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ARITHMETIC $(1; e)$-CURVES AND BELYĬ MAPS

JEROEN SIJSLING

ABSTRACT. Using the theory of Belyĭ maps, we calculate the algebraic curves associated to the Fuchsian groups of signature $(1; e)$ that are commensurable with a triangle group, along with the Picard-Fuchs differential equations on these curves, which are related to pullbacks of hypergeometric differential equations. We focus particularly on the $(1; e)$-groups that are arithmetic.

Let $e \in \mathbb{Z}_{\geq 2}$ be an integer, let $\mathrm{GL}_2(\mathbb{R})^+$ be the group of invertible real two-by-two matrices with positive determinant, and let $\mathrm{PGL}_2(\mathbb{R})^+$ be the adjoint group of $\mathrm{GL}_2(\mathbb{R})^+$. A $(1; e)$-*group* is a Fuchsian group (i.e., a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})^+$ of finite covolume) whose signature equals $(1; e)$. That is to say that Γ has a presentation

$$\Gamma = \left\langle \alpha, \beta, \gamma \,\middle|\, \gamma = \alpha^{-1}\beta^{-1}\alpha\beta, \gamma^e = 1 \right\rangle . \tag{0.1}$$

Let $\mathcal{H}^+ = \mathcal{H}$ be the complex upper half-plane, and let $\mathcal{H}^- = \overline{\mathcal{H}}$. Then $\mathrm{PGL}_2(\mathbb{R})^+$ acts on $\mathcal{H}^\pm$ through Möbius transformations, and we can define the $(1; e)$-*curves*

$$X^\pm(\Gamma) = \Gamma \backslash \mathcal{H}^\pm. \tag{0.2}$$

The curves $X^\pm(\Gamma)$ are of genus 1, and the branch locus of the canonical projection $\mathcal{H}^\pm \to X^\pm(\Gamma)$ consists of the elliptic point of $X^\pm(\Gamma)$, which is of index e.

The more well-known *triangle groups* are those Fuchsian groups whose signature equals $(0; p, q, r)$ for some triple of integers (p, q, r) satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. We refer to [6] for an overview of the geometry surrounding triangle groups. Algebraically, $(1; e)$-groups can be characterized as the Fuchsian groups that are generated by two elements yet are not instances of triangle groups; cf. [15]. In a sense, therefore, $(1; e)$-groups Γ constitute the simplest families of Fuchsian groups whose associated quotients $X^\pm(\Gamma)$ are of genus 1, which makes them a natural object of study.

Let π be the canonical projection map $\mathcal{H}^\pm \to X^\pm(\Gamma)$. The inverse map π^{-1} is a Γ-multivalued function on $X^\pm(\Gamma)$. Choose a Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3 \tag{0.3}$$

for $X^\pm(\Gamma)$, placing the elliptic point at infinity. Then π^{-1} can be obtained as a quotient of two solutions of the *Lamé differential equation*

$$(y\frac{d}{dx})^2 u = (n(n+1)x + A)u \tag{0.4}$$

on $X^\pm(\Gamma)$ ([32], § 5.2).

The Lamé equation (0.4) is a Fuchsian differential equation. The parameter $n = \frac{1}{2e} - \frac{1}{2}$ equals the difference of the local exponents of (0.4) at its unique regular singular point, which is the elliptic point of $X^\pm(\Gamma)$. Unfortunately, the *accessory*

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parameter A in (0.4) cannot be determined from the isomorphism class of $X^{\pm}(\Gamma)$ and this local exponent difference alone, an aspect that makes Lamé equations somewhat harder to deal with than the hypergeometric equations associated to triangle groups (for which this problem does not occur).

Consider the following condition on a $(1;e)$-group Γ:

(a) Γ is contained in a triangle group Δ.

Assuming (a), one obtains Belyǐ maps

$$X^{\pm}(\Gamma) \longrightarrow X^{\pm}(\Delta). \tag{0.5}$$

Using the covers (0.5) greatly simplifies the determination of $X^{\pm}(\Gamma)$. Moreover, the differential equations (0.4) are pullbacks of hypergeometric differential equations on $X^{\pm}(\Delta)$ through the maps (0.5), which enables one to determine the accessory parameters A.

In this paper, we consider the following slightly less restrictive version of (a):

(b1) Γ is commensurable with a triangle group.

Moreover, we impose the following condition:

(b2) Γ is arithmetic.

Assuming condition (b1), the differential equation (0.4) is a Picard-Fuchs equation and hence allows a basis of solutions consisting of G-functions (cf. [1]).

We refer to Section 1 for an explanation of condition (b2). Loosely speaking, it means that the curves $X^{\pm}(\Gamma)$ are close to being Shimura curves. This results in a rich arithmetic theory for $X^{\pm}(\Gamma)$ that was previously explored in [10] and [12] in the case where the base field F equals $\mathbb{Q}$. To give an idea of the power of these arithmetic results, we mention that we use this theory in [23] to determine explicit equations for $X^{\pm}(\Gamma)$ for the more complicated class of arithmetic $(1;e)$-groups Γ that do not satisfy condition (b1).

Our motivation for the somewhat extraneous condition (b2) on top of (b1) was the classification of the $(1;e)$-groups Γ satisfying both (b1) and (b2) by Takeuchi in [26] and [27]; there are 27 such groups up to $\mathrm{PGL}_2(\mathbb{R})$-conjugacy. More agreeably, we show in Section 3 that condition (b2) is not unduly restrictive in the sense that it holds in all sufficiently non-trivial cases where (b1) is satisfied.

Combining conditions (b1) and (b2), it turns out that it is again possible to determine a model for $X^{\pm}(\Gamma)$ over $\mathbb{C}$ by using Belyǐ maps. Having calculated these maps, the accessory parameter A in (0.4) is again relatively straightforward to determine. In the end, apart from one remaining conjectural accessory parameter, the methods of this paper and [23] allow us to fully classify and calculate the Lamé equations associated to the $(1;e)$-groups satisfying (b1), which in particular yields explicit examples of Lamé equations of Picard-Fuchs type.

Relaxing the condition that Γ has signature $(1;e)$, one obtains more Fuchsian groups satisfying conditions (b1) and (b2). For some results on uniformizations and Fuchsian differential equations in these related settings, we refer to [2], [19], and [30] for the triangular case, and to [17] and [28] for an exhaustive study of the genus 0 cases with four singularities satisfying condition (a). Throughout this paper, however, we restrict to curves of signature $(1;e)$.

This paper is organized as follows. In Sections 1 and 2, we briefly review the theory on quaternion algebras, covers, and Belyǐ maps that we need. Section 3 explores when condition (b1) implies condition (b2). Section 4 shows how a $(1;e)$-group Γ satisfying conditions (b1) and (b2) above gives rise to a Belyĭ map, and in Section 5 we discuss how to calculate this map. Section 6 summarizes how to calculate the accessory parameters A in the Lamé equation (0.4) associated with Γ once one has calculated the associated Belyĭ map. Our final results are given in Table 1 in Section 7.

1. Quaternion algebras

Let F be a totally real number field, and let B be a quaternion algebra over F, i.e., a central simple algebra over F whose dimension as a vector space over F equals 4. Let $\mathbb{Z}_F$ be the ring of integers of F, and for a prime $\mathfrak{p}$ of $\mathbb{Z}_F$, denote the completion of F at $\mathfrak{p}$ by $F_\mathfrak{p}$. The *finite discriminant* of B is given by

$$\mathfrak{D}(B)^f = \prod_{\mathfrak{p}\in R_B} \mathfrak{p},$$

where

$$R_B = \{(0) \neq \mathfrak{p} \text{ prime} \subset \mathbb{Z}_F : B \otimes_F F_\mathfrak{p} \ncong M_2(F_\mathfrak{p})\}.$$

An *order* of B is a $\mathbb{Z}_F$-submodule $\mathcal{O}$ of B that is also a unital subring and for which the canonical map $\mathcal{O} \otimes_{\mathbb{Z}_F} F \to B$ is an isomorphism. Let $\mathcal{O}$ be an order of B. By Proposition I.4.2 in [29], $\mathcal{O}$ is contained in a maximal order $\mathcal{O}(1)$. The quotient $\mathcal{O}(1)/\mathcal{O}$ is a finite $\mathbb{Z}_F$-module, hence there exist $\mathbb{Z}_F$-ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ such that

$$\mathcal{O}(1)/\mathcal{O} \cong \prod_{k=1}^{n} \mathbb{Z}_F/\mathfrak{a}_k. \tag{1.1}$$

We define the *level* of of $\mathcal{O}$ to be the $\mathbb{Z}_F$-ideal $\prod_k \mathfrak{a}_k$. It does not depend on the choice of $\mathcal{O}(1)$. Indeed, we could also have defined it in terms of the discriminant of $\mathcal{O}$, as in Section III.5.A of [29].

An *Eichler order* of B is the intersection of two maximal orders. We need the following proposition.

Proposition 1.1. *Let $\mathcal{O}$ be an order of a quaternion algebra B, contained in a maximal order $\mathcal{O}(1)$ of B. Suppose that the level of $\mathcal{O}$ is a prime $\mathfrak{p}$.*

(i) ([9]) *If $\mathfrak{p} \nmid \mathfrak{D}(B)^f$, then $\mathcal{O}$ is an Eichler order.*
(ii) ([13]) *If $\mathfrak{p} \mid \mathfrak{D}(B)^f$, then $\mathcal{O}$ is the unique level $\mathfrak{p}$ suborder of $\mathcal{O}(1)$, and $\mathcal{O}(1)$ is the unique maximal order containing $\mathcal{O}$.*

From now on, we assume that B satisfies

$$B \otimes_\mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R}) \times \mathbb{H} \times \ldots \times \mathbb{H}, \tag{1.2}$$

where $\mathbb{H}$ is the Hamilton quaternion algebra over $\mathbb{R}$, and we choose a corresponding embedding $\iota : B \hookrightarrow M_2(\mathbb{R})$. Using the embedding ι, we view $B^\times$ and $\mathcal{O}^\times$ as subgroups of $\mathrm{GL}_2(\mathbb{R})$. We let $B^+ = B^\times \cap \mathrm{GL}_2(\mathbb{R})^+$ and $B^1 = B^\times \cap \mathrm{SL}_2(\mathbb{R})$, and we define $\mathcal{O}^+$ and $\mathcal{O}^1$ analogously. In this way, we obtain subgroups $\mathrm{P}\mathcal{O}^+$ and $\mathrm{P}\mathcal{O}^1$ of $\mathrm{PGL}_2(\mathbb{R})^+$ and the Riemann surfaces

$$X^\pm(\mathcal{O}^+) = \mathrm{P}\mathcal{O}^+\backslash\mathcal{H}^\pm$$

and

$$X^\pm(\mathcal{O}^1) = \mathrm{P}\mathcal{O}^1\backslash\mathcal{H}^\pm.$$

Given an order $\mathcal{O}$, we can consider its normalizer

$$N(\mathcal{O}) = N_{B^+}(\mathcal{O}) = \left\{ b \in B^+ : b\mathcal{O} = \mathcal{O}b \right\}.$$

Now suppose that $\mathcal{O} = \mathcal{O}(\mathfrak{N})$ is a level $\mathfrak{N}$ Eichler order. Let $\mathfrak{a}$ be an ideal dividing $\mathfrak{D}(B)^f \mathfrak{N}$ that is trivial in the narrow class group of F. Then there exists an element $n(\mathfrak{a})$ of B^+ whose reduced norm generates $\mathfrak{a}$. As in [29], Section IV.3.B, one shows that every element of the group $N_{B^+}(\mathcal{O}(\mathfrak{N}))/\mathcal{O}(\mathfrak{N})^+$ is represented by some $n(\mathfrak{a})$. Through conjugation, $n(\mathfrak{a})$ induces a non-trivial *Atkin–Lehner involution* on $X^\pm(\mathcal{O}(\mathfrak{N})^+)$ and $X^\pm(\mathcal{O}(\mathfrak{N})^1)$.

The following lemma relates the normalizer of $\mathcal{O}$ to that of $\mathcal{O}^1$.

Lemma 1.2. *Let $\mathcal{O}$ be an order of B such that*

$$\mathcal{O} = \mathbb{Z}_F[\mathcal{O}^1]. \tag{1.3}$$

Then

$$N_{\mathrm{PGL}_2(\mathbb{R})^+}(\mathrm{P}\mathcal{O}^1) = N_{\mathrm{PGL}_2(\mathbb{R})^+}(\mathrm{P}\mathcal{O}^+) = \mathrm{P}N_{B^+}(\mathcal{O}).$$

Proof. The inclusion $\mathrm{P}N_{B^+}(\mathcal{O}) \subseteq N_{\mathrm{PGL}_2(\mathbb{R})^+}(\mathrm{P}\mathcal{O}^1)$ is trivial. Conversely, the proof of Théorème IV.3.5 of [29] shows that any element x of $N_{\mathrm{PGL}_2(\mathbb{R})^+}(\mathrm{P}\mathcal{O}^1)$ also normalizes $B^\times \subset \mathrm{GL}_2(\mathbb{R})$. The Skolem–Noether theorem ([29], Théorème I.2.1) allows us to conclude that x lifts to an element $\widetilde{x}$ of $B^\times$. The hypothesis (1.3) then yields that $\widetilde{x}$ normalizes the order $\mathcal{O}$. □

Remark 1.3. In our calculations (e.g. the first case e2d5D4iii below), the hypothesis (1.3) is usually satisfied. Let us mention here, though, that it does not always hold; a counterexample is given by the case e2d1D6ii in [23].

The arithmeticity condition (b2) from the introduction can be phrased as follows. Let $\mathcal{O}(1)$ be a maximal order of a quaternion algebra B satisfying (1.2). Then $\mathrm{P}\mathcal{O}(1)^+ \subset \mathrm{PGL}_2(\mathbb{R})^+$ is a Fuchsian group. Arithmetic Fuchsian groups are those subgroups of $\mathrm{PGL}_2(\mathbb{R})^+$ that are commensurable with such a group $\mathrm{P}\mathcal{O}(1)^+$ for some choice of F, B, $\mathcal{O}(1)$, and ι. Here we recall that two subgroups G and H of $\mathrm{PGL}_2(\mathbb{R})^+$ are called *commensurable* if both indices $[G : G \cap H]$ and $[H : G \cap H]$ are finite; moreover, we use the notation

$$\mathrm{Comm}(G) = \{g \in \mathrm{PGL}_2^+(\mathbb{R}) : gGg^{-1} \text{ commensurable with } G\}$$

for the *commensurator* of G in $\mathrm{PGL}_2^+(\mathbb{R})$. Note that $\mathrm{Comm}(G)$ only depends on the commensurability class of G.

The complete list of arithmetic $(1; e)$-groups up to $\mathrm{PGL}_2(\mathbb{R})$-conjugacy, along with explicit elements of $\mathrm{PGL}_2(\mathbb{R})^+$ generating these groups, was calculated by Takeuchi in [27].

2. Covers

We now give a brief exposition of the theory of covers. For a general exposition, we refer to [14]; another reference, more tailored to our situation, is [24].

Throughout this paper, by a *cover* of a connected curve X over $\mathbb{C}$ we mean a branched cover, i.e., a pair (Y, f) consisting of a curve Y and a finite surjective morphism $f : Y \to X$. A *Belyĭ map* is an étale cover of the curve $\mathbb{P}^1_* = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. A Belyĭ map extends to a non-singular cover $\overline{f}$ of $\mathbb{P}^1_{\mathbb{C}}$ in a unique way. In what follows, we identify a Belyĭ map f with this extension $\overline{f}$. We call the triple of tuples giving the ramification indices of $\overline{f}$ above 0, 1, and ∞ the *(ramification) type* of the Belyĭ map. Setting $n = \deg(f)$, we identify a ramification type with the corresponding triple of conjugacy classes in the symmetric group S_n. Moreover, when writing a signature or a ramification type, we will abbreviate a k-fold repitition of a natural number i by i^k. For example, the Belyĭ map $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ given by $z \mapsto z^2$ has ramification type $((2), (1^2), (2))$.

Given a connected curve X, let $\pi_1(X)$ be the topological fundamental group of X. Then the category of étale covers of X is naturally equivalent to the category of finite left $\pi_1(X)$-sets (see [14], Theorem 1.14). Choosing sufficiently small loops γ_0, γ_1, γ_∞ around 0, 1, and ∞, the group $\pi_1(\mathbb{P}^1_*)$ allows the presentation

$$\langle \gamma_0, \gamma_1, \gamma_\infty \mid \gamma_0\gamma_1\gamma_\infty = 1 \rangle$$

by the Seifert–Van Kampen Theorem. In particular, giving a Belyĭ map (Y, f) is the same as giving a simultaneous S_n-conjugacy class of triples of permutations $(\sigma_0, \sigma_1, \sigma_\infty) \in S_n^3$ with $\sigma_0\sigma_1\sigma_\infty = 1$ for some natural number $n \geq 1$.

The *monodromy group* $\mathrm{Mon}(Y, f)$ of an étale cover (Y, f) is the Galois group $\mathrm{Aut}(Z, g)$ of the minimal Galois étale cover $(Z, g)$of X that factorizes through (Y, f). For Belyĭ maps, this group, along with other invariants of (Y, f), can be calculated in terms of the associated triple of permutations $(\sigma_0, \sigma_1, \sigma_\infty) \in S_n$ as follows, by virtue of the aforementioned Galois equivalence of categories in Theorem 1.14 of [14].

Proposition 2.1. *Let $(\sigma_0, \sigma_1, \sigma_\infty) \in S_n^3$, let M be the subgroup of S_n generated by σ_0, σ_1, and σ_∞, and let (Y, f) be the associated Belyĭ map. Then the following statements hold.*

(i) *The degree of f equals n.*

(ii) *The set of connected components of Y is in bijection with the set of orbits of $\{1, \ldots, n\}$ under the action of M.*

(iii) *The ramification type of (Y, f) is the triple of cycle types of σ_0, σ_1, and σ_∞.*

(iv) *The monodromy group of (Y, f) is isomorphic to M.*

(v) *There is an isomorphism*

$$\mathrm{Aut}(Y, f) \cong \bigcap_{P \in \{0,1,\infty\}} \mathrm{Cent}_{S_n}(\sigma_P).$$

Proof (sketch). We briefly discuss the proof, abbreviating $G = \pi_1(\mathbb{P}^1_*)$.

(i)-(iii): Clear from the fact that the G-set associated to an étale cover (Y, f) of $\mathbb{P}^1_*$ is the fiber $f^{-1}(x)$ for some point $x \in \mathbb{P}^1_*$, the permutations σ_0, σ_1 and σ_∞ describing the action of γ_0, γ_1 and γ_∞ by path-lifting.

(iv): We can restrict to the case where the cover (Y, f) is connected. It then corresponds to a G-set of the form G/H. The minimal Galois cover (Z, g) factoring through (Y, f) then corresponds to the G-set G/N, where N is the largest normal subgroup of G contained in H. That is,

$$N = \bigcap_{g \in G} gHg^{-1} = \ker(G \longrightarrow \mathrm{Sym}(G/H)),$$

whence $\mathrm{Gal}(Z, g) \cong G/N \cong M$.

(v): An automorphism of the G-set $\{1, \ldots, n\}$ corresponding to σ_0, σ_1, σ_∞ is given by a bijection $\{1 \ldots n\} \to \{1, \ldots, n\}$ commuting with the action of G. The claim then follows because γ_0, γ_1 and γ_∞ generate G. $\square$

In what follows, we will be concerned with the covers $X^{\pm}(H) \to X^{\pm}(G)$ arising from inclusions of Fuchsian groups $H \subset G$. Recall (cf. Theorem 10.4.3 of [3]) that a subgroup G of $\mathrm{PGL}_2(\mathbb{R})^+$ with signature $(g; e_1, \ldots, e_n)$ has *(arithmetic) covolume* $\mathrm{Covol}(G)$ equal to

$$\mathrm{Covol}(G) = 2g - 2 + \sum_{i=1}^{n} \left(1 - \frac{1}{e_i}\right).$$

Given a Fuchsian group G, we can consider the covers *subordinate* to G. These are the covers of $X^{\pm}(G)$ whose restriction to the complements of the elliptic points of $X^{\pm}(G)$ is étale and with the additional property that for any elliptic point p of $X^{\pm}(G)$, the ramification indices at the points above p all divide the index of p. The covers subordinate to G again form a category, which we denote by $\mathfrak{SCov}_{X^{\pm}(G)}$. The category of all covers of $X^{\pm}(G)$ is denoted by $\mathfrak{Cov}_{X^{\pm}(G)}$.

Let S be the set of elliptic points of $X^{\pm}(G)$. For $p \in S$, let e_p be the index of p and let γ_p be a sufficiently small counterclockwise loop around p. Then as abstract groups we have

$$G = \pi_1(X^{\pm}(G) - S)/N, \tag{2.1}$$

where N is the smallest normal subgroup of G containing the elements $\gamma_p^{e_p}$ for all $p \in S$.

It is clear that any cover $X^{\pm}(H) \to X^{\pm}(G)$ arising from an inclusion $H \subset G$ is a subordinate cover of $X^{\pm}(G)$. Moreover, we have the following.

Theorem 2.2. *Let G be a Fuchsian group. Through the isomorphism* (2.1)*, the equivalence of Galois categories*

$$\mathfrak{Cov}_{X^{\pm}(G)} \longrightarrow \pi_1(X^{\pm}(G) - S)\text{-}\mathfrak{Sets}$$

induces an equivalence

$$\mathfrak{SCov}_{X^{\pm}(G)} \longrightarrow G\text{-}\mathfrak{Sets}.$$

In particular, when G is a triangle group of signature $(0; p, q, r)$, there are equivalences between the categories of

(i) *connected subordinate covers of $X^{\pm}(G)$,*
(ii) *conjugacy classes of subgroups of G of finite index, and*
(iii) *simultaneous S_n-conjugacy classes of transitive triples $(\sigma_0, \sigma_1, \sigma_\infty)$ satisfying*

$$\sigma_0\sigma_1\sigma_\infty = 1,\ \sigma_0^p = \sigma_1^q = \sigma_\infty^r = 1.$$

Given a subgroup $H \subset G$ of finite index, the corresponding cover is given by the canonical map $X^{\pm}(H) \to X^{\pm}(G)$. Its degree equals

$$[G : H] = \mathrm{Covol}(H)/\mathrm{Covol}(G).$$

Proof. The equivalence of categories follows from Section 6.4 in [18]. The statement relating degrees to covolumes is Lemme IV.1.3 in [29]. □

The following algorithm determines the G-set G/H corresponding to an inclusion of groups $H \subset G$.

Algorithm 2.3. *Let H be a finite index subgroup of a group G, and let S be a set of generators for G. Given a procedure to test whether an element $g \in G$ is in H, the following algorithm determines a set C of right coset representatives of G/H and calculates the induced G-set structure of C.*

1. *Initialize: set* $T = \{1\}$, $C = \emptyset$.
2. *Choose a* $t \in T$ *and remove it from* T. *If there exists no* $c \in C$ *such that* $c^{-1}t \in H$, *then add* t *to* C *and adjoin the set* $St = \{st : s \in S\}$ *to* T.
3. *If* T *is empty, go to step 4. Otherwise, go to step 2.*
4. *For every* s *in* S, *construct the following permutation* σ_s *of* C. *Given* $c \in C$, *find a* $c' \in C$ *such that* $c'^{-1}sc \in H$. *Set*

$$\sigma_s(c) = c'.$$

5. *Return the* G*-set structure on* C *given by* $s \mapsto \sigma_s$.

Proof of correctness. Let us first prove that C is indeed a set of representatives for G/H. On the one hand, we cannot have $cH = c'H$ for two distinct elements $c, c' \in C$. For suppose that of these two elements, c was the first to be appended to C. Then we would have had $c^{-1}c' \in H$ when we appended c' to C, which does not square with step 2.

On the other hand, because we kept adjoining the sets St in step 2, we know that the elements $SC = \{sc : s \in S, c \in C\}$ are all right H-equivalent to elements of C. Since S is a set of generators of G, in fact, all elements of $GC = G$ are right H-equivalent to some element of C.

Likewise, the fact that the algorithm terminates follows because the index $[G : H]$ is finite and S is a set of generators of G: a given right coset in G/H is represented by a finite word in S, hence after some finite time, the algorithm will find a representative for it.

The correctness of step 4 and step 5 is also clear: indeed, asking that $c'^{-1}sc$ be an element of H is equivalent to demanding $scH = c'H$. □

We implemented this algorithm at [22] in the following case:

- G is a subgroup of $\mathrm{PGL}_2(\mathbb{R})^+$ of the form $\mathrm{P}\mathcal{O}(1)^1$ for some maximal order $\mathcal{O}(1)$ in a quaternion algebra B satisfying (1.2); and
- $H \subset G$ is of the form $\mathrm{P}\mathcal{O}^1$ for some quaternion order $\mathcal{O}$ contained in $\mathcal{O}(1)$.

Representatives in $\mathcal{O}(1)^1$ for generators of G can then be found using the `Magma` function `Group`. It is straightforward to check whether a given element of $\mathrm{P}\mathcal{O}(1)^1$ represented by an element b of $\mathcal{O}(1)^1$ is in fact in $\mathrm{P}\mathcal{O}^1$, since this boils down to testing whether b is in $\mathcal{O}$. A substantial advantage of this approach is that it is exact, since all operations can be performed in the $\mathbb{Q}$-vector space B.

Remark 2.4. In some cases, it is possible to give a simple description of the monodromy group. We briefly mention the following examples.

- An inclusion $\mathcal{O}(\mathfrak{N}) \subset \mathcal{O}(1)$ of a level $\mathfrak{N}$ Eichler order into a maximal order gives rise to covers $X^{\pm}(\mathcal{O}(\mathfrak{N})^+) \to X^{\pm}(\mathcal{O}(1)^+)$ of degree $|\mathbb{P}^1(\mathbb{Z}_F/\mathfrak{N})|$. The monodromy group of these covers is isomorphic to $\mathrm{PSL}_2(\mathbb{Z}_F/\mathfrak{N})$. The corresponding minimal Galois covers are $X^{\pm}(\mathcal{O}^+) \to X^{\pm}(\mathcal{O}(1)^+)$, where $\mathcal{O}$ is the order $\mathbb{Z}_F + \mathfrak{N}\mathcal{O}(1)$.
- Let $\mathfrak{p}$ be prime dividing $\mathfrak{D}(B)^f$, and let $\mathcal{O}(\mathfrak{p})$ be the unique level $\mathfrak{p}$ suborder of a maximal order $\mathcal{O}(1)$ (cf. Proposition 1.1(ii)). Then the projections $X^{\pm}(\mathcal{O}(\mathfrak{p})^+) \to X^{\pm}(\mathcal{O}(1)^+)$ are cyclic Galois covers of degree $|\mathbb{P}^1(\mathbb{Z}_F/\mathfrak{p})|$.

In our calculations, however, we quite often encountered non-Eichler orders of large level, in which case the analysis becomes more subtle, even when the narrow class group of F is trivial (so that $\mathrm{P}\mathcal{O}^+ = \mathrm{P}\mathcal{O}^1$ and therefore $X^{\pm}(\mathcal{O}^+) \cong X^{\pm}(\mathcal{O}^1)$). In this case, we have calculated the corresponding monodromy groups using Algorithm 2.3.

Returning to general covers, Theorem 2.2 suggests the following naive algorithm to determine the Belyĭ maps of fixed ramification type.

Algorithm 2.5. *Let $R = (C_0, C_1, C_\infty)$ be a triple of conjugacy classes of S_n. The following algorithm determines a set S of representatives for the simultaneous S_n-conjugacy classes of triples $(\sigma_0, \sigma_1, \sigma_\infty)$ that satisfy $\sigma_0\sigma_1\sigma_\infty = 1$ and whose ramification type equals R.*

1. *Set $S = \emptyset$ and $T = \emptyset$. Choose a $\sigma_0 \in C_0$.*
2. *Run through the pairs $(\sigma_1, \sigma_\infty) \in C_1 \times C_\infty$. If $\sigma_0\sigma_1\sigma_\infty = 1$, then add $(\sigma_0, \sigma_1, \sigma_\infty)$ to T.*
3. *Run through the elements t of T. Set $C = \mathrm{Cent}_{S_n}(\sigma_0)$. If $\{ctc^{-1} : c \in C\}$ has empty intersection with S, then add t to S.*

Proof of correctness. We justify step 1 and step 3. As for step 1, we fix $\sigma_0 \in C_0$ to decrease the run time; since we are only interested in the triples $(\sigma_0, \sigma_1, \sigma_\infty)$ up to simultaneous S_n-conjugation, this is unproblematic.

Step 3 sifts out solutions in S that are simultaneously conjugate; indeed, if such a simultaneous conjugation between two elements of T occurs, then it is induced by an element of $\mathrm{Cent}_{S_n}(\sigma_0)$, since all elements of S have first coordinate σ_0 by construction. □

Remark 2.6. (i) Given a triple $(\sigma_0, \sigma_1, \sigma_\infty)$ of elements of S_n satisfying $\sigma_0\sigma_1\sigma_\infty = 1$, the invariants in Proposition 2.1 are easy to calculate. Our explicit implementation at [22] returns these invariants as well.

(ii) In this paper, we typically use Algorithm 2.5 to prove non-existence or uniqueness of a Belyĭ map of given ramification type. We also use it to distinguish between Belyĭ maps of identical type by calculating their invariants as in (i).

We conclude this section with the following proposition, which will be of great use in descending genus 1 covers to genus 0 covers.

Proposition 2.7. *Let*

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array} \tag{2.2}$$

be a cartesian diagram of non-singular covers. Suppose $\deg(X_1|X_0) = \deg(Y_1|Y_0) = m$*. Then for some divisor d of m we have*

$$|\mathrm{Mon}(Y_0|X_0)| = d|\mathrm{Mon}(Y_1|X_1)|.$$

Proof. To prove the proposition, we use the correspondence between non-singular complex projective curves X and their function fields $\mathbb{C}(X)$. The condition that (2.2) be cartesian translates into $\mathbb{C}(Y_1) = \mathbb{C}(X_1)\mathbb{C}(Y_0)$. Furthermore, the condition on the equality of degrees gives that the extensions $\mathbb{C}(Y_0)|\mathbb{C}(X_0)$ and $\mathbb{C}(X_1)|\mathbb{C}(X_0)$ are linearly disjoint.

The Galois group of the Galois closure L_i of the extension $\mathbb{C}(Y_i)|\mathbb{C}(X_i)$ is isomorphic to the monodromy group of the cover $Y_i \to X_i$. Because of the linear disjointness proved above, we have $L_1 = L_0\mathbb{C}(X_1)$. The degree d of the extension $L_0\mathbb{C}(X_1)|L_0$ divides the degree m of the extension $\mathbb{C}(X_1)|\mathbb{C}(X_0)$, and we have

$$\begin{aligned} d|\mathrm{Mon}(Y_0|X_0)| &= [L_0\mathbb{C}(X_1) : L_0][L_0 : \mathbb{C}(X_0)] = [L_0\mathbb{C}(X_1) : \mathbb{C}(X_0)] \\ &= [L_1 : \mathbb{C}(X_0)] = [L_1 : \mathbb{C}(X_1)][\mathbb{C}(X_1) : \mathbb{C}(X_0)] = |\mathrm{Mon}(Y_1|X_1)|m. \end{aligned}$$

Since m/d is also a divisor of m, the proposition is proved. $\square$

Remark 2.8. An alternative proof uses the fact that d equals the index $[S_0 : S_1]$, where the S_i are the images of the fundamental groups $\pi_1(X_i, x_i)$ in the symmetric groups on the fiber of Y_1 over x_0. Here x_0 is a point of X_0 that is not in the branch locus of the cover $Y_1 \to X_0$, and $x_1 \in X_1$ is a point above x_0.

3. $(1;e)$-curves from covers

In this section, we prove the following result by using the classification of arithmetic triangle groups in [26].

Theorem 3.1. *Let Γ be a $(1;e)$-group, and let $J^{\pm}(\Gamma) = \mathrm{Jac}(X^{\pm}(\Gamma))$. If Γ is commensurable with a triangle group, then either Γ is arithmetic or $j(J^{\pm}(\Gamma)) \in \{0, 1728\}$.*

Proof. We first reduce to the case where Γ is in fact included in a triangle group Δ. This follows from [16, Chapter IX, Theorem 1.16]: if Γ is not arithmetic, then its commensurator $\mathrm{Comm}(\Gamma)$ is again a Fuchsian group. But under our hypotheses, $\mathrm{Comm}(\Gamma)$ contains a triangle group, hence it is itself a triangle group. This implies our claim since clearly $\Gamma \subset \mathrm{Comm}(\Gamma)$.

It remains to classify the $(1;e)$-groups Γ that are contained in a triangle group Δ. Though [24] could be used, we give a more elementary proof. The inclusion $\Gamma \subset \Delta$ induces a cover

$$X^{\pm}(\Gamma) \xrightarrow{f} X^{\pm}(\Delta). \tag{3.1}$$

Set $n = \deg(f)$. Then we have $n < 42$. Indeed, it is well known that the arithmetic covolume of a triangle group is at least equal to $1/42$, this minimum being attained by the triangle group with signature $(0; 2, 3, 7)$. On the other hand, a $(1;e)$-group has arithmetic covolume $1 - (1/e) < 1$.

Let $(0; p, q, r)$ be the signature of Δ, and let $x, y, z \in X^{\pm}(\Delta)$ be the corresponding elliptic points of $X^{\pm}(\Delta) \cong \mathbb{P}^1_{\mathbb{C}}$. We may suppose that the elliptic point of $X^{\pm}(\Gamma)$ is over z. Then we are in one of the following two situations:

(i) The fiber over z consists of more than one point;
(ii) The fiber over z consists of a single point;

First suppose that we are in case (i). Then the fiber over z contains a non-elliptic point. As such a point has ramification index r, we have $r < n$. The n/p (respectively n/q) points in the fiber over x (respectively y) all have ramification index p (respectively q). Suppose that the elliptic point of $X^{\pm}(\Gamma)$ has ramification index s. Then the remaining $(n - s)/r$ points over z all have ramification index r. The Riemann-Hurwitz formula yields the diophantine equation

$$n\left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right) = 1 - \frac{s}{r},$$

with $n, p, q, r, s \in \mathbb{N}$ satisfying

$$p|n,\ q|n,\ r|n-s,\ s|r,\ (1-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}) > 0.$$

The solutions of this equation are easily enumerated, and one checks that all possibilities for the signature $(0;p,q,r)$ are arithmetic by invoking [26].

Now suppose that (ii) holds. Then we have $e = nr$. Moreover, n equals the ramification index of the single (elliptic) point in the fiber of z. Proceeding as above, we get the diophantine equation

$$n(1-\frac{1}{p}-\frac{1}{q}) = 1.$$

There are three corresponding ramification types: $((3),(3),(3))$, $((2^2),(4),(4))$, and $((2^3),(3^2),(6))$. By [5], we have $j(J^{\pm}(\Gamma)) \in \{0, 1728\}$ in all cases. $\square$

Remark 3.2. Case (ii) of the proof above gives rise to families of $(1;e)$-covers of curves with signature $(0;3,3,3e)$, $(0;2,4,4e)$, and $(0;2,3,6e)$. Choosing a Weierstrass equation (0.3) for $X^{\pm}(\Gamma)$ with $g_2 = 0$ (respectively $g_3 = 0$) if $j(J^{\pm}(\Gamma)) = 0$ (respectively $j(J^{\pm}(\Gamma)) = 1728$), one shows that $A = 0$ in (0.4) by applying (7.1) to the non-trivial automorphisms of $X^{\pm}(\Gamma)$).

4. Covers from arithmetic $(1;e)$-curves

Let Γ be an arithmetic Fuchsian group. Then Section 3 of [27] shows that Γ lifts to a subgroup $\widetilde{\Gamma}$ of $\mathrm{SL}_2(\mathbb{R})$ with presentation

$$\widetilde{\Gamma} = \left\langle \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} \,\middle|\, \widetilde{\gamma} = -\widetilde{\alpha}^{-1}\widetilde{\beta}^{-1}\widetilde{\alpha}\widetilde{\beta}, \widetilde{\gamma}^e = -1 \right\rangle.$$

Let

$$\Gamma^{(2)} = \langle \gamma^2 : \gamma \in \Gamma \rangle$$

and define $\widetilde{\Gamma}^{(2)}$ similarly. By [25], if we let $F = \mathbb{Q}(\mathrm{tr}(\widetilde{\Gamma}^{(2)}))$, then F is a totally real number field and the group $\widetilde{\Gamma}^{(2)}$ generates a quaternion algebra $B = F(\widetilde{\Gamma}^{(2)})$ over F. The algebra B satisfies (1.2), and the ring $\mathbb{Z}_F[\widetilde{\Gamma}^{(2)}]$ is an order of B. Moreover (cf. [26], Proposition 1), two Fuchsian groups Γ and Γ' are commensurable up to conjugacy if and only if the quaternion algebras thus associated to them are isomorphic over $\mathbb{Q}$. In what follows, we slightly abuse notation by denoting $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\widetilde{\Gamma}^{(2)}]$.

The quaternion algebras associated to arithmetic triangle groups (respectively $(1;e)$-groups) can be found in [26] (respectively [27]). As such, it is straightforward to determine those arithmetic $(1;e)$-groups in Theorem 4.1 of [27] that are commensurable with triangle groups.

The order $\mathcal{O} = \mathbb{Z}_F[\Gamma^{(2)}]$ is contained in a maximal order $\mathcal{O}(1)$ of B, and clearly $\Gamma^{(2)} \subset \mathcal{O}^1$. Inspecting Theorem 4.1 in [27], it turns out that if Γ is both an arithmetic triangle group and commensurable with a triangle group, then the adjoint group $\mathrm{P}N(\mathcal{O}(1))$ of the normalizer of $\mathcal{O}(1)$ is a triangle group. One can find the signature of $X^{\pm}(N(\mathcal{O}(1)))$ in Table (1) of [26].

Remark 4.1. Note that a triangle group Δ such as $\mathrm{P}N(\mathcal{O}(1))$ is determined up to $\mathrm{PGL}_2(\mathbb{R})^+$-conjugacy by its signature. In particular, there is an isomorphism

$$X^+(\Delta) \cong X^-(\Delta), \tag{4.1}$$

whereas for general Fuchsian Γ, there will only exist an antiholomorphic bijection $X^+(\Gamma) \cong X^-(\Gamma)$. This is our reason for considering the curves $X^-(\Gamma)$ as well as the curves $X^+(\Gamma)$: since Theorem 4.1 of [27] only classifies arithmetic $(1;e)$-groups up to $\mathrm{PGL}_2(\mathbb{R})$-conjugacy, there will be instances (such as the case e4d8D2ii below) where $X^-(\Gamma)$ is not isomorphic to $X^+(\Gamma)$. However, note that

$$X^-(\Gamma) = X^+(W\Gamma W^{-1}),$$

where

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{R}) - \mathrm{PGL}_2(\mathbb{R})^+.$$

By Proposition 3.1 of [27], we have an isomorphism $X^\pm(\Gamma^{(2)}) \cong X^\pm(\Gamma)$, and the inclusion $\Gamma^{(2)} \subset \Gamma$ induces the multiplication-by-2 map on the Jacobian of $X^\pm(\Gamma)$. To sum up, we have a diagram of covers

$$X^\pm(\Gamma) \xleftarrow{\text{“[2]”}} X^\pm(\Gamma^{(2)}) \longrightarrow X^\pm(\mathcal{O}^1) \longrightarrow X(N(\mathcal{O}(1))) \tag{4.2}$$

where $X(N(\mathcal{O}(1)))$ is a genus 0 curve with three elliptic points. Here and in what follows, we have abbreviated $X(\Gamma) = X^+(\Gamma)$ for a Fuchsian group Γ.

Remark 4.2. Note that $\Gamma^{(2)}$ need not equal $\mathcal{O}^1$. To calculate the signature of $\mathcal{O}^1$, one calculates the ramification type of the cover

$$X^\pm(\mathcal{O}^1) \longrightarrow X^\pm(\mathcal{O}(1)) \tag{4.3}$$

as in the discussion after Algorithm 2.3. Usually, the curve $X^\pm(\mathcal{O}^1)$ is of genus 1. In this case there are inclusions

$$\Gamma^{(2)} \subseteq \mathrm{P}\mathcal{O}^1 \subseteq \Gamma.$$

The group $\mathrm{P}\mathcal{O}^1$ is therefore given by $\Gamma^{(2)}$, $\langle\Gamma^{(2)}, \alpha\rangle$, $\langle\Gamma^{(2)}, \beta\rangle$, $\langle\Gamma^{(2)}, \alpha\beta\rangle$, or Γ. These five cases can be deduced by checking which of the elements α, β, and $\alpha\beta$ are in $\mathrm{P}\mathcal{O}^1$.

Considering (4.1) and (4.2), we obtain subordinate covers

$$X^\pm(\Gamma^{(2)}) \longrightarrow X(N(\mathcal{O}(1))). \tag{4.4}$$

Since $N(\mathcal{O}(1))$ is a triangle group, we can consider the maps (4.4) as Belyĭ maps, seeing as how the action of $\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ on triples of distinct points in $\mathbb{P}^1(\mathbb{C})$ is transitive.

The covers (4.4) need not factor through Γ. However, whenever possible, we have found a triangle group Δ containing Γ, realizing not merely $X^\pm(\Gamma^{(2)})$ but also $X^\pm(\Gamma)$ as a Belyĭ cover $X^\pm(\Gamma) \to X(\Delta)$. In the cases where we did not manage to find such an inclusion $\Gamma \subset \Delta$, we have proved that it cannot exist.

In the next section, we proceed to calculate some of these Belyĭ maps and the resulting geometric models of $X^\pm(\Gamma)$. We have not included the details of all calculations, since these were performed in a rather ad hoc manner. However, our most frequently applied techniques are the following.

(i) We will often descend a genus 1 Belyĭ map $Y_1 \to X_1 \cong \mathbb{P}^1_{\mathbb{C}}$ to a genus 0 Belyĭ map $Y_0 \to X_0 \cong \mathbb{P}^1_{\mathbb{C}}$ by constructing a diagram as in Proposition 2.7. The cover $Y_0 \to X_0$ is often easier to calculate, and it will always be possible to determine Y_1 given Y_0.

(ii) Arguing in the opposite direction, we can construct $(1;e)$-groups Γ from triangle groups Δ by using Theorem 2.2. This is especially useful if there is a *unique* $(1;e)$-group in the commensurability class of Δ. Knowing e and the signature of Δ, the degree of the corresponding covers can be determined using Theorem 2.2, which also puts a restriction on the possible ramification types. These covers can then be described using Algorithm 2.5.

(iii) Finally, let us mention that the *Atkin–Swinnerton-Dyer differentiation trick* (as described in [5]) is of great use in computing genus 0 Belyĭ maps.

Although calculating Belyĭ maps can be rather involved, it is conversely easy enough to verify that the Belyĭ maps given below indeed have the properties that we claim them to have.

Remark 4.3. In what follows, we often need to determine the signature of some Atkin–Lehner quotient of a Shimura curve. One can derive general formulas for these signatures using the methods of [31] and Section III.5.C of [29]. However, we have not included the details of these formulas here because it was possible in all cases to circumvent these calculations using geometric arguments (which could in principle be applied to a broader class of covers, for example, those not coming from arithmetic groups).

5. The calculations

Starting with an arithmetic $(1;e)$-group Γ, this section will calculate models over $\mathbb{C}$ for the curves $X^{\pm}(\Gamma)$. These model are determined up to isomorphism by the j-invariants $j(J^{\pm}(\Gamma))$ of the elliptic curves

$$J^{\pm}(\Gamma) = \mathrm{Jac}(X^{\pm}(\Gamma)).$$

To facilitate our discussions, we have assigned labels to the arithmetic $(1;e)$-groups Γ in Theorem 4.1 of [27]. Such labels are of the form

$$\mathrm{e}n_e\, \mathrm{d}n_d\, \mathrm{D}n_D\, \mathrm{r},$$

where

- n_e is the index of the unique elliptic point of Γ;
- n_d is the discriminant of the center $F = \mathbb{Q}(\mathrm{tr}(\widetilde{\Gamma}^{(2)}))$ of the quaternion algebra associated to Γ;
- n_D is the norm of the finite discriminant $\mathfrak{D}(B)^f$ of the quaternion algebra $B = F[\widetilde{\Gamma}^{(2)}]$ over F; and
- r is a roman numeral indicating the position at which Γ occurs in Theorem 4.1 of [27] among the Γ with the same n_e, n_d and n_D.

In what follows. we also denote a prime of F over a rational prime p by $\mathfrak{p}_p$, and as in [27], we let

$$w_d = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4; \\ \sqrt{d} & \text{otherwise.} \end{cases}$$

$\boxed{\textbf{e2d5D4:}}$ These first three cases are complicated and varied enough to deserve a rather detailed look, so as to illustrate the techniques involved in the calculations. The considerations for the other curves below will for reasons of space be somewhat more terse.

Let B be a quaternion algebra over $F = \mathbb{Q}(w_5)$ for which $\mathfrak{D}(B)^f = \mathfrak{p}_2 = (2)$. By [27], there are three $(1;2)$-groups whose associated quaternion algebra is isomorphic to B. Before going into detail for the individual cases, let us note the following.

Preamble. Since the narrow class group of F is trivial, every totally positive unit of $\mathbb{Z}_F$ is a square. Hence $\mathrm{P}\mathcal{O}^1 = \mathrm{P}\mathcal{O}^+$ for all orders $\mathcal{O}$ of B. By [26], given a maximal order $\mathcal{O}(1)$ of B, the signature of $X(N(\mathcal{O}(1)))$ equals $(0;2,4,5)$. Moreover, up to conjugacy, there is only one other maximal arithmetic triangle group Δ' in the commensurability class of Δ. This group Δ' has signature $(0;2,4,10)$.

In light of Theorem 2.2, one way to obtain arithmetic $(1;2)$-curves is to construct subordinate covers

$$X(\Gamma) \longrightarrow X(\Delta) = X(N(\mathcal{O}(1))). \tag{5.1}$$

Calculating covolumes as in Theorem 2.2 shows that (5.1) has degree equal to 10. As (5.1) is subordinate, its ramification type equals $((2^5),(4^2,2),(5^2))$.

By Algorithm 2.5, there are two subordinate Belyĭ maps (5.1). The monodromy groups of these covers Belyĭ are of order 120 and 160, respectively. Both have an automorphism group of order 2, and they factor through the unique covers of type $((2,1^3),(4,1),(5))$ and $((2^2,1),(2^2,1),(5))$, respectively. Since $X^+(\Delta)$ is isomorphic to $X^-(\Delta)$ by (4.1), we have $X^+(\Gamma) \cong X^-(\Gamma)$ for the resulting $(1;e)$-groups Γ, considering the fact that these covers are uniquely determined by their ramification indices and monodromy group.

Calculating covolumes also shows that there are no subordinate covers of $X(\Delta')$ by curves of signature $(1;2)$.

For reasons of exposition, we now tackle the subcases in reverse order.

$\boxed{\textbf{e2d5D4iii:}}$ We have

$$\langle \Gamma^{(2)}, \alpha \rangle = \mathrm{P}\mathcal{O}(\mathfrak{p}_3)^1 \tag{5.2}$$

where $\mathcal{O}(\mathfrak{p}_3)$ is the level $\mathfrak{p}_3 = (3)$ order $\mathbb{Z}_F[\Gamma^{(2)}]$. Indeed, let $\mathcal{O}(1)$ be a maximal order containing $\mathcal{O}(\mathfrak{p}_3)$. We obtain an inclusion $\mathcal{O}(\mathfrak{p}_3)^1 \subset \mathcal{O}(1)^1$. The order $\mathcal{O}(\mathfrak{p}_3)$ is Eichler by Proposition 1.1(i). By Remark 2.4(i) or Algorithm 2.3, we get that

$$[\mathrm{P}\mathcal{O}(1)^1 : \mathrm{P}\mathcal{O}(\mathfrak{p}_3)^1] = |\mathrm{P}(\mathbb{Z}_F/\mathfrak{p}_3)| = 10.$$

The group $\mathrm{P}\mathcal{O}(1)^1$ has signature $(0;2,5,5)$ by [26] or [31], hence its arithmetic covolume equals $1/10$. Therefore $\mathrm{P}\mathcal{O}(\mathfrak{p}_3)^1$ has covolume equal to 1. It contains $\Gamma^{(2)}$ by construction, and it is straightforward to check that it contains α as well. But $\langle \Gamma^{(2)}, \alpha \rangle$ has signature $(1;2,2)$ and covolume 1, therefore (5.2) holds.

Note that by the reasoning in the preamble we have

$$X^+(\mathcal{O}(\mathfrak{p}_3)^1) \cong X^-(\mathcal{O}(\mathfrak{p}_3)^1)$$

and

$$X^+(\mathcal{O}(1)^1) \cong X^-(\mathcal{O}(1)^1).$$

(For less ad hoc proofs of these isomorphisms, see [23].)

By Algorithm 2.3, the cover

$$X(\mathcal{O}(\mathfrak{p}_3)^1) \longrightarrow X(\mathcal{O}(1)^1)$$

has trivial automorphism group. Its monodromy group has cardinality $360 = |\mathrm{PSL}_2(\mathbb{Z}_F/\mathfrak{p}_3)|$ (cf. Remark 2.4(i)). The cover

$$X(\mathcal{O}(\mathfrak{p}_3)^1) = X(\langle \Gamma^{(2)}, \alpha \rangle) \longrightarrow X(\Gamma)$$

is a 2-isogeny by (4.2). By Lemma 1.2, the $(1;2)$-curve $X(\Gamma)$ is an Atkin–Lehner quotient of $X(\mathcal{O}(\mathfrak{p}_3)^1)$.

Claim. Γ is not contained in a triangle group.

Proof. Suppose that the claim does not hold. Considering the preamble, Γ would then be contained in $\mathrm{P}N(\mathcal{O}(1))$, which would give rise to the descent (5.3).

$$\begin{tikzcd} X(\mathcal{O}(\mathfrak{p}_3)^1) \arrow[r, "2"] \arrow[d, "10"] & X(\Gamma) \arrow[d, "10"] \\ X(\mathcal{O}(1)^1) \arrow[r, "2"] & X(N(\mathcal{O}(1))) \end{tikzcd} \tag{5.3}$$

Diagram (5.3) would be cartesian because Γ cannot be contained in $\mathrm{P}\mathcal{O}(1)^1$. Indeed, by the discussion preceding Lemma 1.2, there are index 2 inclusions

$$\mathrm{P}\mathcal{O}(\mathfrak{p}_3)^1 \subsetneq \Gamma \subsetneq \mathrm{P}N_{B^+}(\mathcal{O}(\mathfrak{p}_3)).$$

But all subgroups of $N_{B^+}(\mathcal{O}(\mathfrak{p}_3)) \subset B^+$ properly containing $\mathcal{O}(\mathfrak{p}_3)^1$ contain elements whose norm is not a square in F. Hence we cannot have $\Gamma \subset \mathrm{P}\mathcal{O}(1)^1$.

Applying Proposition 2.7, we see that the degree 10 cover

$$X(\Gamma) \longrightarrow X(N(\mathcal{O}(1)))$$

would have a monodromy group of order 360 or 720. But we have seen in the preamble that only cardinalities 120 and 160 are possible. Therefore Γ is not contained in a triangle group. $\square$

Dividing out the Atkin–Lehner involution $w(\mathfrak{p}_2)$ results in Diagram (5.4).

$$\begin{tikzcd} X(\mathcal{O}(\mathfrak{p}_3)^1) \arrow[r, "2"] \arrow[d, "10"] & X(\mathcal{O}(\mathfrak{p}_3)^1)/w(\mathfrak{p}_2) \arrow[d, "10"] \\ X(\mathcal{O}(1)^1) \arrow[r, "2"] & X(N(\mathcal{O}(1))) \end{tikzcd} \tag{5.4}$$

As mentioned in Remark 4.3, the signature of $X(\mathcal{O}(\mathfrak{p}_3)^1)/w(\mathfrak{p}_2)$ can be calculated directly. Alternatively, one excludes that $X(\mathcal{O}(\mathfrak{p}_3)^1)/w(\mathfrak{p}_2)$ has signature $(1;2)$ or $(0;2^5)$ by using Algorithm 2.5 and Proposition 2.7 as in the proof of the claim above, and then the only remaining possibility is $(0;2,2,4,4)$ as the covers in Diagram (5.4) are subordinate. Either way, we see that the signatures in Diagram (5.4) are given by Diagram (5.5).

$$\begin{tikzcd} (1;2,2) \arrow[r, "2"] \arrow[d, "10"] & (0;2,2,4,4) \arrow[d, "10"] \\ (0;2,5,5) \arrow[r, "2"] & (0;2,4,5) \end{tikzcd} \tag{5.5}$$

The Belyĭ map on the right side of (5.5) is of genus 0. The ramification type of this map either equals $((2^4,1^2),(4^2,1^2),(5^2))$ or $((2^5),(4,2^2,1^2),(5^2))$ by subordinateness. Algorithm 2.5 shows that the latter type gives rise to covers whose monodromy groups have order 160, so we can exclude this type by using Proposition 2.7 once more.

The former type has five Belyĭ maps associated to it, only one of which has a monodromy group of the correct order. There is also a more tangible feature distinguishing this Belyĭ map from its four compeers: it is the only one among these five whose automorphism group is trivial.

We place the elliptic points of $X(N(\mathcal{O}(1)))$ of index $2, 4, 5$ at $1, \infty, 0$, respectively. By solving the resulting equations numerically and recognizing the solutions as algebraic numbers, one finds the following cover:

$$z \longmapsto \frac{4(w-8)(z^2-45)^5}{3^5 5^5 (z-5)^4 (z^2+(6-2w)z+(15w-75))}, \tag{5.6}$$

where $w = w_{-15}$. The Belyĭ map (5.6) indeed has trivial automorphism group, which can be checked by verifying that no automorphism of $\mathbb{P}^1_{\mathbb{C}}$ exchanging the zeroes of $z^2+(6-2w)z+(15w-75)$ has the additional property that it fixes the set of zeroes of z^2-45 and the set $\{5,\infty\}$ as well. Note that every automorphism should fix these pairs of points by Proposition 2.1(v), since they are the only unramified points in their respective fibers.

Remark. The Belyĭ map (5.6) is isomorphic to its complex conjugate by the automorphism of $\mathbb{P}^1_{\mathbb{C}}$ associated to the real matrix

$$\begin{pmatrix} 5 & -45 \\ 1 & -5 \end{pmatrix}.$$

As in [7], §5, it can be shown that although (5.6) is defined over $\mathbb{Q}$ (in the sense that it can be obtained by the base extension of a rational function on a conic over $\mathbb{Q}$), there is no totally real number field K such that there exists a rational function $\mathbb{P}^1_K \to \mathbb{P}^1_K$ realizing (5.6).

In terms of equation (5.6), the elliptic points of $X(\mathcal{O}(\mathfrak{p}_3)^1)/w(\mathfrak{p}_2)$ are given by the unramified points above ∞ and 1. The former are given by the zeroes of $z^2+(6-2w)z+(15w-75)$; these have index 4. The latter, which have index 2, are the simple zeroes of the difference of the numerator and the denominator of (5.6), and are given by the zeroes of $z^2+(5w+5)z+(-135w-225)/4$.

By Diagram (5.5), $X(\Gamma)$ can be constructed by taking the degree 2 elliptic cover of $X(\mathcal{O}(\mathfrak{p}_3)^1)/w(\mathfrak{p}_2)$ branched in these 4 elliptic points to construct $X(\mathcal{O}(\mathfrak{p}_3)^1)$ and then identifying the elliptic points of index 2 on the resulting curve by a suitable 2-isogeny. Note that these elliptic points are simply the preimages of the zeroes of $z^2+(6-2w)z+(15w-75)$. We obtain

$$j(J^+(\mathcal{O}(\mathfrak{p}_3)^1)) = \frac{7949^3}{2^5 3^{10}} = j(J^-(\mathcal{O}(\mathfrak{p}_3)^1))$$

and

$$j(J^+(\Gamma)) = \frac{-269^3}{2^{10}3^5} = j(J^-(\Gamma)).$$

The preamble shows that in the remaining two subcases, the group Γ is contained in a triangle group and $X^+(\Gamma) \cong X^-(\Gamma)$. Hence we only consider the curves $X(\Gamma) = X^+(\Gamma)$.

$\boxed{\textbf{e2d5D4ii:}}$ Let $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma]$. Then $\mathcal{O}(1)$ is maximal, and a calculation of covolumes yields that $\mathrm{P}\mathcal{O}(1)^1$ contains Γ as a subgroup of index 5. The ramification type of the corresponding cover

$$X(\Gamma) \longrightarrow X(\mathcal{O}(1)^1) \tag{5.7}$$

equals $((2^2, 1), (5), (5))$ by subordinateness. By Algorithm 2.5, there is a unique Belyĭ map of this type. Its monodromy group has order 60.

Since Γ does not equal $\mathrm{P}\mathcal{O}(1)^1$, we cannot a priori be certain that the Atkin–Lehner involution of $X(\mathcal{O}(1)^1)$ lifts to $X(\Gamma)$. However, one can write down an explicit normalizing element that shows that it does.

Indeed, let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be as in Section 4. Consider the matrix $S \in \mathrm{SL}_2(\mathbb{R})$ given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

One verifies that no element of $\mathcal{O}(1)$ is a scalar multiple of S. Therefore S is not in $\mathcal{O}(1)^1$, and hence not in Γ either.

The matrix S normalizes Γ, as $\widetilde{\alpha}S = S\widetilde{\alpha}^{-1}$ and $\widetilde{\beta}S = S\widetilde{\beta}^{-1}$. Note that S is also in the normalizer of all intermediate groups $\Gamma^{(2)} \subset G \subset \Gamma$.

Clearly, S normalizes the order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma]$ as well. Hence by Lemma 1.2, it induces the unique non-trivial Atkin–Lehner involution of $X(\mathcal{O}(1)^1)$. Furthermore, S fixes the elliptic point of $X(\Gamma)$. We obtain the descent (5.8).

$$\begin{tikzcd} X(\Gamma) \arrow[r, "2"] \arrow[d, "5"] & X(\langle \Gamma, S \rangle) \arrow[d, "5"] \\ X(\mathcal{O}(1)^1) \arrow[r, "2"] & X(N(\mathcal{O}(1))) \end{tikzcd} \tag{5.8}$$

The signatures in Diagram (5.8) can be determined using [26] and the remarks above. They are given in Diagram (5.9).

$$\begin{tikzcd} (1; 2) \arrow[r, "2"] \arrow[d, "5"] & (0; 2, 2, 2, 4) \arrow[d, "5"] \\ (0; 2, 5, 5) \arrow[r, "2"] & (0; 2, 4, 5) \end{tikzcd} \tag{5.9}$$

A priori, the map on the right can have ramification type $((2^2, 1), (2^2, 1), (5))$ or $((2, 1^3), (4, 1), (5))$. By Algorithm 2.5, both of these types give rise to unique Belyĭ maps. These covers have monodromy group of order 10 and 120, respectively. Proposition 2.7 therefore shows that the correct type is in fact the latter. Placing the elliptic points of index $2, 4, 5$ at $1, \infty, 0$, respectively, the corresponding cover is given by

$$z \longmapsto \frac{z^5}{5z - 4}. \tag{5.10}$$

By Diagram (5.9), we can construct a geometric model of $X(\Gamma)$ by taking the degree 2 elliptic cover of $\mathbb{P}^1_{\mathbb{C}}$ branched in the unramified preimages of 1 and ∞ under (5.10). We obtain

$$j(J^+(\Gamma)) = \frac{5^1 211^3}{2^{15}} = j(J^-(\Gamma)).$$

Remark. There is an alternative way to derive this result that does not make use of the explicit matrix S. Consider the function fields

$$K = \mathbb{C}(X(N(\mathcal{O}(1)))),\ L = \mathbb{C}(X(\mathcal{O}(1)^1)),\ M = \mathbb{C}(X(\Gamma)).$$

Let N_K be the normal closure of the extension $M|K$, and let N_L be the normal closure of the extension $M|L$. Since N_K, being normal over K, is also normal over L, we have a chain of inclusions

$$K \subset L \subset M \subset N_L \subset N_K.$$

Hence

$$|\mathrm{Mon}(X(\Gamma) \longrightarrow X(N(\mathcal{O}(1))))| = [N_K : K]$$

is a multiple of

$$|\mathrm{Mon}(X(\Gamma) \longrightarrow X(\mathcal{O}(1)^1))| = [N_L : L].$$

We saw at the beginning of this case that the latter cardinality equals 60.

We conclude that of the two Belyĭ maps $X(\Gamma) \to X(N(\mathcal{O}(1)))$ constructed in the preamble, we are in fact considering the former. This Belyĭ map factorizes through an automorphism (which is of course that induced by S). It can be calculated as above.

$\boxed{\textbf{e2d5D4i:}}$ Note that the cover

$$X(\Gamma) \longrightarrow X(N(\mathcal{O}(1)))$$

is isomorphic to the cover in the preamble whose monodromy group has order 160. Indeed, by our previous results, no other entry in Theorem 4.1 of [27] qualifies. By a similar uniqueness argument, we see that $X(\Gamma)$ can be constructed by the following two-step process:

- First take the cover of $X(N(\mathcal{O}(1))$ of ramification type $((2^2,1),(2^2,1),(5))$ above the elliptic points of index $2,4,5$;
- Then take the degree 2 cover ramifying above the 4 elliptic points of the resulting curve. Note that these elliptic points are given by the elements in the fiber above the elliptic point of index 4 along with the unique unramified point in the fiber above the elliptic point of index 2.

The first degree 5 cover was calculated in [5]; placing the elliptic points of index $2,4,5$ at $-4,0,\infty$, respectively, it is given by

$$z \longmapsto (z-2)(z^2+z-1)^2.$$

One calculates that the unique unramified point above -4 has z-coordinate equal to -2. The second degree 2 cover therefore ramifies above the zeroes of the rational function $(z+2)(z-2)(z^2+z-1)$. We end up with a genus 1 curve whose Jacobian has j-invariant

$$j(J^+(\Gamma)) = 2^4 17^3 = j(J^-(\Gamma)).$$

$\boxed{\textbf{e2d8D2:}}$ This time $F = \mathbb{Q}(w_2)$ and $\mathfrak{D}(B)^f = \mathfrak{p}_2$. In [26], it is shown that there is a triangle group Δ with signature $(0;2,4,8)$ in the commensurability class associated to B. Taking the unique subordinate cover

$$X(\Gamma') \longrightarrow X(\Delta) \tag{5.11}$$

with ramification type $((2^2),(4),(4))$ amounts to constructing a $(1;2)$-curve by Theorem 2.2, which by uniqueness is isomorphic to the $(1;2)$-curves $X^+(\Gamma)$ and $X^-(\Gamma)$ whose associated quaternion algebra is isomorphic to B.

The Galois cover (5.11) is given by

$$(x,y) \longmapsto x^2$$

from the curve

$$y^2 = x^3 - x$$

if we set the x-coordinates of the elliptic points of index 2, 4, 8 equal to 1, 0, ∞, respectively. Hence

$$j(J^+(\Gamma)) = 1728 = j(J^-(\Gamma)).$$

e2d12D2: Let $\mathcal{O}(\mathfrak{p}_2) = \mathbb{Z}_F[\Gamma]$. Then $\Gamma = \mathcal{O}(\mathfrak{p}_2)^1$. Furthermore, the non-Eichler order $\mathcal{O}(\mathfrak{p}_2)$ is the unique level $\mathfrak{p}_2$ suborder of the unique maximal order $\mathcal{O}(1)$ containing $\mathcal{O}(\mathfrak{p}_2)$ (cf. Proposition 1.1(ii)). By [26] or [31], the curve $X(\mathcal{O}(1)^1)$ has signature $(0; 3, 3, 6)$, and we get degree 3 Galois covers

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1) \longrightarrow X(\mathcal{O}(1)^1). \tag{5.12}$$

The covers (5.12) have ramification type $((3), (3), (3))$. They are isomorphic, and are realized by the morphism

$$(x, y) \longmapsto (1 + y)/2$$

from the curve

$$y^2 = x^3 + 1$$

if we set the x-coordinates of the elliptic points of index $3, 3, 6$ equal to $0, 1, \infty$, respectively. Therefore

$$j(J^+(\Gamma)) = 0 = j(J^-(\Gamma)).$$

e2d12D3: As in the previous case, the narrow class group of F has order 2.

Claim. Γ is not contained in a triangle group.

Proof. Indeed, if it were, Γ would be contained in a triangle group with signature $(0; 2, 4, 12)$, because [26] shows that this is the signature of the unique maximal triangle group in the commensurability class of Γ. Calculating covolumes gives that the corresponding subordinate cover would have degree 3. This manifestly cannot give rise to a group of signature $(1; 2)$. $\square$

We use the curves $X^{\pm}(\Gamma^{(2)})$ instead. If we let $\mathcal{O} = \mathbb{Z}_F[\Gamma^{(2)}]$, then applying Algorithm 2.3, we see that $\mathrm{P}\mathcal{O}^1 = \Gamma^{(2)}$. Choosing a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$, the signature of $X(N(\mathcal{O}(1)))$ equals $(0; 2, 4, 12)$ by [26]. Note that

$$\mathrm{P}N(\mathcal{O}(1)) = \mathrm{P}\mathcal{O}(1)^+$$

as $\mathrm{P}\mathcal{O}(1)^+ \subseteq \mathrm{P}N(\mathcal{O}(1))$ and both groups contain $\mathrm{P}\mathcal{O}(1)^1$ as a subgroup of index 2.

Calculating the orders inbetween $\mathcal{O}$ and $\mathcal{O}(1)$ yields that the level $\mathfrak{p}_2^4$ order $\mathcal{O}$, though non-Eichler itself, is contained in a level $\mathfrak{p}_2$ Eichler order $\mathcal{O}(\mathfrak{p}_2)$. Using [31], we see that $X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1)$ has signature $(0; 2^6)$. The curve $X^{\pm}(\mathcal{O}^1)$ is a degree 2 genus 1 cover of $X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1)$.

We now determine the two covers in the degree 12 composition

$$X^{\pm}(\mathcal{O}^1) \longrightarrow X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1) \longrightarrow X(N(\mathcal{O}(1))). \tag{5.13}$$

As for the cover

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1) \longrightarrow X(N(\mathcal{O}(1))), \tag{5.14}$$

its possible ramification types are $((2,1^4),(4,2),(6))$ and $((2^2,1^2),(2^3),(6))$. Algorithm 2.5 shows that first of these types gives rise to a unique cover. Its automorphism group is trivial. Hence it cannot be the cover (5.14), which after all factors through the degree 2 quotient

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1) \longrightarrow X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^+).$$

Therefore the type of (5.14) equals $((2^2,1^2),(2^3),(6))$. Again there is a unique Belyĭ map of this type. Putting the elliptic points of index $2,4,12$ at $1,0,\infty$, respectively, it is given by

$$z \longmapsto \frac{((z^2-2)-1)^2((z^2-2)+2)}{4}.$$

The second cover

$$X^{\pm}(\mathcal{O}^1) \longrightarrow X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1) \tag{5.15}$$

in (5.13) is of degree 2. It ramifies above four of the six index 2 elliptic points of $X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1)$. A priori, there are many possibilities for such a cover. However, by (4.2), the resulting curve of signature $(1;2^4)$ should have the property that all elliptic points differ by a translation by a 2-torsion point on its Jacobian. It turns out that up to automorphisms of $\mathbb{P}^1_{\mathbb{C}}$, there is only one quadruple of elliptic points that does the trick, given by the set of zeroes of $(z^2-3)(z^2-4)$.

We obtain the isomorphic degree 12 covers $X^{\pm}(\mathcal{O}^1) \to X(N(\mathcal{O}(1)))$ by composing the explicit equations for the covers (5.14) and (5.15) calculated above. This yields

$$j(J^+(\Gamma)) = j(J^+(\mathcal{O}^1)) = \frac{2^2 193^3}{3} = j(J^-(\mathcal{O}^1)) = j(J^-(\Gamma)).$$

$\boxed{\textbf{e2d81D1:}}$ We have $\Gamma = \mathcal{O}(\mathfrak{p}_2)^1$, where $\mathcal{O}(\mathfrak{p}_2) = \mathbb{Z}_F[\Gamma]$ is a level $\mathfrak{p}_2 = (2)$ Eichler order. Let $\mathcal{O}(1)$ be a maximal order containing $\mathcal{O}(\mathfrak{p}_2)$. We obtain a curve $X(\mathcal{O}(1)^1)$ of signature $(0;2,3,9)$. Perforce the covers

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1) \longrightarrow X(\mathcal{O}(1)^1) \tag{5.16}$$

have ramification type $((2^4,1),(3^3),(9)$. Algorithm 2.5 shows that there is a unique cover of this type, whence $X^+(\mathcal{O}(\mathfrak{p}_2)^1) \cong X^-(\mathcal{O}(\mathfrak{p}_2)^1)$. As mentioned in the case e2d5D4iii, this isomorphism can be derived by a less ad hoc method (cf. [23]).

Sander Dahmen has explicitly determined an equation for the cover (5.16). It is given by

$$(x,y) \longmapsto (Ay+B)/2^{23}$$

from the curve

$$y^2 = x^3 - 1515x - 46106.$$

Here

$$A = x^3 - 9x^2 - 597x - 3851$$

and

$$B = -9x^4 + 132x^3 + 11250x^2 + 117108x + 218895.$$

Another equation was determined in [11]. Regardless, we get

$$j(J^+(\Gamma)) = \frac{-3^2 5^3 101^3}{2^{21}} = j(J^-(\Gamma)).$$

$\boxed{\textbf{e3d5D5:}}$ Because this Γ is the unique $(1;3)$-group for the quaternion algebra associated to it, $X^+(\Gamma) \cong X^-(\Gamma)$ is unique subordinate $(1;3)$-cover

$$X(\Gamma) \longrightarrow X(N(\mathcal{O}(1))). \tag{5.17}$$

Since $X(N(\mathcal{O}(1)))$ has signature $(0;2,3,10)$, the subordinate cover (5.17) has ramification type $((2^5),(3^3,1),(10))$. There is a unique Belyĭ map of this type by Algorithm 2.5. As in the previous case, this cover was calculated by Sander Dahmen. It is given by

$$(x,y) \longmapsto (Ay+B)/2^8 3^{22} 5^5$$

from the curve

$$y^2 = x^3 - 3564675x - 4863773250.$$

Here

$$A = 2^1 3^1 5^2(-x^3 + 405x^2 + 1414125x + 405300375)$$

and

$$\begin{aligned} B = & -x^5 - 5475x^4 + 7206750x^3 + 19533521250x^2 \\ & + 5715377971875x - 1221071756709375. \end{aligned}$$

We get

$$j(J^+(\Gamma)) = -\frac{5281^3}{3^{16}5^1} = j(J^-(\Gamma)).$$

$\boxed{\textbf{e3d5D9:}}$ Let $\mathcal{O}(\mathfrak{p}_2) = \mathbb{Z}_F[\Gamma^{(2)}]$. This is a level $\mathfrak{p}_2 = (2)$ Eichler order for which $\mathcal{O}(\mathfrak{p}_2)^1 = \langle \Gamma^{(2)}, \alpha\beta \rangle$. As in the case e2d5D4iii, $X^\pm(\Gamma)$ is an Atkin–Lehner quotient of $X^\pm(\mathcal{O}(\mathfrak{p}_2)^1)$.

Claim. Γ is not contained in a triangle group.

Proof. This is proved as in the case e2d12D3. □

The curve $X(\mathcal{O}(1)^1)$ has signature $(0;3,5,5)$. We therefore consider the covers

$$X^\pm(\mathcal{O}(\mathfrak{p}_2)^1) \longrightarrow X(\mathcal{O}(1)^1)$$

for a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}(\mathfrak{p}_2)$. There is an Atkin–Lehner involution $w(\mathfrak{p}_3)$ acting on both $X^\pm(\mathcal{O}(\mathfrak{p}_2)^1)$ and $X(\mathcal{O}(1)^1)$, which yields Diagram (5.18).

$$\begin{tikzcd} X^\pm(\mathcal{O}(\mathfrak{p}_2)^1) \arrow[r, "2"] \arrow[d, "5"] & X^\pm(\mathcal{O}(\mathfrak{p}_2)^1)/w(\mathfrak{p}_3) \arrow[d, "5"] \\ X(\mathcal{O}(1)^1) \arrow[r, "2"] & X(\mathcal{O}(1)^1)/w(\mathfrak{p}_3) \end{tikzcd} \tag{5.18}$$

Calculating signatures (we will discuss how to circumvent this step later), we see that the signatures in the diagram above are given by Diagram (5.19).

$$\begin{tikzcd} (1;3,3) \arrow[r, "2"] \arrow[d, "5"] & (0;2,2,6,6) \arrow[d, "5"] \\ (0;3,5,5) \arrow[r, "2"] & (0;2,5,6) \end{tikzcd} \tag{5.19}$$

The map on the right of the (5.19) is a Belyĭ map. By subordinateness, its ramification type equals $((2^2,1),(5),(3,1^2))$. Algorithm 2.5 gives that there is a unique Belyĭ map of this type; putting the elliptic points of index $2, 5, 6$ at $1, \infty, 0$, respectively, it is given by

$$z \longmapsto 4z^3(36z^2 + 15z + 10).$$

The elliptic points of the resulting cover are at the zeroes of $36z^2 + 15z + 10$, giving the two elliptic points of index 6, at 0, an elliptic point of index 2, and at the simple zero $1/4$ of $4z^3(36z^2 + 15z + 10) - 1$, also of index 2. One constructs $X^+(\mathcal{O}(\mathfrak{p}_2)^1) \cong X^-(\mathcal{O}(\mathfrak{p}_2)^1)$ by taking the degree 2 cover ramified above these points. Subsequently identifying the preimages of the zeroes of $36z^2+15z+10$, we obtain $X^+(\Gamma) \cong X^-(\Gamma)$. In the end

$$j(J^+(\mathcal{O}(\mathfrak{p}_2)^1)) = \frac{-269^3}{2^{10}3^5} = j(J^-(\mathcal{O}(\mathfrak{p}_2)^1))$$

and

$$j(J^+(\Gamma)) = \frac{7949^3}{2^5 3^{10}} = j(J^-(\Gamma)).$$

Note that we have seen these j-invariants before (at e2d5D4iii).

Remark. (i) Calculating signatures is not essential for our argument. In fact, we had already excluded that $X^{\pm}(\mathcal{O}(\mathfrak{p}_2)^1)/w(\mathfrak{p}_3)$ has signature $(1;3)$. Hence apart from the signature $(0;2,2,6,6)$ above, only $(0;2^4,3)$ remains as a possibility.

Now although Proposition 2.7 cannot be used to exclude this possibility, a calculation of these covers yields that the two elliptic points on the resulting $(1;3,3)$-curves do not differ by a 2-torsion point on the corresponding Jacobians. Therefore this signature cannot be correct, and we can proceed as above.

(ii) As in the case e2d5D4iii, we refer to [23] for a less ad hoc proof of the isomorphism $X^+(\mathcal{O}(\mathfrak{p}_2)^1) \cong X^-(\mathcal{O}(\mathfrak{p}_2)^1)$.

$\boxed{\textbf{e3d12D3:}}$ Let $\mathcal{O}(\mathfrak{p}_3) = \mathbb{Z}_F[\Gamma]$. Then $\Gamma = \mathcal{O}(\mathfrak{p}_3)^1$. Furthermore, $\mathcal{O}(\mathfrak{p}_3)$ is the unique level $\mathfrak{p}_3$ non-Eichler suborder of the maximal order $\mathcal{O}(1)$ containing $\mathcal{O}(\mathfrak{p}_3)$ (cf. Proposition 1.1(ii)). As in the case e2d12D3, we get Galois $\mathbb{Z}/4\mathbb{Z}$-covers

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_3)^1) \longrightarrow X^{\pm}(\mathcal{O}(1)^1) \longrightarrow X(N(\mathcal{O}(1))) = X(\mathcal{O}(1)^+),$$

and

$$j(J^+(\Gamma)) = j(J^+(\mathcal{O}(\mathfrak{p}_3)^1) = 1728 = j(J^-(\mathcal{O}(\mathfrak{p}_3)^1) = j(J^-(\Gamma)).$$

$\boxed{\textbf{e3d49D1:}}$ This case is analogous to the case e2d81D1. This time the cover

$$X^{\pm}(\Gamma) \longrightarrow X(\mathcal{O}(1)^1) = X(N(\mathcal{O}(1)))$$

has ramification type $((2^{14}), (3^9, 1), (7^4))$. It was calculated in [11].

$\boxed{\textbf{e3d81D1:}}$ This case is analogous to e2d12D2. Indeed, considering [26], there is a triangle group with signature $(0;3,3,9)$ in the commensurability class of Γ. As before, we obtain

$$j(J^+(\Gamma)) = 0 = j(J^-(\Gamma)).$$

$\boxed{\textbf{e4d8D2:}}$ We have $F = \mathbb{Q}(w_2)$, and $\mathfrak{D}(B)^f = \mathfrak{p}_2 = (w_2)$. As for the case e2d5D4, we start with some general considerations.

Preamble. Let $\mathcal{O}(1)$ be a maximal order of B. Then by [26], the signature of $\Delta = \mathrm{P}\mathcal{O}(1)^1$ equals $(0;3,3,4)$, and that of $\Delta' = N(\mathcal{O}(1))$ equals $(0;2,3,8)$. Moreover, Δ' is a maximal triangle group.

There is one more maximal triangle group Δ'', which has signature $(0;2,6,8)$, in the commensurability class of Δ'. As in the case e2d5D4, we will apply Theorem 2.3 to construct $(1;4)$-curves in this commensurability class by taking subordinate covers of the covers $X^{\pm}(\Delta')$ and $X^{\pm}(\Delta'')$.

First consider the triangle group Δ of signature $(0;3,3,4)$. Calculating covolumes and using Theorem 2.3, one sees that a subordinate cover

$$X(\Gamma) \longrightarrow X(\Delta) \tag{5.20}$$

of the curve $X(\Delta)$ by a $(1;4)$-curve has degree equal to 9. The ramification type of (5.20) equals $((3^3),(3^3),(4^2,1))$ by subordinateness. Algorithm 2.5 yields two Belyĭ maps of this type.

Now consider the maximal triangle group $\Delta' \supset \Delta$ of signature $(0;2,3,8)$. It gives rise to a degree two projection map $X(\Delta) \to X(\Delta')$. The subordinate covers

$$X(\Gamma) \longrightarrow X(\Delta') \tag{5.21}$$

of $X(\Delta')$ by $(1;4)$-curves are of degree 18. Again the type is uniquely determined; it is given by $((2^9),(3^6),(8^2,2))$.

Claim. The two composed covers $X(\Gamma) \to X(\Delta) \to X(\Delta')$ obtained from (5.20) are distinct. Moreover, they are the only two subordinate $(1;4)$-covers of $X(\Delta')$.

Proof. In this case, our implementation of Algorithm 2.5 took too long to run. However, let S be the set of isomorphism classes of covers of type $((2^9),(3^6),(8^2,2))$. Then by Theorem 7.2.1 of [18] we have

$$\sum_{(Y,f)\in S} \frac{1}{\mathrm{Aut}(Y,f)} = \frac{|C_2||C_3||C_8|}{|S_{18}|^2} \sum_{\chi} \frac{\chi(C_2)\chi(C_3)\chi(C_8)}{\chi(1)} = 1. \tag{5.22}$$

Here C_2, C_3, C_8 are the conjugacy classes in S_{18} corresponding to the tuples (2^9), (3^6), $(8^2,2)$ constituting the given ramification type. In the sum on the right-hand side of (5.22), χ runs over the characters of the irreducible representations of S_{18}.

The two covers $X(\Gamma) \to X(\Delta')$ constructed in the previous paragraph are not isomorphic, as will follow from the explicit calculation in the case e4d8D2i below. This can also be shown in a simpler way by drawing the associated *dessins d'enfants* as in [7]. Since no Belyĭ map of type $((2^9),(3^6),(8^2,2))$ has more than two automorphisms by Proposition 2.1(v), we see that no covers of this type can arise other than the two obtained from (5.20). □

A covolume calculation yields that no subordinate covers of the curve $X(\Delta'')$ by $(1;4)$-curves exist.

We now proceed to treat the individual cases, again in an order differing from that in [27].

$\boxed{\textbf{e4d8D2i/iii:}}$ These two cases give rise to Galois conjugate curves (cf. [8]). We consider the first. Then $\langle \Gamma^{(2)}, \alpha \rangle = \mathcal{O}(\mathfrak{p}_{17})^1$, where $\mathcal{O}(\mathfrak{p}_{17}) = \mathbb{Z}_F[\Gamma^{(2)}]$ is a level $\mathfrak{p}_{17}$ Eichler order for one of the two primes $\mathfrak{p}_{17}$ of F above 17. Note that the Eichler orders whose level is given by the other prime of F above 17 give rise to the Galois conjugate second case.

Let $\mathcal{O}(1)$ be a maximal order containing $\mathcal{O}(\mathfrak{p}_{17})$. We obtain covers

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_{17})^1) \longrightarrow X(\mathcal{O}(1)^1) \tag{5.23}$$

of degree 18. As usual, $X^{\pm}(\Gamma)$ is an Atkin–Lehner quotient of $X^{\pm}(\mathcal{O}(\mathfrak{p}_{17})^1)$. In particular, as in the case e2d5D4iii, we see that Γ is not contained in $\mathrm{P}\mathcal{O}(1)^1$. Hence

Claim. Γ is not contained in a triangle group.

Proof. This follows by combining the observation preceding the claim with the preamble. □

The covers (5.23) have trivial automorphism group, which makes them rather hard to calculate. We refer to [23] for a discussion on how to find an equation for the curve $X^+(\mathcal{O}(\mathfrak{p}_{17})^1) \cong X^-(\mathcal{O}(\mathfrak{p}_{17})^1)$ using modular methods.

$\boxed{\textbf{e4d8D2ii:}}$ We have $\Gamma \subsetneq \mathcal{O}(1)^1$, where $\mathcal{O}(1)$ is the maximal order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$. By elimination, this case corresponds to the two covers of $X(\mathcal{O}(1)^1)$ of ramification type $((3^3), (3^3), (4^2, 1))$.

We can descend as in the case e2d5D4ii, yielding Diagram (5.24).

$$\begin{tikzcd} X^{\pm}(\Gamma) \arrow[r, "2"] \arrow[d, "9"] & X^{\pm}(\langle \Gamma, S \rangle) \arrow[d, "9"] \\ X(\mathcal{O}(1)^1) \arrow[r, "2"] & X(N(\mathcal{O}(1))) \end{tikzcd} \tag{5.24}$$

By subordinateness of the covers involved, the corresponding diagram of signatures is given by Diagram (5.25).

$$\begin{tikzcd} (1;4) \arrow[r, "2"] \arrow[d, "9"] & (0;2,2,2,8) \arrow[d, "9"] \\ (0;3,3,4) \arrow[r, "2"] & (0;2,3,8) \end{tikzcd} \tag{5.25}$$

In a similar manner, we conclude that the type of the Belyĭ map on the right equals $((2^3, 1^3), (3^3), (8, 1))$. Explicitly, placing the elliptic points of index $2, 3, 8$ at $1, 0, \infty$, we found the two conjugate covers

$$z \longmapsto \frac{(z^3 + 24z^2 + 12(11 \pm w)z + 8(5 \pm w))^3}{177147(7 \pm 4w)z}. \tag{5.26}$$

where $w = w_{-2}$. The covers (5.26) are not isomorphic. Indeed, Proposition 2.1(v) shows that an isomorphism between them would have to fix the branch points 0 and ∞ in the fiber over ∞. Hence it would be given by some scalar multiplication, which is easily ruled out.

We can now construct the curves $X^{\pm}(\Gamma)$ by taking the degree 2 elliptic cover ramified above the unique elliptic point of index 8 (given by 0) and the three elliptic points of index 2 (given by the unramified preimages of 1 or, more explicitly, by the zeroes of $z^3 + 42z^2 + 3(191 \pm 10w)z + 512(5 \pm w)$). We thus obtain the two conjugate j-invariants

$$j(J^{\pm}(\Gamma)) = \frac{119421866 \pm 241123607w}{2^{14}}$$

of norm $3^1 11^3 41^3 691^3 / 2^{27}$. Note that these j-invariants also show that the two covers (5.26) cannot be isomorphic.

Remark. A way to derive the descent (5.24) without using the explicit matrix S is as follows. Using Algorithm 2.5, one verifies that there are exactly two subordinate covers

$$X(\Gamma_1), X(\Gamma_2) \longrightarrow X(N(\mathcal{O}(1))) = X(\Delta')$$

by curves $X(\Gamma_i)$ of signature $(0; 2^3, 8)$. Both of these covers have ramification type $((2^3, 1^3), (3^3), (8, 1))$. (Note that a priori, $((2^4, 1), (3^3), (4^2, 1))$ is also possible, but there are no Belyǐ maps of this type by Algorithm 2.5.)

We saw in the preamble that there were also two subordinate $(1; 4)$-covers of $X(\Delta')$. By elimination, these are given by

$$X^{\pm}(\Gamma) \longrightarrow X(\Delta'). \tag{5.27}$$

There is no other possibility than that these are the two covers obtained by composing the covers $X(\Gamma_i) \to X(\Delta')$ with the degree 2 Galois covers ramifying above the 4 elliptic points of the $X(\Gamma_i)$. Now the preamble shows that the two covers (5.27) factor through the cover $X(\Delta) \to X(\Delta')$, whence the descent (5.24) as $\Delta = \mathrm{P}\mathcal{O}(1)^1$ and $\Delta' = \mathrm{P}N(\mathcal{O}(1))$.

e4d2304D2: By [26], there is a triangle group with signature $(0; 3, 3, 12)$ in the commensurability class of Γ. As in the case e2d12D2, we get

$$j(J^+(\Gamma)) = 0 = j(J^-(\Gamma)).$$

e5d5D5: We have $F = \mathbb{Q}(w_5)$, and B has discriminant $\mathfrak{D}(B)^f = \mathfrak{p}_5 = (2w_5 - 1)$. There are three $(1; 5)$-groups in the corresponding commensurability class.

Preamble. Let $\mathcal{O}(1)$ be a maximal order of B. Then by [26], the signatures of the groups $\mathcal{O}(1)^1$ and $N(\mathcal{O}(1))$ are $(0; 3, 3, 5)$ and $(0; 2, 3, 10)$, respectively. Moreover, $N(\mathcal{O}(1))$ is a maximal triangle group, and in fact the only maximal triangle group in its commensurability class.

By Theorem 2.2, we can construct $(1; 5)$-curves by taking subordinate covers

$$X(\Gamma) \longrightarrow X(N(\mathcal{O}(1))). \tag{5.28}$$

Such a cover has degree 12, and its ramification type is given by $((2^6), (3^4), (10, 2))$. By Algorithm 2.5, there exists a unique Belyǐ map of this type.

Analogously, one shows that the subordinate covers

$$X(\Gamma) \longrightarrow X(\mathcal{O}(1)^1) \tag{5.29}$$

are of type $((3^2), (3^2), (5, 1))$. Again there is a unique Belyǐ map of this type. Hence the cover (5.28) will factor through (5.29).

e5d5D5iii: We have $\Gamma \subsetneq \mathcal{O}(1)^1$, where $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma]$ is a maximal order. In particular, we get covers

$$X^{\pm}(\Gamma) \longrightarrow X(\mathcal{O}(1)^1).$$

By uniqueness, these are isomorphic to the subordinate cover $X(\Gamma') \to X(\mathcal{O}(1)^1)$ of ramification type $((3^2), (3^2), (5, 1))$ described in the preamble. We descend as in the case e2d5D4ii to obtain Diagram (5.30).

$$\begin{tikzcd} X(\Gamma) \arrow[r, "2"] \arrow[d, "6"] & X(\langle \Gamma, S\rangle) \arrow[d, "6"] \\ X(\mathcal{O}(1)^1) \arrow[r, "2"] & X(N(\mathcal{O}(1))) \end{tikzcd} \tag{5.30}$$

The corresponding diagram of signatures is given by Diagram (5.31).

$$\begin{tikzcd} (1;5) \arrow[r, "2"] \arrow[d, "6"] & (0;2,2,2,10) \arrow[d, "6"] \\ (0;3,3,5) \arrow[r, "2"] & (0;2,3,10) \end{tikzcd} \tag{5.31}$$

Remark. The descent (5.30) also follows from the uniqueness in the preamble along with the existence of a degree 6 subordinate cover of $X(N(\mathcal{O}(1)))$ with signature equal to $(0;2^3,10)$, which in turn follows from Algorithm 2.5.

The ramification type of the cover

$$X(\langle \Gamma, S\rangle) \longrightarrow X(N(\mathcal{O}(1)))$$

equals $((2^2,1^2),(3^2),(5,1))$. By Algorithm 2.5, there is a unique Belyĭ map of this type, which is given by the rational function

$$z \longmapsto -\frac{(z^2-10z+5)^3}{1728z}$$

if we place the elliptic points of index $2,3,10$ at $1,0,\infty$.

To construct $X(\Gamma)$, one takes a degree 2 cover ramified above the unramified point $z=0$ above ∞, which is elliptic of index 10, the unramified point $z=\infty$ above ∞, which is elliptic of index 2, and the unramified preimages of 1, which are elliptic of index 2 and are given by the zeroes of $z^2-22z+125$. In the end,

$$j(J^+(\Gamma)) = \frac{-2^4 109^3}{5^6} = j(J^-(\Gamma)).$$

$\boxed{\textbf{e5d5D5i/ii:}}$ As in the cases e4d8D2i/iii, these two cases are Galois conjugate (cf. [8]); we consider the first.

Claim. Γ is not contained in a triangle group.

Proof. Clear from the preamble and the previous case. □

One has $\langle \Gamma^{(2)}, \beta\rangle = \mathcal{O}(\mathfrak{p}_{11})^1$, where $\mathcal{O}(\mathfrak{p}_{11}) = \mathbb{Z}_F[\Gamma]$ is a level $\mathfrak{p}_{11}$ Eichler order. Let $\mathcal{O}(1)$ be a maximal order containing $\mathcal{O}(\mathfrak{p}_{11})$. Then we can consider the degree 12 covers

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_{11})^1) \longrightarrow X(\mathcal{O}(1)^1).$$

Algorithm 2.3 yields that the monodromy groups of these covers have order 660. By Lemma 1.2, $X^{\pm}(\Gamma)$ is an Atkin–Lehner quotient of $X^{\pm}(\mathcal{O}(\mathfrak{p}_{11}))$.

We can descend via the Atkin–Lehner involution $w(\mathfrak{p}_5)$. This results in Diagram (5.32).

$$\begin{tikzcd} X^{\pm}(\mathcal{O}(\mathfrak{p}_{11})^1) \arrow[r, "2"] \arrow[d, "12"] & X^{\pm}(\mathcal{O}(\mathfrak{p}_{11})^1)/w(\mathfrak{p}_5) \arrow[d, "12"] \\ X(\mathcal{O}(1)^1) \arrow[r, "2"] & X(\mathcal{O}(1)^1)/w(\mathfrak{p}_5) \end{tikzcd} \tag{5.32}$$

By direct calculation, the signatures in Diagram (5.32) are given by Diagram (5.33).

$$\begin{tikzcd} (1;5,5) \arrow[r, "2"] \arrow[d, "12"] & (0;2,2,10,10) \arrow[d, "12"] \\ (0;3,3,5) \arrow[r, "2"] & (0;2,3,10) \end{tikzcd} \tag{5.33}$$

Remark. Alternatively, [26] shows that the curve

$$X(\mathcal{O}(1)^1)/w(\mathfrak{p}_5) = X(N(\mathcal{O}(1)))$$

has signature $(0;2,3,10)$. This enables us to exclude the other two a priori possible signatures $(1;5)$ and $(0;2^4,5)$ of $X^{\pm}(\mathcal{O}(\mathfrak{p}_{11})^1)/w(\mathfrak{p}_5)$. Indeed, Theorem 2.2 and Algorithm 2.5 show that there exist subordinate Belyĭ maps for both of these signatures, but none of these has the monodromy group of order 660 or 1320 demanded by Proposition 2.7.

The ramification type of the subordinate cover on the right of Diagram (5.33) is given by $((2^5,1^2),(3^4),(10,1^2))$. Algorithm 2.5 gives that there are four Belyĭ maps of this type. Two of these have monodromy group of order 1320; for the other two, this order is 3840. These two pairs of Belyĭ maps are also distinguished by the fact that the Belyĭ maps (Y,f) in the former pair have $|\mathrm{Aut}(Y,f)| = 1$ while those in the latter have $|\mathrm{Aut}(Y,f)| = 2$. Considering Proposition 2.7, the two Belyĭ maps realizing the cases e5d5D5i and e5d5D5ii are the former pair.

Let $w = w_5$. Placing the elliptic points of index $2,3,10$ at $1,0,\infty$, respectively, the former pair of Belyĭ maps is given by

$$z \longmapsto \frac{44281^5 f_4^3}{6912(164w-587)^5 C(z^2-500w+875)} \tag{5.34}$$

and its conjugate. Here

$$\begin{aligned} f_4 &= z^4+10z^3+1160z^2+17550z+326175-20w(z^3+31z^2+585z+9915), \\ C &= 2232924308430846135w - 3603199856376900322. \end{aligned}$$

It is straightforward to verify that (5.34) has a trivial automorphism group. Indeed, by Proposition 2.1(v), a non-trivial automorphism would fix ∞ since this is the only point with ramification index 10. Moreover, it would interchange the other two points in the fiber above ∞ (given by the zeroes of $z^2-500w+875$). These demands determine a unique automorphism of $\mathbb{P}^1_{\mathbb{C}}$, but this is not an automorphism of (5.34). A similar calculation shows that (5.34) is not isomorphic to its Galois conjugate.

From (5.34), the curve $X^+(\mathcal{O}(\mathfrak{p}_{11})^1) \cong X^-(\mathcal{O}(\mathfrak{p}_{11})^1)$ can be constructed by taking the degree 2 cover ramifying above the elliptic points of index 10 (given by the zeroes of $z^2 - 500w + 875$), and the unramified preimages of 1 (given by the zeroes of $z^2 + (16 - 32w)z + (587 - 164w)$). This yields

$$j(J^+(\mathcal{O}(\mathfrak{p}_{11})^1)) = \frac{1485675267531w + 2666389392178}{5^3 11^6} = j(J^-(\mathcal{O}(\mathfrak{p}_{11})^1))$$

and, identifying the elliptic points of index 5 on $X^{\pm}(\mathcal{O}(\mathfrak{p}_{11})^1)$,

$$j(J^+(\Gamma)) = \frac{4560282420936767w + 2818578140804845}{5^6 11^3} = j(J^-(\Gamma)),$$

The norms of these j-invariants equal $19^3 90019^3/5^6 11^6$ and $59^3 167809^3/5^{12}$, respectively.

$\boxed{\textbf{e5d5D9:}}$ For this case, $\langle \Gamma^{(2)}, \alpha\beta\rangle = \mathcal{O}(\mathfrak{p}_5)^1$, where $\mathcal{O}(\mathfrak{p}_5) = \mathbb{Z}_F[\Gamma^{(2)}]$ is a level $\mathfrak{p}_5$ Eichler order. By Lemma 1.2, $X^{\pm}(\Gamma)$ is an Atkin–Lehner quotient of $X^{\pm}(\mathcal{O}(\mathfrak{p}_5)^1)$.

There is a unique subordinate $(1;5)$-cover

$$X(\Gamma) \to X(N(\mathcal{O}(1))) \tag{5.35}$$

which perforce yields the $(1;5)$-group under consideration. The cover (5.35) has ramification type $((2^3), (5,1), (6))$. Algorithm 2.5 shows that it is in fact determined by its type. Placing the elliptic points of order $2, 5, 6$ at $1, 0, \infty$, respectively, an explicit morphism realizing (5.35) is given by

$$(x,y) \longmapsto \frac{2^2}{5^5}(9xy - x^3 - 15x^2 - 36x + 32)$$

from the curve

$$y^2 + xy + y = x^3 + x^2 + 35x - 28.$$

One deduces that

$$j(J^+(\Gamma)) = \frac{23^3 73^3}{3^2 5^8} = j(J^-(\Gamma)).$$

$\boxed{\textbf{e5d1125D5:}}$ By [26], this $(1;5)$-group is commensurable with a triangle group of signature $(0;3,3,15)$. As in the case e2d12D2, we have

$$j(J^+(\Gamma)) = 0 = j(J^-(\Gamma)).$$

$\boxed{\textbf{e7d49D1:}}$ We have $\langle \Gamma^{(2)}, \alpha\beta\rangle = \mathcal{O}(\mathfrak{p}_2\mathfrak{p}_7)^1$, where $\mathcal{O}(\mathfrak{p}_2\mathfrak{p}_7) = \mathbb{Z}_F[\Gamma^{(2)}]$ is a level $\mathfrak{p}_2\mathfrak{p}_7$ Eichler order. Let $\mathcal{O}(1)$ be a maximal order containing $\mathcal{O}(\mathfrak{p}_2\mathfrak{p}_7)$. We get covers

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_2\mathfrak{p}_7)^1) \longrightarrow X(\mathcal{O}(1)^1) \tag{5.36}$$

of the signature $(0;2,3,7)$ curve $X(\mathcal{O}(1)^1)$. The Belyĭ maps (5.36) have degree 72, making them quite intractable. However, [26] yields that the commensurability class of Γ also contains a triangle group Δ of signature $(0;2,4,7)$. The curve $X(\Delta)$ allows a degree 8 subordinate $(1;7)$-cover

$$X(\Gamma') \longrightarrow X(\Delta) \tag{5.37}$$

which Algorithm 2.5 shows to be uniquely determined by its ramification type $((2^4), (4^2), (7,1))$. The cover $X(\Gamma')$ is isomorphic to the arithmetic $(1;7)$-curves $X^+(\Gamma) \cong X^-(\Gamma)$ since Γ is the only $(1;7)$-group in its commensurability class.

The Belyĭ map (5.37) was computed by Frits Beukers. It is given by

$$(x,y) \longmapsto \frac{A + By}{C}$$

from the curve

$$y^2 = (x+78)(x^2 - 78x - 39951)$$

if we place the elliptic points of index $2, 4, 7$ at $1, 0, \infty$, respectively. Here

$$\begin{aligned}
A &= x^3 - 405x^2 + 6291x + 3297321, \\
B &= -3(7x^4 - 3108x^3 + 9450x^2 + 62545500x + 3089292615), \\
C &= 2^{15}3^7(x - 363).
\end{aligned}$$

In the end,

$$j(J^+(\Gamma)) = \frac{5^3 11^3 31^3}{2^3 7^6} = j(J^-(\Gamma)).$$

$\boxed{\textbf{e11d14641D1:}}$ There is an equality $\Gamma = \mathcal{O}(\mathfrak{p}_{11})^1$, where $\mathcal{O}(\mathfrak{p}_{11}) = \mathbb{Z}_F[\Gamma]$ is a level $\mathfrak{p}_{11}$ Eichler order. The inclusion of $\mathcal{O}(\mathfrak{p}_{11})$ into a maximal order $\mathcal{O}(1)$ gives rise to degree 12 covers

$$X^{\pm}(\mathcal{O}(\mathfrak{p}_{11})^1) \longrightarrow X(\mathcal{O}(1)^1) \tag{5.38}$$

of the signature $(0; 2, 3, 11)$ curve $X(\mathcal{O}(1)^1)$. The ramification type of (5.38) equals $((2^6), (3^4), (11, 1))$ by subordinateness. Algorithm 2.5 shows that there is a unique corresponding cover, whence $X^+(\mathcal{O}(\mathfrak{p}_{11})^1) \cong X^-(\mathcal{O}(\mathfrak{p}_{11})^1)$. The covers (5.38) are isomorphic over F to the classical modular cover $X_0(11) \to X_0(1)$. We therefore forgo the calculation of (5.38), since classical modular methods are available for this (such as using q-expansions). Suffice to conclude that since $X^{\pm}(\mathcal{O}(\mathfrak{p}_{11})^1)$ is geometrically isomorphic to the strong Weil curve of conductor 11, one has

$$j(J^+(\Gamma)) = j(J^+(\mathcal{O}(\mathfrak{p}_{11})^1)) = \frac{-2^{12}31^3}{11^5} = j(J^-(\mathcal{O}(\mathfrak{p}_{11})^1)) = j(J^-(\Gamma)).$$

The rational cases. There are 4 cases where B has center $\mathbb{Q}$. We refer to [10] for the cases e2d1D6i and e3d1D6i, in which the $(1; e)$-group Γ is not contained in a triangle group. It remains to determine the cases e2d1D6ii and e3d1D6ii.

$\boxed{\textbf{e2d1D6ii:}}$ The order $\mathcal{O} = \mathbb{Z}[\Gamma^{(2)}]$ is of level 2^3 and non-Eichler, with $\Gamma^{(2)} = \mathcal{O}^1$. By Proposition 1.1(ii), there is a unique maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$. The group $\Delta = N(\mathcal{O}(1))$ has signature $(0; 2, 4, 6)$, and there is a unique subordinate cover

$$X(\Gamma') \longrightarrow X(\Delta) \tag{5.39}$$

for which $X(\Gamma')$ has signature $(1; 2)$. Its ramification type is $((2^3), (4, 2), (6))$.

The subordinate cover (5.39) is in [5]; putting the elliptic points of index $2, 4, 6$ at $4, 0, \infty$, it is given by

$$(x, y) \longmapsto (x-1)^2(x+2)$$

from the elliptic curve

$$y^2 = (x-1)(x-2)(x+2).$$

We have not yet proved the isomorphisms $X(\Gamma') \cong X^+(\Gamma) \cong X^-(\Gamma)$. But these follow because otherwise $X(\Gamma')$ would correspond to the case e2d1D6i, and the explicit equations in [10] show that this is not the case.

We get the j-invariants

$$j(J^+(\Gamma)) = \frac{2^4 13^3}{3^2} = j(J^-(\Gamma)).$$

Remark. An alternative way to prove that Γ' does not come from the case e2d1D6i, using the results in [12], is the following. Supposing the contrary, Γ' would correspond to a subgroup of the normalizer $N(\mathcal{O}(5))$ of a level 5 Eichler order $\mathcal{O}(5)$. However, [12] shows that the unique subgroup of $N(\mathcal{O}(5))$ having signature $(1;2)$ is not a subgroup of $N(\mathcal{O}(1))$.

e3d1D6ii: This case is completely analogous to the previous. In this case, Γ gives rise to a subordinate cover

$$X(\Gamma) \longrightarrow X(\Delta) \tag{5.40}$$

with ramification type $((2^4),(4^2),(6,2))$. By Algorithm 2.5, there is a unique cover of this type. It has automorphism group $\mathbb{Z}/2\mathbb{Z}$, and decomposes as a degree 2 genus 1 cover of the unique genus 0 cover of $X(\Delta)$ of type $((2,1,1),(4),(3,1))$ (see [5]).

Putting the elliptic points of index $2,4,6$ at $-27,\infty,0$, respectively, we end up with the cover

$$(x,y) \longmapsto (x+3)^3(x-1)$$

from the genus 1 curve

$$y^2 = (x-1)(x+3)(x^2+8x+18).$$

Our final two j-invariants therefore equal

$$j(J^+(\Gamma)) = \frac{2^1 47^3}{3^8} = j(J^-(\Gamma)).$$

6. Pullbacks and parameters

This section will briefly expound the computation of the accessory parameter A figuring in the Lamé equation (0.4) associated with the arithmetic $(1;e)$-groups Γ from the previous section.

First we give a description of the general method. Consider the Fuchsian differential operators living on the curves in Diagram (4.2). As we saw in Section 4, the group $N(\mathcal{O}(1))$ is triangular. Consequently, the Fuchsian differential operator associated to $N(\mathcal{O}(1))$ is given by the hypergeometric differential operator

$$z(z-1)\frac{d^2}{dz^2} + ((a+b+1)z-c)\frac{d}{dz} + ab. \tag{6.1}$$

The singular points of this differential operator are at $0,1,\infty$. These are regular singular points, with local exponents given by $\{0,1-c\},\{0,c-a-b\},\{a,b\}$, respectively.

Starting with a triangle group Δ of signature $(0;p,q,r)$, suitable values of a,b,c in (6.1) can therefore be determined from the system of equations (6.2).

$$\left\{\begin{array}{cl} 1-c & = 1/p, \\ c-a-b & = 1/q, \\ a-b & = 1/r. \end{array}\right. \tag{6.2}$$

Pulling back the corresponding hypergeometric differential operator through the cover $X^{\pm}(\Gamma^{(2)}) \to X(N(\mathcal{O}(1)))$, one obtains an explicit differential operator D_0 with four regular singular points on the genus 1 curve $X^{\pm}(\Gamma^{(2)})$. Similarly, we can pull back the Lamé differential operator

$$(y\frac{d}{dx})^2 - (n(n+1)x+A) \tag{6.3}$$

through the map $X^{\pm}(\Gamma^{(2)}) \to X^{\pm}(\Gamma)$. This results in an operator D_1 whose expression depends on the as yet unknown parameter A.

The equations D_0 and D_1 need not be identical up multiplication by a rational function, as their local exponents will in general be different. However, there is always a unique equation D_0' that has the same local exponents as D_1 and that is *projectively equivalent* to D_0 in the sense that

$$D_0' = \vartheta^{-1} D_0 \vartheta,$$

for some radical function ϑ on $X^{\pm}(\Gamma^{(2)})$ (cf. Section 1 of [4]). Having thus equated the local exponents, we obtain a linear equation for A by inspecting the global expressions of both pullbacks, which allows us to determine A explicitly in all our cases.

Example 6.1. We illustrate the general picture above by calculating an example, namely the case e3d5D9 from Section 5. In this case $X^+(\Gamma)$ and $X^-(\Gamma)$ are isomorphic, so we drop the $\pm$ from our discussion above. We have seen that there is a cover

$$X(\langle \Gamma^{(2)}, \alpha\beta \rangle) \to X(N(\mathcal{O}(1)) \tag{6.4}$$

given by

$$f : (x, y) \longmapsto 4x^3(36x^2 + 15x + 10)$$

from the genus 1 curve

$$y^2 = x(4x - 1)(36x^2 + 15x + 10). \tag{6.5}$$

We put the elliptic points of order $2, 5, 6$ of $X(N(\mathcal{O}(1))$ at $0, 1, \infty$, which amounts to using $g = 1/f$ instead of f as our Belyĭ map. As for the cover

$$X(\langle \Gamma^{(2)}, \alpha\beta \rangle) \to X(\Gamma), \tag{6.6}$$

the curve $X(\langle \Gamma^{(2)}, \alpha\beta \rangle)$ allows a Weierstrass equation

$$y^2 = 4x^3 - 119235x + 7897825 \tag{6.7}$$

for which the cover (6.6) is given by a 2-isogeny, acting on x-coordinates by

$$x \longmapsto \frac{-7740x^2 + 6495x + 1900}{72x^2 + 30x + 20}. \tag{6.8}$$

The pullback D_1 of the operator (6.3) on (6.7) through the 2-isogeny (6.8) equals

$$6y^2 \frac{d^2}{dx^2} + 6(288x^3 + 36x^2 + 25x - 5)\frac{d}{dx} + (\frac{-7740x^2 + 6495x + 1900}{324x^2 + 135x + 90} - A). \tag{6.9}$$

It is straightforward to calculate that the pullback D_0 to (6.5) of the hypergeometric differential operator obtained from (6.2) through the morphism g has the same local exponents as D_1, except at the points mapping to ∞ under g. More precisely, D_0 has local exponents $2/5$ and $7/5$ at the double zero $(0, 0)$ of the rational function x, while D_1 has local exponents 0 and 1 at this point, and whereas D_0 has local exponents $2/15$ and $7/15$ at the support of $36x^2+15x+10$, the corresponding local exponents for D_1 are given by $1/3$ and $2/3$. At the former point, the local exponents of D_0 differ from those of D_0 by $2/5$, while at the latter two points, the exponents of D_1 are off by $-1/5$.

TABLE 1. Equations and accessory parameters associated with arithmetic $(1;e)$-groups commensurable with triangle groups.

Label	g_2	g_3	A
e2d1D6i	48027	2021723	$-237/32$
e2d1D6ii	39	35	$-3/32$
e2d5D4i	255	775	$-15/32$
e2d5D4ii	-3165	15535	$3/32$
e2d5D4iii	4035	237025	$-15/32$
e2d8D2	1	0	0
e2d12D2	0	1	0
e2d12D3	579	2681	$-15/16$
e2d81D1	1515	23053	$-15/32$
e3d1D6i	14979	124865	$-67/72$
e3d1D6ii	-141	2359	$1/18$
e3d5D5	15843	720559	$-35/18$
e3d5D9	119235	7897825	$-245/18$
e3d12D3	1	0	0
e3d49D1	32844	1146439	$-70/9$
e3d81D1	0	1	0
e4d8D2i/iii	$\pm1266\sqrt{2}+801$	$\pm7187\sqrt{2}+16243$	$(\mp81\sqrt{2}-99)/2^7$ (?)
e4d8D2ii	$\pm1116\sqrt{-2}+147$	$\pm6966\sqrt{-2}-6859$	$(\mp78\sqrt{-2}-123)/2^7$
e4d2304D2	0	1	0
e5d5D5i/ii	$(\pm28059\sqrt{5}+88641)/2$	$(\pm1679762\sqrt{5}+4259192)/2$	$(\mp279\sqrt{5}-927)/2^2 5^2$
e5d5D5iii	327	1727	$-9/25$
e5d5D9	-5037	36631	$9/50$
e5d1125D5	0	1	0
e7d49D1	5115	57707	$-165/98$
e11d14641D1	372	2501	$42/121$

However, the previous paragraph shows equally well that if we define the radical function ϑ by

$$\vartheta = \left(\frac{36x^2+15x+10}{x^2}\right)^{1/10}$$

and let $D_0' = \vartheta^{-1}D_0\vartheta$, then D_0' has the same local exponents as D_1. Now up to multiplication by a rational function on $X(\langle\Gamma^{(2)},\alpha\beta\rangle)$, the operator D_0' is given by

$$6y^2\frac{d^2}{dx^2} + 6(288x^3+36x^2+25x-5)\frac{d}{dx} + \left(\frac{-2700x^2+1035x+150}{72x^2+30x+20}\right). \tag{6.10}$$

Comparing (6.9) with (6.10), we conclude that $A = 245/18$.

Remark 6.2. Note that passing to the curve $X(\Gamma^{(2)})$ was not necessary in this example, since $X(\langle\Gamma^{(2)},\alpha\beta\rangle)$ was already a common cover of $X(N(\mathcal{O}(1)))$ and $X(\Gamma)$.

The calculations for the remaining cases can be found at [22]. We refer to [17] for similar applications of the methods in this section.

7. Results

Proceeding as in the previous sections, we computed Table 1. We have chosen the equation (0.3) in such a way that g_2 and g_3 are integral elements of the field of moduli of $X^{\pm}(\Gamma)$. Moreover, we have minimized the absolute values of g_2 and g_3, and we have taken g_3 to be positive; these demands determine the pair (g_2, g_3).

The models for $X^{\pm}(\Gamma)$ given below are not necessarily canonical in the sense of [20]: for this, one needs to twist appropiately (cf. [23]). Note that a quadratic twist by α changes the triple (g_2, g_3, A) by the rule

$$(g_2, g_3, A) \longmapsto (g_2\alpha^4, g_3\alpha^6, A\alpha^2). \tag{7.1}$$

For one case in Table 1, the accessory parameter is marked with a question mark. In this case, we have not managed to find equations for $X^{\pm}(\Gamma)$ and the parameter A using the methods in this paper. Resorting to the modular methods in [23], we still managed to find an equation for $X^{\pm}(\Gamma)$. The parameter A was then approximated by using a `Maple` program kindly shared with us by Yifan Yang. We hope to return to the exact determination of A at some future occasion.

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Along with [23], this paper summarizes the results in the author's Ph.D. thesis [21], which was written under the supervision of Frits Beukers at the Universiteit Utrecht.

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Mathematisch Instituut Universiteit Utrecht, Postbus 80010, 3508TA Utrecht, The Netherlands

Current address: IRMAR–Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cédex, France

E-mail address: `sijsling@gmail.com`