)\left[\frac{\mathrm{L}_1^2-2\mathrm{L}_1-8}{\mathrm{L}_1^2-4\mathrm{L}_1-8}\right] \\
&\cong \frac{\sqrt{x}}{\mathrm{L}_1-2-4/\mathrm{L}_1} \cong \frac{\sqrt{x}}{\mathrm{L}_1-2} \cong \frac{\sqrt{x}}{\mathrm{L}_1}\left[1+\frac{2}{\mathrm{L}_1}\right] \cong \frac{\sqrt{x}}{\mathrm{L}_1}.
\end{aligned}$$

We also find $\log(\mathrm{R}(x))/\mathrm{L}_1$ for large x, using the following logarithmic approximation:

$$\log\left(1+\frac{\mathrm{a}}{x}\right) \approx \frac{2\mathrm{a}}{2x+\mathrm{a}} + O\left(\frac{\mathrm{a}}{x}\right)^3 \text{ for } x > a \tag{A.6}$$

$$\begin{aligned}
\frac{\log \mathrm{R}(x)}{\mathrm{L}_1} &\cong \frac{1}{2}-\frac{\mathrm{L}_2}{\mathrm{L}_1}+\left(\frac{2/\mathrm{L}_1}{2x^{\frac{1}{6}}+1}\right)+\frac{2}{\mathrm{L}_1^2-3L_1-8} \cong \frac{1}{2}-\frac{\mathrm{L}_2}{\mathrm{L}_1}+\frac{2}{\mathrm{L}_1^2-3\mathrm{L}_1-8} \\
&\cong \frac{1}{2}-\frac{\mathrm{L}_2}{\mathrm{L}_1}+\frac{2}{\mathrm{L}_1^2}.
\end{aligned}$$

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