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# TRACTABILITY INDEX OF HYBRID EQUATIONS FOR CIRCUIT SIMULATION

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Abstract. Modern modeling approaches for circuit simulation such as the modified nodal analysis (MNA) lead to differential-algebraic equations (DAEs). The index of a DAE is a measure of the degree of numerical difficulty. In general, the higher the index is, the more difficult it is to solve the DAE.

In this paper, we consider a broader class of analysis methods called the hybrid analysis. For nonlinear time-varying circuits with general dependent sources, we give a structural characterization of the tractability index of DAEs arising from the hybrid analysis. This enables us to determine the tractability index efficiently, which helps to avoid solving higher index DAEs in circuit simulation.

## 1. Introduction

In circuit simulation, we set up a system of equations by using circuit analysis methods such as the modified nodal analysis (MNA), the tableau analysis, the loop analysis, the cutset analysis, and the hybrid analysis. MNA is the most popular method and adopted in SPICE [31]. This is mainly because it allows an automatic setup of model equations. In contrast, the hybrid analysis retains flexibility, which can be exploited to find a model description that reduces the numerical difficulties. The purpose of this paper is to clarify inherent advantage of the hybrid analysis in the ease of numerical solution.

Circuit analysis methods lead to differential-algebraic equations (DAEs), which consist of algebraic equations and differential operations. DAEs often present numerical and analytical difficulties which do not occur with ordinary differential equations (ODEs). The numerical difficulty of DAEs is measured by the *index*. In general, the higher the index is, the more difficult it is to solve the DAE. Many different concepts exist to assign an index to a DAE such as the *differentiation index* [5, 7, 13], the *perturbation index* [13], the *strangeness index* [22], the *tractability index* [9, 24], the *geometric index* [35], and the *Kronecker index* [36]. These indices are closely related to each other. The index more than one is called a *higher index*.

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The difficulties of DAEs with higher index are much greater than DAEs with index zero or one.

In this paper, we focus on the tractability index, which is analyzed by projector-based techniques developed in [2, 11, 24]. The tractability index of a DAE obtained by applying MNA to nonlinear time-invariant RLC circuits does not exceed two [39]. It is also shown in [39] that MNA leads to a DAE with index at most one if and only if a circuit contains neither L-J cutsets nor C-V loops (except for C-loops), where an *L-J cutset* means a cutset consisting only of inductors and/or current sources, a *C-V loop* means a cycle consisting only of capacitors and/or voltage sources, and a *C-loop* means a cycle consisting only of capacitors. This implies that the index of a DAE arising from MNA is determined uniquely by the network. Furthermore, these results in [39] are generalized for nonlinear time-varying electric circuits that may contain a wide class of dependent sources [9]. The results in [9, 39] suggest that DAEs arising from MNA often have higher index. Reißig [33, 36] has obtained similar results for DAEs arising from the tableau analysis.

The hybrid analysis is a common generalization of the loop analysis and the cutset analysis. Kron [21] proposed the hybrid analysis in 1939, and Amari [1] and Branin [4] developed it further in the 1960s. While the procedure of MNA is uniquely determined, the hybrid analysis starts with selecting a *partition* of elements and a *reference tree* in the network. This selection determines DAEs, called the *hybrid equations*, to be solved numerically. Thus it is natural to search for an optimal selection that makes the hybrid equations easy to solve. In fact, the problem of obtaining the minimum size hybrid equations was solved in [16, 20, 30] in 1968. This turned out to be an application of matroid intersection [17]. See also [29, 32] for matroid theoretic approach to circuit analysis.

Recently, the index of the hybrid equations has been analyzed to make a comparison with MNA theoretically. An algorithm for finding an optimal hybrid analysis which minimizes the index of the hybrid equations is proposed in [18] for linear time-invariant circuits that may contain dependent sources. For linear time-invariant RLC circuits, it is proved in [38] that the index of the hybrid equations is at most one. A structural characterization of circuits with index zero is also given in [38]. These results indicate that the index of the hybrid equations never exceeds the index of DAEs arising from MNA.

The results in [38] are extended to nonlinear time-varying circuits which may contain a certain restricted class of dependent sources in [19]. In particular, it is proved that the tractability index of the hybrid equations is at most one. From a practical point of view, however, it is important to deal with general dependent sources, which often result in higher index DAEs.

In this paper, extending the results in [19] to circuits with general dependent sources, we give structural characterizations of circuits with tractability index zero and at most one. This also leads to the characterization of circuits with index one. The structural characterizations enable us to determine efficiently whether the hybrid equations of a circuit have higher index or not. The proof exploits properties of skew-symmetric matrices.

The organization of this paper is as follows. In Section 2, we describe nonlinear time-varying circuits and present the procedure of the hybrid analysis. Section 3 is devoted to the definition of the tractability index of DAEs. We analyze the hybrid

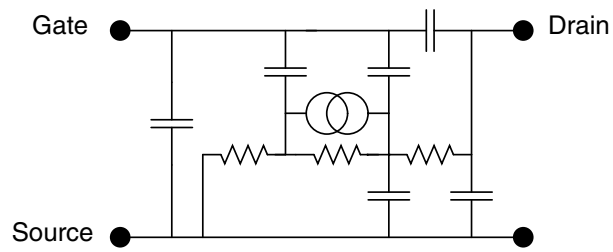


FIGURE 1. A MOSFET model.

equations in Section 4. Section 5 gives structural characterizations of the hybrid equations with index zero and at most one. Finally, Section 6 concludes this paper.

## 2. Hybrid analysis of nonlinear time-varying circuits

In this paper, we consider nonlinear time-varying circuits composed of resistors, inductors, capacitors, and independent/dependent voltage/current sources.

We denote the vector of currents through all branches of the circuit by $\boldsymbol{i}$, and the vector of voltages across all branches by $\boldsymbol{u}$. Let $V$, $J$, $C$, $L$, and $R$ denote the sets of independent voltage sources, independent current sources, capacitors, inductors, and resistors, respectively. Dependent voltage/current sources, denoted by $S_U$ and $S_I$, are controlled by voltages across or currents through other branches.

The vectors of currents through $V$, $J$, $C$, $L$, $R$, $S_U$, and $S_I$ are denoted by $\boldsymbol{i}_V$, $\boldsymbol{i}_J$, $\boldsymbol{i}_C$, $\boldsymbol{i}_L$, $\boldsymbol{i}_R$, $\boldsymbol{i}_U$, and $\boldsymbol{i}_I$. Similarly, the vectors of voltages are denoted by $\boldsymbol{u}_V$, $\boldsymbol{u}_J$, $\boldsymbol{u}_C$, $\boldsymbol{u}_L$, $\boldsymbol{u}_R$, $\boldsymbol{u}_U$, and $\boldsymbol{u}_I$. The physical characteristics of elements determine *constitutive equations*. Independent voltage and current sources simply read as

$$\boldsymbol{u}_V = \boldsymbol{v}_s(t) \quad \text{and} \quad \boldsymbol{i}_J = \boldsymbol{j}_s(t). \tag{2.1}$$

We assume that the constitutive equations of capacitors and inductors are described by

$$\boldsymbol{i}_C = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}(\boldsymbol{u}_C, t) \quad \text{and} \quad \boldsymbol{u}_L = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(\boldsymbol{i}_L, t). \tag{2.2}$$

Resistors are given in the form of

$$\boldsymbol{i}_R = \boldsymbol{\sigma}(\boldsymbol{u}_R, t). \tag{2.3}$$

Moreover, dependent current sources and dependent voltage sources are modeled by

$$\boldsymbol{i}_I = \boldsymbol{j}_I(\boldsymbol{i}_L, \boldsymbol{i}_J, \boldsymbol{u}_V, \boldsymbol{u}_C, t) \quad \text{and} \quad \boldsymbol{u}_U = \boldsymbol{v}_U(\boldsymbol{i}_L, \boldsymbol{i}_J, \boldsymbol{u}_V, \boldsymbol{u}_C, t). \tag{2.4}$$

This includes a wide class of dependent current/voltage sources.

**Example 2.1.** Consider a MOSFET model [10] depicted in Figure 1. The dependent current source is controlled by voltages across other branches. Since this circuit has a spanning tree which consists only of capacitors, the dependent current source can be described by a constitutive equation with argument $\boldsymbol{u}_C$, that is to say, equation (2.4).

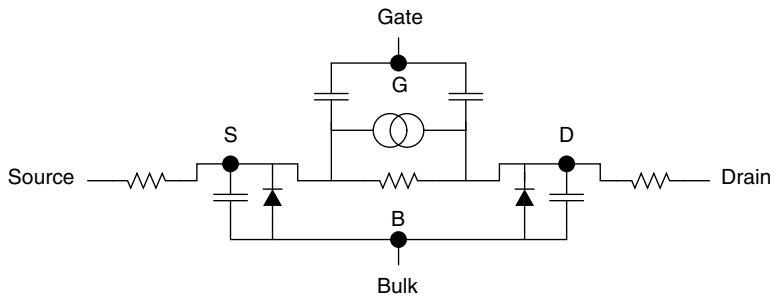


FIGURE 2. Another MOSFET model.

**Example 2.2.** Consider another MOSFET model [9] depicted in Figure 2. The dependent current source is controlled by $u_{\mathrm{GS}}$, $u_{\mathrm{DS}}$, and $u_{\mathrm{BS}}$, where $u_{\mathrm{GS}}$ is a branch voltage between G and S, $u_{\mathrm{DS}}$ is a branch voltage between D and S, and $u_{\mathrm{BS}}$ is a branch voltage between B and S. Since these voltages are expressed by voltages across capacitors, the dependent current source can be described by equation (2.4).

A vector $(\boldsymbol{i}, \boldsymbol{u})$ satisfying (2.1), (2.3), (2.4), and Kirchhoff's current/voltage laws at a given time $t$ is called an *operating point* at $t$ [36]. For a matrix $A$, we denote the $(i, j)$ entry of $A$ by $(A)_{ij}$. For a vector-valued function $\boldsymbol{f}$, we denote the $i$th component of $\boldsymbol{f}$ by $(\boldsymbol{f})_i$. The capacitance matrix $C$, the inductance matrix $L$, and the conductance matrix $K$ are given by

$$(C)_{ij} = \frac{\partial(\boldsymbol{q})_i}{\partial(\boldsymbol{u}_C)_j}, \quad (L)_{ij} = \frac{\partial(\boldsymbol{\phi})_i}{\partial(\boldsymbol{i}_L)_j}, \quad \text{and} \quad (K)_{ij} = \frac{\partial(\boldsymbol{\sigma})_i}{\partial(\boldsymbol{u}_R)_j}.$$

A square matrix $A$ is called *positive definite* if $\boldsymbol{x}^\top A\boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq 0$. In this paper, we assume the following conditions.

**Assumption 2.3.** *The capacitance matrix $C$ and the inductance matrix $L$ are positive definite at all operating points.*

**Assumption 2.4.** *The conductance matrix $K$ is symmetric and positive definite at all operating points.*

Assumption 2.3 means that capacitors and inductors are strictly locally passive elements at all operating points. Assumption 2.4 indicates that resistors are reciprocal and strictly locally passive at all operating points [8, 36].

Let $\Gamma = (W, E)$ be the network graph with vertex set $W$ and edge set $E$. An edge in $\Gamma$ corresponds to a branch that contains one element in the circuit. For a consistent model description, $\Gamma$ contains no cycles consisting only of independent voltage sources and no cutsets consisting only of independent current sources, where a *cutset* is a set of edges whose deletion increases the number of connected components in $\Gamma$. We denote the set of edges corresponding to independent voltage sources and independent current sources by $E_v$ and $E_j$, respectively. We split $E_* := E \setminus (E_v \cup E_j)$ into $E_y$ and $E_z$, i.e., $E_y \cup E_z = E_*$ and $E_y \cap E_z = \emptyset$. A partition $(E_y, E_z)$ is called an *admissible partition*, if $E_y$ includes all the capacitors and all the dependent current sources, and $E_z$ includes all the inductors and all the dependent voltage sources.

We now rewrite constitutive equations with respect to an admissible partition $(E_y, E_z)$. We split $\boldsymbol{i}$ and $\boldsymbol{u}$ into

$$\boldsymbol{i} = (\boldsymbol{i}_V, \boldsymbol{i}_C, \boldsymbol{i}_I, \boldsymbol{i}_Y, \boldsymbol{i}_Z, \boldsymbol{i}_U, \boldsymbol{i}_L, \boldsymbol{i}_J)^\top \quad \text{and} \quad \boldsymbol{u} = (\boldsymbol{u}_V, \boldsymbol{u}_C, \boldsymbol{u}_I, \boldsymbol{u}_Y, \boldsymbol{u}_Z, \boldsymbol{u}_U, \boldsymbol{u}_L, \boldsymbol{u}_J)^\top,$$

where the subscripts $Y$ and $Z$ correspond to the resistors in $E_y$ and $E_z$. Resistors are modeled by constitutive equations in the form of

$$\boldsymbol{i}_Y = \boldsymbol{g}(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t) \quad \text{and} \quad \boldsymbol{u}_Z = \boldsymbol{h}(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t). \tag{2.5}$$

The matrices $Z$, $H$, $G$, $Y$ are defined by

$$(Z)_{ij} = \frac{\partial(\boldsymbol{h})_i}{\partial(\boldsymbol{i}_Z)_j}, \quad (H)_{ij} = \frac{\partial(\boldsymbol{h})_i}{\partial(\boldsymbol{u}_Y)_j}, \quad (G)_{ij} = \frac{\partial(\boldsymbol{g})_i}{\partial(\boldsymbol{i}_Z)_j}, \quad (Y)_{ij} = \frac{\partial(\boldsymbol{g})_i}{\partial(\boldsymbol{u}_Y)_j}.$$

With the aid of the conductance matrix in the form of $K = \begin{pmatrix} K_Y & K_G \\ K_H & K_Z \end{pmatrix}$, where the row/column sets of $K_Y$ and $K_Z$ are the sets of resistors in $E_y$ and $E_z$, the four Jacobian matrices above are expressed by

$$Z = {K_Z}^{-1}, \quad H = -{K_Z}^{-1}K_H, \quad G = K_G{K_Z}^{-1}, \quad Y = K_Y - K_G{K_Z}^{-1}K_H.$$

Then, by Assumption 2.4, one can show that the hybrid immittance matrix $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$ satisfies the following conditions at all operating points:

(i) The hybrid immittance matrix $\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}$ is positive definite.
(ii) The principal submatrices $Z$ and $Y$ are symmetric.
(iii) $H = -G^\top$ holds.

A *spanning tree* in a connected graph is a maximal set of edges which contains no cycles. We call a spanning tree $T$ of $\Gamma$ a *reference tree* if $T$ contains all edges in $E_v$, no edges in $E_j$, and as many edges in $E_y$ as possible. Note that a reference tree $T$ may contain some edges in $E_z$. A reference tree is called *normal* if it contains as many edges as possible in the order corresponding to $V$, $C$, $S_I$, $Y$, $Z$, $S_U$, and $L$. The cotree of $T$ is denoted by $\overline{T} = E \setminus T$. Normal trees have already been used in [6] for state approaches for linear RLC networks. The results have been extended in [34] for linear circuits containing ideal transformers, nullors, independent/dependent sources, resistors, inductors, capacitors, and, under a topological restriction, gyrators.

The hybrid equations are determined by an admissible partition $(E_y, E_z)$ and a reference tree $T$, which is not necessarily normal. In this paper, we adopt a normal reference tree. With respect to a normal reference tree $T$, we further split $\boldsymbol{i}$ and $\boldsymbol{u}$ into

$$\boldsymbol{i} = (\boldsymbol{i}_V, \boldsymbol{i}_C^\tau, \boldsymbol{i}_I^\tau, \boldsymbol{i}_Y^\tau, \boldsymbol{i}_Z^\tau, \boldsymbol{i}_U^\tau, \boldsymbol{i}_L^\tau, \boldsymbol{i}_C^\lambda, \boldsymbol{i}_I^\lambda, \boldsymbol{i}_Y^\lambda, \boldsymbol{i}_Z^\lambda, \boldsymbol{i}_U^\lambda, \boldsymbol{i}_L^\lambda, \boldsymbol{i}_J)^\top$$

and

$$\boldsymbol{u} = (\boldsymbol{u}_V, \boldsymbol{u}_C^\tau, \boldsymbol{u}_I^\tau, \boldsymbol{u}_Y^\tau, \boldsymbol{u}_Z^\tau, \boldsymbol{u}_U^\tau, \boldsymbol{u}_L^\tau, \boldsymbol{u}_C^\lambda, \boldsymbol{u}_I^\lambda, \boldsymbol{u}_Y^\lambda, \boldsymbol{u}_Z^\lambda, \boldsymbol{u}_U^\lambda, \boldsymbol{u}_L^\lambda, \boldsymbol{u}_J)^\top,$$

where the superscripts $\tau$ and $\lambda$ designate the tree $T$ and the cotree $\overline{T}$. With respect to a normal reference tree $T$, the vector-valued function $\boldsymbol{g}$ is also split into $\boldsymbol{g}^\tau$ and $\boldsymbol{g}^\lambda$. This means $\boldsymbol{i}_Y^\tau = \boldsymbol{g}^\tau(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t)$ and $\boldsymbol{i}_Y^\lambda = \boldsymbol{g}^\lambda(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t)$. Similarly, we split $\boldsymbol{h}$,

$\boldsymbol{q}$, and $\boldsymbol{\phi}$. The matrix $Y$ is written in the form of $\begin{pmatrix} Y_\tau^\tau & Y_\lambda^\tau \\ Y_\tau^\lambda & Y_\lambda^\lambda \end{pmatrix}$, where

$$(Y_\tau^\tau)_{ij} = \frac{\partial(\boldsymbol{g}^\tau)_i}{\partial(\boldsymbol{u}_Y^\tau)_j}, \quad (Y_\lambda^\tau)_{ij} = \frac{\partial(\boldsymbol{g}^\tau)_i}{\partial(\boldsymbol{u}_Y^\lambda)_j}, \quad (Y_\tau^\lambda)_{ij} = \frac{\partial(\boldsymbol{g}^\lambda)_i}{\partial(\boldsymbol{u}_Y^\tau)_j}, \quad (Y_\lambda^\lambda)_{ij} = \frac{\partial(\boldsymbol{g}^\lambda)_i}{\partial(\boldsymbol{u}_Y^\lambda)_j}.$$

In a similar way, the matrices $C$, $L$, $Z$, $H$, $G$ are written in the form of

$$\begin{pmatrix} C_\tau^\tau & C_\lambda^\tau \\ C_\tau^\lambda & C_\lambda^\lambda \end{pmatrix}, \quad \begin{pmatrix} L_\tau^\tau & L_\lambda^\tau \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix}, \quad \begin{pmatrix} Z_\tau^\tau & Z_\lambda^\tau \\ Z_\tau^\lambda & Z_\lambda^\lambda \end{pmatrix}, \quad \begin{pmatrix} H_\tau^\tau & H_\lambda^\tau \\ H_\tau^\lambda & H_\lambda^\lambda \end{pmatrix}, \quad \begin{pmatrix} G_\tau^\tau & G_\lambda^\tau \\ G_\tau^\lambda & G_\lambda^\lambda \end{pmatrix}.$$

By the definition of a normal reference tree, the *fundamental cutset matrix* $F$ is given by

$$F = \begin{array}{c} \begin{array}{cccccccccccccc} \boldsymbol{i}_V & \boldsymbol{i}_C^\tau & \boldsymbol{i}_I^\tau & \boldsymbol{i}_Y^\tau & \boldsymbol{i}_Z^\tau & \boldsymbol{i}_U^\tau & \boldsymbol{i}_L^\tau & \boldsymbol{i}_C^\lambda & \boldsymbol{i}_I^\lambda & \boldsymbol{i}_Y^\lambda & \boldsymbol{i}_Z^\lambda & \boldsymbol{i}_U^\lambda & \boldsymbol{i}_L^\lambda & \boldsymbol{i}_J \end{array} \\ \begin{pmatrix} I & O & O & O & O & O & O & A_{VC} & A_{VI} & A_{VY} & A_{VZ} & A_{VU} & A_{VL} & A_{VJ} \\ O & I & O & O & O & O & O & A_{CC} & A_{CI} & A_{CY} & A_{CZ} & A_{CU} & A_{CL} & A_{CJ} \\ O & O & I & O & O & O & O & O & A_{II} & A_{IY} & A_{IZ} & A_{IU} & A_{IL} & A_{IJ} \\ O & O & O & I & O & O & O & O & O & A_{YY} & A_{YZ} & A_{YU} & A_{YL} & A_{YJ} \\ O & O & O & O & I & O & O & O & O & O & A_{ZZ} & A_{ZU} & A_{ZL} & A_{ZJ} \\ O & O & O & O & O & I & O & O & O & O & O & A_{UU} & A_{UL} & A_{UJ} \\ O & O & O & O & O & O & I & O & O & O & O & O & A_{LL} & A_{LJ} \end{pmatrix} \end{array}.$$

Then Kirchhoff's current law is written as $F\boldsymbol{i} = \boldsymbol{0}$. Performing the hybrid analysis described in [18], we obtain the *hybrid equations* (or *hybrid equation system*)

$$\begin{aligned} -A_{VL}^\top \boldsymbol{v}_s(t) - A_{CL}^\top \boldsymbol{u}_C^\tau - A_{IL}^\top \boldsymbol{u}_I^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau - A_{ZL}^\top \boldsymbol{h}^\tau - A_{UL}^\top \boldsymbol{v}_U^\tau(\cdot) - A_{LL}^\top \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\tau + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\lambda &= \boldsymbol{0}, \\ -A_{VU}^\top \boldsymbol{v}_s(t) - A_{CU}^\top \boldsymbol{u}_C^\tau - A_{IU}^\top \boldsymbol{u}_I^\tau - A_{YU}^\top \boldsymbol{u}_Y^\tau - A_{ZU}^\top \boldsymbol{h}^\tau - A_{UU}^\top \boldsymbol{v}_U^\tau(\cdot) + \boldsymbol{v}_U^\lambda(\cdot) &= \boldsymbol{0}, \\ -A_{VZ}^\top \boldsymbol{v}_s(t) - A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{IZ}^\top \boldsymbol{u}_I^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau - A_{ZZ}^\top \boldsymbol{h}^\tau + \boldsymbol{h}^\lambda &= \boldsymbol{0}, \\ \boldsymbol{g}^\tau + A_{YY}\boldsymbol{g}^\lambda + A_{YZ}\boldsymbol{i}_Z^\lambda + A_{YU}\boldsymbol{i}_U^\lambda + A_{YL}\boldsymbol{i}_L^\lambda + A_{YJ}\boldsymbol{j}_s(t) &= \boldsymbol{0}, \\ \boldsymbol{j}_I^\tau(\cdot) + A_{II}\boldsymbol{j}_I^\lambda(\cdot) + A_{IY}\boldsymbol{g}^\lambda + A_{IZ}\boldsymbol{i}_Z^\lambda + A_{IU}\boldsymbol{i}_U^\lambda + A_{IL}\boldsymbol{i}_L^\lambda + A_{IJ}\boldsymbol{j}_s(t) &= \boldsymbol{0}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\tau + A_{CC}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\lambda + A_{CI}\boldsymbol{j}_I^\lambda(\cdot) + A_{CY}\boldsymbol{g}^\lambda + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CU}\boldsymbol{i}_U^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{j}_s(t) &= \boldsymbol{0} \end{aligned}$$

with

$$\begin{aligned} &\boldsymbol{q}^\tau = \boldsymbol{q}^\tau(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t), \quad \boldsymbol{q}^\lambda = \boldsymbol{q}^\lambda(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t), \\ &\boldsymbol{g}^\tau = \boldsymbol{g}^\tau(\boldsymbol{\alpha}, \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, \boldsymbol{\beta}, t), \quad \boldsymbol{g}^\lambda = \boldsymbol{g}^\lambda(\boldsymbol{\alpha}, \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, \boldsymbol{\beta}, t), \\ &\boldsymbol{h}^\tau = \boldsymbol{h}^\tau(\boldsymbol{\alpha}, \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, \boldsymbol{\beta}, t), \quad \boldsymbol{h}^\lambda = \boldsymbol{h}^\lambda(\boldsymbol{\alpha}, \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, \boldsymbol{\beta}, t), \\ &\boldsymbol{\phi}^\tau = \boldsymbol{\phi}^\tau(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t), \quad \boldsymbol{\phi}^\lambda = \boldsymbol{\phi}^\lambda(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t), \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\alpha} &= -A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZU}\boldsymbol{i}_U^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \\ \boldsymbol{\beta} &= A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{IY}^\top \boldsymbol{u}_I^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau. \end{aligned}$$

The idea of its derivation is to use all constitutive equations so that Kirchhoff's current/voltage laws provide a system that depends only on $\boldsymbol{u}_C^\tau$, $\boldsymbol{u}_I^\tau$, $\boldsymbol{u}_Y^\tau$ and $\boldsymbol{i}_Z^\lambda$, $\boldsymbol{i}_U^\lambda$, $\boldsymbol{i}_L^\lambda$.

## 3. DAEs with properly stated leading term

In this section, we briefly explain DAEs with properly stated leading term and the tractability index. Consider a DAE in the form of

$$A(\boldsymbol{x}(t),t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{d}(\boldsymbol{x}(t),t)+\boldsymbol{b}(\boldsymbol{x}(t),t)=\boldsymbol{0} \tag{3.1}$$

for $\boldsymbol{x}\in\mathcal{D}\subseteq\mathbb{R}^m$ and $t\in\mathcal{I}\subseteq\mathbb{R}$. Let $A(\boldsymbol{x}(t),t)$ be an $m\times n$ matrix. We define

$$D(\boldsymbol{x},t)=\frac{\partial\boldsymbol{d}(\boldsymbol{x},t)}{\partial\boldsymbol{x}},\quad B(\boldsymbol{x},t)=\frac{\partial\boldsymbol{b}(\boldsymbol{x},t)}{\partial\boldsymbol{x}},\quad\text{and}\quad M(\boldsymbol{x},t)=A(\boldsymbol{x},t)D(\boldsymbol{x},t).$$

A matrix $Q(\boldsymbol{x},t)$ satisfying $Q(\boldsymbol{x},t)^2=Q(\boldsymbol{x},t)$ is called a *projector*. Moreover, a projector $Q(\boldsymbol{x},t)$ is called a *projector onto* a subspace $\Pi$ if $\operatorname{Im}Q(\boldsymbol{x},t)=\Pi$.

**Definition 3.1** ([14, Definition 26, Lemma A.1])**.** The equation (3.1) is said to be a DAE with properly stated leading term if the size of $D(\boldsymbol{x},t)$ is $n\times m$, the three conditions

$$\operatorname{Im}M(\boldsymbol{x},t)=\operatorname{Im}A(\boldsymbol{x},t),\ \operatorname{Ker}M(\boldsymbol{x},t)=\operatorname{Ker}D(\boldsymbol{x},t),\ \operatorname{Ker}A(\boldsymbol{x},t)\cap\operatorname{Im}D(\boldsymbol{x},t)=\{\boldsymbol{0}\}$$

hold for all $\boldsymbol{x}\in\mathcal{D}$ and $t\in\mathcal{I}$, and there is an $n\times n$ projector function $P(t)$ continuously differentiable with respect to $t$ such that $\operatorname{Ker}P(t)=\operatorname{Ker}A(\boldsymbol{x},t)$, $\operatorname{Im}P(t)=\operatorname{Im}D(\boldsymbol{x},t)$, and $\boldsymbol{d}(\boldsymbol{x},t)=P(t)\boldsymbol{d}(\boldsymbol{x},t)$ for all $\boldsymbol{x}\in\mathcal{D}$ and $t\in\mathcal{I}$.

A DAE with properly stated leading term (3.1) arises in circuit simulation via circuit analysis methods such as MNA [14]. Since this concept was first introduced in [2], the analysis of such DAEs has been developed in [14, 26, 27, 37].

Obviously, the DAE (3.1) represents a regular ODE if and only if the matrix $M(\boldsymbol{x},t)$ is nonsingular for all $\boldsymbol{x}\in\mathcal{D}$ and $t\in\mathcal{I}$. In this case we say that the DAE (3.1) has index zero. The definition of the index to be one is as follows.

**Definition 3.2** ([25, Definition 3.3])**.** The DAE (3.1) has *tractability index one* if $M(\boldsymbol{x},t)$ is singular and

$$\operatorname{Ker}M(\boldsymbol{x},t)\cap\{\boldsymbol{z}\in\mathbb{R}^m|\ B(\boldsymbol{x},t)\boldsymbol{z}\in\operatorname{Im}M(\boldsymbol{x},t)\}=\{\boldsymbol{0}\}$$

for all $\boldsymbol{x}\in\mathcal{D}$ and $t\in\mathcal{I}$.

*Remark* 3.3 ([11]). The following three conditions (a)–(c) are equivalent.

(a) It holds that $\operatorname{Ker}M(\boldsymbol{x},t)\cap\{\boldsymbol{z}\in\mathbb{R}^m|\ B(\boldsymbol{x},t)\boldsymbol{z}\in\operatorname{Im}M(\boldsymbol{x},t)\}=\{\boldsymbol{0}\}$ for all $\boldsymbol{x}\in\mathcal{D}$ and $t\in\mathcal{I}$.
(b) For some projector $Q(\boldsymbol{x},t)$ onto $\operatorname{Ker}M(\boldsymbol{x},t)$, $M(\boldsymbol{x},t)+B(\boldsymbol{x},t)Q(\boldsymbol{x},t)$ is nonsingular for all $\boldsymbol{x}\in\mathcal{D}$ and $t\in\mathcal{I}$.
(c) For any projector $Q(\boldsymbol{x},t)$ onto $\operatorname{Ker}M(\boldsymbol{x},t)$, $M(\boldsymbol{x},t)+B(\boldsymbol{x},t)Q(\boldsymbol{x},t)$ is nonsingular for all $\boldsymbol{x}\in\mathcal{D}$ and $t\in\mathcal{I}$.

## 4. Hybrid equations with properly stated leading term

In this section, we rewrite the hybrid equation system as a DAE with properly stated leading term. A reflexive generalized inverse [3] of a matrix $A$ is a matrix $A^-$ which satisfies $AA^-A=A$ and $A^-AA^-=A^-$. We now define

$$A = \begin{pmatrix} O & -A_{LL}^\top & I & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & I & A_{CC} & O \end{pmatrix}, \ \boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{i}_L^\lambda \\ \boldsymbol{i}_U^\lambda \\ \boldsymbol{i}_Z^\lambda \\ \boldsymbol{u}_Y^\tau \\ \boldsymbol{u}_I^\tau \\ \boldsymbol{u}_C^\tau \end{pmatrix}, \ \boldsymbol{d}(\boldsymbol{x},t) = A^- A \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\phi}^\tau \\ \boldsymbol{\phi}^\lambda \\ \boldsymbol{q}^\tau \\ \boldsymbol{q}^\lambda \\ \boldsymbol{0} \end{pmatrix},$$

and

$$\boldsymbol{b}(\boldsymbol{x},t) = \begin{pmatrix} -A_{YL}^\top \boldsymbol{v}_s(t) - A_{CL}^\top \boldsymbol{u}_C^\tau - A_{IL}^\top \boldsymbol{u}_I^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau - A_{ZL}^\top \boldsymbol{h}^\tau - A_{UL}^\top \boldsymbol{v}_U^\tau(\cdot) \\ -A_{YU}^\top \boldsymbol{v}_s(t) - A_{CU}^\top \boldsymbol{u}_C^\tau - A_{IU}^\top \boldsymbol{u}_I^\tau - A_{YU}^\top \boldsymbol{u}_Y^\tau - A_{ZU}^\top \boldsymbol{h}^\tau - A_{UU}^\top \boldsymbol{v}_U^\tau(\cdot) + \boldsymbol{v}_U^\lambda(\cdot) \\ -A_{YZ}^\top \boldsymbol{v}_s(t) - A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{IZ}^\top \boldsymbol{u}_I^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau - A_{ZZ}^\top \boldsymbol{h}^\tau + \boldsymbol{h}^\lambda \\ \boldsymbol{g}^\tau + A_{YY}\boldsymbol{g}^\lambda + A_{YZ}\boldsymbol{i}_Z^\lambda + A_{YU}\boldsymbol{i}_U^\lambda + A_{YL}\boldsymbol{i}_L^\lambda + A_{YJ}\boldsymbol{j}_s(t) \\ \boldsymbol{j}_I^\tau(\cdot) + A_{II}\boldsymbol{j}_I^\lambda(\cdot) + A_{IY}\boldsymbol{g}^\lambda + A_{IZ}\boldsymbol{i}_Z^\lambda + A_{IU}\boldsymbol{i}_U^\lambda + A_{IL}\boldsymbol{i}_L^\lambda + A_{IJ}\boldsymbol{j}_s(t) \\ A_{CI}\boldsymbol{j}_I^\lambda(\cdot) + A_{CY}\boldsymbol{g}^\lambda + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CU}\boldsymbol{i}_U^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{j}_s(t) \end{pmatrix}.$$

By $A = AA^-A$, this gives the hybrid equation system in the form of (3.1). Then $\mathcal{D}$ denotes the set of $\boldsymbol{x}(t)$ such that $(\boldsymbol{i}, \boldsymbol{u})$ is an operating point at $t$.

*Remark* 4.1. The matrix $A$ and the vector-valued function $\boldsymbol{d}(\boldsymbol{x},t)$ coincide with the case for the circuits without $S_I$ or $S_U$ discussed in [19], while $\boldsymbol{b}(\boldsymbol{x},t)$ contains new terms.

Let us define

$$\Omega(\boldsymbol{x},t) = \begin{pmatrix} O & O & O & O & O & O \\ O & L_\tau^\tau & L_\lambda^\tau & O & O & O \\ O & L_\tau^\lambda & L_\lambda^\lambda & O & O & O \\ O & O & O & C_\tau^\tau & C_\lambda^\tau & O \\ O & O & O & C_\tau^\lambda & C_\lambda^\lambda & O \\ O & O & O & O & O & O \end{pmatrix}.$$

The matrices $D(\boldsymbol{x},t)$ and $M(\boldsymbol{x},t)$ are given by $D(\boldsymbol{x},t) = A^- A\Omega(\boldsymbol{x},t)A^\top$ and

$$M(\boldsymbol{x},t) = A\Omega(\boldsymbol{x},t)A^\top = \begin{pmatrix} M_L(\boldsymbol{x},t) & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & O \\ O & O & O & O & O & M_C(\boldsymbol{x},t) \end{pmatrix},$$

where

$$M_L(\boldsymbol{x},t) = \begin{pmatrix} -A_{LL}^\top & I \end{pmatrix} \begin{pmatrix} L_\tau^\tau & L_\lambda^\tau \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix} \begin{pmatrix} -A_{LL} \\ I \end{pmatrix} \quad \text{and}$$

$$M_C(\boldsymbol{x},t) = \begin{pmatrix} I & A_{CC} \end{pmatrix} \begin{pmatrix} C_\tau^\tau & C_\lambda^\tau \\ C_\tau^\lambda & C_\lambda^\lambda \end{pmatrix} \begin{pmatrix} I \\ A_{CC}^\top \end{pmatrix}.$$

**Lemma 4.2.** *Under Assumption* 2.3, $M_L(\boldsymbol{x},t)$ *and* $M_C(\boldsymbol{x},t)$ *are positive definite.*

*Proof.* Since $\begin{pmatrix} L_\tau^\tau & L_\lambda^\tau \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix}$ is positive definite and $\begin{pmatrix} -A_{LL} \\ I \end{pmatrix}$ is of full column rank, $M_L(\boldsymbol{x},t)$ is positive definite. Similarly, $M_C(\boldsymbol{x},t)$ is positive definite. □

**Lemma 4.3.** *Under Assumption* 2.3, $\operatorname{Im} M(\boldsymbol{x},t) = \operatorname{Im} A$ *and* $\operatorname{Ker} M(\boldsymbol{x},t) = \operatorname{Ker} D(\boldsymbol{x},t) = \operatorname{Ker} A^\top$ *hold.*

*Proof.* Since Lemma 4.2 ensures that $M_L(\boldsymbol{x},t)$ and $M_C(\boldsymbol{x},t)$ are nonsingular, $\operatorname{Im} M(\boldsymbol{x},t) = \operatorname{Im} A$ holds. We now have

$$\operatorname{Ker} M(\boldsymbol{x},t) = \operatorname{Ker} AD(\boldsymbol{x},t) \supseteq \operatorname{Ker} D(\boldsymbol{x},t) = \operatorname{Ker} A^- A\Omega(\boldsymbol{x},t)A^\top \supseteq \operatorname{Ker} A^\top.$$

Let $\boldsymbol{z}$ be an element in $\operatorname{Ker} M(\boldsymbol{x},t)$. Since $M_L(\boldsymbol{x},t)$ and $M_C(\boldsymbol{x},t)$ are nonsingular by Lemma 4.2, $\boldsymbol{z}$ is in the form of $\boldsymbol{z} = \left(\boldsymbol{0}^\top \ * * * * \ \boldsymbol{0}^\top\right)^\top$. Hence $A^\top \boldsymbol{z} = \boldsymbol{0}$ holds, which implies $\boldsymbol{z} \in \operatorname{Ker} A^\top$. Thus we obtain $\operatorname{Ker} M(\boldsymbol{x},t) = \operatorname{Ker} D(\boldsymbol{x},t) = \operatorname{Ker} A^\top$. □

**Lemma 4.4.** *Under Assumption* 2.3, $\operatorname{Ker} A \cap \operatorname{Im} D(\boldsymbol{x},t) = \{\boldsymbol{0}\}$ *holds.*

*Proof.* Let $\boldsymbol{z}$ be an element in $\operatorname{Ker} A \cap \operatorname{Im} D(\boldsymbol{x},t)$. Then we have $A\boldsymbol{z} = \boldsymbol{0}$ and $\boldsymbol{z} = D(\boldsymbol{x},t)\boldsymbol{y}$ for some $\boldsymbol{y}$. Hence $AD(\boldsymbol{x},t)\boldsymbol{y} = \boldsymbol{0}$ holds, which implies that $\boldsymbol{y} \in \operatorname{Ker} AD(\boldsymbol{x},t) = \operatorname{Ker} D(\boldsymbol{x},t)$ by Lemma 4.3. Thus we obtain $\boldsymbol{z} = D(\boldsymbol{x},t)\boldsymbol{y} = \boldsymbol{0}$. □

With the use of a reflexive generalized inverse, we define a constant projector $P = A^- A$. Then the projector $P$ has the following property.

**Lemma 4.5.** *Under Assumption* 2.3, *we have* $\operatorname{Ker} P = \operatorname{Ker} A$ *and* $\operatorname{Im} P = \operatorname{Im} D(\boldsymbol{x},t)$ *for the projector* $P = A^- A$.

*Proof.* By the definition of the reflexive generalized inverse, $\operatorname{Ker} P = \operatorname{Ker} A$ obviously holds. We prove $\operatorname{Im} P = \operatorname{Im} D(\boldsymbol{x},t)$. It clearly holds that

$$\operatorname{Im} D(\boldsymbol{x},t) = \operatorname{Im} A^- A\Omega(\boldsymbol{x},t)A^\top \subseteq \operatorname{Im} P.$$

By Lemma 4.3, $\operatorname{Ker} D(\boldsymbol{x},t) = \operatorname{Ker} A^\top$ holds. Hence we have

$$\begin{aligned}\dim \operatorname{Im} D(\boldsymbol{x},t) &= m - \dim \operatorname{Ker} D(\boldsymbol{x},t) = m - \dim \operatorname{Ker} A^\top = \dim \operatorname{Im} A^\top \\ &= \dim \operatorname{Im} A = \dim \operatorname{Im} A^- A = \dim \operatorname{Im} P.\end{aligned}$$

Thus we obtain $\operatorname{Im} P = \operatorname{Im} D(\boldsymbol{x},t)$. □

By Lemmas 4.3–4.5, we obtain the following proposition.

**Proposition 4.6.** *Under Assumption* 2.3, *the hybrid equation system in the form of* (3.1) *is a DAE with properly stated leading term.*

## 5. Index of hybrid equations

This section gives two main theorems concerning the index of the hybrid equations. We present a structural characterization for index zero in Section 5.1, and for index at most one in Section 5.2. These characterizations lead to an algorithm for determining the index of the hybrid equations, which is given in Section 5.3.

### 5.1. Necessary and sufficient condition for index zero.

We now introduce the *Resistor-Acyclic condition* for an admissible partition $(E_y, E_z)$, which is proved in Theorem 5.2 to be a necessary and sufficient condition for the hybrid equations to have index zero.

**[Resistor-Acyclic condition]**

- Each resistor in $Y$ and each dependent current source in $S_I$ belong to a cycle consisting of independent voltage sources, capacitors, and itself.

- Each resistor in $Z$ and each dependent voltage source in $S_U$ belong to a cutset consisting of inductors, independent current sources, and itself.

This is an extension of the Resistor-Acyclic condition discussed in [19] for the circuits without $S_I$ or $S_U$. The Resistor-Acyclic condition can be expressed as follows.

**Lemma 5.1.** *An admissible partition $(E_y, E_z)$ satisfies the Resistor-Acyclic condition if and only if there exists a normal reference tree $T$ such that $S_I \cup Y \subseteq \overline{T}$ and $Z \cup S_U \subseteq T$.*

We obtain the necessary and sufficient condition for index zero as follows.

**Theorem 5.2.** *Under Assumption* 2.3*, the index of the hybrid equations is zero if and only if the admissible partition $(E_y, E_z)$ satisfies the Resistor-Acyclic condition.*

*Proof.* The index of the hybrid equations is zero if and only if $M(\boldsymbol{x}, t)$ is nonsingular. Since $M_L(\boldsymbol{x}, t)$ and $M_C(\boldsymbol{x}, t)$ are nonsingular by Lemma 4.2, this is equivalent to the condition that we have no variables $\boldsymbol{i}_Z^\lambda$, $\boldsymbol{i}_U^\lambda$, and $\boldsymbol{u}_I^\tau$, $\boldsymbol{u}_Y^\tau$. In other words, $S_I \cup Y \subseteq \overline{T}$ and $Z \cup S_U \subseteq T$ hold. This is equivalent to the Resistor-Acyclic condition by Lemma 5.1. □

5.2. **Necessary and sufficient condition for index at most one.** For a matrix $A$, we denote the submatrix of $A$ with row set $W_R$ and column set $W_C$ by $A[W_R, W_C]$. For a square matrix $A$, we denote by $A[W]$ the principal submatrix of $A$ with row/column set $W$. A square matrix $A$ is said to be *skew-symmetric* if $A = -A^\top$. In order to derive a necessary and sufficient condition for index at most one, we use the following lemma from linear algebra. The proof is given in Appendix A.

**Lemma 5.3.** *Let $A$ be a skew-symmetric matrix with row/column set $X$, and $D$ be a diagonal matrix with nonnegative entries. Then $A + D$ is nonsingular if and only if $A[S, X]$ is of full row rank, where $S$ is a row/column set of $D$ corresponding to zero diagonals.*

By using Lemma 5.3, we analyze the hybrid equations. Let us define

$$A_Z = \begin{pmatrix} -A_{ZU}^\top & O \\ -A_{ZZ}^\top & I \end{pmatrix}, \quad A_Y = \begin{pmatrix} I & A_{YY} \\ O & A_{IY} \end{pmatrix}, \quad N = \begin{pmatrix} A_{YU} & A_{YZ} \\ A_{IU} & A_{IZ} \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} - \begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ G & Y \end{pmatrix} \begin{pmatrix} -A_Z^\top & O \\ O & -A_Y^\top \end{pmatrix}.$$

Then we have the following lemma.

**Lemma 5.4.** *Under Assumption* 2.3*, the index of the hybrid equations is at most one if and only if $\Lambda$ is nonsingular.*

*Proof.* With a projector

$$Q = \begin{pmatrix} O & O & O & O & O & O \\ O & I & O & O & O & O \\ O & O & I & O & O & O \\ O & O & O & I & O & O \\ O & O & O & O & I & O \\ O & O & O & O & O & O \end{pmatrix}$$

onto $\operatorname{Ker} M(\boldsymbol{x},t)$, we have

$$M(\boldsymbol{x},t)+B(\boldsymbol{x},t)Q=\begin{pmatrix} M_L(\boldsymbol{x},t) & * & * & O \\ O & B_{ZZ}(\boldsymbol{x},t) & -N^\top+B_{ZY}(\boldsymbol{x},t) & O \\ O & N+B_{YZ}(\boldsymbol{x},t) & B_{YY}(\boldsymbol{x},t) & O \\ O & * & * & M_C(\boldsymbol{x},t) \end{pmatrix},$$

where

$$\begin{aligned} B_{ZZ}(\boldsymbol{x},t)&=A_ZZA_Z^\top, \quad B_{ZY}(\boldsymbol{x},t)=A_ZHA_Y^\top,\\ B_{YZ}(\boldsymbol{x},t)&=A_YGA_Z^\top, \quad B_{YY}(\boldsymbol{x},t)=A_YYA_Y^\top. \end{aligned}$$

By Definition 3.2 and Remark 3.3, the index of the hybrid equations is at most one if and only if $M(\boldsymbol{x},t)+B(\boldsymbol{x},t)Q$ is nonsingular. The matrix $M(\boldsymbol{x},t)+B(\boldsymbol{x},t)Q$ is nonsingular if and only if

$$\begin{aligned} &\begin{pmatrix} B_{ZZ}(\boldsymbol{x},t) & -N^\top+B_{ZY}(\boldsymbol{x},t) \\ N+B_{YZ}(\boldsymbol{x},t) & B_{YY}(\boldsymbol{x},t) \end{pmatrix}\\ &\quad=\begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix}+\begin{pmatrix} A_ZZA_Z^\top & A_ZHA_Y^\top \\ A_YGA_Z^\top & A_YYA_Y^\top \end{pmatrix}\\ &\quad=\begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix}-\begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix}\begin{pmatrix} Z & H \\ G & Y \end{pmatrix}\begin{pmatrix} -A_Z^\top & O \\ O & -A_Y^\top \end{pmatrix} \end{aligned}$$

is nonsingular by Lemma 4.2. □

We now obtain the following lemma.

**Lemma 5.5.** *Under Assumptions* 2.3 *and* 2.4*, the index of the hybrid equations is at most one if and only if* $\begin{pmatrix} A_Z & N^\top \end{pmatrix}$ *and* $\begin{pmatrix} N & A_Y \end{pmatrix}$ *are of full row rank.*

*Proof.* By Lemma 5.4, the index is at most one if and only if $\Lambda$ is nonsingular. Under Assumption 2.4, there exists an orthogonal matrix $\Theta$ such that $\Sigma=\Theta^\top\begin{pmatrix} Z & O \\ O & Y \end{pmatrix}\Theta$ is a diagonal matrix. Then all the diagonal entries of $\Sigma$ are positive. By setting

$$\tilde{A}=\begin{pmatrix} A_Z & O \\ O & A_Y \end{pmatrix}\Theta \quad\text{and}\quad \tilde{N}=\begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix}+\tilde{A}\Theta^\top\begin{pmatrix} O & H \\ G & O \end{pmatrix}\Theta\tilde{A}^\top,$$

we can express $\Lambda$ by $\Lambda=\tilde{N}-\tilde{A}\Sigma(-\tilde{A}^\top)$. Then $\Lambda$ is nonsingular if and only if $\begin{pmatrix} \Sigma^{-1} & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix}$ is nonsingular by the property of the Schur complement [15].

In the rest of the proof, we find a necessary and sufficient condition for the nonsingularity of the matrix

$$\begin{pmatrix} \Sigma^{-1} & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix}=\begin{pmatrix} O & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix}+\begin{pmatrix} \Sigma^{-1} & O \\ O & O \end{pmatrix}, \tag{5.1}$$

which is the sum of a skew-symmetric matrix and a diagonal matrix with nonnegative entries. Let $X$ be the row/column set of the matrix in (5.1), and $S\subseteq X$ be the row/column set of $\tilde{N}$. It follows from Lemma 5.3 that the matrix in (5.1) is nonsingular if and only if $\begin{pmatrix} O & -\tilde{A}^\top \\ \tilde{A} & \tilde{N} \end{pmatrix}[S,X]=\begin{pmatrix} \tilde{A} & \tilde{N} \end{pmatrix}$ is of full row rank. Since

we have

$$\begin{aligned}(\,\tilde{A} \mid \tilde{N}\,) &= \left(\tilde{A} \;\middle|\; \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix} + \tilde{A}\Theta^\top \begin{pmatrix} O & H \\ G & O \end{pmatrix} \Theta \tilde{A}^\top \right) \\ &\xrightarrow[\text{operations}]{\text{column}} \left(\tilde{A} \;\middle|\; \begin{pmatrix} O & -N^\top \\ N & O \end{pmatrix}\right) \xrightarrow[\text{operations}]{\text{column}} \left(\begin{array}{cc|cc} A_Z & O & O & -N^\top \\ O & A_Y & N & O \end{array}\right) \\ &\xrightarrow{\text{permutations}} \left(\begin{array}{cc|cc} A_Z & -N^\top & O & O \\ \hline O & O & N & A_Y \end{array}\right),\end{aligned}$$

$\begin{pmatrix} \tilde{A} & \tilde{N} \end{pmatrix}$ is of full row rank if and only if $\begin{pmatrix} A_Z & N^\top \end{pmatrix}$ and $\begin{pmatrix} N & A_Y \end{pmatrix}$ are of full row rank. □

For the network graph $\Gamma = (W, E)$, *contracting* $e \in E$ means deleting $e$ and identifying its end-vertices. Let $\Gamma^\circ$ denote the graph obtained by contracting all edges in $V \cup C$ and deleting all edges in $L \cup J$. The fundamental cutset matrix $F^\circ$ is given by

$$F^\circ = \begin{array}{c} \begin{array}{cccccccc} \boldsymbol{i}_I^\tau & \boldsymbol{i}_Y^\tau & \boldsymbol{i}_Z^\tau & \boldsymbol{i}_U^\tau & \boldsymbol{i}_I^\lambda & \boldsymbol{i}_Y^\lambda & \boldsymbol{i}_Z^\lambda & \boldsymbol{i}_U^\lambda \end{array} \\ \begin{pmatrix} I & O & O & O & A_{II} & A_{IY} & A_{IZ} & A_{IU} \\ O & I & O & O & O & A_{YY} & A_{YZ} & A_{YU} \\ O & O & I & O & O & O & A_{ZZ} & A_{ZU} \\ O & O & O & I & O & O & O & A_{UU} \end{pmatrix} \end{array}.$$

Then Lemma 5.5 leads to the following main theorem.

**Theorem 5.6.** *Under Assumptions* 2.3 *and* 2.4*, the index of the hybrid equations is at most one if and only if* $\Gamma^\circ$ *contains neither a cycle consisting of dependent voltage sources nor a cutset consisting of dependent current sources.*

*Proof.* By Lemma 5.5, the index is at most one if and only if $\begin{pmatrix} A_Z & N^\top \end{pmatrix}$ and $\begin{pmatrix} N & A_Y \end{pmatrix}$ are of full row rank. The condition that

$$\begin{pmatrix} A_Z & N^\top \end{pmatrix} = \begin{pmatrix} -A_{ZU}^\top & O & A_{YU}^\top & A_{IU}^\top \\ -A_{ZZ}^\top & I & A_{YZ}^\top & A_{IZ}^\top \end{pmatrix}$$

is of full row rank is equivalent to the fact that

$$\begin{pmatrix} O & A_{IU} \\ O & A_{YU} \\ O & A_{ZU} \\ I & A_{UU} \end{pmatrix} \tag{5.2}$$

is of full column rank. This is a submatrix of $F^\circ$ with the column set corresponding to $S_U$, and hence it is of full column rank if and only if $\Gamma^\circ$ contains no cycles that consist of dependent voltage sources. The condition that

$$\begin{pmatrix} N & A_Y \end{pmatrix} = \begin{pmatrix} A_{YU} & A_{YZ} & I & A_{YY} \\ A_{IU} & A_{IZ} & O & A_{IY} \end{pmatrix}$$

is of full row rank is equivalent to the fact that

$$\begin{pmatrix} O & O & O & A_{IY} & A_{IZ} & A_{IU} \\ I & O & O & A_{YY} & A_{YZ} & A_{YU} \\ O & I & O & O & A_{ZZ} & A_{ZU} \\ O & O & I & O & O & A_{UU} \end{pmatrix} \tag{5.3}$$

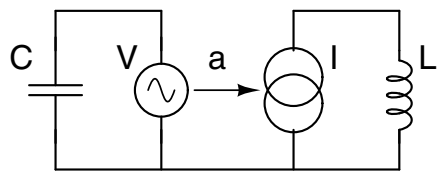


FIGURE 3. A circuit with a dependent current source.

is of full row rank. This is a submatrix of $F^\circ$ with the column set corresponding to $Y \cup Z \cup S_U$, and hence it is of full row rank if and only if $\Gamma^\circ$ has a spanning forest consisting of edges in $Y$, $Z$, and $S_U$, where a spanning forest is a maximal set of edges which contains no cycles. The condition is equivalent to that $\Gamma^\circ$ contains no cutsets consisting of dependent current sources. □

Let us define a *CVU-loop* as a cycle consisting of capacitors, independent voltage sources, and/or dependent voltage sources, and an *LJI-cutset* as a cutset consisting of inductors, independent current sources, and/or dependent current sources. Theorem 5.6 is rewritten as follows.

**Corollary 5.7.** *Under Assumptions* 2.3 *and* 2.4*, the index of the hybrid equations is at most one if and only if the network graph* $\Gamma$ *contains neither CVU-loops with at least one dependent voltage source nor LJI-cutsets with at least one dependent current source.*

Theorem 5.2 implies that an admissible partition that leads to the hybrid equations with index zero is unique if it exists. This is because a resistor which belongs to a cycle consisting of independent voltage sources, capacitors, and itself never belongs to a cutset consisting of inductors, independent current sources, and itself, and vice versa. On the other hand, Theorem 5.6 indicates that it does not depend on the choice of an admissible partition whether the index exceeds one or not. Moreover, Theorem 5.6 leads to the statement that the index of hybrid equations is at most one for the circuits without $S_I$ or $S_U$, which is proved in [19].

**Example 5.8.** Consider the circuit depicted in Figure 3, which contains a dependent current source $I$ controlled by the voltage across $C$. While MNA results in a DAE with index three [12], the hybrid analysis with admissible partition

$$E_g = \{V\}, \quad E_h = \emptyset, \quad E_y = \{C, I\}, \quad E_z = \{L\}$$

results in a DAE with index two. In fact, the network graph $\Gamma$ of this circuit depicted in Figure 4 contains an LJI-cutset with one dependent current source, which implies by Corollary 5.7 that the index must be at least two.

5.3. **Algorithm for index determination.** We now provide an algorithm for determining the index of the hybrid equations. The correctness of this algorithm follows from Theorems 5.2 and 5.6.

A *coloop* is an edge whose deletion increases the number of connected components in the graph. The following algorithm first determines whether the minimum index

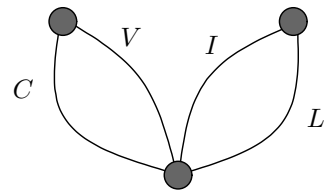


FIGURE 4. The graph $\Gamma$ of Example 5.8.

$\nu$ is at most one or not. If it turns out to be at most one, the algorithm further checks whether it can be zero for some choice of an admissible partition. If it can, the algorithm also finds such an admissible partition $(E_y, E_z)$.

**Algorithm for index determination**

**1.** Set $E_y \leftarrow \{e \mid e \in C \cup S_I\}$ and $E_z \leftarrow \{e \mid e \in S_U \cup L\}$.

**2.** Contract all edges in $V \cup C$ and delete all edges in $L \cup J$ from $\Gamma = (W, E)$. Then we obtain graph $\Gamma^\circ$.

**3.** If $\Gamma^\circ$ contains a cycle consisting of dependent voltage sources $S_U$ or a cutset consisting of dependent current sources $S_I$, then return $\nu \geq 2$ and halt.

**4.** If $\Gamma^\circ$ satisfies at least one of the following three conditions, then return $\nu = 1$ and halt:
- $S_I \neq \emptyset$ and $S_I$ does not consist of selfloops,
- $S_U \neq \emptyset$ and $S_U$ does not consist of coloops,
- resistors form a cycle except selfloops.

**5.** Set $E_y \leftarrow E_y \cup \{e \mid e : \text{selfloop of a resistor}\}$ and $E_z \leftarrow E_* \setminus E_y$. Return $\nu = 0$ and $(E_y, E_z)$, and halt.

This is an extension of Algorithm for index minimization in RLC circuit given in [38]. The algorithm runs in linear time in $|E|$, i.e., the number of elements in the circuit.

## 6. Conclusion

For nonlinear time-varying circuits composed of resistors, inductors, capacitors, independent voltage/current sources, and dependent voltage/current sources, we have given structural characterizations of the tractability index of the hybrid equations. This enables us to determine efficiently whether the hybrid equations have higher index or not, which helps to avoid solving higher index DAEs in circuit simulation. Analysis of other indices is left for future investigation. In particular, we anticipate similar structural characterizations for the differentiation index.

This paper focuses on the modeling step which derives model equations. When we solve DAEs with higher index, index reduction methods [23, 28] are available. Comparing the practical effect of the hybrid analysis with the combination of MNA and index reduction methods will also be interesting.

## Appendix A. Proof of Lemma 5.3

In order to prove Lemma 5.3, we give the following two lemmas.

**Lemma A.1.** *Let $A$ be a skew-symmetric matrix, and $D$ be a diagonal matrix with nonnegative entries. Then $A + D$ is nonsingular if and only if there exists a*

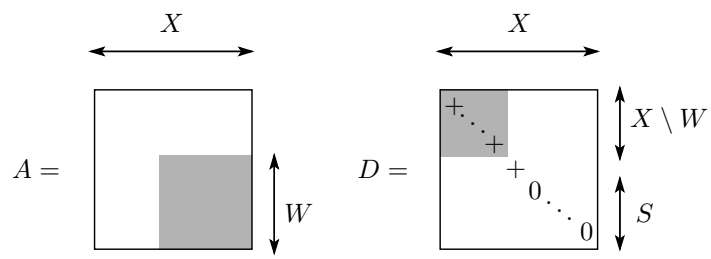


FIGURE 5. A skew-symmetric matrix $A$ and a diagonal matrix $D$.

*nonsingular principal submatrix of $A$ containing $A[S]$, where $S$ is a row/column set of $D$ corresponding to zero diagonals.*

*Proof.* Let $X$ be the row/column set of $A$ and $D$. Since $D$ is a diagonal matrix, we have

$$\begin{aligned} \det(A+D) &= \sum_{W \subseteq X} \det A[W] \cdot \det D[X \setminus W] \\ &= \sum_{W \supseteq S} \det A[W] \cdot \det D[X \setminus W] \end{aligned} \tag{A.1}$$

by the definition of $S$. Figure 5 shows submatrices $A[W]$ and $D[X \setminus W]$. Since $A[W]$ is skew-symmetric, $\det A[W] \geq 0$ holds. Moreover, we have $\det D[X \setminus W] > 0$ for $W \supseteq S$. Hence each term of (A.1) is nonnegative. Thus, $A + D$ is nonsingular if and only if there exists $W \supseteq S$ such that $A[W]$ is nonsingular. □

**Lemma A.2.** *Let $A$ be a skew-symmetric matrix with row/column set $X$. For a subset $S$ of $X$, there exists a nonsingular principal submatrix containing $A[S]$ if and only if $A[S, X]$ is of full row rank.*

*Proof.* If there exists a nonsingular principal matrix containing $A[S]$, it is obvious that $A[S, X]$ is of full row rank. To show the converse, we now assume that $A[S, X]$ is of full row rank. Consider a subset $S^\circ \subseteq S$ such that $B = A[S^\circ]$ is a maximum size nonsingular principal submatrix of $A[S]$. By the property of a skew-symmetric matrix, there exists a nonsingular matrix $D$ such that $D^\top A[S]D = \begin{pmatrix} B & O \\ O & O \end{pmatrix}$. Using this $D$ with row set indexed by $S$, we transform $A$ into $\tilde{A} = \begin{pmatrix} I & O \\ O & D^\top \end{pmatrix} A \begin{pmatrix} I & O \\ O & D \end{pmatrix}$. Since $A[S, X]$ is of full row rank, so is $\tilde{A}[S, X]$. Then there exists a subset $X^\circ \subseteq X \setminus S$ such that $C = \tilde{A}[S \setminus S^\circ, X^\circ]$ is nonsingular. Thus $\tilde{A}$ is in the form of

$$\tilde{A} = \left(\begin{array}{cc|cc} * & * & * & -C^\top \\ * & * & * & * \\ \hline * & * & B & O \\ C & * & O & O \end{array}\right).$$

Therefore, $\tilde{A}[X^\circ \cup S]$ is a nonsingular principal submatrix, and so is $A[X^\circ \cup S]$. □

By combining Lemmas A.1 and A.2, we obtain Lemma 5.3.

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## References

1. S. Amari, *Topological foundations of Kron's tearing of electric networks*, RAAG Memoirs **3** (1962), 322–350.
2. K. Balla and R. März, *A unified approach to linear differential algebraic equations and their adjoint equations*, Zeitschrift für Analysis und ihre Anwendungen **21** (2002), 783–802. MR1929432 (2003g:34002)
3. A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Springer-Verlag, New York, 2003, 2nd edition. MR1987382 (2004b:15008)
4. F. H. Branin, *The relation between Kron's method and the classical methods of network analysis*, The Matrix and Tensor Quarterly **12** (1962), 69–115.
5. K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, SIAM, Philadelphia, 1996, 2nd edition. MR1363258 (96h:65083)
6. P. R. Bryant, *The order of complexity of electrical networks*, Proceedings of the Institution of Electrical Engineers, Part C **106** (1959), 174–188.
7. S. L. Campbell and C. W. Gear, *The index of general nonlinear DAEs*, Numerische Mathematik **72** (1995), 173–196. MR1362259 (96i:34008)
8. L. O. Chua, *Dynamic nonlinear networks: state-of-the-art*, IEEE Transactions on Circuits and Systems **27** (1980), 1059–1087. MR594151 (82j:94024)
9. D. Estévez Schwarz and C. Tischendorf, *Structural analysis of electric circuits and consequences for MNA*, International Journal of Circuit Theory and Applications **28** (2000), 131–162.
10. W. Fischer, *Equivalent circuit and gain of MOS field effect transistors*, Solid-State Electronics **9** (1966), 71–81.
11. E. Griepentrog and R. März, *Differential-Algebraic Equations and Their Numerical Treatment*, Teubner, Leipzig, 1986. MR881052 (88e:65105)
12. M. Günther and P. Rentrop, *The differential-algebraic index concept in electric circuit simulation*, Zeitschrift für angewandte Mathematik und Mechanik **76, supplement 1** (1996), 91–94.
13. E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II*, Springer-Verlag, Berlin, 1996, 2nd edition. MR1439506 (97m:65007)
14. I. Higueras, R. März, and C. Tischendorf, *Stability preserving integration of index-1 DAEs*, Applied Numerical Mathematics **45** (2003), 175–200. MR1967573 (2004b:34008)
15. R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985. MR832183 (87e:15001)
16. M. Iri, *A min-max theorem for the ranks and term-ranks of a class of matrices: An algebraic approach to the problem of the topological degrees of freedom of a network* (in Japanese), Transactions of the Institute of Electronics and Communication Engineers of Japan **51A** (1968), 180–187. MR0258849 (41:3495a)
17. ———, *Applications of matroid theory*, Mathematical Programming — The State of the Art, Springer-Verlag, Berlin, 1983, pp. 158–201. MR717401 (85a:05022)
18. S. Iwata and M. Takamatsu, *Index minimization of differential-algebraic equations in hybrid analysis for circuit simulation*, Mathematical Programming **121** (2010), 105–121. MR2520408 (2010j:65107)
19. S. Iwata, M. Takamatsu, and C. Tischendorf, *Hybrid analysis of nonlinear time-varying circuits providing DAEs with index at most one*, Scientific Computing in Electrical Engineering SCEE 2008 (J. Roos and L. R. J. Costa, eds.), Mathematics in Industry, vol. 14, Springer, 2010, pp. 151–158.
20. G. Kishi and Y. Kajitani, *Maximally distinct trees in a linear graph* (in Japanese), Transactions of the Institute of Electronics and Communication Engineers of Japan **51A** (1968), 196–203. MR0252270 (40:5491)

21. G. Kron, *Tensor Analysis of Networks*, John Wiley and Sons, New York, 1939.
22. P. Kunkel and V. Mehrmann, *Canonical forms for linear differential-algebraic equations with variable coefficients*, Journal of Computational and Applied Mathematics **56** (1994), 225–251. MR1335565 (96m:34005)
23. ———, *Index reduction for differential-algebraic equations by minimal extension*, Zeitschrift für angewandte Mathematik und Mechanik **84** (2004), 579–597. MR2083283 (2005d:34008)
24. R. März, *Numerical methods for differential-algebraic equations*, Acta Numerica **1** (1992), 141–198. MR1165725 (93e:65096)
25. ———, *Nonlinear differential-algebraic equations with properly formulated leading term*, Tech. Report 01-3, Department of Mathematics, Humboldt-Universität zu Berlin, 2001, http://www.mathematik.hu-berlin.de/publ/pre/2001/P-01-3.ps.
26. ———, *The index of linear differential algebraic equations with properly stated leading term*, Results in Mathematics **42** (2002), 308–338. MR1946748 (2003i:34004)
27. R. März and R. Riaza, *Linear differential-algebraic equations with properly stated leading term: Regular points*, Journal of Mathematical Analysis and Applications **323** (2006), 1279–1299. MR2260180 (2007f:34009)
28. S. E. Mattsson and G. Söderlind, *Index reduction in differential-algebraic equations using dummy derivatives*, SIAM Journal on Scientific Computing **14** (1993), 677–692. MR1214776 (93m:65094)
29. H. Narayanan, *Submodular Functions and Electrical Networks*, North-Holland Publ., Amsterdam, 1997. MR1453578 (98g:94053)
30. T. Ohtsuki, Y. Ishizaki, and H. Watanabe, *Network analysis and topological degrees of freedom* (in Japanese), Transactions of the Institute of Electronics and Communication Engineers of Japan **51A** (1968), 238–245. MR0265068 (41:9655)
31. J. M. Rabaey, *The spice page*, `http://bwrc.eecs.berkeley.edu/Classes/icbook/SPICE/`.
32. A. Recski, *Matroid Theory and Its Applications in Electric Network Theory and in Statics*, Springer-Verlag, Berlin, 1989.
33. G. Reißig, *The index of the standard circuit equations of passive RLCTG-networks does not exceed 2*, Proceedings of the 1998 IEEE International Symposium on Circuits and Systems (ISCAS '98) **3** (1998), 419–422.
34. ———, *Extension of the normal tree method*, International Journal of Circuit Theory and Applications **27** (1999), 241–265.
35. W. C. Rheinboldt, *Differential-algebraic systems as differential equations on manifolds*, Mathematics of Computation **43** (1984), 473–482. MR758195 (86c:58131)
36. R. Riaza, *Differential-Algebraic Systems: Analytical Aspects and Circuit Applications*, World Scientific Publishing Company, Singapore, 2008. MR2426820 (2009e:34003)
37. R. Riaza and R. März, *Linear index-1 DAEs: Regular and singular problems*, Acta Applicandae Mathematicae **84** (2004), 29–53. MR2104183 (2005k:34010)
38. M. Takamatsu and S. Iwata, *Index characterization of differential-algebraic equations in hybrid analysis for circuit simulation*, International Journal of Circuit Theory and Applications **38** (2010), 419–440.
39. C. Tischendorf, *Topological index calculation of differential-algebraic equations in circuit simulation*, Surveys on Mathematics for Industry **8** (1999), 187–199. MR1737412 (2000i:94093)

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