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ODD PERFECT NUMBERS ARE GREATER THAN 10^{1500}

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ABSTRACT. Brent, Cohen, and te Riele proved in 1991 that an odd perfect number N is greater than 10^{300}. We modify their method to obtain $N > 10^{1500}$. We also obtain that N has at least 101 not necessarily distinct prime factors and that its largest component (i.e. divisor p^a with p prime) is greater than 10^{62}.

1. INTRODUCTION

A natural number N is said *perfect* if it is equal to the sum of its positive divisors (excluding N). It is well known that an even natural number N is perfect if and only if $N = 2^{k-1}(2^k - 1)$ for an integer k such that $2^k - 1$ is a Mersenne prime. On the other hand, it is a long-standing open question whether an odd perfect number exists.

In order to investigate this question, several authors gave necessary conditions for the existence of an odd perfect number N. Euler proved that $N = p^e q^2$ for a prime p, with $p = e = 1 \pmod 4$ and $\gcd(p, q) = 1$. More recent results show that N must be greater than 10^{300} [1], it must have at least 75 prime factors (counting multiplicities) [4], and it must have at least 9 distinct prime factors [5]. Moreover, the largest prime factor of N must be greater than 10^8 [3], and N must have a component greater than 10^{20} [2] (i.e. N must have a divisor p^a with p prime, and $p^a > 10^{20}$).

We improve in this paper some of these results. In Section 3 we show that N must be greater than 10^{1500}. We use for this the approach of Brent *et al.* [1], with a method to by-pass deadlocks similar to the method used by Hare [4]. With a slight modification of the approach, we show that N must have at least 101 prime factors in Section 4, and that N must have a component greater than 10^{62} in Section 5. These results are outcomes of some improvements in the used techniques, and of factorization efforts. We discuss that in Section 6.

2. PRELIMINARIES

Let n be a natural number. Let $\sigma(n)$ denote the sum of the positive divisors of n, and let $\sigma_{-1}(n) = \frac{\sigma(n)}{n}$ be the *abundancy* of n. Clearly, n is perfect if and only if $\sigma_{-1}(n) = 2$. We first recall some easy results on the functions σ and σ_{-1}. If p is prime, $\sigma(p^q) = \frac{p^{q+1}-1}{p-1}$, and $\sigma_{-1}(p^\infty) = \lim_{q\to+\infty} \sigma_{-1}(p^q) = \frac{p}{p-1}$. If $\gcd(a, b) = 1$, then $\sigma(ab) = \sigma(a)\sigma(b)$ and $\sigma_{-1}(ab) = \sigma_{-1}(a)\sigma_{-1}(b)$.

Euler proved that if an odd perfect number N exists, then it is of the form $N = p^e m^2$ where $p = e = 1 \pmod 4$ and $\gcd(p, m) = 1$. The prime p is said to be the *special prime*.

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Many results on odd perfect numbers are obtained using the following argument. Suppose that N is an odd perfect number, and that p is a prime factor of N. If $p^q \parallel N$ for a $q > 0$, then $\sigma(p^q) \mid 2N$. Thus if we have a prime factor $p' > 2$ of $\sigma(p^q)$, we can recurse on the factor p'. We make all suppositions that for q up we get a contradiction (e.g. p^q is greater than the limit we want to prove). Moreover, since $\sigma(p^a) \mid \sigma(p^b)$ if $a+1 \mid b+1$, we can only suppose that $p^q \parallel N$ for q such that $q+1$ is prime. Major changes between the approaches to get the theorems are the supposition we make on the hypothetical odd perfect number, the order of exploration of prime factors, and the contradictions we use.

3. Size of an odd perfect number

Theorem 1. *An odd perfect number is greater than* 10^{1500}.

We use factor chains as described in [1] to forbid the factors in $S = \{127, 19, 7, 11, 331, 31, 97, 61, 13, 398581, 1093, 3, 5, 307, 17\}$, in this order. These chains are constructed using *branchings*. To branch on a prime p means that we sequentially branch on all possible components p^a. To branch on a component p^a for p prime means that we suppose $p^a \parallel N$, and thus $p^a \times \sigma(p^a) \mid 2N$ since $\gcd(p^a, \sigma(p^a)) = 1$. Then, if we do not reach a contradiction at this point, we recursively branch on a prime factor of N that has not yet been branched on. If there is no known other factor of N, we have a situation called *roadblock*, which is discussed below. Two types of the latter branching are also discussed below.

In this section, we branch on the overall largest available prime factor and use the following contradictions:

- The abundancy of the current number is strictly greater than 2.
- The current number is greater than 10^{1500}.

When branching on a prime p, we have to consider various cases depending on the multiplicity of p in N. We stop when the multiplicity a of p is such that $p^a > 10^{1500}$ and, except in the cases described below, we consider only the multiplicities a such that $a+1$ is prime. This is because $\sigma(p^a) \mid \sigma\left(p^{(a+1)t-1}\right)$, so any contradiction obtained thanks to the factors of $\sigma(p^a)$ when supposing $p^a \parallel N$ also gives a contradiction in the case $p^{(a+1)t-1} \parallel N$. So p^a is a representative for all $p^{(a+1)t-1}$, and to compute lower bounds on the abundancy or the size to test for contradictions, we suppose that the multiplicity of p is exactly a.

By-passing roadblocks. A *roadblock* is a situation such that there is no contradiction and no possibility to branch on a prime. This happens when we have already made suppositions for the multiplicity of all the known primes and the other numbers are composites. We use a method to circumvent roadblocks similar to the one used by Hare [4].

This method requires us to know an upper bound on the abundancy of the current number that is strictly smaller than 2. An obvious upper bound on the contribution of the component p^a to the abundancy is $\sigma_{-1}(p^\infty) = \frac{p}{p-1}$, but it might not always ensure that the bound on the abundancy of the current number is strictly smaller than 2. In order to obtain good enough upper bounds on the abundancy, we distinguish between *exact branchings* and *standard branchings*. Exact branchings concern the special component p^1, as well as 3^2, 3^4, and 7^2. Standard branchings concern everything else.

In the case of an exact branching on p^a, we suppose that $p^a \parallel N$, we use $\sigma_{-1}(p^a)$ for the abundancy, and we use an additional contradiction, occurring when p appears at least $a+1$ times in the factors of $\prod_{i=1}^{k} \sigma(p_i^{q_i})$, where $(p_1^{q_1}, \ldots, p_k^{q_k})$ is the sequence of considered branchings. In the case of a standard branching on p^a, we suppose that $p^{(a+1)t-1} \parallel N$ for a $t \geq 1$, and we use $\sigma_{-1}(p^\infty) = \frac{p}{p-1}$ as an upper bound on the abundancy.

Due to these exact branchings, we have to add standard branchings on 3^8, 3^{14}, 3^{24}, and 7^8 in order to cover all possible exponents for 3 and 7. Let us detail this for the base 3: we make exact branchings on 3^2 and 3^4, and standard branchings on 3^8, 3^{14}, 3^{24}, and 3^{p-1} for every prime $p \geq 7$. Then the case $3^{m-1} \parallel N$ for m odd is handled by 3^2 if $m=3$, by 3^4 if $m=5$, by 3^8 if $3^2 \mid m$, by 3^{14} if $3 \times 5 \mid m$, by 3^{24} if $5^2 \mid m$, and by 3^{p-1} if $p \mid m$. Note that we suppose that the branching for the special prime p^1 is always an exact branching, since if $p^{4k+1} \parallel N$ with $k \geq 1$, then this case will be handled by the standard branching p^{q-1}, where q is a factor of $2k+1$.

Finally, we have to consider abundancy of nonfactored composites. We check that the composite C has no factors less than α (we used $\alpha = 10^8$ for our computations), thus C has at most $\left\lfloor \frac{\ln(C)}{\ln(\alpha)} \right\rfloor$ different prime factors, each greater than α. Thus the abundancy contributed by C is at most $\left(\frac{\alpha}{\alpha-1}\right)^{\left\lfloor \frac{\ln(C)}{\ln(\alpha)} \right\rfloor}$.

Given a roadblock M, we compute an upper bound a on the abundancy. Our method to by-pass the roadblock only works if $a < 2$. That is why the exact branchings were suitably chosen to ensure that $a < 2$ for every roadblock.

Suppose that $a < 2$ and that there is an odd perfect number N divisible by M. Let p be the smallest prime which divides N and not M. Thus N has at least $t_a(p) := \left\lceil \frac{\ln\left(\frac{2}{a}\right)}{\ln\left(\frac{p}{p-1}\right)} \right\rceil$ distinct prime factors which do not divide M. Each of these factors has multiplicity at least 2, except for at most one (special) prime with multiplicity at least one. Thus, if $p^{2t_a(p)-1}$ is greater than $\frac{10^{1500}}{M}$, N is clearly greater than 10^{1500}.

Let $b = \max\left\{p : p^{2t_a(p)-1} \leq \frac{10^{1500}}{M}\right\}$, which is defined since $p \to p^{2t_a(p)-1}$ is strictly growing. To prove that there is no odd perfect number $N < 10^{1500}$ such that M divides N, we branch on every prime factor up to b to rule them out. We start to branch on the primes in S, since we already have good factor chains for theses numbers. We do not branch on a prime that divides M or that is already forbidden. When applying this method, we might encounter other roadblocks, because of composite number or because every "produced" prime already divides M. So we have to apply the method recursively.

Example. An example of by-pass two nested roadblocks is shown in Figure 1.

We first try to rule out 127 as a factor and encounter as a first roadblock $\sigma\left(127^{192}\right)$, which is a composite number with no known factors and no factors less than 10^8. Here, $M = 127^{192} \times \sigma\left(127^{192}\right) > 7 \times 10^{807}$. This composite number has at most $\left\lfloor \frac{\ln\left(\sigma\left(127^{192}\right)\right)}{\ln(10^8)} \right\rfloor = 50$ factors who contribute to the abundancy up to at most $C = (1+10^{-8})^{50} < 1 + 6 \times 10^{-7}$. As an upper bound on the abundancy, we thus have $a = \sigma_{-1}\left(127^\infty\right) \times \left(1 + 6 \times 10^{-7}\right) < 1.008$. We try every number until we

$$\begin{aligned}127^{192} &\Longrightarrow \sigma\left(127^{192}\right) \quad \text{Roadblock}\\ 19^2 &\Longrightarrow 3\times 127\\ 3^2 &\Longrightarrow 13\\ 13^1 &\Longrightarrow 2\times 7\\ 7^2 &\Longrightarrow 3\times 19 \quad \text{Roadblock 2}\end{aligned}$$

FIGURE 1. Example of two nested roadblock circumvents.

get $t_a(220) = 151$ and $220^{301} > 10^{705} > \frac{10^{1500}}{M}$. So, to get around this roadblock, we have to branch on every prime $p < 220$ except 127.

We start with 19, which is the next number in S, and then we get stuck with another roadblock ("Roadblock 2").

Here, $M' = 3^2 \times 7^2 \times 13^1 \times 19^2 \times 127^{192} \times \sigma\left(127^{192}\right) > 10^{814}$. As an upper bound on the abundancy, we have $a' = \sigma_{-1}\left(3^2 \times 7^2 \times 13^1 \times 19^\infty \times 127^\infty\right) \times C$. We thus have an upper bound $a' < 1.92522$. We try every number until we get $t_{a'}(2625) = 101$ and $2625^{201} > 10^{687} > \frac{10^{1500}}{M'}$. So, to get around this roadblock, we have to branch on every prime p such that $p < 2625$, except 3, 7, 13, 19 and 127. We continue to branch on other primes in S, and then on all other primes smaller than 2625.

This last example shows that exact branchings on 3^2 and 7^2 are necessary since $\sigma_{-1}\left(3^\infty \times 7^\infty \times 13^1 \times 19^\infty \times 127^\infty\right) > 2$. Notice also the exact branching on the special prime 13.

When N has no factors in S. Finally, we have to show that if N has no divisor in S, then $N > 10^{1500}$. We use the following argument, which is an improved version of the argument in [1]. For a prime p and an integer a, we define the *efficiency* $f(p,a)$ of the component p^a as $f(p,a) = \frac{\ln(\sigma_{-1}(p^a))}{\ln(p^a)}$. The efficiency is the ratio between the contribution in abundancy and the contribution in size of the component p^a. Both contributions are multiplicative increasings, which explains the logarithms.

Remark.

- $a < b \implies f(p,a) > f(p,b)$.
- $p < q \implies f(p,a) > f(q,a)$.

Notice that the best way to reach abundancy 2 and to keep N small is to take components with highest efficiency f:

- For each allowed prime p, we find the smallest exponent a such that $\sigma(p^a)$ is not divisible by 4 nor a factor in S. Example: Consider $p = 23$. $\sigma(23^1)$, $\sigma(23^2)$, $\sigma(23^3)$, are respectively divisible by 4, 7, 4. So the exponent of 23 is at least 4.
- We sort these components p^a by decreasing efficiency f to get an ordering $p_1,\ p_2,\ p_3, \ldots$ such that $f(p_1,a_1) \ge f(p_2,a_2) \ge f(p_3,a_3) \ge \ldots$.
- The product $\Pi_{i=1}^{200} \frac{p_i}{p_i-1} = 1.99785\ldots$ is smaller than 2, whereas the product $\Pi_{i=1}^{200} p_i^{a_i}$ is greater than 10^{1735}.

4. Total number of prime factors of an odd perfect number

Hare proved that an odd perfect number has at least 75 prime factors (counting multiplicities) [4].

Theorem 2. *The total number of prime factors of an odd perfect number is at least* 101.

We use the following contradictions:

- The abundancy of the current number is strictly greater than 2.
- The current number has at least 101 prime factors.

We forbid the factors in $S' = \{3, 5, 7, 11\}$, in this order. We branch on the smallest available prime. We still use a combination of exact branchings (for p^1, 3^2, and 3^4) and standard branchings, as in the previous section.

By-passing roadblocks. Given a roadblock M with at least g not necessarily distinct prime factors, we compute an upper bound a on the abundancy, as described in the previous section.

Suppose that $a < 2$ and that there is an odd perfect number N divisible by M. Let p be the smallest prime which divides N and not M. Thus N has at least $t_a(p)$ distinct prime factors which do not divide M. Each of these factors has multiplicity at least 2, except for at most one (special) prime with multiplicity at least one. Thus, if $2t_a(p) - 1$ is greater than $101 - g$, N has more than 101 not necessarily distinct prime factors. So we have a contradiction.

For the lower bound g of the not necessarily distinct prime factors, we compute the sum g_p of the exponents of the primes that have been branched on, and we add four times the number g_c of composites. Since we have checked that a composite is not a perfect power, it must be divided by two different primes, each having multiplicity at least two, except for at most one (the special prime). So we take $g = g_p + 4g_c$ or $g = g_p + 4g_c - 1$, depending on whether we have already branched on the special prime.

By the above, we can compute an upper bound on the smallest prime dividing N but not M. So, to prove that there is no odd perfect number with fewer than 101 not necessarily distinct prime factors such that M divides N, we branch on every prime factor up to this bound to rule them out. We do not branch on a prime that divides M or that is already forbidden. We have to resort to exact branchings as in the previous section, but this time only on 3^2 and 3^4.

When N has no factors in S'. We use a suitable notion of efficiency defined as $f'(p, a) = \frac{\ln(\sigma_{-1}(p^a))}{a}$. It is the ratio between the multiplicative contribution in abundancy and the additive contribution to the number of primes of the component p^a.

Remark.

- $a < b \implies f'(p, a) > f'(p, b)$.
- $p < q \implies f'(p, a) > f'(q, a)$.

Notice that the best way to reach abundancy 2 with the fewest primes is to take components with highest efficiency f':

- For each allowed prime p, we find the smallest exponent a such that $\sigma(p^a)$ is not divisible by 4 nor a factor in S'.

- We sort these components p^a by decreasing efficiency f' to get an ordering $p_1,\ p_2,\ p_3, \ldots$ such that $f'(p_1, a_1) \geq f'(p_2, a_2) \geq f'(p_3, a_3) \geq \ldots$.
- The product $\Pi_{i=1}^{49} \frac{p_i}{p_i - 1} = 1.99601\ldots$ is smaller than 2, whereas $\Sigma_{i=1}^{49} a_i = 118$.

5. Largest component of an odd perfect number

Cohen [2] proved in 1987 that an odd perfect number has a component greater than 10^{20}.

Theorem 3. *The largest component of an odd perfect number is greater than* 10^{62}.

We use the same algorithm as in the previous section to forbid every prime less than 10^8 using the following contradictions:

- The abundancy of the current number is strictly greater than 2.
- The current number has a component greater than 10^{62}.

Since we want to quickly reach a large component, we branch on the largest available prime. There is no unfactored composite here, and thus no roadblock, since every number is less than 10^{62} and thus has been easily factored.

Suppose now that N is an odd perfect number with no prime factor less than 10^8 and no component $p^e > 10^{62}$. First, the exponent e of any prime factor p is less than 8, since otherwise $p^e > (10^8)^8 > 10^{62}$. The exponent of the special prime p_1 is thus 1, because $3 \mid \sigma(p^5)$ and $3 \nmid N$. So N has a prime decomposition $N = p_1 \prod_{i=1}^{n_2} p_{i,2}^2 \prod_{i=1}^{n_4} p_{i,4}^4 \prod_{i=1}^{n_6} p_{i,6}^6$.

Let $\pi(x)$ denote the number of primes less than or equal to x. In the following, we will use these known values of $\pi(x)$ [8]:

- $\pi(10^8) = 5761455$,
- $\pi(3 \times 10^{10}) = 1300005926$,
- $\pi(32 \times 10^{14}) = 92295556538011$,
- $\pi(98 \times 10^{14}) = 273808176380030$.

It is well known (see [6]) that for primes q, r, and s such that $q \mid \sigma(r^{s-1})$, either $q = s$ or $q \equiv 1 \bmod s$. So if $p_{j,e'} \mid \sigma(p_{i,e}^e)$, then $p_{j,e'} \equiv 1 \bmod (e+1)$, since $(e+1) \nmid N$. We thus have $e' \neq e$, since otherwise $e+1$ would divide $\sigma(p_{j,e}^e)$ (that is, $\sigma(p_{j,e'}^{e'})$), but not N. Moreover, $\sigma(p_{i,e}^e)$ cannot be prime unless it is the special prime p_1. Suppose to the contrary that $\sigma(p_{i,e}^e) = p_{j,e'}$. Then $p_{j,e'}^{e'}$ is a component of N. Since $e' \neq e$, we have that $ee' \geq 8$, so that $p_{j,e'}^{e'} = (\sigma(p_{i,e}^e))^{e'} > (p_{i,e}^e)^{e'} = (p_{i,e})^{ee'} > (10^8)^8 > 10^{62}$. So each $\sigma(p_{i,e}^e)$ produces at least two factors or the special prime.

Let $n_{2,2}$ be the number of primes $p_{i,2}$ such that $\sigma(p_{i,2}^2) = q \times r$ where $q < r$, q and r primes. Let $n_{2,3}$ be the number of primes $p_{i,2}$ such that $\sigma(p_{i,2}^2)$ factors into at least three not necessarily distinct primes.

By the above, we have

$$n_2 \leq n_{2,2} + n_{2,3} + 1. \tag{1}$$

By counting the number of primes produced by the factors $\sigma(p_{i,2}^2)$, we obtain

$$2n_{2,2} + 3n_{2,3} \leq 4n_4 + 6n_6 + 1. \tag{2}$$

For $e \in \{4, 6\}$, we have $p_{i,e} < 32\times10^{14}$, since otherwise $p_{i,e}^e > (32\times10^{14})^4 > 10^{62}$. Suppose that a prime $p_{i,2}$ is such that $\sigma(p_{i,2}^2) = q \times r$ where $q < r$, q and r primes.

Then we have that $r > p_{i,2}$, and by previous discussion, either $r = p_1$ or $r = p_{i',e}$ for $e \in \{4, 6\}$. This implies that at least $(n_{2,2} - 1)$ primes $p_{i,2}$ are smaller than the largest prime $p_{i,e}$ for $e \in \{4, 6\}$. So, $n_{2,2} - 1 + n_4 + n_6 \leq \pi(32 \times 10^{14}) - \pi(10^8) = 92295550776556$ which gives

$$n_{2,2} + n_4 + n_6 \leq 92295550776557. \tag{3}$$

Similarly, $p_{i,6} < 3 \times 10^{10}$ since otherwise $p_{i,6}^6 > 10^{62}$. So, $n_6 \leq \pi(3 \times 10^{10}) - \pi(10^8)$, which gives

$$n_6 \leq 1294244471. \tag{4}$$

Now, we consider an upper bound on the abundancy of primes greater than 10^8. We use equation (3.29) in [7],

$$\prod_{\substack{p<x \\ p \text{ prime}}} \frac{p}{p-1} < e^\gamma \ln(x) \left(1 + \frac{1}{2\ln^2(x)}\right)$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant. We compute that

$$\prod_{\substack{p<10^8 \\ p \text{ prime}}} \frac{p}{p-1} > c_1 = 32.80869860873870116$$

and we obtain

$$\prod_{\substack{10^8<p<98\times 10^{14} \\ p \text{ prime}}} \frac{p}{p-1} < e^\gamma \ln(98 \times 10^{14}) \left(1 + \frac{1}{2\ln^2(98 \times 10^{14})}\right) / c_1 < 2.$$

By the above, we have $1+n_2+n_4+n_6 > \pi(98\times 10^{14}) - \pi(10^8) = 273808170618575$, which gives,

$$273808170618575 \leq n_2 + n_4 + n_6. \tag{5}$$

The combination $3 \times \mathbf{(1)} + 1 \times \mathbf{(2)} + 7 \times \mathbf{(3)} + 2 \times \mathbf{(4)} + 3 \times \mathbf{(5)}$ gives $6n_{2,2} + 175353067930880 \leq 0$, a contradiction.

6. Improvements over previous methods

This paper provides a unified framework to obtain lower bounds on three parameters of an odd perfect number: the OPN itself, the total number of prime factors, and the largest component. These parameters are well-suited because a bound on the parameter implies an obvious and reasonable bound on the exponent of a prime factor of an OPN. That is not the case for other parameters of interest, such as the largest prime factor or the number of distinct prime factors.

The most useful new tool is the way to get around roadblocks in the proof of Theorem 1. The argument to obtain a bound on the smallest not-yet-considered prime is an adaptation of the one in [4]. In both cases it implies a bound b, an exponent t, an inequality related to the abundancy, and an inequality related to the corresponding parameter. The argument is more sophisticated in the context of a bound on the size rather than on the total number of primes, because both b and t are involved in both inequalities.

Brent *et al.* [1] used standard branchings and Hare [4] used exact branchings. We introduce the use of a combination of standard and exact branchings to reduce

	# Branch.	# Branch. (circ.)	Approx. time
Theorem 1	22 514 255	10 406 935	12 hours
Theorem 2	447 019 005	444 022	93 hours
Theorem 3	6 574 758	0	30 minutes

FIGURE 2. Total number of branchings, number of branchings in roadblocks circumventing, and approximate time.

the size of the proof tree. Standard branchings are economical but exact branchings are sometimes unavoidable when we have to by-pass a roadblock.

In the final phase of the proof of Theorems 1 and 2, we have to argue that an odd perfect number with no factors in a set of small forbidden primes necessarily violate the corresponding bound. When the bound increases, the set of forbidden primes must get larger. Suitable notions of efficiency of a component are introduced in order to restrain the growth of this set. They allow a better use of the fact that some primes are forbidden, by considering the exponent of the remaining potential prime factors.

Finally, we give a proof of Theorem 3 using a system of inequalities. The idea behind it is as follows. If all primes up to B are forbidden, then the largest prime factor must be at least B^2 in order to reach abundancy 2. Then we use various arguments and inequalities in order to show that a not too small proportion C of the prime factors have exponent at least 4. Then we conclude that a component of size at least $(C \times B^2)^4 = C' \times B^8$ exists.

We would like to point out the importance of separating the search for factors, with efficient dedicated software, from the generation of the proof tree. In particular, this generates most of the improvement to Theorem 2.

Credit and acknowledgements

The program was written in C++ and uses GMP. The program and the factors are available at `http://www.lri.fr/~ochem/opn/`.

We present in Figure 2 the number of branchings on prime factors (overall and needed in circumvents of roadblocks), and the time needed on a AMD Phenom(tm) II X4 945 to process the tree of suppositions for each theorem. Of course, this does not take into account the time needed to find the factors.

Various software and algorithms were used for the factorizations:

- GMP-ECM for P-1, P+1 and ECM,
- msieve and yafu for MPQS,
- msieve combined with GGNFS for NFS (both general and special).

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