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# NUMERICAL IDENTIFICATION OF A ROBIN COEFFICIENT IN PARABOLIC PROBLEMS

BANGTI JIN AND XILIANG LU

Abstract. This paper studies a regularization approach for an inverse problem of estimating a spatially-and-temporally dependent Robin coefficient arising in the analysis of convective heat transfer. The parameter-to-state map is analyzed, especially a differentiability result is established. A regularization approach is proposed, and the properties, e.g., existence and optimality system, of the functional are investigated. A finite element method is adopted for discretizing the continuous optimization problem, and the convergence of the finite element approximations as the mesh size and temporal step size tend to zero is established. Numerical results by the conjugate gradient method for one- and two-dimensional problems are presented.

## 1. Introduction

In this paper, we are interested in an inverse problem arising in transient convective heat transfer. It consists of estimating a spatially-and-temporally dependent heat transfer coefficient, also known as a Robin coefficient, from noisy measurements of the temperature and flux on a part of the boundary.

The Robin coefficient characterizes thermal properties of the conductive material on the interface and also profiles certain physical processes near the boundary, e.g., corrosion [JZ09]. Therefore, the value of the coefficient is of significant practical interest in thermal engineering, e.g., quenching processes, safety analysis of nuclear reactors and thermal protection of space shuttles [BBSC85], as well as in nondestructive evaluation, e.g., corrosion detection and imaging of contact between metal and silicon in transistors. However, the accurate values of the coefficient are experimentally difficult to acquire because they depend strongly on at least twelve variables or eight dimensionless groups [Whi88]. The conventional approach in thermal engineering employs the empirical correlations which only represent curve fitting through experimental data in a limited range of flow-field parameters [Whi88]. This greatly constrains the range of practical applications of convective heat transfer since it only allows modeling with one single value of the Robin coefficient on the whole surface exposed to a moving fluid. Recently, engineers seek to estimate a distributed Robin coefficient from internal/boundary measurements as an inverse problem [OB88, KJ01, SH04]. This is practically advantageous in that only temperature data measured at some interior/boundary locations of the test material are required, and that information of the exterior conditions for the test material, such as flow field, physical properties and temperature distribution of the

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fluid, is not required [CW08]. Hence this technique is believed to greatly broaden the scope of convective heat conduction provided that the inverse problem can be solved reliably.

As is typical for many distributed parameter identification problems, the Robin inverse problem suffers from ill-posedness, in particular, small errors in the data can lead to large deviations in the solutions. Therefore, specialized techniques are needed for its efficient, accurate and stable solution. Numerically, one widely used engineering approach is the sequential function specification method [BBSC85], which has been applied to estimate the temporally-dependent Robin coefficient from the measured temperature data for quenching process [OB88, Cha99]. However, the approach is generally sensitive to the noise in the data, and cannot yield accurate solutions in the presence of data noises. Theoretically, its rigorous justification seems difficult, and remains completely open. Another popular engineering approach is of variational type [Ali94, KJ01, SH04, Jin07, JZ08, OIL08, YYW09]. Typically, an objective functional involving $L^2$ data fitting, i.e., the squared $L^2$-norm of the difference between the measured data and that due to an assumed Robin coefficient (model output), possibly with an appropriate regularization term incorporated, is first defined, then discretized by the finite difference method or the boundary element method, and finally iteratively minimized using standard optimization algorithms. However, these works make no attempt to analyze the well-posedness of the formulation and the convergence of its discrete approximations. To the best of our knowledge, the only existing works partially considering these important theoretical issues are [LW93, SV02]. In [LW93], Lenhart and Wilson presented an optimal control formulation of the inverse problem, established the well-posedness of the formulation, and derived the optimal system. They also proposed an algorithm based on maximum principle. However, no numerical results were presented, and the algorithm is not applicable to general parabolic problems due to the lack of maximum principle necessary for constructing the solution. Slodička and Van Keer [SV02] studied the recovery of a temporally-dependent Robin coefficient in a semilinear parabolic equation from an over-specified nonlocal boundary condition, and proposed a time discretization by Rothe's method with some convergence analysis. However, spatial discretization, which is necessary for practical computations, was not considered. Recently, Slodička et al. [SLO10] extended the analysis to estimating a temporally-dependent Robin coefficient in a nonlinear boundary condition for one-dimensional heat equation, and showed the existence and uniqueness of the solution.

This paper attempts to provide a complete analysis of the above-mentioned numerical issues, i.e., to analyze a regularization approach to the Robin inverse problem as well as its finite-element discretization. A complete numerical analysis for transient inverse heat transfer problems is often challenging, and thus there are only a very limited number of relevant studies available in the literature. Amongst existing works, [XZ05] is of particular relevance, where the authors analyzed the finite element method for a linear inverse problem of reconstructing heat flux in an annular domain, and established the convergence of finite element approximations. In contrast to this work [XZ05], the Robin inverse problem under consideration is highly nonlinear, and thus the analysis from there is not directly applicable. In this paper, we shall present such an analysis for a variational approach to this inverse problem. In particular, we shall investigate the parameter-to-state map, especially

$L^q$ differentiability properties, well-posedness of the variational formulation, the finite element discretization as well as its convergence. The present work can be regarded as a sequel to our earlier works for the stationary counterpart [JZ10, JZ09].

The remaining part of the paper is structured as follows. In Section 2, we investigate analytical properties, e.g., continuity and differentiability, of the parameter-to-state map, and in Section 3 we propose a regularization formulation of Tikhonov type, establish its well-posedness, and derive the optimality system. The finite element discretization is described in Section 4, and the convergence of finite element approximations is shown. Finally, numerical results for several one- and two-dimensional problems are presented to illustrate the accuracy and efficiency of the proposed approach.

We end this section with some useful notation. Let $\Omega$ be an open bounded polyhedral domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with a boundary $\Gamma = \partial\Omega$. The Sobolev spaces $L^p(\Omega)$ and $H^1(\Omega)$ are endowed with the norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$, respectively [AF03]. Similarly, we define the norm $\|\cdot\|_{L^2(\Gamma)}$ for the Sobolev space $L^2(\Gamma)$. We shall denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$, and similarly $(H^{\frac{1}{2}}(\Gamma))'$. For a Banach space $B$, we define

$$L^2(0,T;B) = \{u(t) \in B \text{ for a.e. } t \in (0,T) \text{ and } \|u\|_{L^2(0,T;B)} < \infty\},$$

and the norm is given by

$$\|u\|^2_{L^2(0,T;B)} = \int_0^T \|u(t)\|^2_B.$$

We shall frequently use the space $\mathcal{W}$ defined below

$$\mathcal{W} = \left\{u : u \in L^2(0,T;H^1(\Omega)), u_t \in L^2(0,T;(H^1(\Omega))')\right\},$$

where $\mathcal{W}$ is equipped with the norm

$$\|u\|^2_{\mathcal{W}} = \|u\|^2_{L^2(0,T;H^1(\Omega))} + \|u_t\|^2_{L^2(0,T;(H^1(\Omega))')}.$$

In what follows, we shall frequently use the notation $C$ to denote a generic constant, which may differ at different occurrences but does not depend on the finite element mesh size $h$, temporal step size $\tau$ and any functions involved in the analysis. Whenever no confusion can arise, we will suppress the symbols $dx$, $ds$ and $dt$ in the integrals for notational simplicity.

## 2. Properties of parameter-to-state map

Let the boundary $\Gamma$ consist of two disjoint parts $\Gamma = \Gamma_i \cup \Gamma_c$, with $\Gamma_i$ and $\Gamma_c$ being the union of some $(d-1)$-dimensional polyhedral domains. The boundaries $\Gamma_i$ and $\Gamma_c$ refer to the experimentally inaccessible and accessible parts, respectively. To consider a spatially-and-temporally dependent problem, we define the spatial-temporal domains $Q$ and $\Sigma_i$, respectively, by

$$Q = (0,T) \times \Omega \quad \text{and} \quad \Sigma_i = (0,T) \times \Gamma_i.$$

The domain $\Sigma_c$ is defined similarly. Then transient heat transfer can often be described by the following parabolic equation

$$u_t - \nabla \cdot (\alpha \nabla u) = f \quad \text{in } Q,$$

where $f \in L^2(0,T;L^2(\Omega))$ is the source term, and the thermal conductivity $\alpha(x) \in C(\bar{\Omega})$ satisfies $0 < \alpha_0 \le \alpha \le \alpha_1 < \infty$ for some positive constants $\alpha_0$ and $\alpha_1$. The equation is equipped with the initial condition

$$u(\cdot,0) = u_0 \quad \text{in } \Omega,$$

and the following boundary conditions on the boundary $\Gamma_c$:

$$u = g \quad \text{and} \quad \alpha\frac{\partial u}{\partial n} = q.$$

Throughout the paper, we assume the given functions (heat flux) $q$ and $u_a$ have following regularity:

$$q \in L^2(0,T;(H^{\frac{1}{2}}(\Gamma_c))') \quad \text{and} \quad u_a \in L^2(0,T;(H^{\frac{1}{2}}(\Gamma_i))'). \tag{2.1}$$

Through the remaining part of the boundary, i.e., $\Gamma_i$, occurs the convective heat transfer, which according to Newton's law [Whi88] can be described by

$$\alpha\frac{\partial u}{\partial n} + \gamma u = u_a,$$

where $u_a$ is a known function. The inverse problem is to estimate the spatially-and-temporally dependent Robin coefficient $\gamma(t,x)$ on the boundary $\Gamma_i$. The admissible set $\mathcal{A}$ for the Robin coefficient $\gamma$ is taken to be

$$\mathcal{A} = \{\gamma : 0 < c_0 \le \gamma(t,x) \le c_1 < \infty,\ a.e.\ (t,x) \in \Sigma_i\}, \tag{2.2}$$

where $c_0$ and $c_1$ are two known positive constants. The parameters $c_0$ and $c_1$ may be specified based on the concrete application at hand; see, e.g., Table 4.3.10 in the handbook [Kre00] for typical values of the film heat transfer coefficients for shell-and-tube heat exchangers.

Observe that both Dirichlet and Neumann boundary conditions are specified on the boundary $\Gamma_c$, whereas no boundary condition is specified on the remaining part $\Gamma_i$. It is often known as the lateral Cauchy problem for parabolic equations, and it usually suffers severely from ill-posedness. Consequently, the Robin inverse problem is also ill-posed. Despite the extensive literature on the lateral Cauchy problem [NH94, NHR97] or stationary Robin inverse problems (see [CJ99, CFJL04, JZ10] and references therein), there exists only a very limited number of theoretical studies [Cho99, BCC08] on the transient Robin inverse problem under consideration. In [Cho99] the unique identification of a nonlinear law on the boundary was discussed. Recently, Bellassoued et al. [BCC08] established some conditional stability estimates.

We shall consider a variational approach to the inverse problem in Section 3, which reformulates the problem as an optimization problem of minimizing a certain functional over the admissible set $\mathcal{A}$. Therefore, we need to define an appropriate forward operator and to study its properties. The forward operator $\gamma \mapsto u(\gamma)$ is defined by

$$\begin{cases} u_t - \nabla\cdot(\alpha\nabla u) = f & \text{in } Q, \\ \alpha\frac{\partial u}{\partial n} = q & \text{on } \Sigma_c, \\ \alpha\frac{\partial u}{\partial n} + \gamma u = u_a & \text{on } \Sigma_i, \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

The weak formulation of system (2.3) is given by: find $u \in \mathcal{W}$ such that

$$\int_Q u_t v + \int_Q \alpha \nabla u \cdot \nabla v + \int_{\Sigma_i} \gamma u v = \int_{\Sigma_c} q v + \int_{\Sigma_i} u_a v + \int_Q f v \quad \forall v \in L^2(0,T;H^1(\Omega))$$

and $u(0) = u_0$. Here the integral $\int_Q u_t v$ should be understood in the sense of duality pairing between the spaces $L^2(0,T;(H^1(\Omega))')$ and $L^2(0,T;H^1(\Omega))$.

We recall the following important embedding results for the spaces $\mathcal{W}$ and $H^1(\Omega)$.

**Lemma 2.1.** *The following embedding results hold:*

(a) $\mathcal{W} \hookrightarrow C(0,T;L^2(\Omega))$, $\mathcal{W} \hookrightarrow L^2(Q)$ *compactly,* $\mathcal{W} \hookrightarrow C(0,T;(H^1(\Omega))')$ *compactly;*

(b) $H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\Gamma) \hookrightarrow \begin{cases} L^p(\Gamma), p<\infty & d=2, \\ L^4(\Gamma) & d=3, \end{cases}$ $\quad L^2(0,T;H^1(\Omega)) \hookrightarrow L^2(\Sigma_i)$*;*

(c) $\mathcal{W} \hookrightarrow H^\sigma(0,T;H^{\frac{1}{2}+\epsilon}(\Omega))$ *for every* $\sigma \in (0,\frac{1}{4})$ *and* $\epsilon \in (0, \frac{1-4\sigma}{2})$*;*

(d) $\mathcal{W} \hookrightarrow L^{2+\theta}(\Sigma_i)$ *compactly for some positive* $\theta$*;*

(e) $\|u\|^2_{L^2(\Gamma)} \le C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}$.

*Proof.* The embedding results in (a) and (b) are standard; cf. [AF03, Eva98]. The third embedding can be found in [CGH06, Lem. 3.6, Thm. 3.7]. Assertion (d) is an immediate corollary of (b) and (c), and (e) can be found at [LU68, pp. 50, eq. 2.27]. □

Let $q$ be the real number which satisfies $\frac{1}{2+\theta} + \frac{1}{q} = \frac{1}{2}$. Throughout, we reserve the notation $q$ for the above-defined value, and always endow the set $\mathcal{A}$ with the $L^q(\Sigma_i)$-norm. Note that the set $\mathcal{A}$ contains no interior point with respect to $L^q(\Sigma_i)$-topology, and hence all the results presented below should be understood with respect to the relative topology.

The following norm equivalence lemma will be needed; cf. [JZ10].

**Lemma 2.2.** *The norm* $\|\cdot\|_{1,\Gamma_i}$ *defined by*

$$\|u\|^2_{1,\Gamma_i} = \int_\Omega |\nabla u|^2 + \int_{\Gamma_i} |u|^2$$

*is equivalent to the standard norm* $\|\cdot\|_{H^1(\Omega)}$.

We will not distinguish the norm $\|\cdot\|_{1,\Gamma_i}$ with $\|\cdot\|_{H^1(\Omega)}$ in later discussions. A standard argument (see, e.g., [Eva98, Sect. 7.1] and [CGH06, Thm. 2.4]) shows the following a priori estimate of the solution $u(\gamma)$ to system (2.3).

**Lemma 2.3.** *For any* $\gamma \in \mathcal{A}$*, there exists a unique solution* $u(\gamma) \in \mathcal{W}$ *to system* (2.3)*, and it satisfies the following a priori estimate:*

(2.4)
$$\|u\|_{\mathcal{W}} \le C\left(\|f\|_{L^2(Q)} + \|q\|_{L^2(0,T;(H^{\frac{1}{2}}(\Gamma_c))')} + \|u_a\|_{L^2(0,T;(H^{\frac{1}{2}}(\Gamma_i))')} + \|u_0\|_{L^2(\Omega)}\right),$$

*where the constant* $C = C(c_0, c_1, \alpha, \Omega)$.

We deduce from Lemma 2.3 that the forward map $\gamma \mapsto u(\gamma)$ is well-defined as a mapping from $\mathcal{A}$ to $\mathcal{W}$. Next we prove its continuity and differentiability.

**Lemma 2.4.** *The mapping $\gamma \mapsto u(\gamma)$ is Lipschitz continuous from $\mathcal{A}$ to $\mathcal{W}$, i.e., for any $\gamma, \gamma+\zeta \in \mathcal{A}$, there holds*

$$\|u(\gamma+\zeta) - u(\gamma)\|_{\mathcal{W}} \leq C\|\zeta\|_{L^q(\Sigma_i)}.$$

*Proof.* Denote by $\delta u = u(\gamma+\zeta) - u(\gamma)$. Then $\delta u$ satisfies

$$\begin{cases} \delta u_t - \nabla\cdot(\alpha\nabla\delta u) = 0 & \text{in } Q, \\ \alpha\frac{\partial \delta u}{\partial n} = 0 & \text{on } \Sigma_c, \\ \alpha\frac{\partial \delta u}{\partial n} + (\gamma+\zeta)\delta u = -\zeta u(\gamma) & \text{on } \Sigma_i, \\ \delta u(\cdot, 0) = 0 & \text{in } \Omega. \end{cases}$$

Multiplying the first equation with $\delta u$ and integrating on the domain $\Omega$ gives

$$\frac{1}{2}\frac{d}{dt}\|\delta u\|^2_{L^2(\Omega)} + \int_\Omega \alpha|\nabla\delta u|^2 + \int_{\Gamma_i}(\gamma+\zeta)|\delta u|^2 = -\int_{\Gamma_i}\zeta u(\gamma)\delta u.$$

Upon integrating the above equation from 0 to $T$, and noting the fact $\gamma+\zeta \in \mathcal{A}$, the initial condition $\delta u(\cdot, 0) = 0$, the generalized Hölder's inequality and Lemma 2.1, we deduce

$$\begin{aligned} \tfrac{1}{2}\|\delta u(T)\|^2 + \alpha_0\|\nabla\delta u\|^2_{L^2(Q)} + c_0\|\delta u\|^2_{L^2(\Sigma_i)} &\leq C\|\zeta\|_{L^q(\Sigma_i)}\|u\|_{L^{2+\theta}(\Sigma_i)}\|\delta u\|_{L^2(\Sigma_i)} \\ &\leq C\|\zeta\|_{L^q(\Sigma_i)}\|\delta u\|_{L^2(0,T;H^1(\Omega))}\|u\|_{\mathcal{W}}, \end{aligned}$$

from which along with Lemma 2.2 it follows directly that

$$\|\delta u\|_{L^2(0,T;H^1(\Omega))} \leq C\|\zeta\|_{L^q(\Sigma_i)}\|u\|_{\mathcal{W}}. \tag{2.5}$$

Next we estimate the time derivative $(\delta u)_t$ by the generalized Hölder inequality and Lemma 2.1 as follows:

$$\begin{aligned} &\|(\delta u)_t\|_{L^2(0,T;(H^1(\Omega))')} \\ &\quad= \sup_{\|v\|_{L^2(0,T;H^1(\Omega))}=1} \langle(\delta u)_t, v\rangle_{L^2(0,T;(H^1(\Omega))'),L^2(0,T;H^1(\Omega))} \\ &\quad= \sup_{\|v\|_{L^2(0,T;H^1(\Omega))}=1} \left\{-\int_{\Sigma_i}\zeta uv - \int_Q \alpha\nabla\delta u\cdot\nabla v - \int_{\Sigma_i}(\zeta+\gamma)\delta uv\right\} \\ &\quad\leq \sup_{\|v\|_{L^2(0,T;H^1(\Omega))}=1} \big\{\|\zeta\|_{L^q(\Sigma_i)}\|u\|_{L^{2+\theta}(\Sigma_i)}\|v\|_{L^2(\Sigma_i)} \\ &\qquad\qquad + \alpha_1\|\nabla\delta u\|_{L^2(Q)}\|\nabla v\|_{L^2(Q)} + c_1\|\delta u\|_{L^2(\Sigma_i)}\|v\|_{L^2(\Sigma_i)}\big\} \\ &\quad\leq C(\|\zeta\|_{L^q(\Sigma_i)}\|u\|_{L^{2+\theta}(\Sigma_i)} + \|\delta u\|_{L^2(0,T;H^1(\Omega))}) \\ &\quad\leq C\|\zeta\|_{L^q(\Sigma_i)}\|u\|_{\mathcal{W}}, \end{aligned}$$

where the last inequality follows from inequality (2.5). Combining the above two estimates gives $\|\delta u\|_{\mathcal{W}} \leq C\|\zeta\|_{L^q(\Sigma_i)}\|u\|_{\mathcal{W}}$. Since $\|u\|_{\mathcal{W}}$ can be bounded independent of the element $\gamma \in \mathcal{A}$ (see Lemma 2.3); this concludes the proof. □

Next we show the Fréchet differentiability of the forward map $u(\gamma)$. To this end, we denote the sensitivity problem for $u^1 \equiv u'(\gamma)\zeta \in \mathcal{W}$, which satisfies

$$\begin{cases} u_t^1 - \nabla \cdot (\alpha \nabla u^1) = 0 & \text{in } Q, \\ \alpha \frac{\partial u^1}{\partial n} = 0 & \text{on } \Sigma_c, \\ \alpha \frac{\partial u^1}{\partial n} + \gamma u^1 = -\zeta u(\gamma) & \text{on } \Sigma_i, \\ u^1(\cdot, 0) = 0 & \text{in } \Omega. \end{cases}$$

Inspecting the arguments in Lemma 2.4 shows that the solution $u^1$ satisfies an identical estimate, and thus it defines a bounded linear operator from $L^q(\Sigma_i)$ to $\mathcal{W}$. The sensitivity problem will play a role in deriving the gradient formula for the regularization functional in Section 3 and numerically calculating the step size in gradient-descent type methods; see Appendix A. With the sensitivity solution $u'(\gamma)\zeta$ at hand, we can now show:

**Lemma 2.5.** *The mapping $\gamma \mapsto u(\gamma)$ is Fréchet differentiable in the sense that for any $\gamma, \gamma + \zeta \in \mathcal{A}$ there holds*

$$\lim_{\|\zeta\|_{L^q(\Sigma_i)} \to 0} \frac{\|u(\gamma + \zeta) - u(\gamma) - u'(\gamma)\zeta\|_{\mathcal{W}}}{\|\zeta\|_{L^q(\Sigma_i)}} = 0.$$

*Proof.* By letting $\delta u = u(\gamma + \zeta) - u(\gamma)$ and $w = \delta u - u'(\gamma)\zeta$, we have

$$\begin{cases} w_t - \nabla \cdot (\alpha \nabla w) = 0 & \text{in } Q, \\ \alpha \frac{\partial w}{\partial n} = 0 & \text{on } \Sigma_c, \\ \alpha \frac{\partial w}{\partial n} + \gamma w = -\zeta \delta u & \text{on } \Sigma_i, \\ w(\cdot, 0) = 0 & \text{in } \Omega. \end{cases}$$

Repeating the arguments in Lemma 2.4 yields

$$\|w\|_{\mathcal{W}} \leq C \|\zeta\|_{L^q(\Sigma_i)} \|\delta u\|_{\mathcal{W}}.$$

The differentiability follows from the Lipschitz continuity of the map $\gamma \mapsto u(\gamma)$; see Lemma 2.4. □

The preceding two results are concerned with the strong topology $\|\cdot\|_{L^q(\Sigma_i)}$. We also have the following weak continuity result, which will be useful for establishing the existence of a minimizer to the proposed regularization formulation.

**Lemma 2.6.** *Let $\{\gamma^n\} \subset \mathcal{A}$ be a sequence converging to $\gamma^*$ weakly in $L^q(\Sigma_i)$. Then the sequence $\{u(\gamma^n)\}$ converges to $u(\gamma^*)$ weakly in $\mathcal{W}$.*

*Proof.* The a priori estimate in Lemma 2.3 implies that the sequence $\{u(\gamma^n)\}$ is uniformly bounded in $\|\cdot\|_{\mathcal{W}}$. Therefore, after possibly passing to a subsequence, which is again denoted by $\{u(\gamma^n)\}$, we deduce that there exists some $u^* \in \mathcal{W}$ such that

$$u(\gamma^n) \to u^* \text{ weakly in } \mathcal{W}.$$

By Lemma 2.1, the embedding from $\mathcal{W}$ into $L^{2+\theta}(\Sigma_i)$ is compact and thus we have

$$u(\gamma^n) \to u^* \text{ in } L^{2+\theta}(\Sigma_i).$$

Recall the weak formulation of $u(\gamma^n)$ with $u(\gamma^n)(\cdot,0) = u_0$ and for any $v \in L^2(0,T;H^1(\Omega))$

$$\int_Q u(\gamma^n)_t v + \int_Q \alpha\nabla u(\gamma^n)\cdot\nabla v + \int_{\Sigma_i} \gamma^n u(\gamma^n) v = \int_{\Sigma_c} qv + \int_{\Sigma_i} u_a v + \int_Q fv.$$

By weak convergence of the sequence $\{u(\gamma^n)\}$ to $u^*$ in $\mathcal{W}$, we have

$$\lim_{n\to+\infty}\int_Q u(\gamma^n)_t v = \int_Q u_t^* v,$$
$$\lim_{n\to+\infty}\int_Q \alpha\nabla u(\gamma^n)\cdot\nabla v = \int_Q \alpha\nabla u^*\cdot\nabla v.$$

The weak convergence of $\{\gamma^n\}$ to $\gamma^*$ in $L^q(\Sigma_i)$ and the convergence of $\{u(\gamma^n)\}$ to $u^*$ in $L^{2+\theta}(\Sigma_i)$ implies

$$\lim_{n\to+\infty}\int_{\Sigma_i}\gamma^n u(\gamma^n)v = \lim_{n\to+\infty}\int_{\Sigma_i}[\gamma^n u^* v + \gamma^n(u(\gamma^n)-u^*)v] = \int_{\Sigma_i}\gamma^* u^* v.$$

Therefore, the limit $u^*$ satisfies that for any $v \in L^2(0,T;H^1(\Omega))$, there holds

$$\int_Q u_t^* v + \int_Q \alpha\nabla u^*\cdot\nabla v + \int_{\Sigma_i}\gamma^* u^* v = \int_{\Sigma_c} qv + \int_{\Sigma_i} u_a v + \int_Q fv,$$

which together with $u^*(\cdot,0) = u_0$ (it follows from Lemma 2.1(a): $\mathcal{W} \hookrightarrow C(0,T;(H^1(\Omega))')$ compactly) and the uniqueness of the solution to the forward problem consequently implies $u^* = u(\gamma^*)$. Since every subsequence has a subsubsequence converging weakly to $u(\gamma^*)$, the whole sequence converges weakly. □

## 3. A regularization approach

In this section, we investigate a regularization approach of Tikhonov type to the inverse problem.

As mentioned earlier, like most inverse problems, the Robin inverse problem is ill-posed in the sense of Hadamard. The ill-posedness is mainly reflected in the following aspects: the solution does not necessarily exist for a given lateral Cauchy data, and even if it does exist for a specific set of Cauchy data, it lacks a continuous dependence on the data perturbation. In practice, the Cauchy data are collected via experimental devices, e.g., thermal sensors, and thus inevitably contaminated by measurement errors that may lead to large deviations of the solution from the exact one. This poses significant challenges to its accurate yet stable numerical solution. In order to cope with the numerical instability of the inverse problem, we adopt the classical Tikhonov regularization [TA77]

$$J_\eta(\gamma) = \tfrac{1}{2}\|u(\gamma)-g\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\gamma\|^2_{L^2(\Sigma_i)}. \tag{3.1}$$

In the functional $J_\eta$, the first term incorporates the information contained in the data, the second term stabilizes the formulation, and the regularization parameter $\eta$ compromises the tradeoff between the two terms. Here we use the notation $g$ to denote the measurement, i.e., $g \in L^2(\Sigma_c)$, for notational simplicity. We also remark alternative penalty terms, e.g., $H^1(\Sigma_i)$-norm or seminorm, may be employed, and the derivations below still hold with minor modifications. It is obvious that the constrained optimization problem with $\eta = 0$, denoted by $J_0$, is equivalent to the inverse problem when the Cauchy data is attainable (compatible).

We consider the minimization of this cost functional over the admissible set $\mathcal{A}$ as

**Problem 3.1.** $\min_{\gamma\in\mathcal{A}} J_\eta(\gamma)$.

A first result shows the existence of a minimizer to Problem 3.1. Therefore, the formulation is well-posed. Recall that the inverse problem may not possess a solution for noisy Cauchy data. Thus the inverse problem and the optimization problem may not be equivalent.

**Theorem 3.1.** *There exists at least one optimal solution to Problem* 3.1.

*Proof.* The functional $J_\eta$ is bounded from below by zero, and thus we can find a minimizing sequence $\{\gamma^n\} \subset \mathcal{A}$ such that

$$\lim_{n\to\infty} J_\eta(\gamma^n) = \inf_{\gamma\in\mathcal{A}} J_\eta(\gamma).$$

By the uniform boundedness of the admissible set $\mathcal{A}$, the sequence $\{\gamma^n\}$ is uniformly bounded in $L^q(\Sigma_i)$. Therefore, there exists a weakly convergent subsequence, also denoted by $\{\gamma^n\}$ and some $\gamma^*$ such that

$$\gamma^n \to \gamma^* \text{ weakly in } L^q(\Sigma_i).$$

One can show that the weak limit $\gamma^*$ satisfies $c_0 \le \gamma^*(t,x) \le c_1$ for a.e. $(t,x) \in \Sigma_i$, which implies that $\gamma^* \in \mathcal{A}$. Consequently, from Lemma 2.6, we deduce

$$u(\gamma^n) \to u(\gamma^*) \text{ weakly in } \mathcal{W}.$$

Now from the compactness of the embedding from $\mathcal{W}$ into $L^{2+\theta}(\Sigma_i)$ (see Lemma 2.1), we have

$$u(\gamma^n) \to u(\gamma^*) \text{ in } L^{2+\theta}(\Sigma_c).$$

In particular, this implies

$$\lim_{n\to+\infty} \|u(\gamma^n) - g\|^2_{L^2(\Sigma_c)} = \|u(\gamma^*) - g\|^2_{L^2(\Sigma_c)}. \tag{3.2}$$

Finally, the weak lower semi-continuity of norms gives

$$\begin{aligned} J_\eta(\gamma^*) &= \tfrac{1}{2}\|u(\gamma^*) - g\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\gamma^*\|^2_{L^2(\Sigma_i)} \\ &\le \liminf_{n\to+\infty} \tfrac{1}{2}\|u(\gamma^n) - g\|^2_{L^2(\Sigma_c)} + \liminf_{n\to+\infty} \tfrac{\eta}{2}\|\gamma^n\|^2_{L^2(\Sigma_i)} \\ &\le \liminf_{n\to+\infty} \left(\tfrac{1}{2}\|u(\gamma^n) - g\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\gamma^n\|^2_{L^2(\Sigma_i)}\right) = \inf_{\gamma\in\mathcal{A}} J_\eta(\gamma), \end{aligned}$$

i.e., the weak limit $\gamma^*$ is a minimizer to the functional $J_\eta$. □

The next result shows that the sequence of minimizers of the perturbed optimization problem has a subsequence converging weakly in $L^q(\Sigma_i)$ as the magnitude of the perturbation in the Cauchy data decreases to zero. Therefore, the functional $J_\eta$ indeed merits a certain (weak) stability with respect to the perturbations of the data $g$.

**Theorem 3.2.** *Let $\{g^n\} \subset L^2(\Sigma_c)$ be a sequence converging to $g$, and $\{\gamma^n\}$ be the sequence of minimizers to the perturbed functionals*

$$\tfrac{1}{2}\|u(\gamma) - g^n\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\gamma\|^2_{L^2(\Sigma_i)},$$

*i.e., with $g^n$ in place of $g$ in the functional $J_\eta$, Then the sequence $\{\gamma^n\}$ has a subsequence converging in $L^2(\Sigma_i)$ and weakly in $L^q(\Sigma_i)$, and the limit is a minimizer of the functional $J_\eta$.*

*Proof.* The proof proceeds closely as that of Theorem 3.1 and thus we only give a brief sketch. For the sequence of minimizers $\{\gamma^n\}$, we pass to a subsequence, again denoted by $\{\gamma^n\}$, which converges to some $\gamma^* \in \mathcal{A}$ weakly in $L^q(\Sigma_i)$, and as a consequence of Lemma 2.6, $u(\gamma^n)$ converges to $u(\gamma^*)$ in $L^{2+\theta}(\Sigma_c)$. Since the sequence $\{g^n\}$ converges to $g$ in $L^2(\Sigma_c)$, we get an analogue to (3.2), i.e.,

$$\lim_{n\to+\infty} \|u(\gamma^n) - g^n\|^2_{L^2(\Sigma_c)} = \|u(\gamma^*) - g\|^2_{L^2(\Sigma_c)}.$$

Therefore, the weak limit $\gamma^*$ is a minimizer to the functional $J_\eta$ by repeating the arguments in Theorem 3.1. To see the strong convergence in $L^2(\Sigma_i)$, we observe that by the weak lower semi-continuity

$$\begin{aligned}
J_\eta(\gamma^*) &\le \lim_{n\to+\infty} \tfrac{1}{2}\|u(\gamma^n) - g^n\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2} \liminf_{n\to+\infty} \|\gamma^n\|^2_{L^2(\Sigma_i)} \\
&\le \liminf_{n\to+\infty} \left(\tfrac{1}{2}\|u(\gamma^n) - g^n\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\gamma^n\|^2_{L^2(\Sigma_i)}\right) \\
&\le \limsup_{n\to+\infty} \left(\tfrac{1}{2}\|u(\gamma^n) - g^n\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\gamma^n\|^2_{L^2(\Sigma_i)}\right) \\
&\le \limsup_{n\to+\infty} \left(\tfrac{1}{2}\|u(\gamma^*) - g^n\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\gamma^*\|^2_{L^2(\Sigma_i)}\right) = J_\eta(\gamma^*),
\end{aligned}$$

where the last inequality follows from the minimizing property of $\gamma^n$ for the data $g^n$. Therefore, by combining with the preceding inequality, we have $\lim_{n\to+\infty} \|\gamma^n\|^2_{L^2(\Sigma_i)} = \|\gamma^*\|^2_{L^2(\Sigma_i)}$, which together with the weak convergence in $L^q(\Sigma_i)$ shows the desired assertion. □

Next we derive the gradient formula of the functional $J_\eta$ with respect to the parameter (Robin coefficient), and the first-order necessary optimality condition. The gradient formula enables applying popular gradient-descent type algorithms, whereas the optimality system is useful for designing Newton type methods. We will use the adjoint variable and adjoint equation frequently. Define the adjoint variable $p$ by the following adjoint equation:

$$\begin{cases} -p_t - \nabla\cdot(\alpha\nabla p) = 0 & \text{in } Q, \\ \alpha\frac{\partial p}{\partial n} = u(\gamma) - g & \text{on } \Sigma_c, \\ \alpha\frac{\partial p}{\partial n} + \gamma p = 0 & \text{on } \Sigma_i, \\ p(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{3.3}$$

The next result shows the differentiability of the functional with respect to $L^q(\Sigma_i)$. The gradient calculation via the adjoint method is frequently exploited in numerical algorithms, e.g., the popular conjugate gradient method. Similar formulas have been routinely utilized in the engineering literature, but rarely justified.

**Theorem 3.3.** *The functional $J_\eta$ is Fréchet differentiable with respect to $L^q(\Sigma_i)$ with its derivative given by*

$$J'_\eta(\gamma)\zeta = \int_{\Sigma_i} \zeta[-u(\gamma)p + \eta\gamma], \tag{3.4}$$

*where $p$ is defined in the adjoint equation* (3.3).

*Proof.* By virtue of Lemma 2.1, the generalized Hölder inequality and uniform boundedness of the admissible set $\mathcal{A}$, we deduce

$$\begin{aligned}\left|\int_{\Sigma_i}\zeta[-u(\gamma)p+\eta\gamma]\right| &\le \|\zeta\|_{L^q(\Sigma_i)}\left(\|u(\gamma)\|_{L^{2+\theta}(\Sigma_i)}\|p\|_{L^2(\Sigma_i)}+\eta\|\gamma\|_{L^{q'}(\Sigma_i)}\right)\\ &\le C\|\zeta\|_{L^q(\Sigma_i)}\left(\|u(\gamma)\|_{\mathcal{W}}\|p\|_{L^2(0,T;H^1(\Omega))}+\eta\|\gamma\|_{L^2(\Sigma_i)}\right),\end{aligned}$$

where the exponent $q'$ satisfies $\frac{1}{q}+\frac{1}{q'}=1$. In view of Lemma 2.3, $\|u(\gamma)\|_{\mathcal{W}}$ can be bounded independently of $\gamma$, and similarly $\|p\|_{\mathcal{W}}$. Therefore, the representation (3.4) defines a bounded linear functional on $L^q(\Sigma_i)$. Next we show that it is the Fréchet derivative of the functional $J_\eta$. To this end, we estimate the remainder $R$ as follows:

(3.5)

$$\begin{aligned}R :=& J_\eta(\gamma+\zeta)-J_\eta(\gamma)-\int_{\Sigma_i}\zeta[-u(\gamma)p+\eta\gamma]\\ =&\frac{1}{2}\int_{\Sigma_c}\left[|u(\gamma+\zeta)-g|^2-|u(\gamma)-g|^2\right]+\frac{1}{2}\int_{\Sigma_i}\left[\eta|\gamma+\zeta|^2-\eta|\gamma|^2-2\zeta[-u(\gamma)p+\eta\gamma]\right]\\ =&\frac{1}{2}\int_{\Sigma_c}(u(\gamma+\zeta)+u(\gamma)-2g)(u(\gamma+\zeta)-u(\gamma))+\frac{1}{2}\int_{\Sigma_i}\left[2\zeta u(\gamma)p+\eta|\zeta|^2\right].\end{aligned}$$

By the weak formulations for the adjoint variable $p$, and denote $w=u'(\gamma)\zeta$, then we have

$$\begin{aligned}\int_Q -p_t w+\int_Q \alpha\nabla p\cdot\nabla w+\int_{\Sigma_i}\gamma p w &= \int_{\Sigma_c}(u(\gamma)-g)w,\\ \int_Q w_t p+\int_Q \alpha\nabla p\cdot\nabla w+\int_{\Sigma_i}\gamma p w &= \int_{\Sigma_i}-\zeta u(\gamma)p.\end{aligned}$$

By subtracting these two equations and applying the following identity

$$\int_Q w_t p+\int_Q p_t w=\frac{d}{dt}\int_0^T\langle p,w\rangle_{L^2(\Omega)}=0,$$

which follows from the initial condition for $w$ and terminal condition for $p$, we arrive at

$$\int_{\Sigma_i}-\zeta u(\gamma)p=\int_{\Sigma_c}(u(\gamma)-g)u'(\gamma)\zeta.$$

Combining it with identity (3.5), we get

$$\begin{aligned}|R| =& \left|\frac{1}{2}\int_{\Sigma_c}[2(u(\gamma)-g)(u(\gamma+\zeta)-u(\gamma)-u'(\gamma)\zeta)\right.\\ &\left.+(u(\gamma+\zeta)-u(\gamma))^2]+\int_{\Sigma_i}\eta|\zeta|^2\right|\\ \le& \|u(\gamma)-g\|_{L^2(\Sigma_c)}\|u(\gamma+\zeta)-u(\gamma)-u'(\gamma)\zeta\|_{L^2(\Sigma_c)}\\ &+\|u(\gamma+\zeta)-u(\gamma)\|^2_{L^2(\Sigma_c)}+\eta\|\zeta\|^2_{L^2(\Sigma_i)}.\end{aligned}$$

This together with the embedding results in Lemma 2.1, the Lipschitz continuity of Lemma 2.4, and differentiability of the forward map of Lemma 2.5, yields the desired assertion. □

The following result shows the optimality system of Problem 3.1. Below we denote by $P_{[c_0,c_1]}$ the canonical pointwise projection operator, i.e., $P_{[c_0,c_1]}g(x) = \min(c_0, \max(c_1, g(x)))$.

**Theorem 3.4.** *The optimality system of Problem* 3.1 *is given by*

$$\begin{array}{ll}
\textit{primal equation} & \begin{cases} u(\gamma^*)_t - \nabla\cdot(\alpha\nabla u(\gamma^*)) = f & \textit{in } Q, \\ \alpha\frac{\partial u(\gamma^*)}{\partial n} = q & \textit{on } \Sigma_c, \\ \alpha\frac{\partial u(\gamma^*)}{\partial n} + \gamma^* u(\gamma^*) = u_a & \textit{on } \Sigma_i, \\ u(\gamma^*)(\cdot,0) = u_0 & \textit{in } \Omega; \end{cases} \\
\textit{adjoint equation} & \begin{cases} -p^*_t - \nabla\cdot(\alpha\nabla p^*) = 0 & \textit{in } Q, \\ \alpha\frac{\partial p^*}{\partial n} = u(\gamma^*) - g & \textit{on } \Sigma_c, \\ \alpha\frac{\partial p^*}{\partial n} + \gamma^* p^* = 0 & \textit{on } \Sigma_i, \\ p^*(\cdot,T) = 0 & \textit{in } \Omega. \end{cases} \\
\textit{complementarity condition} & \gamma^* = P_{[c_0,c_1]}(\frac{1}{\eta}u(\gamma^*)p^*).
\end{array}$$

*Proof.* Let $\gamma^*$ be an optimal solution. By Theorem 3.3, the cost function $J_\eta(\gamma)$ is Fréchet differentiable, and the regular point condition is satisfied at point $\gamma^*$, which guarantee the existence of Lagrange multipliers associated to the inequality constraint $c_0 \le \gamma^*(t,x) \le c_1$ for a.e. $(t,x) \in \Sigma_i$; see [ZK79]. We denote the Lagrange multiplier by $\lambda^* \in L^{q'}(\Sigma_i)$ (with $\frac{1}{q'} + \frac{1}{q} = 1$) such that

$$0 = J'_\eta(\gamma^*) + \lambda^*, \quad \langle \lambda^*, \gamma' - \gamma^* \rangle_{L^{q'}(\Sigma_i), L^q(\Sigma_i)} \le 0 \quad \forall \gamma' \in \mathcal{A}.$$

By Theorem 3.3 and the definition of adjoint variable $p^*$, we have

$$\lambda^* = -J'_\eta(\gamma^*) = u(\gamma^*)p^* - \eta\gamma^*.$$

This implies that

$$(\tfrac{1}{\eta}u(\gamma^*)p^* - \gamma^*, \gamma' - \gamma^*) \le 0 \quad \forall \gamma' \in \mathcal{A}.$$

Hence $\gamma^* = P_{\mathcal{A}}\left(\frac{1}{\eta}u(\gamma^*)p^*\right)$, which can be pointwise expressed as

$$\gamma^* = P_{[c_0,c_1]}\left(\tfrac{1}{\eta}u(\gamma^*)p^*\right).$$

This shows the optimality system. □

From the optimality system in Theorem 3.4, we can deduce higher regularity for the adjoint variable $p^*$ and consequently the Robin coefficient $\gamma^*$. First, it follows from the maximal parabolic regularity result [HMRR10, Thm. 3.18] (cf. also [Gr92, Thm. 1]) that $p^* \in H^1(0,T;(W^{1,r'}(\Omega))') \cap L^2(0,T;W^{1,r}(\Omega))$ for some $r > 2$ (where $\frac{1}{r'} + \frac{1}{r} = 1$) which depends on $\Omega$, $\alpha_0$, $\alpha_1$, $c_0$, $c_1$ and $d$ only [Gr92, HMRR10], and $u^* \in \mathcal{W}$.

Let $w = u^*p^*$, then $\nabla w = p^*\nabla u^* + u^*\nabla p^*$. Consequently, for $d = 2$, by Hölder's inequality and Sobolev embedding theorem, there holds

$$\begin{aligned}\int_\Omega |\nabla w|^s &= \int_\Omega |p^*\nabla u^* + u^*\nabla p^*|^s \\ &\le 2^{s-1}\int_\Omega |p^*|^s|\nabla u^*|^s + \int_\Omega |u^*|^s|\nabla p^*|^s \\ &\le C(s)\left(\|p^*\|^s_{L^\infty(\Omega)}\|\nabla u^*\|^s_{L^2(\Omega)} + \|u^*\|^s_{L^{\frac{rs}{r-s}}(\Omega)}\|\nabla p^*\|^s_{L^r(\Omega)}\right) \\ &\le C(s)\|u^*\|^s_{H^1(\Omega)}\|p^*\|^s_{W^{1,r}(\Omega)},\end{aligned}$$

for any $s \in [1, 2]$. Similarly, for $d = 3$ and by implicitly identifying $\frac{2s}{s-2}$ for $s = 2$ with $+\infty$, we have

$$\begin{aligned}\int_\Omega |\nabla w|^s dx &\le 2^{s-1}\left(\|p^*\|^s_{L^{\frac{2s}{2-s}}(\Omega)}\|\nabla u^*\|^s_{L^2(\Omega)} + \|u^*\|^s_{L^{\frac{rs}{r-s}}(\Omega)}\|\nabla p^*\|^s_{L^r(\Omega)}\right) \\ &\le C(s)\|u^*\|^s_{H^1(\Omega)}\|p^*\|^s_{W^{1,r}(\Omega)},\end{aligned}$$

where by Sobolev embedding theorem, the exponent $s \in [1, 2]$ satisfies $s \le \frac{6r}{6+r}$ (and $s < 2$ if $r = 3$). Consequently, we have $w \in W^{1,s}(\Omega)$ for any $s \in [1, \min(2, \frac{6r}{6+r})]$ ($s \in [1, 2)$ in case of $r = 3$). Hence, the Cauchy-Schwarz inequality gives

$$\int_0^T \|u^*p^*\|_{H^s(\Omega)} \le \int_0^T \|u^*\|_{H^1(\Omega)}\|p^*\|_{W^{1,r}(\Omega)} \le \|u^*\|_{L^2(0,T;H^1(\Omega))}\|p^*\|_{L^2(0,T;W^{1,r}(\Omega))}.$$

This together with the continuity of the canonical projection operator $P_{[c_0,c_1]}$ from $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)$ [KS80, pp. 50, Thm. A.1] and the trace theorem [EG92] implies the following enhanced regularity of the minimizer $\gamma^*$.

**Proposition 3.1.** *The optimal solution $\gamma^*$ to Problem* 3.1 *is in the space $L^1(0, T; W^{1-\frac{1}{s},s}(\Gamma_i))$ for $s = 2$ if $d = 2$ and $s = \min(2, \frac{6r}{6+r})$ ($s = 2 - \epsilon$ for any $\epsilon > 0$ if $r = 3$) if $d = 3$.*

The following theorem shows the essential convexity of the functional $J_0$ under some minor conditions. A nonnegative functional $\mathcal{G}$ is called essentially convex if $\mathcal{G}'(p) = 0$ implies $\mathcal{G}(p) = 0$ [Kno98]. Essential convexity is weaker than convexity, but often sufficient as it ensures that all critical points of the functional $J_0$ are global minimizers, and the algorithm would not get trapped into local minima. The nonvanishing condition on the solution $u(t, x)$ to system (2.3) may be verified by means of the maximum principle.

**Theorem 3.5.** *Assume that the* $\mathrm{meas}(\{(t, x) \in \Sigma_i : u(\gamma)(t, x) = 0\}) = 0$, *and the conductivity $\alpha$ is analytic. Then the functional $J_0$ attains its minimum zero at $\gamma \in \mathcal{A}$ if and only if $J_0'(\gamma) = 0$.*

*Proof.* If the functional $J_0$ attains zero at $\gamma$, then the residual $\tilde{r} = u(\gamma) - g = 0$ on $\Sigma_c$ and the solution $p(\gamma)$ of the adjoint problem vanishes identically. Consequently, $J_0'(\gamma)[\zeta] = 0$ for any $\zeta \in L^\infty(\Sigma_i)$, and hence $J_0'(\gamma) = 0$.

Now we assume that $J_0'(\gamma) = 0$ at some $\gamma \in \mathcal{A}$, i.e.,

$$\int_{\Sigma_i} -\zeta u(\gamma)p(\gamma) = 0, \quad \forall\, \zeta \in L^\infty(\Sigma_i),$$

which implies that $-u(\gamma)p(\gamma) = 0$ on $\Sigma_i$. Noting that the assumption $\mathrm{meas}(\{(t,x) \in \Sigma_i : u(\gamma)(t,x) = 0\}) = 0$, we have $p(\gamma) = 0$ almost everywhere on $\Sigma_i$. Thus the Robin boundary condition for the adjoint variable $p(\gamma)$ implies that

$$\alpha \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma_i.$$

Appealing to the unique continuation principle for parabolic problems [Miz58, Che98] yields that $p(\gamma)$ vanishes identically, and consequently, $u(\gamma) - g = 0$ on the boundary $\Sigma_c$. Hence $J_0(\gamma) = 0$, which concludes the proof. □

## 4. Finite element approximation

In this section we describe a finite element discretization of the constrained optimization problem, i.e., Problem 3.1, and analyze the convergence of the approximations.

To this end, we first triangulate the polyhedral domain $\Omega$ into a shape regular quasi-uniform mesh $\mathcal{T}_h$ of simplicial elements. Then we define the piecewise linear finite element space $V_h \subset H^1(\Omega)$ by

$$V_h = \{v_h : v_h \in C(\bar{\Omega}), v_h|_K \in P_1(K) \ \ \forall K \in \mathcal{T}_h\}, \tag{4.1}$$

where the space $P_1(K)$ denotes the space of linear polynomials on the element $K$. The space $V_{\Gamma_i,h}$ is defined as the restriction of $V_h$ on $\Gamma_i$. Next we define the set $U_h$ as

$$U_h = \{\gamma_h : \gamma_h \in V_{\Gamma_i,h}, \quad c_0 \le \gamma_h \le c_1\}.$$

The set $U_h$ will be used for the spatial discretization of the admissible set $\mathcal{A}$. To fully discretize system (2.3), we introduce a uniform partition of the interval $[0,T]$: $0 = t_0 < t_1 < \cdots < t_N = T$, where $\tau = \frac{T}{N}$ is the temporal step size and $\{t_n = n\tau\}$ are the partition points. The backward difference quotient $\partial_\tau u^n = \frac{u^n - u^{n-1}}{\tau}$ will be used for temporal discretization of system (2.3).

We shall also need the following approximations. Given any sequence $\{w_h^n\} \subset V_h$ (or $U_h$), we construct its piecewise constant/linear interpolant $\tilde{w}_{h,\tau}/w_{h,\tau}$ as follows: for $t \in (t_{n-1}, t_n)$,

$$\tilde{w}_{h,\tau}(t,x) = w_h^n \quad \text{and} \quad w_{h,\tau}(t,x) = \tfrac{t - t_{n-1}}{\tau} w_h^n + \tfrac{t_n - t}{\tau} w_h^{n-1}. \tag{4.2}$$

Further, for any $w \in L^2(0,T;B)$ ($B$ may be any spatial Hilbert space, e.g., $L^2(\Omega)$, $(H^{\frac{1}{2}}(\Gamma_i))'$), we construct its piecewise constant interpolant by setting for $t \in (t_{n-1}, t_n)$, $w^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} w(t)dt$. By means of straightforward computations, we have the following estimates.

**Lemma 4.1.** *For the approximations $\tilde{\gamma}_{h,\tau}$, $\tilde{u}_{h,\tau}$, $u_{h,\tau}$ and $w^n$ defined above, the following estimates hold:*

$$\|\tilde{\gamma}_{h,\tau}\|^q_{L^q(\Sigma_i)} = \tau \sum_{n=1}^N \|\gamma_h^n\|^q_{L^q(\Gamma_i)}, \qquad \|\tilde{u}_{h,\tau}\|^2_{L^2(0,T;H^1(\Omega))} = \tau \sum_{n=1}^N \|u_h^n\|^2_{H^1(\Omega)},$$

$$\|u_{h,\tau} - \tilde{u}_{h,\tau}\|^2_{L^2(Q)} = \frac{\tau}{3} \sum_{n=1}^N \|u_h^n - u_h^{n-1}\|^2_{L^2(\Omega)},$$

$$\|(u_{h,\tau})_t\|^2_{L^2(0,T;(H^1(\Omega))')} = \tau \sum_{n=1}^N \|\partial_\tau u_h^n\|^2_{(H^1(\Omega))'},$$

$$\|u_{h,\tau}\|^2_{L^2(0,T;H^1(\Omega))} \le \tau \sum_{n=0}^N \|u_h^n\|^2_{H^1(\Omega)}, \qquad \tau \sum_{n=1}^N \|w^n\|^2_B \le \|w\|^2_{L^2(0,T;B)}.$$

Now we describe the finite element approximation of system (2.3). The finite element system reads as follows: Given a sequence $\{\gamma_h^n\} \subset U_h$ and $u^0 = P_h u_0$, find $\{u_h^n\} \subset V_h$ for $n = 1, \dots, N$ such that

$$\begin{aligned} &\int_\Omega \partial_\tau u_h^n v_h + \int_\Omega \alpha^n \nabla u_h^n \cdot \nabla v_h + \int_{\Gamma_i} \gamma_h^n \tfrac{u_h^n + u_h^{n-1}}{2} v_h \\ &\qquad = \int_{\Gamma_i} u_a^n v_h + \int_{\Gamma_c} q^n v_h + \int_\Omega f^n v_h \quad \forall v \in V_h. \end{aligned} \tag{4.3}$$

Before proceeding to the a priori estimate for the finite element solution $\{u_h^n(\{\gamma_h^n\})\}$, we recall the $L^2(\Omega)$-projection operator $P_h : L^2(\Omega) \mapsto V_h$ and the $H^1(\Omega)$-projection operator $Q_h : H^1(\Omega) \mapsto V_h$ defined by

$$(P_h u, v_h) = (u, v_h) \quad \forall v_h \in V_h,$$
$$(\nabla Q_h u, \nabla v_h) + (Q_h u, v_h) = (\nabla u, \nabla v_h) + (u, v_h) \quad \forall v_h \in V_h,$$

respectively. The following lemma shows that the $L^2(\Omega)$-projection operator $P_h$ is $H^1(\Omega)$-stable.

**Lemma 4.2.** *The $L^2(\Omega)$-projection $P_h$ is $H^1(\Omega)$-stable, i.e.,*

$$\|P_h u\|_{H^1(\Omega)} \le C \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega).$$

*Proof.* The proof for the case of $H_0^1(\Omega)$ and $H^1(\Omega)$ can be found in [BX91] and [BPS02], respectively. □

The solution $\{u_h^n(\{\gamma_h^n\})\}$ to the discrete system (4.3) satisfies the following a priori estimate. It will play a crucial role in establishing the convergence of finite element approximations.

**Lemma 4.3.** *For any sequence $\{\gamma_h^n\} \subset U_h$, the discrete system (4.3) has a unique solution $\{u_h^n(\{\gamma_h^n\})\} \subset V_h$. If the mesh size $h$ and temporal step size $\tau$ satisfy $\tau \le \beta h$ for some $\beta > 0$, then for sufficiently small $\tau$ the following a priori estimate holds,*

$$\begin{aligned} &\|u_{h,\tau}\|^2_{\mathcal{W}} + \|\tilde{u}_{h,\tau}\|^2_{L^2(0,T;H^1(\Omega))} + \tfrac{1}{\tau}\|u_{h,\tau} - \tilde{u}_{h,\tau}\|^2_{L^2(Q)} \\ &\le C_\beta \left( \|u^0\|^2_{H^1(\Omega)} + \|u_a\|^2_{L^2(0,T;(H^{\frac{1}{2}}(\Gamma_i))')} + \|q\|_{L^2(0,T;(H^{\frac{1}{2}}(\Gamma_c))')} + \|f\|^2_{L^2(Q)} \right), \end{aligned}$$

*where $u_{h,\tau}$ and $\tilde{u}_{h,\tau}$ are piecewise linear and constant interpolants of $\{u_h^n(\{\gamma_h^n\})\}$, respectively.*

*Proof.* The existence and uniqueness of $\{u_h^n \equiv u_h^n(\{\gamma_h^n\})\} \subset V_h$ follow directly from the continuity and coercivity of the bilinear form. Upon taking $v_h = u_h^n$ in the discrete system (4.3), appealing to the Cauchy-Schwarz inequality and noting the lower bound for the conductivity $\alpha(x)$ and $\tilde{\gamma}_{h,\tau}$, we deduce

$$\begin{aligned}
&\tfrac{1}{2\tau}\left(\|u_h^n\|_{L^2(\Omega)}^2 - \|u_h^{n-1}\|_{L^2(\Omega)}^2 + \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2\right) \\
&\qquad + \alpha_0\|\nabla u_h^n\|_{L^2(\Omega)}^2 + c_0\|u_h^n\|_{L^2(\Gamma_i)}^2 \\
&\quad \le c_1\|u_h^n - u_h^{n-1}\|_{L^2(\Gamma_i)}\|u_h^n\|_{L^2(\Gamma_i)} + \|f^n\|_{L^2(\Omega)}\|u_h^n\|_{L^2(\Omega)} \\
&\qquad + c_1\|u_a^n\|_{(H^{\frac{1}{2}}(\Gamma_i))'}\|u_h^n\|_{H^{\frac{1}{2}}(\Gamma_i)} + \|q^n\|_{(H^{\frac{1}{2}}(\Gamma_c))'}\|u_h^n\|_{H^{\frac{1}{2}}(\Gamma_c)}.
\end{aligned}$$

By utilizing Young's inequality and Lemma 2.2, we get

$$\begin{aligned}
&\tfrac{1}{\tau}\left(\|u_h^n\|_{L^2(\Omega)}^2 - \|u_h^{n-1}\|_{L^2(\Omega)}^2 + \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2\right) + \|u_h^n\|_{H^1(\Omega)}^2 \\
&\quad \le C\left(\|u_a^n\|_{(H^{\frac{1}{2}}(\Gamma_i))'}^2 + \|q^n\|_{(H^{\frac{1}{2}}(\Gamma_c))'}^2 + \|f^n\|_{L^2(\Omega)}^2\right. \\
&\qquad\qquad \left. + \|u_h^n - u_h^{n-1}\|_{L^2(\Gamma_i)}\|u_h^n\|_{L^2(\Gamma_i)}\right).
\end{aligned} \tag{4.4}$$

Now observe that $u_h^n - u_h^{n-1} \in V_h \subset H^1(\Omega)$, Lemma 2.1, i.e.,

$$\|u_h^n\|_{L^2(\Gamma_i)}^2 \le C\|u_h^n\|_{L^2(\Omega)}\|u_h^n\|_{H^1(\Omega)},$$

inverse estimate for the finite element space $V_h$ [BS08, Thm 4.5.11], i.e., $\|u_h\|_{H^1(\Omega)} \le Ch^{-1}\|u_h\|_{L^2(\Omega)}$, and the condition $\tau \le \beta h$ imply that

$$\begin{aligned}
\|u_h^n - u_h^{n-1}\|_{L^2(\Gamma_i)}^2 &\le C\|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}\|u_h^n - u_h^{n-1}\|_{H^1(\Omega)} \\
&\le Ch^{-1}\|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2 \\
&\le C\beta\tau^{-1}\|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

Combining this inequality with Lemma 2.1 and Young's inequality yields

$$\begin{aligned}
&C\|u_h^n - u_h^{n-1}\|_{L^2(\Gamma_i)}\|u_h^n\|_{L^2(\Gamma_i)} \\
&\quad \le C\tfrac{\sqrt{\beta}}{\sqrt{\tau}}\|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}\|u_h^n\|_{L^2(\Omega)}^{\frac{1}{2}}\|u_h^n\|_{H^1(\Omega)}^{\frac{1}{2}} \\
&\quad \le \tfrac{1}{2}\|u_h^n\|_{H^1(\Omega)}^2 + C(\tfrac{\beta}{\tau})^{\frac{2}{3}}\|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^{\frac{4}{3}}\|u_h^n\|_{L^2(\Omega)}^{\frac{2}{3}} \\
&\quad \le \tfrac{1}{2}\|u_h^n\|_{H^1(\Omega)}^2 + \tfrac{1}{2\tau}\|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2 + C\beta^2\|u_h^n\|_{L^2(\Omega)}^2.
\end{aligned}$$

This together with inequality (4.4) gives

$$\begin{aligned}
&\tfrac{1}{\tau}\left(\|u_h^n\|_{L^2(\Omega)}^2 - \|u_h^{n-1}\|_{L^2(\Omega)}^2 + \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2\right) + \|u_h^n\|_{H^1(\Omega)}^2 \\
&\le C\left(\|u_a^n\|_{(H^{\frac{1}{2}}(\Gamma_i))'}^2 + \|f^n\|_{L^2(\Omega)}^2 + \|q^n\|_{(H^{\frac{1}{2}}(\Gamma_c))'}^2\right) + C\beta^2\|u_h^n\|_{L^2(\Omega)}^2.
\end{aligned}$$

By the discrete Gronwall's inequality and that for sufficiently small $\tau$, i.e., $\frac{1}{\tau} > C\beta^2$, we deduce

$$\begin{aligned}
&\max_{1\le n\le N}\|u_h^n\|^2_{L^2(\Omega)} + \sum_{n=1}^N \|u_h^n - u_h^{n-1}\|^2_{L^2(\Omega)} + \tau\sum_{n=1}^N \|u_h^n\|^2_{H^1(\Omega)} \\
&\le C_\beta\left(\|u^0\|^2_{L^2(\Omega)} + \tau\sum_{n=1}^N\left(\|u_a^n\|^2_{(H^{\frac12}(\Gamma_i))'} + \|q^n\|^2_{(H^{\frac12}(\Gamma_c))'} + \|f^n\|^2_{L^2(\Omega)}\right)\right).
\end{aligned} \tag{4.5}$$

Next we estimate the term $\|\partial_\tau u_h^n\|_{(H^1(\Omega))'}$. To this end, for any $v_h \in V_h$, we have

$$\begin{aligned}
\langle \partial_\tau u_h^n, v_h\rangle &\le \alpha_1\|\nabla u_h^n\|_{L^2(\Omega)}\|\nabla v_h\|_{L^2(\Omega)} + c_1\left\|\tfrac{u_h^n+u_h^{n-1}}{2}\right\|_{L^2(\Gamma_i)}\|v_h\|_{L^2(\Gamma_i)} \\
&\quad + c_1\|u_a^n\|_{(H^{\frac12}(\Gamma_i))'}\|v_h^n\|_{H^{\frac12}(\Gamma_i)} + \|q^n\|_{(H^{\frac12}(\Gamma_c))'}\|v_h^n\|_{H^{\frac12}(\Gamma_c)} + \|f^n\|_{L^2(\Omega)}\|v_h\|_{L^2(\Omega)} \\
&\le C\|v_h\|_{H^1(\Omega)}(\|u_h^n\|_{H^1(\Omega)} + \|u_h^{n-1}\|_{H^1(\Omega)} + \|u_a^n\|_{(H^{\frac12}(\Gamma_i))'} \\
&\qquad\qquad + \|q^n\|^2_{(H^{\frac12}(\Gamma_c))'} + \|f^n\|_{L^2(\Omega)}).
\end{aligned}$$

Consequently, the definition of the $L^2(\Omega)$-projection operator $P_h$ and its $H^1(\Omega)$ stability (see Lemma 4.2) yield

$$\begin{aligned}
\|\partial_\tau u_h^n\|_{(H^1(\Omega))'} &= \sup_{v\in H^1(\Omega)}\frac{\langle\partial_\tau u_h^n, v\rangle}{\|v\|_{H^1(\Omega)}} \\
&= \sup_{v\in H^1(\Omega)}\frac{\langle\partial_\tau u_h^n, P_h v\rangle}{\|P_h v\|_{H^1(\Omega)}}\frac{\|P_h v\|_{H^1(\Omega)}}{\|v\|_{H^1(\Omega)}} \le C\sup_{v_h\in V_h}\frac{\langle\partial_\tau u_h^n, v_h\rangle}{\|v_h\|_{H^1(\Omega)}} \\
&\le C\left(\|u_h^n\|_{H^1(\Omega)} + \|u_h^{n-1}\|_{H^1(\Omega)} + \|u_a^n\|_{(H^{\frac12}(\Gamma_i))'} + \|q^n\|_{(H^{\frac12}(\Gamma_c))'} + \|f^n\|_{L^2(\Omega)}\right).
\end{aligned}$$

This together with estimate (4.5) gives

$$\begin{aligned}
&\tau\sum_{n=1}^N \|\partial_\tau u_h^n\|^2_{(H^1(\Omega))'} \\
&\qquad \le C_\beta\left(\|u^0\|^2_{H^1(\Omega)} + \tau\sum_{n=1}^N\left(\|u_a^n\|^2_{(H^{\frac12}(\Gamma_i))'} + \|q^n\|^2_{(H^{\frac12}(\Gamma_c))'} + \|f^n\|^2_{L^2(\Omega)}\right)\right).
\end{aligned}$$

The desired assertion follows from above estimate, inequality (4.5) and Lemma 4.1. □

*Remark* 4.1. The stability constant $C_\beta$ in Lemma 4.3 is independent of the regularization parameter $\eta$. The constant $C_\beta$ is obtained by discrete Gronwall's inequality in (4.5), and $C_\beta = e^{C\beta^2+C}$. Therefore, we cannot choose a large $\beta$, i.e., the time step $\tau$ has a slight restriction. It violates the absolute stability of the usual implicit scheme for the heat equation. This is due to the mid-point rule employed in the term $\int_{\Gamma_i}\gamma_h^n\frac{u_h^n+u_h^{n-1}}{2}v_h ds$. Nonetheless, the usual setting $\tau = \mathcal{O}(h)$ does satisfy the assumption in Lemma 4.3.

The fully discretized optimization problem of the continuous formulation, i.e., Problem 3.1, now reads

**Problem 4.1.** Find $\{\gamma_h^n\} \subset U_h$ such that

$$\min_{\{\gamma_h^n\}\subset U_h}\{J_{h,\tau}(\{\gamma_h^n\}) = \tfrac12\|u_{h,\tau}(\{\gamma_h^n\}) - g\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\tilde\gamma_{h,\tau}\|^2_{L^2(\Sigma_i)}\},$$

where the approximation $u_{h,\tau}(\{\gamma_h^n\})$ is the piecewise linear interpolant of $\{u_h^n(\{\gamma_h^n\})\}$.

**Lemma 4.4.** *Under the assumption in Lemma* 4.3*, the map* $\{\gamma_h^n\} \mapsto \{u_h^n(\{\gamma_h^n\})\}$ *is continuous.*

*Proof.* Consider an increment $\{\zeta_h^n\}$ of $\{\gamma_h^n\}$ with $\{\gamma_h^n + \zeta_h^n\} \subset U_h$. Let $\delta u_h^n = u_h^n(\{\gamma_h^n + \zeta_h^n\}) - u_h^n(\{\gamma_h^n\})$, then $\delta u_h^n$ satisfies $\delta u^0 = 0$ and for $n = 1, \ldots, N$,

$$\int_\Omega \partial_\tau \delta u_h^n v_h + \int_\Omega \alpha^n \nabla \delta u_h^n \cdot \nabla v_h + \int_{\Gamma_i} (\gamma_h^n + \zeta_h^n) \tfrac{\delta u_h^n + \delta u_h^{n-1}}{2} v_h = \int_{\Gamma_i} -\zeta_h^n u_h^n v_h \quad \forall v_h \in V_h.$$

By taking $v_h = \delta u_h^n$ in the weak formulation and repeating the arguments of Lemma 4.3, we arrive at

$$\tau \sum_{n=1}^N \|\delta u_h^n\|_{H^1(\Omega)}^2 \le C \sup_n \|\zeta_h^n\|_{L^\infty(\Gamma_i)}^2 \tau \sum_{n=1}^N \|u_h^n\|_{H^1(\Omega)}^2.$$

The continuity of the map $\{\gamma_h^n\} \mapsto \{u_h^n(\{\gamma_h^n\})\}$ follows directly from this inequality and norm equivalence in a finite-dimensional space. □

With the help of Lemma 4.4, we can establish the well-posedness of the discrete optimization Problem 4.1.

**Theorem 4.1.** *Under the assumption in Lemma* 4.3*, there exists at least one solution to Problem* 4.1*.*

*Proof.* By Lemma 4.4, the objective functional $J_{h,\tau}$ is continuous. The minimization problem is finite dimensional, and the functional $J_{h,\tau}$ is continuous and bounded from below by zero. The desired assertion on the existence of a minimizer follows directly from the compactness of the discrete admissible set. □

The remaining part of this section is devoted to the convergence analysis of the finite element approximation. To this end, we shall need the following two approximation results in $L^2(0,T;H^1(\Omega))$ and $L^q(\Sigma_i)$.

**Lemma 4.5.** *For any* $v(t,x) \in L^2(0,T;H^1(\Omega))$*, there exist sequences* $\{v_h^n\} \subset V_h$ *such that the piecewise constant interpolant* $\tilde{v}_{h,\tau}$ *satisfies*

$$\|\tilde{v}_{h,\tau} - v\|_{L^2(0,T;H^1(\Omega))} \to 0 \quad \text{as } (h,\tau) \to \vec{0}^+.$$

*Proof.* First we assume $v(t,x) \in C^\infty(Q)$. Then for any fixed $t$, the $H^1(\Omega)$-projection operator $Q_h v(t)$ satisfies the following error estimate (cf. [BS08])

$$\|Q_h v(t) - v(t)\|_{H^1(\Omega)} \le Ch \|v(t)\|_{H^2(\Omega)}.$$

We define a discrete sequence $\{v_h^n\}$ as follows:

$$v_h^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} Q_h v(t).$$

Consequently, by the triangle inequality, Hölder's inequality, Fubini's theorem [EG92] and the error estimate for $Q_h$, we have for $t \in [t_{n-1}, t_n]$,

$$\begin{aligned}
\|v_h^n - v\|_{H^1(\Omega)}^2 &= \left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_n} Q_h v(\xi) - v \right\|_{H^1(\Omega)}^2 \\
&\le 2 \left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_n} (Q_h v(\xi) - v(\xi)) \right\|_{H^1(\Omega)}^2 + 2 \left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \int_t^{\xi} \frac{\partial v}{\partial t}(s) \right\|_{H^1(\Omega)}^2 \\
&\le 2 \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \|(Q_h v(\xi) - v(\xi))\|_{H^1(\Omega)}^2 + 2 \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \left\| \int_t^{\xi} \frac{\partial v}{\partial t}(s) \right\|_{H^1(\Omega)}^2 \\
&\le C \left( h^2 \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \|v(\xi)\|_{H^2(\Omega)}^2 + \tau \int_{t_{n-1}}^{t_n} \left\| \frac{\partial v}{\partial t}(s) \right\|_{H^1(\Omega)}^2 \right).
\end{aligned}$$

Integrating the above inequality from 0 to $T$, we arrive at

$$\|\tilde{v}_{h,\tau} - v\|_{L^2(0,T;H^1(\Omega))} \le C \left( h \|v\|_{L^2(0,T;H^2(\Omega))} + \tau \left\| \frac{\partial v}{\partial t} \right\|_{L^2(0,T;H^1(\Omega))} \right).$$

Therefore, the approximation $\tilde{v}_{h,\tau}$ satisfies the property $\|\tilde{v}_{h,\tau} - v\|_{L^2(0,T;H^1(\Omega))} \to 0$ for $v \in C^\infty(Q)$. The lemma follows from this estimate and the density of $C^\infty(Q)$ in $L^2(0,T;H^1(\Omega))$. □

**Lemma 4.6.** *For any $\gamma \in \mathcal{A}$, there exist sequences $\{\gamma_h^n\} \subset U_h$ such that the piecewise constant interpolant $\tilde{\gamma}_{h,\tau}$ satisfies*

$$\|\tilde{\gamma}_{h,\tau} - \gamma\|_{L^q(\Sigma_i)} \to 0 \text{ as } (h,\tau) \to \vec{0}^+.$$

*Proof.* First assume $\gamma \in C^\infty(\Sigma_i) \cap \mathcal{A}$. Let $\mathcal{I}_h$ be the standard nodal interpolation operator [BS08]. Then, for each fixed $t$, $\mathcal{I}_h \gamma(\cdot, t) \in U_h$, and on each polyhedral surface, there holds [BS08] the following estimate:

$$\|\gamma - \mathcal{I}_h \gamma\|_{L^\infty} \le Ch \|\gamma\|_{C^1}.$$

Next we define

$$\gamma_h^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \mathcal{I}_h \gamma.$$

Clearly, we have $\gamma_h^n \in U_h$. Analogous to the proof of Lemma 4.5, by virtue of the triangle inequality and Hölder's inequality, we deduce that for any $t \in [t_{n-1}, t_n]$,

$$\begin{aligned}
&\|\gamma_h^n - \gamma(t)\|_{L^q(\Gamma_i)}^q \\
&\quad \le C \left( \left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_n} [\mathcal{I}_h \gamma(\xi) - \gamma(\xi)] \right\|_{L^q(\Gamma_i)}^q + \left\| \frac{1}{\tau} \int_{t_{n-1}}^{t_n} [\gamma(\xi) - \gamma(t)] \right\|_{L^q(\Gamma_i)}^q \right) \\
&\quad \le \frac{C}{\tau} \left( \int_{t_{n-1}}^{t_n} \|\mathcal{I}_h \gamma(\xi) - \gamma(\xi)\|_{L^q(\Gamma_i)}^q + \int_{t_{n-1}}^{t_n} \left\| \int_t^{\xi} \frac{\partial \gamma}{\partial t}(s) ds \right\|_{L^q(\Gamma_i)}^q \right) \\
&\quad \le \frac{C}{\tau} \left( h^q \int_{t_{n-1}}^{t_n} \|\gamma(\xi)\|_{C^1(\Gamma_i)}^q + \tau^q \int_{t_{n-1}}^{t_n} \left\| \frac{\partial \gamma}{\partial t}(s) \right\|_{L^q(\Gamma_i)}^q \right).
\end{aligned}$$

Integrating the inequality from $[0,T]$, we get

$$\|\tilde{\gamma}_{h,\tau}-\gamma\|^q_{L^q(\Sigma_i)} \le C\left(h^q\|\gamma\|^q_{L^q(0,T;C^1(\Gamma_i))}+\tau^q\left\|\frac{\partial\gamma}{\partial t}(\eta)\right\|^q_{L^q(0,T;L^q(\Gamma_i))}\right).$$

The desired assertion follows from this estimate and the density of $C^\infty(\Sigma_i)$ in $L^q(\Sigma_i)$. □

Now we can show the following important result. It is a discrete analogue of Lemma 2.6, and will be key to demonstrating the convergence of the finite element approximations.

**Lemma 4.7.** *Let the assumption in Lemma* 4.3 *be fulfilled, and given a sequence* $\{\gamma_h^n\}\subset U_h$*, let* $\{u_h^n(\{\gamma_h^n\})\}$ *be the solution to system* (4.3)*. If the sequence of piecewise constant interpolants* $\tilde{\gamma}_{h,\tau}$ *converges to* $\gamma$ *weakly in* $L^q(\Sigma_i)$ *as* $h,\tau$ *tend to zero, then the sequences of piecewise linear and constant interpolants* $u_{h,\tau}$ *and* $\tilde{u}_{h,\tau}$ *converge to* $u(\gamma)$ *weakly in* $\mathcal{W}$ *and* $L^2(0,T;H^1(\Omega))$*, respectively.*

*Proof.* By virtue of the a priori estimate in Lemma 4.3, $u_{h,\tau}$ and $\tilde{u}_{h,\tau}$ are uniformly bounded in the spaces $\mathcal{W}$ and $L^2(0,T;H^1(\Omega))$, respectively. Therefore, there exist subsequences of them, possibly after relabeling, and some $u\in\mathcal{W}$ and $\tilde{u}\in L^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned}
&\tilde{\gamma}_{h,\tau}\to\gamma && \text{weakly in } L^q(\Sigma_i),\\
&u_{h,\tau}\to u && \text{weakly in } \mathcal{W},\\
&\tilde{u}_{h,\tau}\to\tilde{u} && \text{weakly in } L^2(0,T;H^1(\Omega)).
\end{aligned}$$

By Lemma 2.1, we have

$$u_{h,\tau}\to u \ \text{ in } L^{2+\theta}(\Sigma_i) \text{ and } L^2(Q).$$

Now Lemma 4.3 indicates that $\|\tilde{u}_{h,\tau}-u_{h,\tau}\|^2_{L^2(Q)}\le C\tau$. Consequently, we have $\tilde{u}=u$. Hence, it suffices to show that $u=u(\gamma)$. Recall that $\{u_h^n(\{\gamma_h^n\})\}$ satisfies the discrete system (4.3), or equivalently

$$\begin{aligned}
&\int_Q (u_{h,\tau})_t\tilde{v}_{h,\tau}dxdt+\int_Q \alpha\nabla\tilde{u}_{h,\tau}\cdot\nabla\tilde{v}_{h,\tau}+\int_{\Sigma_i}\tilde{\gamma}_{h,\tau}u_{h,\tau}\tilde{v}_{h,\tau}\\
&\quad=\int_{\Sigma_i}u_a\tilde{v}_{h,\tau}+\int_{\Sigma_c}q\tilde{v}_{h,\tau}dsdt+\int_Q f\tilde{v}_{h,\tau},
\end{aligned}$$

for all $\{v_h^n\}\subset V_h$ with $\tilde{v}_{h,\tau}$ being its piecewise constant interpolant. For any test function $v\in L^2(0,T;H^1(\Omega))$, by Lemma 4.5, we can choose $\{v_h^n\}$ such that $\|\tilde{v}_{h,\tau}-$

$v\|_{L^2(0,T;H^1(\Omega))} \to 0$ as $h, \tau$ tend to zero. By taking the limit on the subsequence as $h, \tau \to 0$, we have

$$\left.\begin{array}{r}(u_{h,\tau})_t \to u_t \text{ weakly in } L^2(0,T;(H^1(\Omega))') \\ \tilde{v}_{h,\tau} \to v \text{ in } L^2(0,T;H^1(\Omega))\end{array}\right\} \Rightarrow \int_Q (u_{h,\tau})_t \tilde{v}_{h,\tau} \to \int_Q u_t v,$$

$$\left.\begin{array}{r}\tilde{u}_{h,\tau} \to u \text{ weakly in } L^2(0,T;H^1(\Omega)) \\ \tilde{v}_{h,\tau} \to v \text{ in } L^2(0,T;H^1(\Omega))\end{array}\right\} \Rightarrow \int_Q \alpha \nabla \tilde{u}_{h,\tau} \cdot \nabla \tilde{v}_{h,\tau} \to \int \alpha \nabla u \cdot \nabla v,$$

$$\left.\begin{array}{r}\tilde{\gamma}_{h,\tau} \to \gamma \text{ weakly in } L^q(\Sigma_i) \\ u_{h,\tau} \to u \text{ in } L^{2+\theta}(\Sigma_i) \\ \tilde{v}_{h,\tau} \to v \text{ in } L^2(\Sigma_i)\end{array}\right\} \Rightarrow \int_{\Sigma_1} \tilde{\gamma}_{h,\tau} u_{h,\tau} \tilde{v}_{h,\tau} \to \int_{\Sigma_i} \gamma u v,$$

$$\left.\begin{array}{r}\tilde{v}_{h,\tau} \to v \text{ weakly in } L^2(0,T;H^{\frac{1}{2}}(\Gamma_i)) \\ u_a \in L^2(0,T;(H^{\frac{1}{2}}(\Gamma_i))')\end{array}\right\} \Rightarrow \int_{\Sigma_i} u_a \tilde{v}_{h,\tau} \to \int_{\Sigma_i} u_a v,$$

$$\left.\begin{array}{r}q \in L^2(0,T;(H^{\frac{1}{2}}(\Gamma_c))') \\ \tilde{v}_{h,\tau} \to v \text{ weakly in } L^2(0,T;H^{\frac{1}{2}}(\Gamma_i))\end{array}\right\} \Rightarrow \int_{\Sigma_c} q \tilde{v}_{h,\tau} \to \int_{\Sigma_c} f v,$$

$$\left.\begin{array}{r}f \in L^2(Q) \\ \tilde{v}_{h,\tau} \to v \text{ in } L^2(Q)\end{array}\right\} \Rightarrow \int_Q f \tilde{v}_{h,\tau} \to \int_Q f v.$$

By combining these identities, we deduce that the limit $u$ satisfies the variational equation

$$\int_Q u_t v + \int_Q \alpha \nabla u \cdot \nabla v + \int_{\Sigma_i} \gamma u v = \int_{\Sigma_i} u_a v + \int_{\Sigma_c} q v + \int_Q f v \quad \forall v \in L^2(0,T;H^1(\Omega)).$$

It remains to show the initial condition $u(0) = u_0$ holds. From Lemma 2.1,

$$u_{h,\tau} \to u \ \text{ in } C(0,T;(H^1(\Omega))').$$

Then for any $v \in H^1$, we have

$$(u(0), v) = \lim_{h \to 0} \int_\Omega u_{h,\tau}(0) v = \lim_{h \to 0} \int_\Omega P_h u_0 v = \int_\Omega u_0 v.$$

This implies that $u(0) = u_0$, and thus $u = u(\gamma)$. Since this holds for every weakly convergent subsequence, the whole sequence converges weakly. This concludes the proof of the lemma. □

Now we can show the main result of this section: the convergence of the finite element approximations.

**Theorem 4.2.** *Let the assumption in Lemma* 4.3 *be fulfilled, and* $\{\gamma_h^{n\,*}\}$ *be a sequence of minimizers to Problem* 4.1. *Then there exists a subsequence, still denoted by* $\{\gamma_h^{n\,*}\}$, *such that its piecewise constant interpolant* $\{\tilde{\gamma}_{h,\tau}^*\}$ *converges weakly in* $L^q(\Sigma_i)$ *to a minimizer of Problem* 3.1 *as* $h, \tau$ *tend to zero.*

*Proof.* Recall the definition of piecewise constant and linear interpolants (see (4.2)) and Problem 4.1. Let $u^*_{h,\tau} = u_{h,\tau}(\{\gamma_h^{n\,*}\})$ and $\tilde{u}^*_{h,\tau} = \tilde{u}_{h,\tau}(\{\gamma_h^{n\,*}\})$. Clearly, the sequence $\{\tilde{\gamma}^*_{h,\tau}\}$ is uniformly bounded in $L^q(\Sigma_i)$. Therefore, there exists a subsequence, also denoted as $\{\tilde{\gamma}^*_{h,\tau}\}$, and some $\gamma^* \in \mathcal{A}$ such that

$$\tilde{\gamma}^*_{h,\tau} \to \gamma^* \qquad \text{weakly in } L^q(\Sigma_i).$$

By Lemma 4.7, $u^*_{h,\tau} \to u(\gamma^*)$ weakly in $\mathcal{W}$ as well as strongly in $L^2(\Sigma_c)$ as a consequence of Lemma 2.1. We need only to show that $\gamma^*$ is a minimizer of Problem 3.1. To this end, we recall

$$J_{h,\tau}(\{\gamma_h^n\}) = \tfrac{1}{2}\|u_{h,\tau} - g\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\tilde{\gamma}_{h,\tau}\|^2_{L^2(\Sigma_i)}.$$

By observing $u^*_{h,\tau} \to u(\gamma^*)$ in $L^2(\Sigma_c)$ and $\tilde{\gamma}^*_{h,\tau} \to \gamma^*$ weakly in $L^2(\Sigma_i)$, and weakly lower semi-continuity of norms, we derive

$$J_\eta(\gamma^*) \le \liminf_{(h,\tau)\to\vec{0}^+} J_{h,\tau}(\{\gamma_h^n\}^*).$$

By Lemma 4.6, for any $\gamma \in \mathcal{A}$, there exists a sequence $\{\{\gamma_h^n\} \subset U_h\}$ such that $\tilde{\gamma}_{h,\tau} \to \gamma$ in $L^q(\Sigma_i)$. Then Lemma 4.7 implies that $u_{h,\tau}(\{\gamma_h^n\}) \to u(\gamma)$ weakly in $\mathcal{W}$, which by virtue of Lemma 2.1 implies $u_{h,\tau}(\{\gamma_h^n\}) \to u(\gamma)$ in $L^2(\Sigma_c)$. Consequently,

$$\begin{aligned} J_{h,\tau}(\{\gamma_h^n\}) &= \tfrac{1}{2}\|u_{h,\tau}(\{\gamma_h^n\}) - g\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\tilde{\gamma}_{h,\tau}\|^2_{L^2(\Sigma_i)} \\ &\to \tfrac{1}{2}\|u(\gamma) - g\|^2_{L^2(\Sigma_c)} + \tfrac{\eta}{2}\|\gamma\|^2_{L^2(\Sigma_i)} \quad \text{as } h,\tau \to 0. \end{aligned}$$

Therefore, by the minimizing property of $\tilde{\gamma}^*_{h,\tau}$, we arrive at

$$J_\eta(\gamma^*) \le \liminf_{h,\tau\to 0} J_{h,\tau}(\{\gamma_h^{n\,*}\}) \le \lim_{h,\tau\to 0} J_{h,\tau}(\{\gamma_h^n\}) = J_\eta(\gamma), \quad \forall \gamma \in \mathcal{A}.$$

Hence the limit $\gamma^*$ is a minimizer of the functional $J_\eta$. □

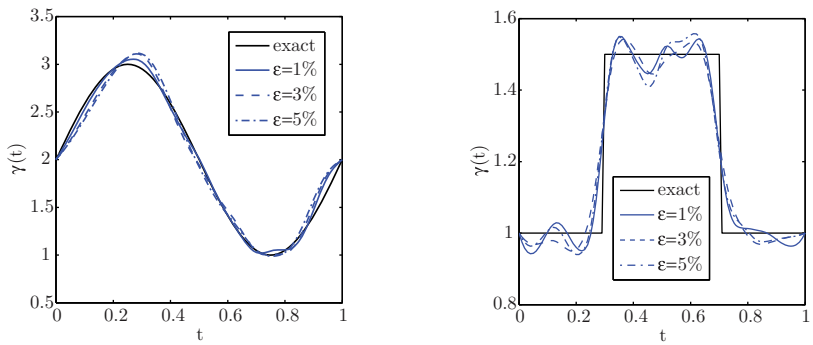
## 5. Numerical experiments and discussions

In this section, we present some numerical results for one- and two-dimensional problems to illustrate the proposed approach. The discrete constrained optimization problem, i.e., Problem 4.1, is minimized by the conjugate gradient method with a smoothed gradient [JZ10]. The algorithm is widely applied in inverse heat conduction problems, and we refer to Appendix A for a detailed description. All the computations were performed using MATLAB 2010a on a personal computer with dual core and 2.00GB memory.

5.1. **One-dimensional example.** In this part, we consider two one-dimensional examples. The domain $\Omega$ is taken to $(0,1)$, and the final time $T$ is fixed at $T = 1$. The temperature measurements are made on the boundary $\Gamma_c = \{x = 0\}$, and a Robin coefficient $\gamma(t)$ on the boundary $\Gamma_i = \{x = 1\}$ is to be estimated. The spatial domain $\Omega$ is discretized into 50 elements, and the number of time steps is 100, i.e. $h = 0.02$ and $\tau = 0.01$. For the inverse problem, the noisy data $g^\delta$ are generated pointwise as

$$g^\delta(t_i) = g(t_i) + \varepsilon \max_i |g(t_i)|\xi_i,$$

where $\{\xi_i\}$ are standard normal random variables, and $\varepsilon$ denotes the relative noise level. For the results presented below, we fixed the regularization parameter $\eta$ at




(a) Example 5.1 (b) Example 5.2

FIGURE 1. Reconstructions for Examples 5.1 and 5.2 with noisy data.

TABLE 1. Numerical results (accuracy error $e$) for 1d examples.

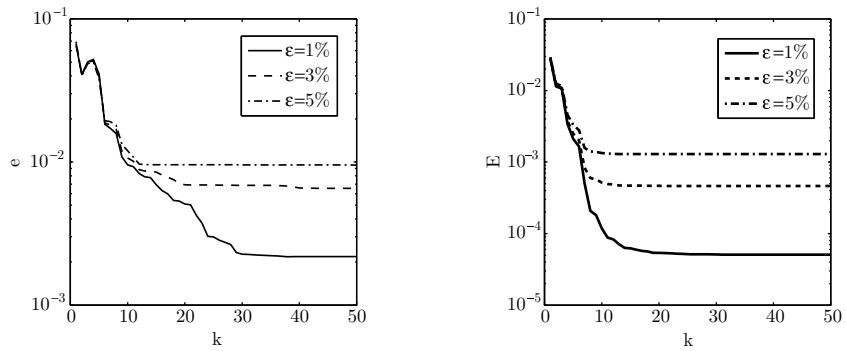
| $\varepsilon$ | 0% | 1% | 3% | 5% |
|---|---|---|---|---|
| Example 5.1 | 1.54e-4 | 2.18e-3 | 6.56e-3 | 9.08e-3 |
| Example 5.1 | 3.79e-3 | 3.88e-3 | 4.85e-3 | 4.73e-3 |

$\eta = 1.0 \times 10^{-6}$. To illustrate the approach, we consider the following two examples with a smooth solution and a discontinuous solution, respectively.

**Example 5.1.** The conductivity $\alpha(x)$ is set to $\alpha = 1$, and the exact Robin coefficient $\gamma^\dagger(t)$ is given by $\gamma^\dagger(t) = 2 + \sin(2\pi t)$. The boundary conditions, source term and initial condition are specified such that the analytical solution to the direct problem (2.3) is given by $u(t) = e^{2x}\sin(t)$.

**Example 5.1.** The conductivity $\alpha(x)$ is set to $\alpha(x) = e^x$, and the exact Robin coefficient $\gamma^\dagger(t)$ is given by $\gamma^\dagger(t) = 1 + \frac{1}{2}\chi_{\frac{3}{10}\le x\le\frac{7}{10}}$, where $\chi_S$ denotes the characteristic function of the set $S$. The boundary conditions, source term and initial condition are specified such that the analytical solution to the direct problem (2.3) is given by $u(t) = x^2\sin(\pi t)$.

The numerical results for the two examples are shown in Figure 1 and Table 1. First we observe that as the noise level $\varepsilon$ decreases from 5% to 0%, the accuracy of the reconstructions $\tilde{\gamma}$ increases accordingly. This holds for both smooth and nonsmooth solutions, although it is not so conspicuous for the latter due to the discontinuity of the exact solution. Therefore, the proposed approach is convergent with respect to noise level. This is more clearly seen from Table 1, where the accuracy error $e$ is defined by relative error $e = \|\tilde{\gamma} - \gamma^\dagger\|_{L^2(\Sigma_i)}/\|\gamma^\dagger\|_{L^2(\Sigma_i)}$, with $\gamma^\dagger$ being the exact Robin coefficient. Second, the reconstructions remain very reasonable approximations for up to $\varepsilon = 5\%$ noise in the data. The result for discontinuous case is less accurate, see Figure 1(b), as the true solution contains discontinuities which




(a) accuracy error $e$ (b) residual $E$

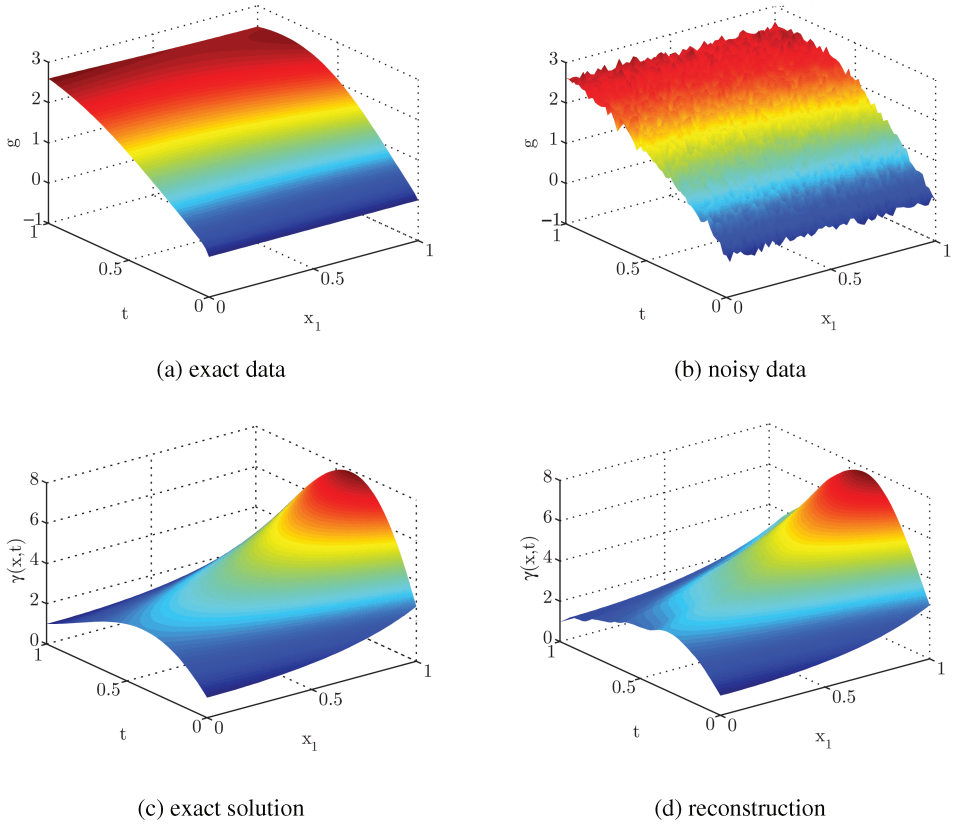
FIGURE 2. Convergence of the method for Example 5.1 with noisy data.

could not be accurately resolved by a smoothing regularization term $\|\gamma\|^2_{L^2(\Sigma_i)}$. This might be overcome by adopting an edge-preserving regularization formulation, e.g. total variation [JZ09]. Third, the conjugate gradient algorithm converges fairly steadily. The accuracy error $e$ of the intermediate iterates first decreases quickly and then levels off as the iteration proceeds, and the residual $E = \|u(\tilde{\gamma}) - g\|^2_{L^2(\Sigma_c)}$ decreases monotonically; see Figure 2.

5.2. **Two-dimensional example.** Finally, we show a two-dimensional example. The domain $\Omega$ is taken to be the unit square $(0,1) \times (0,1)$, and the final time $T$ is fixed at $T = 1$. The boundaries $\Gamma_i$ and $\Gamma_c$ are taken to be $\Gamma_i = \{x = (x_1, x_2) : 0 < x_1 < 1, x_2 = 1\}$ and $\Gamma_c = \Gamma \backslash \Gamma_i$, respectively. The spatial domain $\Omega$ is discretized into a uniform mesh composed of 3600 triangular elements, and the number of time steps is 50. The noisy data is generated as before.

**Example 5.2.** The conductivity $\alpha(x)$ is $\alpha(x) = e^{x_1 - x_2}$, and the exact Robin coefficient $\gamma^\dagger(t,x)$ is taken to be $\gamma^\dagger(t,x) = 3e^{x_1 - t}\sin(\pi t) + e^{x_1^2}$. The boundary conditions, source term and initial condition are specified such that the analytical solution to the direct problem (2.3) is given by $u(t) = \frac{1}{2}(x_1^2 + x_2^2)\sin(\pi t)$.

The numerical result for Example 5.2 with 15% noise (see Figures 3(a) and 3(b) for exact and noisy data, respectively) is shown in Figures 3. The reconstruction is in excellent agreement with the exact solution (cf. Figure 3(d)), especially when taking into account the fairly large amount of noise in the data. The accuracy error $e$ is $e = 3.01 \times 10^{-2}$. The convergence of the algorithm for Example 5.2 remains very steady; see Figure 4, in terms of the accuracy error $e$ and residual $E$. However, the overall convergence is much slower compared to Examples 5.1 and 5.1 since the number of unknowns $(3,060)$ is much larger.



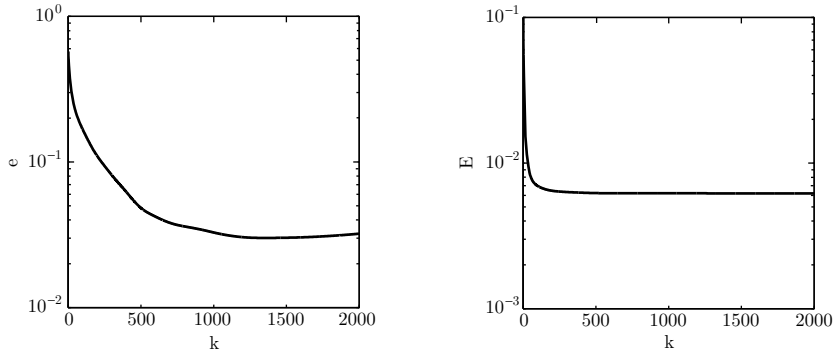

(a) exact data

(b) noisy data

(c) exact solution

(d) reconstruction

FIGURE 3. The exact and noisy (15%) data for Example 5.2, and the exact solution and reconstruction.




(a) accuracy error $e$

(b) residual $E$

FIGURE 4. Convergence of the method for Example 5.2 with 15% noise in the data.

## 6. Conclusions

We have studied the numerical identification of a Robin coefficient in parabolic problems from one single boundary measurement. Some analytical properties, e.g., continuity, differentiability and weak continuity, of the map from the parameter (Robin coefficient) to the state were discussed. A regularization formulation of Tikhonov type was studied, the existence and stability of the minimizers were shown, and the optimality system was derived. A finite element method with non-standard time-direction discretization was suggested for numerically implementing the optimizing problem, and the convergence of the finite element approximations was established, which seems to be the first discrete scheme with a rigorous convergence proof. Numerical results for both one- and two-dimensional examples were also presented to illustrate the feasibility of the approach.

There are several avenues deserving further investigations. First, the adoption of faster algorithms and preconditioning techniques as well as their theoretical justification, is of significant interest. This is especially important in two- and three dimensions, for which the conjugate gradient method is relatively slow and each forward solve is expensive, as we observed in Section 5.2. Second, although we have analyzed the convergence of the finite element discretization, error estimates and convergence rate analysis for such discretization are still missing. The latter knowledge would give an indication of an appropriate discretization level compatible with the noise level and achievable accuracy in order to achieve good computational complexity. Third, the optimal regularity, especially in the time-direction, of the minimizer $\gamma^*$ to the Tikhonov functional remains to be established. This relies crucially on the maximal regularity results for parabolic equations with rough coefficients. One then might also expect a bootstrap argument to further improve the regularity for the dual variable $p^*$ and the minimizer $\gamma^*$. Finally, the Robin boundary condition can be inaccurate for certain high-temperature applications, e.g., in steel production, and instead the Stefan-Boltzmann type nonlinear boundary condition is more suitable. One natural research problem is to extend the analysis and numerics of the proposed approach to these more complicated boundary conditions.

## Appendix A. Numerical algorithm

We adopt the conjugate gradient method for minimizing the constrained optimization problems, i.e., Problem 4.1. A complete description is shown in Algorithm 1. Every iteration of the algorithm requires solving two auxiliary problems, i.e., the adjoint problem and the sensitivity problem, which are needed for calculating the gradient and the step size, respectively.

The algorithm is basically of gradient descent type, i.e., $\gamma^{k+1} = \gamma^k - \beta_k d_k$, where $\beta_k$ is the step size at the $k$th iteration and $d_k$ is a descent direction given by $d_k = J'_\eta(\gamma^k) + \theta_k d_{k-1}$ with the conjugate coefficient $\theta_k$ being determined by (the Fletcher-Reeves choice)

$$\theta_k = \frac{\|J'_\eta(\gamma^k)\|^2_{L^2(\Sigma_i)}}{\|J'_\eta(\gamma^{k-1}))\|^2_{L^2(\Sigma_i)}}$$

with the convention $\theta_0 = 0$. The step size $\beta_k$ is determined by minimizing the functional $J_\eta(\gamma)$ with respect to $\beta$, i.e., solving

$$\beta = \arg\min_{\beta \geq 0} J_\eta(\gamma),$$

**Algorithm 1**. Conjugate gradient algorithm.

Set $k = 0$ and choose $\gamma^0$.
**repeat**
    Solve the direct problem with $\gamma = \gamma^k$, and determine the residual $r_k = u(\gamma^k) - g$;
    Solve the adjoint problem;
    Calculate the gradient $J'_\eta(\gamma^k)$, the conjugate coefficient $\theta_k$, and the descent direction $d_k$;
    Solve the sensitivity problem with $\zeta = d_k$;
    Compute the step length $\beta_k$ in the conjugate direction $d_k$;
    Update the Robin coefficient $\gamma^k$ by $\gamma^k = \gamma^k - \beta_k d_k$;
    Increase $k$ by one;
**until** A stopping criterion is satisfied.

with $\gamma = \gamma^k - \beta d_k$. By means of a Taylor expansion of the forward operator $u(\gamma)$ around $\gamma^k$, we arrive at the following approximate formula for determining $\beta_k$:

$$\beta_k = \frac{\langle r_k, u'(\gamma^k)d_k\rangle_{L^2(\Sigma_c)} + \eta\langle \gamma_k, d_k\rangle_{L^2(\Sigma_i)}}{\|u'(\gamma^k)d_k\|^2_{L^2(\Sigma_c)} + \eta\|d_k\|^2_{L^2(\Sigma_i)}}.$$

In the computation, we have employed this heuristic rule for determining the step size. Our numerical experiments indicate that it works well for the Robin inverse problem.

Finally, we would like to remark on the choice of initial guess $\gamma^0$ for the algorithm. The optimality system (see Theorem 3.4) indicates that the minimizer to the optimization problem (3.1) must vanish on the lateral boundary $t = T$ (and $t = 0$, possibly) due to the vanishing terminal condition for the adjoint problem. Therefore, a direct application of the adjoint method could only poorly recover the value of the Robin coefficient over these parts. In the numerical experiments, we specified its true value. This is reasonable as both the Dirichlet and Neumann boundary conditions are known there, and the coefficient on the boundary can be evaluated directly from the Newton's law. In order to preserve the boundary condition, we smooth the gradient by Laplacian (temporally and/or spatially independently) with a homogeneous Dirichlet boundary condition [JZ10].

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## Added after posting

An error in the numbering of the examples in Section 5 occurred during the production process. Specifically, the two one-dimensional examples were incorrectly both labeled as 5.1, which should be separately labeled as 5.1 and 5.2, and the two-dimensional example was labeled as 5.2, which should be 5.3. We clarify the possible confusion in the tables and figures as follows: The second row of Table 1 is for Example 5.2, Figure 2 is for Example 5.1, and Figures 3 and 4 are for Example 5.3. Similar corrections should be applied to the numbering in the paragraph after Example 5.3.

## References


[AF03] Robert Adams and John Fournier, *Sobolev Spaces*, Elsevier/Academic Press, Amsterdam, 2003. MR2424078 (2009e:46025)

[Ali94] Oleg M. Alifanov, *Inverse heat transfer problems*, Springer-Verlag, Berlin, 1994.

[BBSC85] James V. Beck, Ben Blackwell, and Charles R. St. Clair, *Inverse Heat Conduction: Ill-Posed Problems*, Wiley-Interscience, New York, 1985.

[BCC08] Mourad Bellassoued, Jin Cheng, and Mourad Choulli, *Stability estimate for an inverse boundary coefficient problem in thermal imaging*, J. Math. Anal. Appl. **343** (2008), no. 1, 328–336. MR2412131 (2009e:35299)

[BPS02] James H. Bramble, Joseph E. Pasciak, and Olaf Steinbach, *On the stability of the $L^2$ projection in $H^1(\Omega)$*, Math. Comp. **71** (2002), no. 237, 147–156. MR1862992 (2002h:65175)

[BS08] Susanne C. Brenner and L. Ridgway Scott, *The mathematical theory of finite element methods*, Springer, New York, 2008. MR2373954 (2008m:65001)

[BX91] James H. Bramble and Jinchao Xu, *Some estimates for a weighted $L^2$ projection*, Math. Comp. **56** (1991), no. 194, 463–476. MR1066830 (91k:65140)

[CFJL04] S. Chaabane, I. Fellah, M. Jaoua, and J. Leblond, *Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems*, Inverse Problems **20** (2004), no. 1, 47–59. MR2044605 (2005a:35276)

[CGH06] K. Chrysafinos, M. D. Gunzburger, and L. S. Hou, *Semidiscrete approximations of optimal Robin boundary control problems constrained by semilinear parabolic PDE*, J. Math. Anal. Appl. **323** (2006), no. 2, 891–912. MR2260151 (2007h:49041)

[Cha99] S. Chantasiriwan, *Inverse heat conduction problem of determining time-dependent heat transfer coefficient*, Int. J. Heat Mass Transfer **42** (1999), no. 23, 4275–4285.

[Che98] Xu-Yan Chen, *A strong unique continuation theorem for parabolic equations*, Math. Ann. **311** (1998), no. 4, 603–630. MR1637972 (99h:35078)

[Cho99] Mourad Choulli, *On the determination of an unknown boundary function in a parabolic equation*, Inverse Problems **15** (1999), no. 3, 659–667. MR1696926 (2001b:35291)

[CJ99] Slim Chaabane and Mohamed Jaoua, *Identification of Robin coefficients by the means of boundary measurements*, Inverse Problems **15** (1999), no. 6, 1425. MR1733209 (2000j:35270)

[CW08] Han-Taw Chen and Xin-Yi Wu, *Investigation of heat transfer coefficient in two-dimensional transient inverse heat conduction problems using the hybrid inverse scheme*, Int. J. Numer. Methods Engrg. **73** (2008), no. 1, 107–122. MR2378451 (2009b:65242)

[EG92] Lawrence C. Evans and Ronald F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC, Boca Raton, 1992. MR1158660 (93f:28001)

[Eva98] Lawrence C. Evans, *Partial Differential Equations*, AMS, Providence, R.I., 1998.

[Gr92] Konrad Gröger, *$W^{1,p}$-estimates of solutions to evolution equations corresponding to nonsmooth second order elliptic differential operators*, Nonlin. Anal. **18** (1992), no. 6, 569–577. MR1154481 (93a:35066)

[HMRR10] D Hömberg, C Meyer, J Rehberg, and W Ring, *Optimal control for the thermistor problem*, SIAM J. Control Optim. **48** (2010), no. 5, 3449–3481. MR2599927 (2011b:49009)

[Jin07] Bangti Jin, *Conjugate gradient method for the Robin inverse problem associated with the Laplace equation*, Int. J. Numer. Methods Eng. **71** (2007), no. 4, 433–453. MR2332815 (2008g:65146)

[JZ08] Bangti Jin and Jun Zou, *Inversion of Robin coefficient by a spectral stochastic finite element approach*, J. Comput. Phys. **227** (2008), no. 6, 3282–3306. MR2392734 (2009c:65225)

[JZ09] ———, *Numerical estimation of piecewise constant Robin coefficient*, SIAM J. Control Optim. **48** (2009), no. 3, 1977–2002. MR2516196

[JZ10] ———, *Numerical estimation of the Robin coefficient in a stationary diffusion equation*, IMA J. Numer. Anal. **30** (2010), no. 3, 677–701. MR2670110 (2011f:65241)

[KJ01] Refahi Abou Khachfe and Yvon Jarny, *Determination of heat sources and heat transfer coefficient for two-dimensional heat flow – numerical and experimental study*, Int. J. Heat Mass Transfer **44** (2001), no. 7, 1309–1322.

[Kno98] Ian Knowles, *A variational algorithm for electrical impedance tomography*, Inverse Problems **14** (1998), no. 6, 1513–1325. MR1662464 (99i:65127)

[Kre00] F Kreith (ed.), *The CRC Handbook of Thermal Engineering*, CRC, Bota Raton, 2000.

[KS80] David Kinderlehrer and Guido Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980. MR567696 (81g:49013)

[LU68] O. A. Ladyzhenskaya and N. N Uralceva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York-London, 1968. MR0244627 (39:5941)

[LW93] S. Lenhart and D. G. Wilson, *Optimal control of a heat transfer problem with convective boundary condition*, J. Optim. Theory Appl. **79** (1993), no. 3, 581–597. MR1255288 (94j:49005)

[Miz58] Sigeru Mizohata, *Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques*, Mem. College Sci. Univ. Kyoto Ser. A Math. **31** (1958), no. 3, 219–239. MR0106347 (21:5081)

[NH94] D. Nho Hào, *A non-characteristic Cauchy problem for linear parabolic equations and related inverse problems: I. solvability*, Inverse Problems **10** (1994), no. 2, 295–315. MR1269009 (95b:35228)

[NHR97] Dinh Nho Hào and H. J. Reinhardt, *On a sideways parabolic equation*, Inverse Problems **13** (1997), no. 2, 297–309. MR1445920 (97m:35273)

[OB88] A. M. Osman and J. V. Beck, *Nonlinear inverse problem for the estimation of time-and-space dependent heat transfer coefficients*, J. Thermophys. **3** (1988), no. 2, 146–152.

[OIL08] T. T. M. Onyango, D. B. Ingham, and D. Lesnic, *Reconstruction of heat transfer coefficients using the boundary element method*, Comput. Math. Appl. **56** (2008), no. 1, 114–126. MR2427690 (2009f:80017)

[SH04] Jian Su and Geoffrey Hewitt, *Inverse heat conduction problem of estimating time-varying heat transfer coefficient*, Numer. Heat Transfer Part A **45** (2004), no. 8, 777–789.

[SLO10] Marián Slodička, D. Lesnic, and T. T. M. Onyango, *Determination of a time-dependent heat transfer coefficient in a nonlinear heat conduction problems*, Inv. Probl. Sci. Eng. **18** (2010), no. 1, 65–81. MR2598682 (2011a:65304)

[SV02] Marián Slodička and Roger Van Keer, *Determination of a Robin coefficient in semilinear parabolic problems by means of boundary measurements*, Inverse Problems **18** (2002), no. 1, 139. MR1893587 (2003d:35271)

[TA77] Andrey N. Tikhonov and Vasiliy Y. Arsenin, *Solutions of Ill-Posed Problems*, John Wiley & Sons, New York, 1977. MR0455365 (56 #13604)

[Whi88] F. M. White, *Heat and mass transfer*, Addison-Wesley, Reading, 1988.

[XZ05] Jianli Xie and Jun Zou, *Numerical reconstruction of heat fluxes*, SIAM J. Numer. Anal. **43** (2005), no. 4, 1504–1535. MR2182138 (2006e:35344)

[YYW09] Fenglian Yang, Liang Yan, and Ting Wei, *The identification of a Robin coefficient by a conjugate gradient method*, Int. J. Numer. Methods Engrg. **78** (2009), no. 7, 800–816. MR2514342

[ZK79] J. Zowe and S. Kurcyusz, *Regularity and stability for the mathematical programming problem in Banach spaces*, Applied Math. Optim. **5** (1979), no. 1, 49–62. MR526427 (82a:90153)

Department of Mathematics and Institute for Applied Mathematics and Computational Science, Texas A&M University, College Station, Texas 77843-3368
*E-mail address*: btjin@math.tamu.edu

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People's Republic of China
*E-mail address*: xllv.math@whu.edu.cn