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ON CLASSIFYING MINKOWSKIAN SUBLATTICES

WOLFGANG KELLER, JACQUES MARTINET, AND ACHILL SCHÜRMANN,
WITH AN APPENDIX BY MATHIEU DUTOUR SIKIRIĆ

ABSTRACT. Let Λ be a lattice in an n-dimensional Euclidean space E and let Λ' be a Minkowskian sublattice of Λ, that is, a sublattice having a basis made of representatives for the Minkowski successive minima of Λ. We extend the classification of possible $\mathbb{Z}/d\mathbb{Z}$-codes of the quotients Λ/Λ' to dimension 9, where $d\mathbb{Z}$ is the annihilator of Λ/Λ'.

1. INTRODUCTION

Let E be an n-dimensional Euclidean space, with scalar product $x \cdot y$. The *norm of* $x \in E$ is $N(x) = x \cdot x$ (the square of the "classical norm" $\|x\|$). Let Λ be a *lattice in* E *of rank* n, that is, a full rank discrete subgroup of E and a $\mathbb{Z}$-module in E of rank n. Let $m_1, \ldots, m_n$ be its *successive minima* in the sense of Minkowski: each m_i is the smallest real number such that the span of the set of vectors in Λ of norm $N \le m_i$ is of dimension at least i. A *Minkowskian sublattice*

$$\Lambda' = \langle e_1, \ldots, e_n \rangle = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$$

of Λ is one having a *basis* consisting of linearly independent representatives $e_1, \ldots, e_n$ of $m_1, \ldots, m_n$. Let $d\mathbb{Z}$ be the annihilator of Λ/Λ'. Then

$$\Lambda = \langle \Lambda', f_1, \ldots, f_k \rangle$$

is generated by the vectors e_i together with some vectors $f_1, \ldots, f_k \in \Lambda$ of the form

$$f_i = \frac{a_1^{(i)} e_1 + \cdots + a_n^{(i)} e_n}{d},$$

where the vectors $a^{(i)} = (a_1^{(i)}, \ldots, a_n^{(i)}) \mod d$ are the words of a *code over* $\mathbb{Z}/d\mathbb{Z}$. We attach in this way to Λ a collection of codes over $\mathbb{Z}/d\mathbb{Z}$ which depend on the choice of the e_i. We consider the problem of classifying for a given dimension n the set of codes which arise for some lattice $\Lambda \in E$ (up to equivalence).

This problem was first considered by Watson in [Wat71b], who obtained, in particular, the classification for $n \le 6$. This theory of Watson was then extended by Ryshkov (see [Ryš76]) to $n = 7$. Zahareva ([Zah80]) considered the problem for $n = 8$. Her results were completed by the second author in [Mar01], where also

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new concepts, such as the perfection rank or the minimal class of a lattice, were introduced. This latter paper will be our basic reference for what follows.

The index theory has various applications. The results of this paper will help to gain a better understanding of lattices in dimension 9 and above. For example, we shall consider in a forthcoming paper [MS10] the question of the existence of a basis of minimal vectors for lattices generated by their minimal vectors. Based on the classification of this paper, it appears possible to resolve this question in the currently open cases of dimensions 9 and 10. Another future application may be a computer assisted classification of perfect forms in dimension 9.

It should be noted that the two mentioned applications make use only of results for *well-rounded lattices*, that is, for lattices with minimal vectors spanning E. In other words, these lattices have equal successive minima $m_1, \dots, m_n$. Indeed, a deformation argument (see [Mar01, Theorem 1.5]) shows that all codes can be realized using well-rounded lattices. So from now on, we shall no longer work with the successive minima. Moreover, since the set of codes associated with a lattice Λ only depends on the similarity class of Λ, we shall in general work with lattices of minimum 1, except that the lattices we shall exhibit will be scaled for convenience to the smallest minimum which makes them integral.

Any code C of length n can be trivially extended to all dimensions $n + k$ by adding k columns of zeros to a generator matrix for C. On the side of lattices, these codes can be realized by convenient direct sums of both Λ and Λ' and k copies of $\mathbb{Z}$. In particular, we may consider $(\Lambda \perp \mathbb{Z}^k, \Lambda' \perp \mathbb{Z}^k)$. For this reason, we shall systematically restrict ourselves to codes which do not extend trivially a code of smaller length, as was done in [Mar01, Table 11.1].

A complete list of the existing codes for $n = 9$ can be found in Sections 6 and 7; in all cases we give the most important invariants. There are 137 codes in dimension 9, whereas only 42 codes exist in dimensions $n \le 8$ all together. Our results are too complex to be shortly described in this introduction, so that we shall content ourselves here with a crude result, namely the list of possible structures of Λ/Λ', merely viewed as an abstract Abelian group. By the comments above, it suffices to list for each dimension n the group structures which exist in this dimension but not in dimension $n-1$. We use the standard convention for quoting Abelian groups by their elementary divisors, writing for example for short 8, $4\cdot 2$, 2^3 for the groups of order 8 isomorphic to $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.1. *The possible structures for quotients Λ/Λ' as above up to dimension $n = 9$ are as follows:*
$\underline{n=1}$: 1 ;
$\underline{n=4}$: 2 ;
$\underline{n=6}$: 3, 2^2 ;
$\underline{n=7}$: 4, 2^3 ;
$\underline{n=8}$: 5, 6, $4\cdot 2$, 3^2, 2^4 ;
$\underline{n=9}$: 7, 8, 9, 10, 12, $6\cdot 2$, 4^2, $4\cdot 2^2$.
Moreover, all structures which exist in dimension $n = 4$, 7, 8 exist for the lattices $\mathbb{D}_4$, $\mathbb{E}_7$, $\mathbb{E}_8$, respectively, but no such "universal lattice" exists in dimensions 6 and 9. For the laminated lattice Λ_9 only the quotient 4^2 is missing. We refer to Appendix A for more information on the mentioned lattices.

The results for $n \le 8$ were obtained in [Mar01], using essentially calculations by hand. After many codes were *a priori* excluded, a computer was used only to find lattices, proving the existence for the remaining codes. The complication of some proofs, however (e.g., the non-existence of cyclic quotients Λ/Λ' of order 8), clearly shows that the methods of [Mar01] are no longer suitable in higher dimensions, at least when it involves an index $[\Lambda : \Lambda'] \geq 7$. So here we develop a method that also allows us to prove non-existence of codes using computer assistance. Our calculations not only verify all of the previously known results for $n \leq 8$, but also allow us to give a full classification for $n = 9$.

In [Zah80], Zahareva introduces the notion of a *free pair* (Λ, Λ'): a pair of well rounded lattices such that the set of minimal vectors of Λ reduces to the basis vectors $\pm e_i$ of Λ'. For each given structure of the Abelian group Λ/Λ', there are minimal dimensions n_0 and n_1 such that n-dimensional lattices with the given structure exist for all $n \geq n_0$ and some of them are free for all $n \geq n_1$. Table 1 shows information on these minimum dimensions up to index 8 that follows from our classification.

Table 1. Existence and free quotients

$[\Lambda : \Lambda']$	2	3	4	2^2	5	6	7	8	$4 \cdot 2$	2^3
exists	4	6	7	6	8	8	9	9	8	7
free	5	7	7	8	8	9	9	9	9	10

The remainder of the paper is organized as follows:

In Section 2, we set some notation and discuss some bounds for the index $[\Lambda : \Lambda']$, by which it becomes clear that in each dimension only finitely many codes have to be considered. We describe some identities which further allow us to considerably reduce the number of codes which need be considered.

In Section 3, we recall the basic dictionary between lattices and positive definite, real symmetric matrices. We, in particular, review some facts about the Ryshkov polyhedron that parametrizes all lattices whose non-zero vectors are of length at least 1. We establish a connection between its facial structure and the possible minimal classes of a lattice. We show that each code over $\mathbb{Z}/d\mathbb{Z}$ of length n is associated with a unique minimal class and a unique set of faces of the Ryshkov polyhedron.

In Section 4, based on the connection established in Section 3, we give an algorithm that allows us to test whether or not a given $\mathbb{Z}/d\mathbb{Z}$ code can be realized by a pair of lattices (Λ, Λ').

In Section 5 we give criteria due to Watson that easily allow us to exclude many codes from further considerations.

In Section 6, we consider cyclic quotients Λ/Λ' which exist in dimension 9, but not in dimension 8. We give a complete list of corresponding codes (see Table 2). Whereas our results for $d \geq 7$ depend on computer calculations, we give arguments for all "small" cyclic quotients of order $d \leq 6$. We hereby establish the classification in all dimensions for lattices Λ with maximal index $[\Lambda : \Lambda'] \leq 6$, for sublattices Λ' generated by minimal vectors of Λ.

Section 7 is devoted to non-cyclic quotients. We consider for all dimensions, quotients of type 2^k, 3^k and $4 \cdot 2^k$. In order to give a complete list of possible codes in dimension 9, other cases are treated computationally, giving overall a computer assisted proof of Theorem 1.1. All of the existing, 9-dimensional codes are listed in Tables 3, 4, 5, 6, 7, 8, 9, and 10.

In Section 8, we discuss the existence of lattices Λ which are universal in the sense that every quotient L/L' which exists in dimension n indeed exists for $L = \Lambda$; see Theorem 1.1. It turns out that such a lattice does not exist for $n = 9$. However, all structures except quotients of type 4^2 are attained by the lattice Λ_9.

Appendix A is devoted to perfect lattices that occur at several places in the presented "index theory": the root lattices, the laminated lattices, and in particular, the Leech lattice. Appendix B by Mathieu Dutour Sikirić describes a strategy to compute the index system of a given lattice. We used his computations to check our results. It helped to discover problems in an earlier version of this paper.

In addition to the information contained in this article, extra data and `MAGMA` scripts accompanying our classification are available as an "online appendix". To access these, either download the source files for the arXiv paper `arXiv:0904.3110` or download it from the corresponding world wide web page of *Mathematics of Computation*. The file `Gramindex.gp` contains a Gram matrix in `PARI-GP` format for every found lattice type.

2. Bounds and identities

Recall that we consider pairs (Λ, Λ') where Λ is a well-rounded lattice in an n-dimensional Euclidean space E and $\Lambda' \subset \Lambda$ is generated by n independent minimal vectors of Λ. We denote by $x \cdot y$ the scalar product on E and define the *norm of* $x \in E$ by $N(x) = x \cdot x$. The *minimum of* Λ (which is actually attained) is

$$\min \Lambda = \inf_{x \in \Lambda \smallsetminus \{0\}} N(x) \,.$$

The set of *minimal vectors of* Λ is

$$S(\Lambda) = \{x \in \Lambda \mid N(x) = \min \Lambda\} \,,$$

and we define $s = s(\Lambda)$ by $|S(\Lambda)| = 2s$. The *Gram matrix of an ordered set* $\mathcal{E} = (x_1, \dots, x_k)$ *of vectors of* E is the $k \times k$ matrix $\mathrm{Gram}(\mathcal{E}) = (x_j \cdot x_k)$. The *determinant* $\det(\Lambda)$ *of* Λ is the determinant of the Gram matrix of any basis for Λ. Finally, the *Hermite invariant of* Λ and the *Hermite constant of dimension* n are

$$\gamma(\Lambda) = \frac{\min \Lambda}{\det(\Lambda)^{1/n}} \quad \text{and} \quad \gamma_n = \sup_{\dim \Lambda = n} \gamma(\Lambda) \,.$$

The following result is well known:

Proposition 2.1. *With the notation and the hypotheses above, we have*

$$[\Lambda : \Lambda'] \leq \lfloor \gamma_n^{n/2} \rfloor \,.$$

Proof. By the definition of the determinant and the index we have $\det(\Lambda') = [\Lambda : \Lambda']^2 \cdot \det(\Lambda)$. Further, $\det(\Lambda') \leq N(e_1) \cdots N(e_n) = 1$ by the Hadamard inequality and the assumption $\min \Lambda = 1$. (We refer to [Mar03] for the corresponding background.) As $\det(\Lambda) \geq \gamma_n^{-n}$ by definition of the Hermite constant, and as the index is a natural number, the result follows. □

Definition 2.2. The *maximal index* $\imath(\Lambda)$ *of the well-rounded lattice* Λ is the largest value that $[\Lambda : \Lambda']$ may attain when Λ' runs through the set of sublattices of Λ which are generated by n independent minimal vectors of Λ.

The *index system* $\mathcal{I}(\Lambda)$ *of* Λ is the set of all structures of Abelian groups provided by quotients Λ/Λ' as above.

Example. By Theorem 1.1, the index system of Λ_9 is

$$\mathcal{I}(\Lambda_9) = \{1, 2, 3, 4, 2^2, 5, 6, 7, 8, 4 \cdot 2, 2^3, 9, 3^2, 10, 12, 6 \cdot 2, 4 \cdot 2^2, 2^4\}\,.$$

The Hermite constants and the *critical lattices* on which they are attained are known for $n \leq 8$ and $n = 24$. For other values of n, we must content ourselves with upper bounds valid for all sphere packings. The best bounds in print are those of Cohn and Elkies [CE03]. In particular, we have

$$\gamma_9^{9/2} \leq 30.21 \quad \text{and} \quad \gamma_{10}^5 \leq 59.44\,. \tag{1}$$

Note that the conjectural values, namely those of the laminated lattices Λ_9, Λ_{10} (defined in [CS99, Chapter 6]) are

$$\gamma(\Lambda_9)^{9/2} = 2^{9/2} = 22.627\ldots \quad \text{and} \quad \gamma(\Lambda_{10})^5 = \frac{4^5}{\sqrt{768}} = 36.950\ldots\,,$$

which give much smaller bounds for $\imath(\Lambda)$ in these two dimensions.

Remark 2.3. It results from [Mar01] that the bound $\imath(\Lambda) \leq \lfloor \gamma_n^{n/2} \rfloor$ is exact for $n \leq 8$, and that the precise equality $\imath(\Lambda) = \gamma_n^{n/2}$ even holds for $n = 4, 7, 8$, with Λ one of the root lattices $\mathbb{D}_4$, $\mathbb{E}_7$, $\mathbb{E}_8$; the bound is also tight for $n = 24$ with the *Leech lattice* Λ_{24} (see Appendix A). Note that by Theorem 1.1 the bound is strict for $n = 9$, as the largest possible value for $\imath(\Lambda)$ is then 16, the same as for $n = 8$. We conjecture that even the conjectural bound $\imath(\Lambda) \leq 36$ for $n = 10$ is strict, the actual bound being probably 32.

3. Ryshkov polyhedron and minimal classes

In Section 4 we formulate an algorithm for determining whether or not a given code C can be realized. For it we use the language of quadratic forms, or equivalently, of real symmetric matrices. Instead of looking at bases of lattices, we consider their positive definite Gram matrices. Note that there is a well-known dictionary translating between lattice and Gram matrix terminology. There is, in particular, a one-to-one correspondence between n-dimensional lattices up to orthogonal transformations and Gram matrices G up to the $\mathrm{GL}_n(\mathbb{Z})$ action $G \mapsto U^t G U$.

By $\mathcal{S}^n$ we denote the space of real symmetric $n \times n$ matrices. It is turned into a Euclidean space with the usual inner product $\langle A, B\rangle = \operatorname{Tr} AB$. For $G \in \mathcal{S}^n$ and $x \in \mathbb{R}^n$ we write $G[x] = x^t G x$. We note that $G[x] = \langle G, xx^t\rangle$ is a linear function on $\mathcal{S}^n$ for a fixed $x \in \mathbb{R}^n$.

Let $\mathcal{S}^n_{>0}$ denote the set of positive definite matrices within $\mathcal{S}^n$. It is well known that $\mathcal{S}^n_{>0}$ is an open convex cone whose closure is the set of positive semi-definite matrices. In accordance with the definition for lattices, we define the *minimum of* G by

$$\min G = \min_{x \in \mathbb{Z}^n \setminus \{0\}} G[x]$$

and its set of *minimal vectors* by

$$S(G) = \{x \in \mathbb{Z}^n \mid G[x] = \min G\}.$$

Within $\mathcal{S}^n_{>0}$, the set of Gram matrices G with minimum $\min(\Lambda)$ at least 1 form a locally finite polyhedron — the *Ryshkov polyhedron*

$$\mathcal{R} = \{G \in \mathcal{S}^n \mid G[x] \geq 1 \text{ for all } x \in \mathbb{Z}^n \setminus \{0\}\}.$$

Here "locally finite" means: for Gram matrices G in any fixed, bounded part of $\mathcal{R}$, all except finitely many of the inequalities $G[x] \geq 1$ are strict (see [Sch09a] for a proof). As a consequence, bases of lattices with minimum 1 are identified with a piecewise linear surface in $\mathcal{S}^n_{>0}$ (the boundary of $\mathcal{R}$). Its *faces* form a cell complex, naturally carrying the structure of a *combinatorial lattice* with respect to inclusion. Note that the relative interiors of faces are disjoint, whereas the closed faces themselves may meet at their boundaries. For basic terminology and results from the theory of polyhedra we refer to [Zie97].

The group $\mathrm{GL}_n(\mathbb{Z})$ acts by $G \mapsto U^t G U$ on the Ryshkov polyhedron and its boundary. All bases of a given lattice Λ with minimum 1 yield Gram matrices that lie in the relative interior of faces of the same dimension k. This invariant is the *perfection co-rank* of Λ; its *perfection rank* is

$$\operatorname{perf} \Lambda = \dim \mathcal{S}^n - k.$$

By a well-known theorem of Voronoi (see [Vor07]), we know that up to the action of $\mathrm{GL}_n(\mathbb{Z})$, there exist only finitely many vertices (0-dimensional faces) of the Ryshkov polyhedron. They are called *perfect*, as are the corresponding lattices, which are the lattices having full perfection rank $\dim \mathcal{S}^n = \binom{n+1}{2}$. As a consequence of Voronoi's finiteness result, there exist only finitely many orbits of faces of any dimension. Thus we obtain an abstract finite complex (quotient complex) from the face lattice of $\mathcal{R}$ modulo the action of $\mathrm{GL}_n(\mathbb{Z})$.

Under the action of $\mathrm{GL}_n(\mathbb{Z})$, the relative interiors of faces of $\mathcal{R}$ fall into equivalence classes. The corresponding equivalence classes of lattices are called *minimal classes*. The inclusion of faces $\mathcal{F}' \subseteq \mathcal{F}$ induces a (reversed) ordering relation on corresponding minimal classes, denoted by $\mathcal{C} \prec \mathcal{C}'$. With respect to this ordering, the minimal classes form a combinatorial lattice that is anti-isomorphic to the face lattice of the quotient complex described above. Note that lattices Λ and Λ' in the same minimal class $\mathcal{C}$ are characterized by the fact that there exists a transformation

$$u \in \mathrm{GL}(E) \quad \text{with} \quad u(\Lambda) = \Lambda' \text{ and } u(S(\Lambda)) = S(\Lambda'). \tag{2}$$

Inclusion of minimal vector sets $u(S(\Lambda)) \subseteq S(\Lambda')$ induces the same ordering relation $\mathcal{C} \prec \mathcal{C}'$ on minimal classes.

Given $\Lambda' \subset \Lambda$ having a basis of minimal vectors of Λ, all lattices L of the minimal class of Λ contain a sublattice L' such that Λ/Λ' and L/L' define the same code over $\mathbb{Z}/d\mathbb{Z}$, where $d\mathbb{Z}$ is the annihilator of Λ/Λ'. This follows from the existence of a transformation u as in (2). As a consequence, given a minimal class $\mathcal{C}$, *the set of codes attached to pairs* (Λ, Λ') *as above with* $\Lambda \in \mathcal{C}$ *is an invariant of* $\mathcal{C}$; and more generally, the set of codes and index system of a class $\mathcal{C}' \succ \mathcal{C}$ contain those of $\mathcal{C}$. This implies that for the classification of possible codes and index systems, it would suffice to study the finitely many perfect lattices of minimum 1. However, in dimension 9 these are not fully known and it appears that there exist too many of them for such an approach (see [DSV07]).

Given a $\mathbb{Z}/d\mathbb{Z}$-code C, we consider the set of minimal classes of well-rounded lattices Λ such that Λ/Λ' defines the code C for a suitably chosen sublattice Λ' of Λ, having a basis of minimal vectors of Λ. As shown by the following proposition, all of these *well-rounded minimal classes* are attached to a uniquely determined *minimal class* $\mathcal{C}_C$ *of* C.

Proposition 3.1. *Let C be a $\mathbb{Z}/d\mathbb{Z}$-code. Then there exists a unique well-rounded minimal class $\mathcal{C}_C$, such that $\mathcal{C}(\Lambda) \succ \mathcal{C}_C$, for every well-rounded lattice Λ with sublattice Λ' generated by n minimal vectors of Λ such that Λ/Λ' defines the code C.*

For the proof of the proposition, we give a geometric argument involving the Ryshkov polyhedron $\mathcal{R}$, which leads us to the main idea underlying the algorithm that we treat in the next section. We show that there exists a uniquely determined orbit of a face $\mathcal{F}$ of the Ryshkov polyhedron $\mathcal{R}$ for every code C that exists.

Proof of Proposition 3.1. Assume the code C is generated by k code words $a^{(i)}$, $i = 1, \dots, k$. So we may assume the lattice Λ' has a basis of minimal vectors $e_1, \dots, e_n$ of Λ and $\Lambda = \langle \Lambda', f_1, \dots, f_k \rangle$ with

$$f_i = \frac{a_1^{(i)} e_1 + \cdots + a_n^{(i)} e_n}{d},$$

for $i = 1, \dots, k$. Choose a basis $B = (b_1, \dots, b_n)$ of Λ. Then e_i has coordinates $\bar{e}^{(i)} \in \mathbb{Z}^n$ with respect to the chosen basis B. Note that these coordinates can be expressed in terms of the $a_j^{(i)}$ and d, independently of the specific lattices Λ and Λ'.

Assuming the minimum of Λ is 1, we know that the Gram matrix of B is contained in the affine subspace

$$T_C = \{G \in \mathcal{S}^n \mid G[\bar{e}^{(i)}] = 1 \text{ for } i = 1, \dots, n\} \tag{3}$$

of $\mathcal{S}^n$, respectively, in its intersection with the Ryshkov polyhedron $\mathcal{R}$. This intersection is a face $\mathcal{F}$ of $\mathcal{R}$ that is determined by C, up to the choice of the basis B. Choosing another basis B', we find a matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ with $B' = BU$ and a corresponding face $\mathcal{F}'$ of $\mathcal{R}$ with $\mathcal{F}' = U^t \mathcal{F} U$. Thus up to the action of $\mathrm{GL}_n(\mathbb{Z})$, the face $\mathcal{F}$ is uniquely determined by the code C. The orbit of the relative interior of $\mathcal{F}$ corresponds to a uniquely determined minimal class $\mathcal{C}_C$. It has the desired property, as every pair of lattices (Λ, Λ') satisfying the assumption of the proposition has a basis with Gram matrix in $\mathcal{F}$. □

Let us note that the face $\mathcal{F}$ of the Ryshkov polyhedron described in the proof is bounded. In fact, it can be shown that the bounded faces of the Ryshkov polyhedron are precisely the ones coming from lattices having n linearly independent minimum vectors (attaining the minimum 1). So the classification of possible codes is equivalent to the classification of bounded faces of $\mathcal{R}$ up to the action of $\mathrm{GL}_n(\mathbb{Z})$. For this it is enough to determine bounded faces of maximal dimension, that is, those bounded faces that are themselves not contained in the boundary of other bounded faces.

An important tool that we use, to show that certain codes cannot be realized, is the estimation of the Hermite constant on a given minimal class. The minimum of the Hermite constant may not be attained on a given minimal class, but if it is attained, then it is attained at a *weakly eutactic* lattice. These lattices are characterized by the fact that a corresponding Gram matrix G satisfies

$$G^{-1} = \sum_{x \in S(G)} \lambda_x x x^t \tag{4}$$

for real coefficients λ_x. A lattice is called *eutactic* if there exists such a relation with strictly positive coefficients λ_x and *strongly eutactic* if they are additionally all equal. The above-mentioned result is due to Anne-Marie Bergé and the second author (see [Mar03, Section 9.4]). They also show that there exists at most one weakly eutactic lattice in a given minimal class $\mathcal{C}$, respectively, in its *closure*

$$\overline{\mathcal{C}} = \bigcup_{\mathcal{C} \prec \mathcal{C}'} \mathcal{C}' .$$

An easy consequence is the following result for orthogonal sums of weakly eutactic lattices.

Proposition 3.2. *Let $\mathcal{C}_1, \mathcal{C}_2$ be minimal classes of dimensions n_1, n_2 and define $\mathcal{C} := \mathcal{C}_1 \oplus \mathcal{C}_2$, of dimension $n = n_1 + n_2$, by*

$$\mathcal{C} = \{\Lambda = \Lambda_1 \oplus \Lambda_2 \mid \Lambda_i \in \mathcal{C}_i,\ S(\Lambda) = S(\Lambda_1) \cup S(\Lambda_2)\} .$$

Then the weakly eutactic lattices in $\overline{\mathcal{C}_1} \oplus \overline{\mathcal{C}_2}$ are the orthogonal sums $\Lambda_1 \perp \Lambda_2$ of weakly eutactic lattices $\Lambda_i \in \mathcal{C}_i$. In particular, the minimum of γ on $\mathcal{C}$ is attained on an orthogonal sum $\Lambda_1 \perp \Lambda_2$.

The following lemma derived from the proposition will allow us to show that certain codes are impossible for $n = 9$.

Lemma 3.3. *Let Λ be a well-rounded lattice of dimension $n = 9$ having an $\mathbb{E}_8$-section with the same minimum. Then no lattice having the same minimum as Λ strictly contains Λ.*

Proof. Assume that some lattice L with $\min L = \min \Lambda$ contains Λ to an index $d \geq 2$. Let us scale for convenience all lattices to minimum 2. By Proposition 3.2, we have $\gamma(\Lambda) \geq \gamma(\mathbb{E}_8 \perp \mathbb{A}_1) = 2 \cdot 2^{-1/9}$, hence $\gamma(L)^{9/2} \geq 2 \cdot \gamma(\mathbb{E}_8 \perp \mathbb{A}_1)^{9/2} = 32$, which contradicts the upper bound (1). ∎

4. An algorithm to check the existence of a code

The basic idea of the following algorithm is motivated by the geometric situation described in the proof of Proposition 3.1. Given a $\mathbb{Z}/d\mathbb{Z}$-code C, we either show that the intersection of T_C (as in (3)) with $\mathcal{R}$ is empty, or we show that it is non-empty by finding a corresponding Gram matrix. A problem we have to deal with is the fact that $\mathcal{R}$ is given by infinitely many inequalities. The idea is to start with a finite set of inequalities and then successively add inequalities until either we find a point in the intersection or have a proof for infeasibility. For the starting set of inequalities we take a finite set of vectors $V \subset \mathbb{Z}^n$ such that the linear function $\operatorname{Tr} G = \langle \mathrm{Id}_n, G \rangle$ is bounded from above on the polyhedron

$$P = \{G \in T_C \mid G[v] \geq 1 \text{ for all } v \in V \}. \tag{5}$$

Note, if P is empty, we have a proof that the minimal class $\mathcal{C}_C$ is empty. The assumption on the bounded trace allows us to find a solution of the *linear programming problem*

$$\max_{Q \in P} \operatorname{Tr} Q. \tag{6}$$

Depending on whether or not the found solution G of this linear program is positive definite or not we have a different strategy for obtaining additional inequalities, respectively, vectors for the description of P. In the first case we compute the minimum of G. If it is 1, the Gram matrix G proves the existence of the minimal class $\mathcal{C}_C$ and corresponding lattices. If the minimum is less than 1 we add $S(G)$ to V. In the second case, if G is not positive definite, we add some vector(s) $v \in \mathbb{Z}^n$

to V with $G[v] \leq 0$. Such vectors can be found for example by an eigenvector computation. Having enlarged V, we can go back and solve the linear program (6), now with respect to a smaller polyhedron P. Again, if P is non-empty, we obtain a new solution G and proceed as described above. See Algorithm 1 for a schematic description of the described procedure. Note that all of the steps can be realized with the help of a Computer Algebra System. We used `MAGMA` [MAGMA] for our computations, in conjunction with `lrs` [lrs] to perform polyhedral computations.

Input: n, d, $C = \{a^{(i)} \in (\mathbb{Z}/d\mathbb{Z})^n \mid i = 1, \dots, k\}$
Output: ("true" and a corresponding Gram matrix G)
          or ("false" and $V \subset \mathbb{Z}^n$ such that P in (5) is empty)
V = initial set of integral, non-zero vectors
do
    $P = \{G \in T_C : G[v] \geq 1 \text{ for all } v \in V\}$
    if $P = \emptyset$
    then
        return ("false", V)
    else
        determine $G \in P$ with $\operatorname{Tr} G = \max_{G' \in P} \operatorname{Tr} G'$
        if $G \in \mathcal{S}^n_{>0}$
        then
            if $\min(G) \geq 1$
            then
                return ("true", G)
            else
                $V = V \cup S(G)$
            end if
        else
            compute finite $\mathrm{NV} \subset \mathbb{Z}^n$ with $G[v] \leq 0$ for $v \in \mathrm{NV}$
            $V = V \cup \mathrm{NV}$
        end if
    end if
end do

Algorithm 1. Determines feasibility of a given code C

It is not *a priori* clear that this procedure is in fact an algorithm, that is, if it stops after finitely many steps. As long as it does in all cases we consider, we may not even care. In order to guarantee that the computation finishes after finitely many steps, depending on T, we can restrict the vectors to be added to V to some large finite subset of $\mathbb{Z}^n$, for example, by bounding the absolute value of coordinates. An explicit bound for the coordinates of vectors $x \in \mathbb{Z}^n$, with $G[x] = 1$ for G in $\mathcal{R}$ with $\operatorname{Tr} G$ bounded by some constant, is derived in [Sch09a, Section 3.1].

Algorithm 1 yields a vertex G of the face $\mathcal{F}_C = T_C \cap \mathcal{R}$ of $\mathcal{R}$, associated with the code C through the choice of specific coordinates $\bar{e}^{(i)}$ (see the proof of Proposition 3.1). If we want to know a description of the whole face $\mathcal{F}_C$, we can compute all of its vertices by exploring neighboring vertices of vertices found so far, until no new vertices are discovered. As the face $\mathcal{F}_C$ is bounded this traversal search on the graph of vertices and edges (one-dimensional faces) of $\mathcal{F}_C$ ends after finitely many steps. Given a vertex G, the neighboring vertices are found as follows. We consider the polyhedral cone

$$\{G' \in T_C \mid G'[x] \geq 1 \text{ for all } x \in S(G)\} \tag{7}$$

with apex G. Thus the elements of $S(G)$ yield a polyhedral description with linear inequalities. Using standard methods (cf. for example [Sch09a, Appendix A]), we can convert it into a description,

$$\{G' \in \mathcal{S}^n \mid G' = G + \lambda_1 R_1 + \ldots + \lambda_k R_k, \lambda_i \geq 0\},$$

with *extreme rays* given by generators $R_1, \ldots, R_k \in \mathcal{S}^n \setminus \{0\}$. For each of these generators R_i we can find a neighboring vertex of G in $\mathcal{F}_C$ by a procedure similar to the one of finding contiguous perfect forms (cf. [Sch09a, Section 3.1]). See Algorithm 2.

Input: Vertex G of $\mathcal{F}_C$ and generator R of an extreme ray of (7)
Output: $\rho > 0$ with $\min(G + \rho R) = \min(G)$ and $S(G + \rho R) \not\subseteq S(G)$.

$(l, u) \leftarrow (0, 1)$
while $G + uR \notin \mathcal{S}^n_{>0}$ or $\min(G + uR) = \min(G)$ **do**
  if $G + uR \notin \mathcal{S}^n_{>0}$ **then** $u \leftarrow (l + u)/2$
  else $(l, u) \leftarrow (u, 2u)$
  end if
end while
while $S(G + lR) \subseteq S(G)$ **do**
  $\gamma \leftarrow \frac{l+u}{2}$
  if $\min(G + \gamma R) \geq \min(G)$ **then** $l \leftarrow \gamma$
  else
    $u \leftarrow \min \left\{ \dfrac{\min(G) - G[x]}{R[x]} \mid x \in S(G + \gamma R), R[x] < 0 \right\} \cup \{\gamma\}$
  end if
  if $\min(G + \gamma R) = \min(G)$ **then** $l \leftarrow u$ **end if**
end while
$\rho \leftarrow l$

Algorithm 2. Determination of neighboring vertices of $\mathcal{F}_C$.

Once we know all the vertices of $\mathcal{F}_C$, we can easily compute a relative interior point that carries information on several invariants of the class $\mathcal{C}_C$, like its perfection rank r and the number s of minimal vectors of Λ. To obtain just any interior point, it is actually enough to know an initial vertex and the generating rays of the polyhedral cone (7). If we know all the vertices $G_1, \ldots, G_k$ of $\mathcal{F}_C$, we can compute the vertex barycenter $\frac{1}{k}\sum_{i=1}^{k} G_i$ of $\mathcal{F}_C$ that carries even more information. For example, its automorphism group is equal to the automorphism group

$$\operatorname{Aut} \mathcal{F}_C = \{U \in \mathrm{GL}_n(\mathbb{Z}) \mid U^t \mathcal{F}_C U = \mathcal{F}_C\}$$

of $\mathcal{F}_C$. This is due to the fact that any automorphism of the face $\mathcal{F}_C$ permutes its vertices, and hence leaves the vertex barycenter fixed. On the other hand, the vertex barycenter is a relative interior point of $\mathcal{F}_C$, that is, a face of the Ryshkov polyhedron on which any element of $\mathrm{GL}_n(\mathbb{Z})$ acts. Therefore, for topological reasons, any automorphism of the vertex barycenter has to be an automorphism of $\mathcal{F}_C$.

In higher dimensions, i.e., for $n = 9$, depending on the face $\mathcal{F}_C$, the polyhedral computations necessary to find an initial vertex G or even all vertices may not be feasible (within a reasonable amount of time). In these cases, we can try to exploit available symmetries, that is, use the group $\operatorname{Aut}\mathcal{F}_C$. It can be computed from the coordinate vectors $\bar{e}^{(i)}$ which define the linear space T_C (see (3)). In fact,

$$\operatorname{Aut}\mathcal{F}_C = \{U \in \mathrm{GL}_n(\mathbb{Z}) \mid U\bar{e}^i \in \{\bar{e}^{(1)}, \dots, \bar{e}^{(n)}\} \text{ for all } i = 1, \dots, n\}.$$

As at least the vertex barycenter of $\mathcal{F}_C$ is invariant with respect to $\mathcal{G} = \operatorname{Aut}\mathcal{F}_C$, it is contained in the *$\mathcal{G}$-invariant linear subspace*

$$T_{\mathcal{G}} = \{G \in \mathcal{S}^n \mid U^t G U = G \text{ for all } U \in \mathrm{GL}_n(\mathbb{Z})\}.$$

So if we just want to check the feasibility of a given code C and want to compute its invariants from the vertex barycenter, then we can restrict the search to the linear space $T_{\mathcal{G}}$, respectively, to the affine space $T_{\mathcal{G}} \cap T_C$. In practice, in many cases the computation time is reduced tremendously by this kind of symmetry reduction.

Note that $T_{\mathcal{G}} \cap \mathcal{R}$, like $\mathcal{R}$ itself, is a locally finite polyhedron. Its vertices (and corresponding lattices) are called *$\mathcal{G}$-perfect*. We refer to [Sch09b] for a detailed account and interesting examples. If there is only one Gram matrix up to scaling in $T_{\mathcal{G}} \cap T_C \cap \mathcal{R}$ it is *$\mathcal{G}$-eutactic* and therefore *eutactic* (see [Mar03] for details). By the discussion at the end of Section 3, we can then conclude that the minimum of γ for the minimal class $\mathcal{C}_C$ is attained on it.

5. Restricting the number of codes under consideration

The computations proposed in the last sections are quite involved, so it is desirable to *a priori* exclude as many cases as possible. An efficient basic tool to restrict the number of possible codes is the following identity.

Proposition 5.1 (Watson, [Wat71b]). *Let $e_1, \dots, e_n$ be independent vectors in E, let $a_1, \dots, a_n$ and $d \geq 2$ be integers, and let*

$$f = \frac{a_1 e_1 + \cdots + a_n e_n}{d}.$$

Denote by $\operatorname{sgn}(x)$ *the sign of the real number x. Then,*

$$\left(\left(\sum_{i=1}^{n} |a_i|\right) - 2d\right) N(f) = \sum_{i=1}^{n} |a_i| \left(N\left(f - \operatorname{sgn}(a_i) e_i\right) - N(e_i)\right).$$

Proof. Just develop both sides of the displayed formula. □

Corollary 5.2 (Watson, [Wat71b]). *With the notation above, assume that the e_i are minimal vectors of a lattice Λ, that f belongs to Λ and that the a_i are non-zero. Then we have*

$$\sum_i |a_i| \geq 2d,$$

and equality holds if and only if the n vectors $e_i' = f - e_i$ are minimal.

When adding vectors f as above to the lattice $\Lambda' = \langle e_1, \dots, e_n \rangle$, one may always reduce the a_i modulo d. When there is only one such vector, i.e., when we may write $\Lambda = \langle \Lambda', f \rangle$, then by negating some e_i if needed, we may, moreover, assume that all a_i are non-negative. By reducing the dimension, we may even assume they are strictly positive. In this case, we adopt the following notation:

Notation 5.3. Suppose that Λ/Λ' is cyclic of order $d \geq 2$, and that

$$\Lambda = \langle \Lambda', f \rangle \quad \text{with} \quad f = \frac{a_1 e_1 + \cdots + a_n e_n}{d}$$

and $a_i \in \{\pm 1, \dots, \pm \lfloor \frac{d}{2} \rfloor\}$. For $i = 1, \dots, \lfloor \frac{d}{2} \rfloor$ we then set

$$m_i = |\{a_j \mid a_j = \pm i\}|$$

and say that Λ *is of type* $(m_1, \dots, m_{\lfloor d/2 \rfloor})_d$, or simply $(m_1, \dots, m_{\lfloor d/2 \rfloor})$.

Note, when we use this notation, we have $m_1 + \cdots + m_{\lfloor d/2 \rfloor} = n$. It will be generally assumed that d and the a_i are coprime, because otherwise, we could replace d by one of its strict divisors.

It should be noted that we also have $\Lambda = \langle \Lambda', a\, f \rangle$ for any a coprime to d. This induces an action of $(\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}$ on the set of admissible types $(m_1, \dots, m_{\lfloor d/2 \rfloor})_d$. If one of the types in an orbit does not satisfy Watson's criterion in Corollary 5.2, we know that a corresponding code does not exist.

6. Classifying cyclic quotients

In this section we give complete results on cyclic quotients for dimension 9. The results are displayed in Table 2, with coordinates $(a_1, \dots, a_9)$ of a generator, together with three basic invariants of lattices in the corresponding minimal class (see Proposition 3.1): $s = s(\Lambda)$, $r = \mathrm{perf}(\Lambda)$ and $s' = s(\Lambda')$. Note that we list only one admissible type $(m_1, \dots, m_{\lfloor d/2 \rfloor})$ of each $(\mathbb{Z}/d\mathbb{Z})^\times/\{\pm 1\}$ orbit, as explained at the end of the previous section.

Our results show that cyclic quotients exist for $n = 9$ only with $d \leq 10$ and $d = 12$. They were obtained using an implementation of Algorithm 1, using `MAGMA` scripts in conjunction with `lrs`. Our source code can be obtained from the online appendix of this paper, contained in the source files of its arXiv version `arXiv:0904.3110`. We used a `C++` program to systematically generate a list of possible cases satisfying the conditions of Watson, described in Section 5. We checked all cases $d \leq 30$, left by Proposition 2.1 and the known bound (1) on γ_9. In this way, we also confirmed all of the previously known results for $n \leq 8$ in [Mar01]. In dimension 9, we found several new possible indices. Below we exemplary give a detailed account of our computational result for $d = 12$. For $d \leq 6$ we give a derivation.

6.1. Cyclic cases with $\mathbf{d = 12}$.

Among the most interesting cases are the cyclic quotients with $d = 12$. There are four different types listed in Table 2. Three of them occur only for the laminated lattice Λ_9; it is the unique lattice in dimension 9 with $s = 136$. The fourth entry shows that there is also one type that occurs for some lattices with $s \geq 87$. One of them with $s = 87$ is the lattice L_{87} with Gram matrix

$$\begin{pmatrix} 4 & 2 & -2 & 2 & -2 & 0 & 2 & -2 & 2 \\ 2 & 4 & -1 & 0 & -2 & 2 & 0 & 0 & 2 \\ -2 & -1 & 4 & -2 & 2 & -1 & -1 & 1 & -2 \\ 2 & 0 & -2 & 4 & 0 & 0 & 2 & -1 & 2 \\ -2 & -2 & 2 & 0 & 4 & -2 & 0 & 1 & -2 \\ 0 & 2 & -1 & 0 & -2 & 4 & 0 & 1 & 2 \\ 2 & 0 & -1 & 2 & 0 & 0 & 4 & -2 & 0 \\ -2 & 0 & 1 & -1 & 1 & 1 & -2 & 4 & 0 \\ 2 & 2 & -2 & 2 & -2 & 2 & 0 & 0 & 4 \end{pmatrix}.$$

The perfection rank of L_{87} and its Gram matrix G is 42. Hence, by the discussion in Section 3, it is in the relative interior of a three-dimensional face of the Ryshkov polyhedron. A closer analysis reveals that this face is an octahedron with centroid G. Its six vertices come in opposite pairs. Two of these pairs contain Gram matrices of Λ_9. They are obtained as $G \pm R$ and $G \pm R'$, with R and R' being symmetric and having entries 0 everywhere, except at the positions determined by the conditions $R_{32} = R_{36} = R_{37} = 1$ and $R'_{84} = R'_{85} = -R'_{86} = 1$ (together with the symmetric ones). The other pair $G \pm R''$, with $R''_{38} = R''_{83} = 1$ and 0 elsewhere, contains two Gram matrices of another perfect lattice with $s = 99$. We call this special perfect lattice L_{99} in the sequel. It is characterized by the fact that it is the only perfect lattice aside from Λ_9 that has a Minkowskian sublattice with cyclic quotient of order $d = 12$. Note that any Gram matrix of a lattice with cyclic quotient of order $d = 12$ can be obtained as a convex combination of suitable Gram matrices of L_{99} and Λ_9.

A computer assisted calculation shows that both lattices L_{87} and L_{99} are eutactic, but not strongly eutactic (see (4)). For example, for a Gram matrix G of L_{99} we compute (using `MAGMA`) the set of minimal vectors $S(G) \subset \mathbb{Z}^n$ and find that it falls into five orbits under the action of the automorphism group of G. Each orbit O yields a barycenter $\sum_{x \in O} xx^t$ and the so obtained five barycenters $b_1, \ldots, b_5$ satisfy a relation $G^{-1} = \lambda_1 b_1 + \ldots + \lambda_5 b_5$ for positive coefficients λ_i, as can be checked easily for example with the Maple package `Convex` [Convex]. See the comments in the LaTeX source file of the arXiv paper `arXiv:0904.3110` for further details. For the perfect lattice L_{99}, its eutaxy implies (by a theorem of Voronoi; see [Mar03]) that it is extreme, that is, it attains a local maximum of the Hermite invariant.

6.2. **Cyclic cases with $\mathbf{d \leq 6}$.** We give arguments for the 9-dimensional cases with $d \leq 6$ below. Actually, we shall see that it is possible to give complete results for all dimensions with little extra work once we know the results up to $n = 8$ for $\imath \leq 5$, and up to $n = 9$ for $\imath = 6$. The general strategy to deal with cyclic quotients of order $\imath = d = 3$ to 6 is as follows (the notation is that of 5.3): taking into account Corollary 5.2 and the action of $(\mathbb{Z}/5\mathbb{Z})^\times$ which allows us to exchange m_1 and m_2 when $d = 5$, we obtain the inequalities $n = m_1 \geq 6$ for $d = 3$, $m_1 \geq 4$ and $m_1 + 2m_2 \geq 8$ for $d = 4$, $m_1 \geq \frac{n}{2}$ and $m_1 + 2m_2 \geq 10$ for $d = 5$ (and $n \geq 8$ is known; see [Mar01]), and finally $m_1 + m_2 \geq 4$, $m_1 + m_3 \geq 6$ and $m_1 + 2m_2 + 3m_3 \geq 12$ for $d = 6$.

As a next step we apply the following averaging argument, justified by the discussion on symmetry at the end of Section 4:

Remark 6.1. Assuming a lattice of type $(m_1, \ldots, m_{d/2})_d$ exists, with the notation of 5.3, we may assume that the scalar products of minimum vectors e_k and e_l associated with m_i and m_j are equal. For $i = j$, that is, for e_k and e_l associated with the same m_i, we denote this scalar product by x_i; for $i \neq j$, we denote it by

TABLE 2. Cyclic cases for $n = 9$

d	generator	s	r	s'
2	(1,1,1,1,1,1,1,1,1)	9	9	9
3	(1,1,1,1,1,1,1,1,1)	9	9	9
4	(1,1,1,1,1,1,1,1,1)	9	9	9
4	(1,1,1,1,1,1,1,1,2)	9	9	9
4	(1,1,1,1,1,1,1,2,2)	9	9	9
4	(1,1,1,1,1,1,2,2,2)	9	9	9
4	(1,1,1,1,1,2,2,2,2)	9	9	9
4	(1,1,1,1,2,2,2,2,2)	17	15	9
5	(1,1,1,1,1,1,1,1,2)	18	17	9
5	(1,1,1,1,1,1,1,2,2)	9	9	9
5	(1,1,1,1,1,1,2,2,2)	9	9	9
5	(1,1,1,1,1,2,2,2,2)	9	9	9
6	(1,1,1,1,1,1,2,2,2)	18	17	9
6	(1,1,1,1,1,2,2,2,2)	9	9	9
6	(1,1,1,1,2,2,2,2,2)	23	20	9
6	(1,1,1,1,1,1,1,2,3)	27	25	9
6	(1,1,1,1,1,1,2,2,3)	9	9	9
6	(1,1,1,1,1,2,2,2,3)	9	9	9
6	(1,1,1,1,2,2,2,2,3)	9	9	9
6	(1,1,1,2,2,2,2,2,3)	17	15	9
6	(1,1,1,1,1,1,1,3,3)	9	9	9
6	(1,1,1,1,1,1,2,3,3)	9	9	9
6	(1,1,1,1,1,2,2,3,3)	9	9	9
6	(1,1,1,1,2,2,2,3,3)	9	9	9
6	(1,1,1,2,2,2,2,3,3)	9	9	9
6	(1,1,2,2,2,2,2,3,3)	17	15	9
6	(1,1,1,1,1,1,3,3,3)	15	14	9
6	(1,1,1,1,1,2,3,3,3)	23	20	9
6	(1,1,1,1,2,2,3,3,3)	15	14	9
6	(1,1,1,2,2,2,3,3,3)	15	14	9
6	(1,1,2,2,2,2,3,3,3)	15	14	9
6	(1,2,2,2,2,2,3,3,3)	23	20	9
7	(1,1,1,1,1,2,2,2,3)	33	31	9
7	(1,1,1,1,2,2,2,2,3)	18	17	9
7	(1,1,1,1,1,1,2,3,3)	136	45	24
7	(1,1,1,1,1,2,2,3,3)	9	9	9
7	(1,1,1,1,2,2,2,3,3)	9	9	9
7	(1,1,1,1,1,2,3,3,3)	30	26	9
7	(1,1,1,1,2,2,3,3,3)	9	9	9

d	generator	s	r	s'
7	(1,1,1,2,2,2,3,3,3)	18	17	9
8	(1,1,1,1,2,2,2,3,3)	136	45	18
8	(1,1,1,2,2,2,2,3,3)	9	9	9
8	(1,1,2,2,2,2,2,3,3)	35	28	9
8	(1,1,1,2,2,2,3,3,3)	50	37	12
8	(1,1,1,1,1,2,2,3,4)	136	45	19
8	(1,1,1,1,2,2,2,3,4)	40	34	9
8	(1,1,1,2,2,2,2,3,4)	37	32	9
8	(1,1,1,1,2,2,3,3,4)	25	24	9
8	(1,1,1,2,2,2,3,3,4)	9	9	9
8	(1,1,2,2,2,2,3,3,4)	17	15	9
8	(1,1,1,1,2,3,3,3,4)	31	29	9
8	(1,1,1,2,2,3,3,3,4)	27	25	9
8	(1,1,1,1,3,3,3,3,4)	34	30	9
8	(1,1,1,1,2,2,3,4,4)	32	28	9
8	(1,1,1,2,2,2,3,4,4)	38	29	9
8	(1,1,1,1,2,3,3,4,4)	42	34	10
8	(1,1,1,2,2,3,3,4,4)	9	9	9
8	(1,1,2,2,2,3,3,4,4)	33	27	9
8	(1,1,1,2,3,3,3,4,4)	23	21	9
9	(1,1,1,2,2,2,3,3,4)	84	43	13
9	(1,1,1,2,2,3,3,3,4)	50	37	10
9	(1,1,1,1,2,3,3,4,4)	136	45	16
9	(1,1,1,2,2,3,3,4,4)	53	37	10
9	(1,1,1,2,3,3,3,4,4)	31	27	9
9	(1,1,2,2,3,3,3,4,4)	15	14	9
10	(1,1,2,2,2,2,3,3,5)	136	45	16
10	(1,1,2,2,2,2,3,4,5)	136	45	16
10	(1,1,2,2,2,3,3,4,5)	64	40	10
10	(1,1,1,2,2,3,4,4,5)	136	45	13
10	(1,1,2,2,2,3,4,4,5)	51	36	9
10	(1,1,2,2,3,3,4,4,5)	43	39	9
10	(1,1,1,2,3,4,4,4,5)	84	43	12
10	(1,1,2,2,3,4,4,4,5)	53	37	9
12	(1,1,2,3,3,4,4,5,6)	136	45	12
12	(1,1,3,3,4,4,5,5,6)	136	45	12
12	(1,2,2,3,3,3,4,4,5)	136	45	13
12	(1,2,2,3,3,4,4,5,6)	87	42	10

$y_{i,j}$. These parameters are omitted if $m_i = 0, 1$, respectively, $m_j = 0, 1$, except of $y_{i,j}$ in the case $m_i = m_j = 1$. For even d, we can additionally set $x_{d/2} = y_{d/2,j} = 0$, as we can average the two Gram matrices resulting from a base change, which replaces only e_k by $-e_k$.

By this kind of averaging argument, we assume that Gram matrices for a lattice type depend only on a short list of parameters. In particular, only on x_1 if $d = 3$ or $d = 4$ (and then $x_1 = 0$ if $m_1 = 4$), at most three parameters $x_1, x_2, y_{1,2}$ if $d = 5$ or $d = 6$. In practice, for large enough m_i, the existence and the equalities $s = r = n$

hold taking pairwise orthogonal vectors e_i, so that a finite number of verifications will suffice, which need difficult arguments only in low dimensions.

Index 2. For $\imath = 2$, there is one lattice, namely $\Lambda = \langle \Lambda', f\rangle$ where $f = \frac{e_1+\cdots+e_n}{2}$, which can be constructed for $n \geq 4$ using an orthogonal basis for Λ'. We have $(s,r) = (12,10)$ and $\Lambda \sim \mathbb{D}_4$ if $n = 4$. For $n \geq 5$ we get $s = r = n$.

Index 3. Here, we have $\Lambda = \langle \Lambda', f\rangle$ and $f = \frac{e_1+\cdots+e_n}{3}$. By Corollary 5.2, we must have $n \geq 6$ and $s \geq 12$ when $n = 6$ (because the vectors e_i and $e'_i = f - e_i$ are minimal; this also shows that the index system of $S_6 = \{e_i, e'_i : i = 1,\ldots,6\}$ is $\mathcal{I}(S_6) = \{1,2,3\}$), and [BM09, Proposition 3.5] shows that $\mathrm{perf}(S_6) = 11$. The result is: *Lattices exist if and only if $n \geq 6$, and we have $(s,r) = (12,11)$ if $n = 6$, $s = r = n$ if $n \geq 7$.* For the proof, it suffices to find convenient values for the common value x_1 of scalar products $e_i\cdot e_j$. We may clearly choose $x_1 = 0$ for $n \geq 10$, and we check that for $x_1 = \frac{1}{5}$, we have $s = 12$ if $n = 6$ and $s = n$ if $n = 7,8,9$.

Index 4. Here we have $\Lambda = \langle \Lambda', f\rangle$ and $f = \frac{e_1+\cdots+e_{m_1}+2(e_{m_1+1}+\cdots+e_n)}{4}$. The result is: *Lattices exist if and only if $m_1 \geq 4$, $n \geq 7$ and $(m_1,m_2) \neq (7,0)$, and we have $(s,r) = (n+8,n+6)$ if $m_1 = 4$ and $s = r = n$ if $m_1 \geq 5$, except in the following three cases: $(s,r) = (23,19)$ if $(m_1,m_2) = (4,3)$, $(s,r) = (21,19)$ if $(m_1,m_2) = (6,1)$, and $(s,r) = (16,15)$ if $(m_1,m_2) = (8,0)$.* For the proof, it suffices to choose $x_1 = 0$ if $m_1 = 4$ or $m_1 + 4m_2 \geq 17$ and $x_1 = 1/5$ otherwise.

Index 5. Recall that we assume that $m_1 \geq \frac{n}{2}$. The result is: *Lattices exist if and only if $n = 8$ and $m_1 = 4,5,6$, or $n = 9$ and $m_1 = 5,6,7,8$, or $n \geq 10$ and we then have $s = r = n$ except in the four cases $(m_1,m_2) = (4,4)$, $(6,2)$, $(8,1)$ and $(10,0)$ where $(s,r) = (2n,2n-1)$.* Watson's conditions together with $n \geq 8$ suffice to ensure the existence of lattices, and the special values for (s,r) occur exactly when equality holds in Watson's Proposition 5.1 and an analogue of Zahareva (see [Mar01, proof of Proposition 9.1]).

Index 6. Here the statement of the result is much more complicated, and for the sake of simplicity, we consider separately the case of dimension 8. *Lattices exist if and only if $n = 8$ and (m_1,m_2,m_3) is one of the six sets listed in* [Mar01, Table 11.1]*; see the list below, or*

$$n = 9,\ m_1 + m_2 \geq 6 \text{ and } m_1 + m_3 \geq 4.$$

When these conditions are satisfied, one has $(s,r) = (n+6,n+5)$ if $m_1 + m_2 = 6$, $(s,r) = (n+8,n+6)$ if $m_1 + m_3 = 4$, and $s = r = n$ otherwise, except in the following exceptions, for which we list (m_1,m_2,m_3) and (s,r):

$n = 8$: $(3,4,1)$: $(31,26)$; $(4,3,1)$: $(27,25)$; $(5,2,1)$: $(120,36)$ ($\Lambda = \mathbb{E}_8$); $(2,4,2)$: $(28,22)$; $(4,2,2)$: $(36,28)$.
$n = 9$: $(4,5,0)$: $(23,20)$; $(6,3,0)$: $(18,17)$; $(7,1,1)$: $(27,25)$; $(1,5,3)$: $(23,20)$; $(5,1,3)$: $(23,20)$.
$n = 10$: $(8,2,0)$: $(20,19)$; $(9,0,1)$: $(30,28)$.
$n = 11$: $(10,1,0)$,$(22,21)$.
$n = 12$: $(12,0)$: $(24,23)$.

7. Classifying non-cyclic quotients

In this section we give complete results on non-cyclic quotients for dimension 9. For non-cyclic cases one only needs to consider cases with d being a product of $k \geq 2$ numbers $d_1,\ldots,d_k$ that share a common divisor greater 1. Otherwise we could reduce the minimal number k of necessary generators. By Proposition 2.1

together with bound (1) on γ_9, we only need to consider products $d \leq 30$. We shall consider for all dimensions, quotients of type 2^k, 3^k, and $4 \cdot 2^k$, which we have been able to compute by hand. This leaves us with the following list of additional possible cases: $6 \cdot 2$, $8 \cdot 2$, $10 \cdot 2$, $12 \cdot 2$, $14 \cdot 2$, 4^2, $6 \cdot 3$, $6 \cdot 2^2$, 5^2 and $9 \cdot 3$.

The case $14 \cdot 2$ can be excluded directly from our classification of cyclic quotients in Section 6, as there exists no cyclic quotient of order 14. The cyclic quotients of order 12 occur either on the similarity class of Λ_9 or on a minimal class containing that of the lattice that we baptized L_{87} in Section 6.1.

This lattice is weakly eutactic (an easily checked linear condition) so that $\gamma(L_{87})$ is minimal among all lattices containing its class, hence among all classes having a cyclic quotient of order 12, since $\gamma(L_{87})$ is smaller than $\gamma(\Lambda_9)$. Doubling its density would produce a lattice in dimension 9 with Hermite invariant 2.16..., which contradicts the Cohn-Elkies bound (1).

Using massive computer calculations to be explained below, we were able to exclude the cases $8 \cdot 2$, $10 \cdot 2$, $6 \cdot 3$, $9 \cdot 3$, $5 \cdot 5$, and classifying all possible codes for $6 \cdot 2$ and 4^2. From the classification of $6 \cdot 2$, the last remaining case $6 \cdot 2^2$ can be excluded, as we shall explain.

For the convenience of the reader, as in the cyclic case, all of the existing cases are listed in tables (see Tables 3, 4, 5, 6, 7, 8, 9, and 10) with coordinates for the generators, together with the three basic invariants $s = s(\Lambda)$, $r = \operatorname{perf}(\Lambda)$ and $s' = s(\Lambda')$ of lattices Λ in the corresponding minimal class.

7.1. **2-elementary quotients: preliminary remarks.** Consider a lattice Λ' of minimum 1 equipped with a basis $(e_1, \dots, e_n)$ of minimal vectors. Quotients Λ/Λ' which are 2-elementary are of the form $\Lambda = \langle \Lambda', x/2 : x \in C \rangle$, where the components of x modulo 2 run through the set of words of a binary code C. The condition $\min \Lambda = \min \Lambda' (= 1)$ implies that C is of weight $w \geq 4$ (because index 2 is impossible in dimensions $1, 2, 3$) and that the scalar products $e_i \cdot e_j$ must be zero whenever i, j belong to the support of some word of weight 4 (because the centered cubic lattice $L \sim \mathbb{D}_4$ is the only 4-dimensional lattice with $\imath(L) = 2$). Under these conditions, the averaging argument of Remark 6.1 shows that we may choose $\Lambda' = \mathbb{Z}^n$. We denote the unique lattice Λ obtained in this way by Λ_C. Its minimal vectors are the $\pm e_i \in \Lambda'$ and the vectors of the form $\frac{\pm e_i \pm e_j \pm e_k \pm e_\ell}{2}$ for sets $\{i, j, k, \ell\}$ which are the support of a weight 4 word of C. For the basic terminology of coding theory used here and in the sequel, we refer to [CS99].

Remark 7.1. Let $C \neq 0$. Then Λ_C is not integral, and the smallest minimum which makes it integral is 2 if C is even and the intersections of the support of its words are even sets; then Λ_C scaled to minimum 2 is even if and only if C is doubly even, and 4 otherwise; when scaled to minimum 4, Λ_C is even if and only if C is.

Proposition 7.2. *Let C be a binary code of weight $w \geq 4$. Denote by w_4 the number of its weight-4 words and by t the number of sets $\{i, j\}$ such that i and j do not belong to the support of the same weight-4 word. Then*

$$s(\Lambda_C) = n + 8\, w_4 \quad \textit{and} \quad r(\Lambda_C) = \frac{n(n+1)}{2} - t\,.$$

Proof. Write Λ_C as a union $\bigcup_w \Lambda' + \frac{x_w}{2}$ where x_w runs through a set of representatives with components $0, 1$ of the words of C. It is clear that the minimum of $\Lambda' + x$ is equal to $\frac{\text{weight}(w)}{4}$, which gives the result for s.

For $\epsilon > 0$ small enough, let $\mathcal{V}_\epsilon$ be the set of lattices of the form $\langle e_1, \dots, e_n \rangle$ with $|N(e_i) - 1| \leq \epsilon$ $(1 \leq i \leq n)$ and $|e_j \cdot e_k| \leq \epsilon$ $(1 \leq j < k \leq n)$. Then the set $\mathcal{V}_\epsilon$ is a neighborhood of Λ' in the set $\mathcal{E}_n$ of similarity classes of well-rounded lattices of minimum 1. The set of lattices obtained from lattices $\Lambda' \in \mathcal{V}_\epsilon$ by adjunctions of the vectors $\frac{x_w}{2}$ as above also is a neighborhood of Λ in $\mathcal{E}_n$, and these neighbor lattices will not have minimum 1 unless $e_i \cdot e_j = 0$, whenever i, j lie in the support of a same weight-4 word. When this condition holds, one has $S(L) = S(\Lambda)$ for all $L \in \mathcal{V}_\epsilon$ and small enough ϵ. This proves that lattices in $\mathcal{V}_\epsilon$ depend up to similarity on t independent parameters, which shows that the perfection co-rank of every $L \in \mathcal{V}_\epsilon$ is equal to t. □

Definition 7.3. We say that a binary code C of weight 4 is *complete* if every 2-set $\{i, j\}$ belongs to at least one weight-4 word of C.

Remark 7.4. Lattices generated by $\mathbb{Z}^n$, together with vectors of the form $v = \frac{e_{i_1} + \cdots + e_{i_k}}{2}$ with $k \geq 4$ are eutactic. Indeed, vectors with $k \geq 5$ may be disregarded. When $k = 4$, we have the relation

$$\sum p_{\frac{e_{i_1} \pm \cdots \pm e_{i_4}}{2}} = 2(p_{e_{i_1}} + \cdots + p_{e_{i_4}}).$$

Thus we may modify the basic eutaxy relation $\sum p_{e_i} = \mathrm{Id}$ of $\mathbb{Z}^n$ by inserting a coefficient $1 - \epsilon$ in front of $p_{e_{i_1}} + \cdots + p_{e_{i_4}}$ and adding $\frac{\epsilon}{2}$ times the eight terms sum above; we conclude the proof by induction on the number of vectors with $k = 4$, after having chosen a small enough ϵ.

As a consequence, lattices constructed with a complete binary code are not merely perfect, but also extreme by a theorem of Voronoi.

Note that we avoid the tempting notion of a *perfect code* as it already exists in coding theory. As was pointed out to us by Gilles Zémor, complete codes are the codes having covering radius 2. He suggested also the following first examples to us: Extend the definition of a complete code to codes of weight 3. Then:

(1) the extension of such a code by the parity check is a complete code of weight 4, and
(2) the Hamming code with parameters $(2^p - 1, p, 3)$ is complete as a weight-3 code, so that the extended code with parameters $(2^p, p, 4)$ is a complete code of weight 4.

Applied to the Hamming code $\mathcal{H}_7$, we obtain the extended Hamming code $\mathcal{H}_8$, for which $\Lambda_C \simeq \mathbb{E}_8$. The easy proposition below gives a method for constructing complete codes.

Proposition 7.5. *Let C be a binary $(n, k, 4)$ code, and let C' be its extension to length $n + 2$ by the vector $(0^{n-2}, 1^4)$. Then C' is an $(n + 2, k + 1, 4)$ code, and C' is complete if and only if C is complete and for every $i < n - 1$, there exists $j \neq i$ such that $(i, j, n - 1, n)$ is the support of a weight-4 word of C; in particular, n must be even.*

We omit the proof, as it is not essential for our classification. Applying the proposition, we see that the $(2m, m - 1, 4)$ code generated by the words $(1^4, 0^{2m-4})$, $(0^2, 1^4, 0^{2m-6})$,..., $(0^{2m-4}, 1^4)$ is complete; the corresponding lattice Λ_C is isometric to $\mathbb{D}_{2m}$. Together with $\mathcal{H}_8$ and its $(7, 3, 4)$ subcode, this exhausts the list of complete codes of length $\ell \leq 8$.

7.2. **2-elementary quotients: classification.** The classification of lattices Λ_C up to dimension $n=9$ amounts to that of binary codes of weight $w \geq 4$ and length $\ell \leq 9$.

For type 2^2, we state the result for all dimensions. Binary codes C of dimension 2 contain 3 non-zero words c_1, c_2, c_3 of weights $w_1, w_2, w_3 \geq 4$, and since Λ_C must non-trivially extend a lattice of lower dimension, the supports of two words must cover the set $\{1, \dots, n\}$. Codes are described by a basis c_1, c_2, which may be assumed to satisfy $4 \leq w_1 \leq w_2 \leq w_3 = 2n - w_1 - w_2$. It is then easily checked that quotients of type 2^2 come from 2-dimensional codes which are classified by the rules

$$4 \leq w_1 \leq \frac{2n}{3} \quad \text{and} \quad \max(w_1, 4) \leq w_2 \leq n - \frac{w_1}{2}\,.$$

Here is the list of weight systems for codes of dimension 2 and length $\ell \leq 9$, from which we can read $s(\Lambda_C)$ of the corresponding lattices Λ_C and find with little effort their perfection rank (using Proposition 7.2).

$n=6$: (4^3).
$n=7$: $(4^2,6)$, $(4,5^2)$.
$n=8$: $(4^2,8)$, $(4,5,7)$, $(4,6^2)$, $(5^2,6)$.
$n=9$: $(4,5,9)$, $(4,6,8)$, $(4,7^2)$, $(5,5,8)$, $(5,6,7)$, (6^3).

In Table 3 we give a list with generators and the associated basic invariants for the $n=9$ cases.

TABLE 3. Non-cyclic cases for $n=9$ and $d=2^2$

generators	s	r	s'
(1,1,1,1,0,0,0,0,0),(0,0,0,0,1,1,1,1,1)	17	15	9
(1,1,1,1,0,0,0,0,0),(0,0,0,1,1,1,1,1,1)	17	15	9
(1,1,1,1,0,0,0,0,0),(0,0,1,1,1,1,1,1,1)	17	15	9
(1,1,1,1,1,0,0,0,0),(0,0,0,0,1,1,1,1,1)	9	9	9
(1,1,1,1,1,0,0,0,0),(0,0,0,1,1,1,1,1,1)	9	9	9
(1,1,1,1,1,1,0,0,0),(0,0,0,1,1,1,1,1,1)	9	9	9

It is easily checked that the unique code of length 8 and weight 5 has no extension of dimension 3 and weight 5 to length 9. We may thus from now on restrict ourselves to codes of weight 4, when considering binary codes of dimension at least 3.

TABLE 4. Non-cyclic cases for $n=9$ and $d=2^3$

generators aside from (1,1,1,1,0,0,0,0,0)	s	r	s'
(0,0,1,1,1,1,0,0,0),(0,0,0,0,0,1,1,1,1)	41	30	9
(0,0,1,1,1,1,0,0,0),(0,0,0,0,1,1,1,1,1)	33	24	9
(0,0,1,1,1,1,0,0,0),(0,0,0,1,0,1,1,1,1)	33	24	9
(0,0,1,1,1,1,0,0,0),(0,1,0,1,0,1,1,1,1)	33	24	9
(0,0,0,1,1,1,1,0,0),(0,0,1,0,0,0,1,1,1)	33	27	9
(0,0,0,1,1,1,1,0,0),(0,0,1,0,0,1,1,1,1)	25	21	9
(0,0,1,1,1,1,1,0,0),(0,0,0,0,0,1,1,1,1)	25	21	9
(0,0,1,1,1,1,1,0,0),(0,1,0,1,0,0,1,1,1)	17	15	9

TABLE 5. Non-cyclic cases for $n = 9$ and $d = 2^4$

generators aside from (1,1,1,1,0,0,0,0,0),(0,0,1,1,1,1,0,0,0)	s	r	s'
(0,1,0,1,0,1,1,0,0),(1,1,0,0,0,0,0,1,1)	89	43	9
(0,1,0,1,0,1,1,0,0),(1,1,0,0,0,0,1,1,1)	65	30	9
(0,0,0,0,1,1,1,1,0),(0,1,0,1,0,1,0,1,1)	57	37	9
(0,0,0,1,0,1,1,1,0),(1,0,1,0,0,0,0,1,1)	81	45	9

Next we turn to binary codes of dimension 3. As shown in [Mar01], there is one code if $n = 7$, and three new codes if $n = 8$. From the list of codes of dimension 2 of length $\ell \leq 8$ above, one easily proves that there are eight new codes in dimension 9. See Table 4. The basic invariants can be easily computed using Proposition 7.2.

Extending the four codes of dimension 3 and length $\ell \leq 8$, we prove that there are four codes in dimension 9 besides the trivial extension of the $(8, 4, 4)$ extended Hamming code $\mathcal{H}_8$. See Table 5. Again, the basic invariants can be easily computed using Proposition 7.2.

Since the automorphism of $\mathcal{H}_8$ is 3-fold transitive on the coordinates, it does not extend to a code of dimension 5 and length 9, which completes the classification of 2-elementary codes for $n = 9$. Note that the latter code extends to a code of dimension 5 and length 10, which lifts to the lattice $\langle \mathbb{E}_8, \mathbb{D}_{10} \rangle$; see Appendix A.

7.3. **3-elementary quotients.** Quotients of Λ/Λ' of type 3^k are constructed using ternary codes of weight $w \geq 6$ and dimension k, but the existence of a code C does not imply the existence of a pair (Λ, Λ') defining C, as shown by the lemma and the comment below.

Lemma 7.6. *There do not exist* 9*-dimensional pairs* (Λ, Λ') *with* Λ/Λ' 3*-elementary of order* 27.

Proof. A ternary code C of length 9 and dimension 3 extends a ternary code C_0 of length 8 and dimension 2. There is a unique code C_0, and the lattice Λ_0 defined by C_0 is the $\mathbb{E}_8$ lattice. Hence Λ must contain to index 3 a lattice having an $\mathbb{E}_8$ cross-section, which contradicts Lemma 3.3. $\square$

Note that despite this lemma, there exists a (unique) ternary code with parameters $(9, 3, 6)$, given by the generating matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Its weight system is $(9, 6^{12})$.

It is easily checked that there are three ternary codes with parameters $(9, 2, 6)$ and one with parameters $(8, 2, 6)$. Their respective weight systems are $(6^3, 9)$, $(6^2, 7, 8)$, $(6, 7^3)$ and (6^4); generating matrices for the first three can be read in Table 6; the latter one, referred to in Lemma 7.6, extends a code of length 8.

The averaging argument of Remark 6.1 applied to the first three codes produces matrices depending on two, zero, and three parameters. In the first and third case, we find lattices in this way for which $s(\Lambda)$ takes the smallest possible value compatible with Watson's conditions (see Proposition 5.1). Hence the minimal

TABLE 6. Non-cyclic cases for $n = 9$ and $d = 3^2$

generators	s	r	s'
(1,1,1,1,1,1,0,0,0),(0,0,0,1,1,1,1,1,1)	27	23	9
(1,1,1,1,1,1,0,0,0),(1,1,2,0,0,0,1,1,1)	50	37	10
(1,1,1,1,1,1,0,0,0),(1,1,0,0,2,2,1,1,1)	15	14	9

classes of our three lattices are the smallest possible, with invariants s, r and s' as displayed in Table 6.

Below we give Gram matrices for the three lattices Λ, obtained by replacing e_1 and e_9 in a basis $(e_1, \dots, e_9)$ for Λ', by vectors with denominators 3 and with there numerators containing representatives obtained from two code words in Table 6:

$$\begin{pmatrix} 94 & 47 & 47 & 47 & 47 & 47 & 0 & 0 & 47 \\ 47 & 90 & 18 & 5 & 5 & 5 & -5 & -5 & 0 \\ 47 & 18 & 90 & 5 & 5 & 5 & -5 & -5 & 0 \\ 47 & 5 & 5 & 90 & 18 & 18 & 5 & 5 & 47 \\ 47 & 5 & 5 & 18 & 90 & 18 & 5 & 5 & 47 \\ 47 & 5 & 5 & 18 & 18 & 90 & 5 & 5 & 47 \\ 0 & -5 & -5 & 5 & 5 & 5 & 90 & 18 & 47 \\ 0 & -5 & -5 & 5 & 5 & 5 & 18 & 90 & 47 \\ 47 & 0 & 0 & 47 & 47 & 47 & 47 & 47 & 94 \end{pmatrix}, \begin{pmatrix} 18 & 9 & 9 & 9 & 9 & 9 & -3 & -3 & 0 \\ 9 & 18 & 3 & 3 & 0 & 0 & -2 & -2 & -3 \\ 9 & 3 & 18 & 3 & 0 & 0 & -2 & -2 & -3 \\ 9 & 3 & 3 & 18 & 0 & 0 & -3 & -3 & -9 \\ 9 & 0 & 0 & 0 & 18 & 9 & 0 & 0 & 9 \\ 9 & 0 & 0 & 0 & 9 & 18 & 0 & 0 & 9 \\ -3 & -2 & -2 & -3 & 0 & 0 & 18 & 3 & 9 \\ -3 & -2 & -2 & -3 & 0 & 0 & 3 & 18 & 9 \\ 0 & -3 & -3 & -9 & 9 & 9 & 9 & 9 & 18 \end{pmatrix},$$

$$\begin{pmatrix} 120 & 60 & 60 & 60 & 60 & 60 & 0 & 0 & 0 \\ 60 & 108 & 9 & 9 & 9 & 9 & 0 & 0 & 0 \\ 60 & 9 & 108 & 36 & 9 & 9 & 0 & 0 & 42 \\ 60 & 9 & 36 & 108 & 9 & 9 & 0 & 0 & 42 \\ 60 & 9 & 9 & 9 & 108 & 36 & 0 & 0 & -42 \\ 60 & 9 & 9 & 9 & 36 & 108 & 0 & 0 & -42 \\ 0 & 0 & 0 & 0 & 0 & 0 & 108 & 27 & 54 \\ 0 & 0 & 0 & 0 & 0 & 0 & 27 & 108 & 54 \\ 0 & 0 & 42 & 42 & -42 & -42 & 54 & 54 & 110 \end{pmatrix}.$$

Index systems for the Gram matrices, hence also for the corresponding minimal classes are:

$$\{1, 2, 3, 4, 2^2, 6, 3^2\},\ \{1, 2, 3, 4, 2^2, 5, 6, 7, 8, 4 \cdot 2, 9, 3^2\},\ \{3, 6, 2^2\}.$$

7.4. **Quotients of type 4·2.** Here, $n = 9$. We define integers a_i, b_i, $1 \le i \le n$, m_1, m_2, such that $m_1 \ge 4$, $m := m_1 + m_2 \in \{7, 8, 9\}$, writing Λ in the form $\Lambda = \langle \Lambda', e, f \rangle$, where

$$e = \frac{a_1 e_1 + \cdots + a_n e_n}{4} \quad \text{and} \quad f = \frac{b_1\, e_1 + \cdots + b_n\, e_n}{2}$$

with $a_i \in \{0, 1, 2\}$, $b_i \in \{0, 1\}$, $a_i = 1$ for $i \le m_1$, $a_i = 2$ for $m_1 + 1 \le i \le m_1 + m_2$, $a_i = 1$ for $i > m$. We also consider

$$e' = \frac{a'_1 e_1 + \cdots + a'_n e_n}{4} \quad \text{and} \quad f' = \frac{b'_1\, e_1 + \cdots + b'_n\, e_n}{2},$$

$e' \equiv e + f \mod \Lambda'$, $f' \equiv 2e + f \mod \Lambda'$, $a'_i = \pm 1$ for $i \le m_1$, $a'_i = 0$ or 2 for $i > m_1$, and $b'_i = 0$ or 1. Note that m_1, namely the number of components ± 1 of words attached to denominator 4, is an invariant for all codes of the form $4 \cdot 2^k$.

We first prove that $m_1 = 9$ is impossible. This shows that minimizing $m_1 + m_2$ by exchanging e and e' if needed, we may assume that $m_1 + m_2 \le 8$, i.e., that all codes extend some 7- or 8-dimensional $\mathbb{Z}/4\mathbb{Z}$-code. Then we must have $b_9 = 1$.

All together, there are 26 new codes in dimension 9 displayed in Table 7 (thus with the extensions of the three 8-dimensional codes, there exist 29 codes). They have been classified by first choosing m, then m_1 as small as possible, then choosing f as short as possible. The numbers of codes for given pairs (m_1, m_2) as above are

$$(4, 3)\colon 6;\ (5, 2)\colon 6;\ (6, 1)\colon 5;\ (4, 4),\ (5, 3)\colon 1;\ (6, 2),\ (7, 1)\colon 3;\ (8, 0)\colon 1\ .$$

TABLE 7. Non-cyclic cases for $n = 9$ and $d = 4 \cdot 2$

generators	s	r	s'
(1,1,1,1,2,2,2,0,0),(0,0,0,0,0,1,1,1,1])	41	30	9
(1,1,1,1,2,2,2,0,0),(0,0,0,1,0,1,1,1,1)	33	27	9
(1,1,1,1,2,2,2,0,0),(0,0,1,1,0,1,1,1,1)	33	27	9
(1,1,1,1,2,2,2,0,0),(0,0,0,1,0,0,1,1,1)	33	27	9
(1,1,1,1,2,2,2,0,0),(0,0,1,1,0,0,1,1,1)	25	21	9
(1,1,1,1,2,2,2,0,0),(0,0,1,1,0,0,0,1,1)	41	30	9
(1,1,1,1,1,2,2,0,0),(0,0,0,0,0,1,1,1,1)	17	15	9
(1,1,1,1,1,2,2,0,0),(0,0,0,0,1,1,1,1,1)	23	22	9
(1,1,1,1,1,2,2,0,0),(0,0,0,1,1,1,1,1,1)	56	37	12
(1,1,1,1,1,2,2,0,0),(0,0,0,0,1,0,1,1,1)	17	15	9
(1,1,1,1,1,2,2,0,0),(0,0,0,1,1,0,1,1,1)	9	9	9
(1,1,1,1,1,2,2,0,0),(0,0,0,1,1,0,0,1,1)	24	21	9
(1,1,1,1,1,1,2,0,0),(0,0,0,0,1,1,0,1,1)	46	34	9
(1,1,1,1,1,1,2,0,0),(0,0,0,1,1,1,0,1,1)	23	21	9
(1,1,1,1,1,1,2,0,0),(0,0,0,0,0,1,1,1,1)	35	28	9
(1,1,1,1,1,1,2,0,0),(0,0,0,0,1,1,1,1,1)	42	34	10
(1,1,1,1,1,1,2,0,0),(0,0,0,1,1,1,1,1,1)	23	21	9
(1,1,1,1,2,2,2,2,0),(0,0,1,1,0,0,0,1,1)	33	24	9
(1,1,1,1,1,2,2,2,0),(0,0,1,1,0,0,0,1,1)	17	15	9
(1,1,1,1,1,1,2,2,0),(0,0,0,0,1,1,0,1,1)	17	15	9
(1,1,1,1,1,1,2,2,0),(0,0,0,1,1,1,0,1,1)	9	9	9
(1,1,1,1,1,1,2,2,0),(0,0,0,1,1,1,0,0,1)	38	29	9
(1,1,1,1,1,1,1,2,0),(0,0,0,0,1,1,1,0,1)	37	32	9
(1,1,1,1,1,1,1,2,0),(0,0,0,0,0,1,1,1,1)	41	35	9
(1,1,1,1,1,1,1,2,0),(0,0,0,0,1,1,1,1,1)	9	9	9
(1,1,1,1,1,1,1,1,0),(0,0,0,0,1,1,1,1,1)	32	29	9

They define only 22 minimal classes, as the two codes with $(s,r) = (41,30)$, those with $(s,r) = (33,27)$ in lines 2 and 4 of Table 7, and the three codes with $(s,r) = (9,9)$ define the same minimal classes.

7.5. **Quotients of type $4{\cdot}2^2$.** Here we may write $\Lambda = \langle \Lambda', f, f', f'' \rangle$ where f, f', f'' have denominators 4, 2, and 2. By the results above for quotients of type $4 \cdot 2$, we know that we may assume that f has component zero on e_9, and then replacing f' by f'' or $f' + f''$, that f' also have the same property. Hence the 8-dimensional section $\Lambda_0 = \langle e_1 \ldots, e_8, f, f' \rangle$ is of one of three types, characterized by $(m_1, m_2) = (4,3)$, $(5,2)$ or $(6,1)$ (see [Mar01, Section 10 and Table 11.1]).

The third type defines only $\mathbb{E}_8$, and thus does not extend to a quotient $(4,2)$ in dimension 9 by Lemma 3.3. For the second one, the eutactic lattice in its minimal class is the 8-dimensional *Watson lattice*, that is, the unique integral lattice of minimum 4, with $s = 75$. (We refer to it as the "Watson lattice", as Watson proved that for $n = 8$, either $s = 120$, attained only by the root lattice $\mathbb{E}_8$, or $s \leq 75$; see [Wat71a]). Watson's lattice is the unique weakly eutactic lattice in its minimal class. It has determinant 512, which implies $\gamma(\Lambda) \geq 2 = \gamma(\Lambda_9)$ by Proposition 3.2. Thus conjecturally, Λ is similar to Λ_9, and indeed, we do find only one code, hence

TABLE 8. Non-cyclic cases for $n = 9$ and $d = 4 \cdot 2^2$

generators	s	r	s'
(1,1,1,1,2,2,2,0,0),(0,0,1,1,0,0,1,1,0),(0,0,1,1,0,1,0,0,1)	89	43	9
(1,1,1,1,2,2,2,0,0),(0,0,1,1,0,0,1,1,0),(0,1,0,1,0,1,0,0,1)	81	45	9
(1,1,1,1,2,2,2,0,0),(0,0,1,1,0,0,1,1,0),(0,1,0,1,0,0,1,0,1)	89	43	9
(1,1,1,1,1,2,2,0,0),(1,1,0,0,0,1,0,1,0),(0,0,0,0,0,1,1,1,1)	136	45	12

only the class of Λ_9. Finally, we find three codes extending the first type. Two of them again define the minimal class with $(s, r) = (89, 43)$ already found for quotients of type 2^4 and $4 \cdot 2$, and one the perfect class with $s = 81$, already found for quotients of type 2^4. The Ryshkov polyhedron of the class with $(s, r) = (89, 43)$ is a square with edges belonging to a same minimal class having $(s, r) = (90, 44)$ and vertices belonging to the minimal class of Λ_9. Our findings are subsumed in Table 8.

7.6. **Computer calculations.** As mentioned at the beginning of Section 7, the full classification for $n = 9$ relies on computer calculations, using an implementation of Algorithm 1. In order to keep the necessary computations as low as possible, we used a program to systematically generate a list of possible cases. It uses the classification of codes for cyclic quotients for $n = 9$. Note that for a type $d_1 \cdots d_k$ (with the d_i having a common divisor greater than 1) to be realizable in dimension n, all of the k types $d_1 \cdots d_{i-1} \cdot d_{i+1} \cdots d_n$ have to be realizable.

For the cases to be treated, it suffices to consider $k = 2$, say types $d_1 \cdot d_2$, with cyclic types d_1 and d_2 both existing. We can run through all combinations of possible codes generated by $a = (a_1, \dots, a_n) \in (\mathbb{Z}/d_1\mathbb{Z})$ and by $b = (b_1, \dots, b_n) \in (\mathbb{Z}/d_2\mathbb{Z})$. From our classification of cyclic cases, the a_i and b_i are assumed to be in $\{0, \dots, \lfloor \frac{d_1}{2} \rfloor\}$, respectively, $\{0, \dots, \pm\lfloor \frac{d_2}{2} \rfloor\}$. This is due to the fact that we could exchange e_i and $-e_i$ in a basis. For one of the given vectors, say a, we may assume that this property holds; we may, moreover, assume that the a_i are in non-decreasing order. For the b_i, however, we cannot make this assumption, as we already used possible sign changes and changes of order of the vectors e_i for "the normalization" of a. With each b_i, we therefore need to consider also $d_2 - b_i$ (except when $b_i = 0$ or $b_i = \frac{d_2}{2}$ and d_2 is even). Moreover, we need to consider all orderings of the b_i — up to some symmetry within equal a_i entries. For example, if $a_i = a_{i+1} \dots = a_{i+l}$, we may assume that $b_i, \dots, b_{i+l}$ are in non-decreasing order. Note that we may still be faced with quite a lot of possibilities, depending on the given choice of a and b. Note also, that it may be advisable to change the roles of a and b. Another possibility is to reduce the number of cases to be considered: For each case we can consider linear combinations $f = x\frac{a_1e_1+\cdots+a_ne_n}{d_1} + y\frac{b_1e_1+\cdots+b_ne_n}{d_2}$ with $x, y \in \mathbb{Z}$ and check (based on the classification of cyclic types) if a corresponding lattice $\Lambda = \langle \Lambda', f \rangle$ could exist.

Using the strategy sketched above, we were able to exclude the types $8 \cdot 2$, $10 \cdot 2$, $6 \cdot 3$, $9 \cdot 3$ and $5 \cdot 5$ in dimension 9. For the types $6 \cdot 2$ and 4^2 we were able to show existence. Moreover, we obtained a complete classification of corresponding codes. See Tables 9 and 10. From the classification of $6 \cdot 2$, we can exclude the last remaining type $6 \cdot 2^2$, as we explain below. Our results were obtained using an implementation of Algorithm 1, using `MAGMA` scripts in conjunction with `lrs`. Our

Table 9. Non-cyclic cases for $n = 9$ and $d = 6 \cdot 2$

generators	s	r	s'
(0,1,1,1,1,2,2,2,3),(1,0,0,0,1,0,0,1,1)	136	45	37
(0,1,1,1,1,2,2,2,3),(1,0,0,1,1,0,0,1,0)	136	45	46
(0,1,1,1,1,2,2,3,3),(1,0,0,1,1,0,1,0,1)	99	45	33
(0,1,1,1,1,2,2,3,3),(1,0,0,1,1,1,1,0,1)	99	45	33
(0,1,1,1,1,2,2,3,3),(1,0,0,1,1,0,0,0,1)	87	42	23
(0,1,1,2,2,2,2,3,3),(1,0,1,0,0,0,1,0,1)	72	35	22
(0,1,1,1,2,2,2,2,3),(1,0,0,1,0,0,0,1,1)	64	40	33
(0,1,1,1,2,2,2,2,3),(1,0,1,1,0,0,0,1,0)	64	40	33
(0,1,1,1,2,2,2,3,3),(1,0,0,1,0,0,1,0,1)	41	34	23

Table 10. Non-cyclic case for $n = 9$ and $d = 4^2$

generators	s	r	s'
(1,1,1,1,2,2,2,0,0),(0,1,-1,2,2,0,1,2,1)	81	45	9

source code can be obtained from the online appendix of this paper, contained in the source files of its arXiv version `arXiv:0904.3110`. We used a `C++` program that systematically generated a list of possible cases as sketched above.

For codes of type $d = 6 \cdot 2$, the computer assisted calculation output was 23 codes, which we had to check for equivalence. Write $\Lambda = \langle \Lambda', e, f \rangle$ with

$$e = \frac{a_1 e_1 + \cdots + a_9 e_9}{6} \quad \text{and} \quad f = \frac{b_1 e_1 + \cdots + b_9 e_9}{2},$$

and with $a_i \in \{0, 1, 2, 3\}$ and $b_i \in \{0, 1\}$. Replacing e by $e + f$ or $2e + f$, we obtain (after reduction modulo 6 and sign changes of some e_i) three sets (t_i) where t_i is the number of a_j equal to i in the numerator of e. Two equivalent $\mathbb{Z}/6\mathbb{Z}$-codes must have the same sets (t_i). For codes having the same sets (t_i), we were able to make a canonical choice of an e' among e, $e+f$ and $2e+f$, constructing this way, two new $\mathbb{Z}/6\mathbb{Z}$-codes (if one of them is defined by a pair (e', f'), the other one corresponds to $(e', 3e'+f')$). Given two pairs (e', f') and (e', f''), we checked whether a convenient permutation of the coordinates could transform f'' or $f''+3e'$ into e'. The result is that the 23 codes found by the computer were classified up to equivalence by their three sets (a, b, c, d), which reduced our list to only 9 classes of codes.

In this list, there are three pairs of lattices having the same kissing number ($s = 136$, $s = 99$, $s = 64$). In each case, the matrices found by the computer define lattices which are isometric, thus defining the same minimal class.

Quotients of type $6 \cdot 2^2$ can be easily ruled out by the classification of type $6 \cdot 2$. Now, writing $\Lambda = \langle \Lambda', e, f, f' \rangle$, we see in Table 9 that we may choose e such that $a_1 = 0$. Then replacing if necessary f by f' or $f + f'$, we may assume that $b_1 = 0$. But this implies the existence of an 8-dimensional lattice having a quotient of type $6 \cdot 2$, a contradiction.

The 4^2 case is very special. There is only one lattice L_{81} in the minimal class, which therefore is perfect. A Gram matrix is, for example,

$$\begin{pmatrix} 4 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 \\ 2 & 1 & -1 & 2 & 2 & 0 & 1 & 2 & 4 \end{pmatrix}.$$

The lattice L_{81} and its *dual lattice* are strongly eutactic, that is, their sets of minimal vectors are *spherical* 3-*designs* (see [Mar03]). Since it is perfect, it is also extreme and dual-extreme by Voronoi's theorem. The complete index system of L_{81} is described in Appendix B.

8. Universal lattices

In this section, we consider lattices Λ which are *universal* (for their dimension n) in the following sense: with our usual notation, every quotient L/L' which exists in dimension n exists with $L = \Lambda$ for a convenient choice of Λ' generated by minimal vectors of Λ.

Theorem 8.1. *For dimensions $n = 1, \dots, 9$, the universal lattices in the sense above are as follows (as usual up to similarity):*

(1) *All lattices if $n = 1$, 2 or 3.*
(2) *The root lattice $\mathbb{D}_4$ if $n = 4$.*
(3) *All lattices L with $\imath(L) = 2$ and $s(L) \geq 6$ if $n = 5$.*
(4) *None if $n = 6$ or $n = 9$.*
(5) *The lattices $\mathbb{E}_7$, $\mathbb{E}_8$ if $n = 7, 8$.*

Proof. $\underline{n \leq 3}$. There is nothing to prove since the maximal index is 1.

$\underline{n = 4}$. The maximal index is equal to 2 only for $\mathbb{D}_4$, which also admits index 1 since $\mathbb{D}_4$ has bases of minimal vectors.

$\underline{n = 5}$. The maximal index is again 2 in dimension 5, so that we must have $\imath(L) = 2$. This implies that L may be written in the form $L = L' \cup (f + L')$ where $f = \frac{e_1 + \cdots + e_\ell}{2}$, $\ell = 4$ or 5, and $L' = \langle e_1, \dots, e_5 \rangle$ has index 2 in L.

If $\ell = 4$, then both the conditions "$s \geq 6$" and "1 is an index" are satisfied.

If $\ell = s(L) = 5$, then $S(L) = S(L')$ and 2 is the only index for L. If $\ell = 5$ and $s(L) \geq 6$, there exists some minimal vector $f \neq \pm e_i$. If $f \in L'$, then $\ell < 5$. Hence f belongs to $f + L'$, and is of the form $f = \frac{a_1 e_1 + \cdots + a_5 e_5}{2}$. We have $|a_i| \leq 2$ because $\imath(L) \leq 2$, and $a_i \neq 0, \pm 2$ because $\ell = 5$. Hence (f, e_2, e_3, e_4, e_5) is a basis for L.

$\underline{n = 6}$. The maximal index is 4, attained uniquely on $\mathbb{D}_6$. This lattice has index system $\{1, 2, 2^2\}$. Since there exist lattices with $\imath = 3$ (e.g. $\mathbb{E}_6$), there is no universal lattice in this dimension.

$\underline{n = 7, 8}$. It results from Table 11.1 of [Mar01] that index 8 for $n = 7$ and index 16 for $n = 8$ occur only for $\mathbb{E}_n$. Using the classification of root systems, it is then easy to list all well-rounded sublattices of minimum 2 of $\mathbb{E}_7$ and $\mathbb{E}_8$ (see [Mar01], Section 6) and then to check that they realize all quotients in their dimensions.

$\underline{n=9}$. Quotients 4^2 occur only on the similarity class of the perfect lattice L_{81} described at the end of the previous section, whereas cyclic quotients of order 12 occur only on similarity classes with $s \geq 87$, as described in Section 6.1. See also Table 11 in Appendix B. $\square$

Remark 8.2. The lattice Λ_9 is almost universal. It realizes all types in dimension 9, except 4^2.

Proof. We must show that all quotients listed in Theorem 1.1, except 4^2, do occur as quotients of Λ_9. This is clear for those which belong to the index system of $\mathbb{E}_8$ since Λ_9 has a cross-section proportional to $\mathbb{E}_8$. It thus remains to consider quotients which are either cyclic of order $7, 8, 9, 10$ and 12 or of type $6 \cdot 2$ or $4 \cdot 2^2$. Luckily, this problem can be solved by a mere inspection of the codes found for dimension 9: indeed, for each of these quotients, there exists at least one code for which Λ_9 is the only admissible lattice. Here is a list of such codes, given with the notation of 5.3 in cyclic cases and by the components of generators otherwise.
Type (7): $(6,1,2)_7$;
Type (8): $(4,3,2,0)_8$;
Type (9): $(4,1,2,2)_9$;
Type (10): $(2,4,2,0,1)_{10}$;
Type (12): $(2,1,2,2,1,1)_{12}$;
Type $(6 \cdot 2)$: $(0,1^4,2^3,3)_6$, $(1,0^3,1,0^2,1^2)_2$;
Type $(4 \cdot 2^2)$: $(1^5,2^2,0^2)_4$, $(1^2,0^3,1,0,1,0)_2$, $(1^2,0^4,1,0,1)_2$. $\square$

We do not know any result of this kind for larger dimensions. Note that Remark 2.3 shows that a 24-dimensional universal lattice, if any, must be the Leech lattice. In dimension 10, a possible universal lattice is provided by the lattice $\langle \mathbb{E}_8, \mathbb{D}_{10} \rangle$, which has quotients of order 32 and of the three types 2^5, $4 \cdot 2^3$ and $4^2 \cdot 2$; it has a cross-section Λ_9, but we do not even know whether all quotients of Λ_{10} occur for this lattice.

Appendix A: Some perfect lattices

As usual, the notation $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_n$, $n = 6, 7, 8$ stands for the standard irreducible *root lattices*, the definitions of which we recall below. Their importance stems from Witt's theorem, which asserts that *integral lattices generated by vectors of norm* 2 *are orthogonal sums of lattices isometric to* $\mathbb{A}_n$, $n \geq 1$, $\mathbb{D}_n$, $n \geq 4$, *or* $\mathbb{E}_n$, $n = 6, 7, 8$. Denoting by $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ the canonical basis for $\mathbb{Z}^{n+1}$ and by $(\varepsilon_1, \ldots, \varepsilon_n)$ that of $\mathbb{Z}^n$, we set

$$\mathbb{A}_n = \left\{ x \in \mathbb{Z}^{n+1} \mid \sum_{i=0}^{n+1} x_i = 0 \right\} \text{ and } \mathbb{D}_n = \left\{ x \in \mathbb{Z}^n \mid \sum_{i=1}^{n} x_i \equiv 0 \mod 2 \right\}$$

(we consider $\mathbb{A}_n$ for $n \geq 1$ and $\mathbb{D}_n$ for $n \geq 2$, but $\mathbb{D}_2 \simeq \mathbb{A}_1 \perp \mathbb{A}_1$ and $\mathbb{D}_3 \simeq \mathbb{A}_3$). For all $n \geq 8$ even, we then set

$$\mathbb{D}_n^+ = \langle \mathbb{D}_n, \frac{\varepsilon_1 + \cdots + \varepsilon_n}{2} \rangle = \mathbb{D}_n \cup \left(\frac{\varepsilon_1 + \cdots + \varepsilon_n}{2} + \mathbb{D}_n \right),$$

and $\mathbb{E}_8 = \mathbb{D}_8^+$ (but $\mathbb{D}_n^+$ is not a root lattice for $n > 8$), and finally define $\mathbb{E}_7$ and $\mathbb{E}_6$ as the orthogonal complement in $\mathbb{E}_8$ of the spans of $\varepsilon_7 + \varepsilon_8$ and $\{\varepsilon_6 + \varepsilon_7, \varepsilon_7 + \varepsilon_8\}$, respectively.

Note that $\mathbb{A}_n$ has a nice characterization in terms of its index system, by an 1877 theorem of Korkine and Zolotareff: it is *the* n-dimensional lattice with $s \geq \frac{n(n+1)}{2}$ and maximal index 1. For $\mathbb{D}_n$, we quote the following property:

Proposition. *For all $n \geq 2$, the index system of $\mathbb{D}_n$ is*

$$\mathcal{I}(\mathbb{D}_n) = \{1, 2, \ldots, 2^t\}\ ,$$

where $t = \lfloor \frac{n-2}{2} \rfloor$.

Sketch of proof. By the classification of root systems, a strict sublattice of $\mathbb{D}_n$ of rank n is an orthogonal sum of irreducible root lattices $L_1, \ldots, L_k$ of dimensions $n_1, \ldots, n_k < n$ which add to n. Embeddings $\mathbb{E}_m \hookrightarrow \mathbb{D}_n$ are impossible (see [Mar02, Section 4.6]). For $m \neq 1, 3$, embeddings $\mathbb{A}_m \hookrightarrow \mathbb{D}_n$ are equivalent modulo an automorphism of $\mathbb{D}_n$ to $\mathbb{A}_m \to L = \langle \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_m - \varepsilon_{m+1} \rangle$, and must be discarded because $L^\perp$ is not a root sublattice of $\mathbb{D}_n$. For $m = 3$, there is a second orbit, namely that of $\mathbb{A}_3 \to L = \langle \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 + \varepsilon_3 \rangle$, for which we have $L^\perp \simeq \mathbb{D}_{n-3}$; we denote by $\mathbb{D}_3$ this kind of embedding of $\mathbb{A}_3$. Finally, $\mathbb{A}_1 \perp \mathbb{A}_1$ embeds as $\langle \varepsilon_i \pm \varepsilon_j \rangle$, which yields an orthogonal decomposition $\mathbb{D}_2 \perp \mathbb{D}_{n-2}$ where we denote by $\mathbb{D}_2$ any $\mathbb{A}_1 \perp \mathbb{A}_1$ embedded as $\langle \varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j \rangle$. With these definitions of $\mathbb{D}_2$ and $\mathbb{D}_3$, we prove inductively that root sublattices of $\mathbb{D}_n$ are obtained taking $L_i = \mathbb{D}_i$, and the proof of the proposition is now easily completed. □

The *laminated lattices* Λ_n were defined inductively by Conway and Sloane; see [CS99, Chapter 6]. They have minimum 4, they are integral in the range $1 \leq n \leq 24$, uniquely defined except for $n = 11, 12, 13$, and for $n \leq 8$, they are scaled copies of $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.

For all even $m \geq 8$ and all $n \geq m$, the lattices $\langle \mathbb{D}_m^+, \mathbb{D}_n \rangle$ (Barnes's lattices $\mathbb{D}_{n,m}$; see [Mar03, Section 5.5]) have minimum 2. They are integral if $m = n \equiv 0 \mod 4$, and only half-integral otherwise, hence become integral in the scale which gives them minimum 4. In particular,

$$\Lambda_9 = \langle \mathbb{D}_8^+, \mathbb{D}_9 \rangle = \langle \mathbb{E}_8, \mathbb{D}_9 \rangle \text{ scaled to minimum } 4\,.$$

Here are unified constructions for the three perfect lattices which were found directly in our classification. There are four $[10, 5, 4]$-codes. They lift over $\mathbb{Z}^{10}$ to four 10-dimensional lattices: $L_a \sim \mathbb{D}_{10}^+$ ($s = 90$), $L_b \sim \langle \mathbb{E}_8, \mathbb{D}_{10} \rangle$ ($s = 154$), L_c ($s = 138$, not K_{10}), and $L_d \sim Q_{10}$ ($s = 130$, the Souvignier lattice).

Among the densest cross-sections of L_b, L_c, L_d we find the lattices Λ_9, L_{99} and L_{81}, respectively. (For L_a we obtain the lattice with $s = 57$ of Table 5.)

The three perfect lattices above are indeed eutactic and hence extreme by Voronoi's theorem; this is clear for Λ_9, which contains $\mathbb{D}_9$ scaled to minimum 4 and for L_{81}, which is strongly eutactic. For the lattice L_{99} we have verified eutaxy by using a computer calculation; see Section 6.1.

The laminated lattice Λ_{24} is known as the *Leech lattice* and has many remarkable properties. By a recent theorem of Cohn and Kumar [CK09], we have $\gamma_{24} = 4$ and the only lattice attaining γ_{24} (up to similarity) is the Leech lattice Λ_{24}, an integral lattice of minimum 4 and determinant 1, whence $\gamma_{24}^{12} = 2^{24}$. It follows that Λ_{24} is the unique lattice in dimension 24 that satisfies the index bound of Proposition 2.1 with equality: By the proof of Conway's uniqueness theorem for the Leech lattice (see [CS99, Chapter 12]), every class of $\Lambda_{24} \mod 2$ has a representative of norm at most $2 \min \Lambda_{24} = 8$, and norm 8 vectors occur in 24 pairs $\pm x$, which implies by [Mar02, Theorem 2.5] that Λ_{24} contains a sublattice L which is a scaled copy with minimum 4 of $\mathbb{D}_{24}$. We have $\det(L) = 4 \cdot 2^{24}$, hence $[\Lambda_{24} : L] = 2^{13}$. Now the root lattice $\mathbb{D}_n$, $n = 2m$ even, contains orthogonal frames, which span lattices of index 2^{m-1} in $\mathbb{D}_n$. This shows that L contains to index 2^{11} a lattice Λ' generated by minimal vectors of Λ_{24}, and we have $[\Lambda_{24} : \Lambda'] = 2^{13} \cdot 2^{11} = 2^{24}$ showing that the bound $\imath(\Lambda) \leq \lfloor \gamma_{24}^{12} \rfloor$ is tight. This implies the known result (see [BCS95]) that the Leech lattice can be constructed as the pull-back of a code of length 24 over $\mathbb{Z}/4\mathbb{Z}$.

Appendix B: Enumerating independent subsets of shortest vectors, by Mathieu Dutour Sikirić

In the present paper, the authors consider a pair (Λ, Λ') of a 9-dimensional lattice Λ and one of its Minkowskian sublattices, and classify all the $\mathbb{Z}/d\mathbb{Z}$ codes associated with the quotient Λ/Λ'. In particular, they obtain all possible structures of Λ/Λ' as an Abelian group. However, given a lattice Λ, the question of how to compute its index system $\mathcal{I}(\Lambda)$ (in the sense of Definition 2.2) is left aside.

For a given lattice Λ of dimension n and (half) kissing number s, a straightforward approach would be to consider all of the $\binom{s}{n}$ possible bases of Minkowskian sublattices. Except for the lattice E_8, this approach works for all perfect lattices for $n \leq 8$, where $s \leq 75$ (see [Mar01]). In dimension 9 however, several interesting lattices cannot be handled by this naive approach. This is, in particular, true for the interesting lattices L_{81} and L_{99} described in Sections 6 and 7.

In this appendix, I shall describe shortly an algorithm which outputs the index system of some lattices with a large kissing number. The results I obtained for L_{81} and L_{99} are displayed in Table 11. I also consider the non-perfect lattice L_{87} of perfection rank 42 and maximal index 12 (see Section 6). The minimal class of L_{87} lies below that of L_{99}, so that every index which occurs for L_{99} already occurs for L_{87}.

We denote by $\mathrm{Aut}(\Lambda)$ the group of lattice automorphisms of Λ. We split $S(\Lambda)$ (the set of minimal vectors of Λ) into pairs of antipodal vectors $\{v_1, -v_1\}, \dots, \{v_s, -v_s\}$ and define $S^{1/2}(\Lambda) = \{v_1, \dots, v_s\}$. The group $\mathrm{Aut}(\Lambda)$ induces an action on the s antipodal pairs and thus defines a permutation group $\mathrm{Aut}^{1/2}(\Lambda)$ on s elements of $S^{1/2}(\Lambda)$. If Λ does not admit a decomposition $\Lambda_1 \perp \Lambda_2$ into two orthogonal sublattices, then the order of $\mathrm{Aut}^{1/2}(\Lambda)$ is half the order of $\mathrm{Aut}(\Lambda)$. Denote by $I_k(\Lambda)$ the list of inequivalent representatives of orbits of independent subsets with k elements of $S^{1/2}(\Lambda)$ under $\mathrm{Aut}^{1/2}(\Lambda)$.

We need to determine $I_n(\Lambda)$, but it turns out that the only known method requires enumerating $I_k(\Lambda)$ for $k \leq n$ as well. Given $I_k(\Lambda)$, for all $S \in I_k(\Lambda)$ we consider all possible ways to add one vector to S and get an independent system. By keeping only inequivalent representatives, we get in this way $I_{k+1}(\Lambda)$. The basic problem is to be able to test if two subsets of $S^{1/2}(\Lambda)$ are equivalent under the group $\mathrm{Aut}^{1/2}(\Lambda)$. There exist backtracking methods for this purpose, that are known to work well in practice (see [Ser03, Chapter 9]). Using these techniques, we find $|I_8(\mathsf{E}_8)| = 1943$ for the highly symmetric E_8 lattice.

The basic problem of this method is that we have to store $I_k(\Lambda)$ in memory and that the number of equivalence tests grows quadratically in the size of $I_k(\Lambda)$. To overcome these difficulties we use an "orderly generation" approach, a classic technique of combinatorial enumeration (see for example [McK98]).

TABLE 11. The number of orbits of bases of Minkowskian sublattices, for each of the 19 possible index types in dimension 9 and for the special lattices L_{81}, L_{87} and L_{99}

lattice Λ	L_{81}	L_{87}	L_{99}
$\lvert \mathrm{Aut}^{1/2}(\Lambda)\rvert$	18432	6144	9216
1	3774844	16730092	49301288
2	474881	2657720	8271400
3	28768	198528	681759
4	6634	46390	163090
2^2	4579	28560	88407
5	348	2859	12126
6	205	2171	8462
7	3	59	230
8	7	49	169
$4, 2$	57	212	597
2^3	32	132	309
9	–	4	4
3^2	–	1	12
10	–	5	5
12	–	1	1
$6, 2$	–	1	10
4^2	1	–	–
$4, 2^2$	1	–	–
2^4	1	–	–

If $S \in I_n(\Lambda)$, then we choose S to be lexicographically minimal in its orbit under $\mathrm{Aut}^{1/2}(\Lambda)$ and write it as $S = \{x_1, \dots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$. Then the sets $S_k = \{x_1, \dots, x_k\}$ for $1 \le k \le n$ are lexicographically minimal in their respective orbits as well. Reversely, suppose we have all lexicographically minimal representatives in $I_k(\Lambda)$, then for all $S_k = \{x_1, \dots, x_k\} \in I_k(\Lambda)$ we consider all sets

$$S_k(t) = S_k \cup \{t\} \text{ for } t \in \{x_k + 1, \dots, s\},$$

and we test for all of them if they are minimal in their orbit $O(S_k(t))$ under $\mathrm{Aut}^{1/2}(\Lambda)$ by computing all elements of $O(S_k(t))$. If they are minimal, then they are added to the set $I_{k+1}(\Lambda)$. Obviously, this method is limited by the size of the group and is not appropriate for E_8 or Λ_9.

Once the sets $I_n(\Lambda)$ are built, we use the Smith Normal Form for each element (basis of a Minkowskian sublattice Λ'), to determine the invariant of the Abelian group Λ/Λ'.

In our, obviously non-optimal, implementation we store the sets $I_k(\Lambda)$ on disk and we use the `GMP` library for exact arithmetic and a `C` program that builds $I_{k+1}(\Lambda)$ from $I_k(\Lambda)$. The Smith Normal Form computation is done in `GAP`. The running time is always less than 1 week. The program is part of the `GAP` package `polyhedral` [Polyhedral].

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FACULTY OF MATHEMATICS, OTTO-VON-GUERICKE UNIVERSITÄT, 39106 MAGDEBURG, GERMANY
E-mail address: `Wolfgang.Keller@student.uni-magdeburg.de`

INSTITUT DE MATHÉMATIQUES, 351, COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE
E-mail address: `Jacques.Martinet@math.u-bordeaux1.fr`

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ROSTOCK, 18051 ROSTOCK, GERMANY
E-mail address: `achill.schuermann@uni-rostock.de`

RUDJER BOSKOVIĆ INSTITUTE, BIJENICKA 54, 10000 ZAGREB, CROATIA
E-mail address: `mdsikir@irb.hr`