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MULTILEVEL PRECONDITIONING FOR THE FINITE VOLUME METHOD

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ABSTRACT. We consider the precondition of linear systems which resulted from the finite volume method (FVM) for elliptic boundary value problems. With the help of the interpolation operator from the trial space to the test space of the FVM and the operator induced by the FVM bilinear form, we show that both wavelet preconditioners and multilevel preconditioners designed originally for the finite element method (FEM) of a boundary value problem can be used to precondition the FVM of the same boundary value problem. We prove that such preconditioners ensure that the resulting coefficient matrix of the FVM has a uniformly bounded condition number. We present seven numerical examples to confirm our theoretical findings.

1. INTRODUCTION

The *finite volume method* (FVM) has been intensively studied for several decades. An incomplete list of references for the method includes [1, 3, 4, 6, 7, 9, 10, 11, 12, 21, 26, 27, 28, 29, 30, 31, 36, 37, 48, 49] and the references cited therein. Because of its local conservation of certain physical quantities and its convenience in numerical implementation, it has been widely used in engineering computation. Similarly to the traditional finite element method (FEM), the FVM suffers from the ill-condition of its coefficient matrix. As a result, the numerical solution of the resulting linear system is not stable. To efficiently solve the resulting linear system of FVM, it is crucial to use a preconditioning technique. The main purpose of this paper is to present multilevel preconditioning schemes for the FVM for solving boundary value problems of elliptic partial differential equations of the second order.

The traditional finite element method for solving differential equations has an obvious shortcoming. Its coefficient matrices are not well-conditioned. Preconditioning for such matrices received considerable attention in the last two decades.

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For example, the hierarchical preconditioner was proposed in [46, 47]. The BPX multilevel preconditioner was introduced in [2, 40]. The wavelet preconditioners were studied in [15, 16, 33]. For other aspects of the wavelet method in solving differential equations, see [8, 17, 18, 44, 45] and the references cited therein. Both BPX preconditioner and wavelet preconditioner lead to matrices with uniformly bounded condition numbers.

The basic idea of this paper is to use the wavelet preconditioners developed in [15] and the multilevel preconditioners developed in [2, 38] originally for the Galerkin FEM to precondition the FVM for the same boundary value problem. It is known that the FVM is different from the Galerkin FEM in its test space. In the Galerkin FEM, the test space is the same as the trial space, while in the FVM, the test space is spanned by piecewise polynomials of a lower order with no global continuity requirement. Such a construction of the test space increases flexibility and convenience in computation. Noticing that the trial space and the test space have the same dimensions, a one-to-one mapping was originally introduced in [27] from the trial space to test space. With this mapping, we shall show that a change of basis in the trial space *only* is sufficient to precondition the FVM schemes. The main purpose of this paper is to develop a general theoretical base for this preconditioning and to confirm theoretical results numerically for several examples of practical importance.

Due to the construction in the trial space and the test space, in general the resulting FVM matrix is nonsymmetric. In this paper, we suggest that the GMRES algorithm be used to solve the corresponding nonsymmetric FVM linear system. For this reason, we also consider the preconditioning of the GMRES algorithm for solving the nonsymmetric linear system resulted from the FVM.

This paper is organized in six sections. In Section 2, we describe the finite volume method for elliptic equations of the second order. In Section 3, we precondition the operator induced by the FVM bilinear form, while in Section 4, we precondition the corresponding finite volume matrix. We present two types of preconditioners. One is obtained by a change basis matrix from the nodal basis to a wavelet (or pre-wavelet) basis. The other one is obtained by using the BPX preconditioning technique presented in [2, 38, 40]. The uniform boundedness of the condition numbers of the preconditioned operator and the preconditioned matrix is shown for both methods. In Section 5, we precondition the FVM when the linear element is used for the elliptic equation having discontinuous coefficients. We prove that in this special case the form of the FVM is equivalent to that of FEM, independent of the jumps in the coefficient. Therefore, the efficient preconditioners for the linear FEM are also efficient for the linear FVM. In Sections 6, we consider special cases of the FVM and present seven examples to confirm the theoretical results concerning the preconditioning. Cases one and six are for the two point boundary value problem. Cases two, three and four are concerned with the Laplace equation using linear elements on rectangle grids, linear elements on triangle grids, and quadratic elements on triangular grids, respectively. Cases five and seven consider, respectively, the elliptic equation with discontinuous coefficients and the Laplace equation using a special quadratic FVM scheme constructed recently in [5]. The numerical results presented in this section confirm the uniform boundedness of the condition numbers of the preconditioned matrices.

2. The finite volume method

In this section, we set up a theoretical setting of the FVM for solving elliptic boundary value problems. Let d be a fixed positive integer and $\Omega \subset \mathbf{R}^d$ a bounded open convex polyhedral with boundary $\partial\Omega$. Let $A(x)$, $x \in \Omega$, be a $d \times d$ symmetric matrix whose entries are functions on Ω. We assume that the matrix A satisfies the uniformly elliptic condition, that is, there exists a constant $r > 0$ such that

$$\xi^T A(x)\xi \geq r\xi^T\xi, \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad \xi \in \mathbf{R}^d.$$

We define an elliptic partial differential operator of the second order by

$$Lu := - \bigtriangledown \cdot (A(x) \bigtriangledown u), \tag{2.1}$$

and for a given function $f \in L^2(\Omega)$ we consider the boundary value problem

$$Lu = f \quad \text{in} \quad \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

For the purpose of constructing the trial space and the test space for the FVM, we now present a primary partition and its corresponding dual partition of the domain Ω. For $n \in \mathbf{N}_0 := \{0, 1, \ldots\}$, we let T_n denote a family of *perfect* polyhedral primary partitions of Ω. That is,

$$\Omega = \bigcup_{K \in T_n} K,$$

where $K \in T_n$ is a polyhedral and the intersection of any two elements K_1, $K_2 \in T_n$ is either an empty set, a common vertex, or a common edge (face). For $K \in T_n$, we set

$$h_K := \operatorname{diam}(K) \quad \text{and} \quad h_n := \max\{h_K : K \in T_n\}.$$

We require that the partitions T_n are *quasi-uniform* (cf. [30]) in the sense that there exist positive constants c_1 and c_2 independent of h_n such that for all $K \in T_n$,

$$c_1 h_n^d \leq \operatorname{vol}(K) \leq c_2 h_n^d, \tag{2.4}$$

where $\operatorname{vol}(K)$ denotes the volume of the element K. We also demand the partitions T_n to be *regular* in the usual sense. To explain the regularity condition, we describe the regularity for the triangular partition and the quadrilateral partition in the case when $d = 2$. For $d = 2$ we set

$$\rho_K := \sup\{\operatorname{diam}(S) : S \text{ is a disk contained in } K\},$$

and let h'_K denote the length of the smallest side of K and $\Theta(K)$ the set of interior angles of K. In particular, we say a triangulation partition T_n is regular (cf. [13]), if there is a positive constant σ independent of h_n such that

$$\frac{h'_K}{\rho_K} \leq \sigma, \quad \text{for all} \quad K \in T_n \text{ and for all } n \in \mathbf{N}_0. \tag{2.5}$$

If $\theta_K \in \Theta(K)$ denotes the smallest angle, then (2.5) is equivalent to the following statement that there exists a constant $\theta_0 > 0$ such that $\theta_K \geq \theta_0$ for all $K \in T_n$. Likewise, we say that a quadrilateral partition T_n is regular, if there exist positive constants σ and θ_0 such that for all $K \in T_n$, all $n \in \mathbf{N}_0$ and all $\theta \in \Theta(K)$,

$$\frac{h_K}{h'_K} \leq \sigma \quad \text{and} \quad \theta_0 \leq \theta \leq \pi - \theta_0. \tag{2.6}$$

Associated with a primary partition T_n, we construct a dual partition T_n^* of the domain Ω. The Lagrange interpolation points (or Hermite interpolation points) of any element $K \in T_n$ are chosen as the nodes of K. We denote by Ω_n the set of all the nodes of all elements $K \in T_n$. For each $P \in \Omega_n$, we construct a *control volume* (or dual element) K_p^* which is often a polyhedral surrounding P. Usually, we require that any two different control volumes do not overlap in the sense that

$$\operatorname{meas}(K_P \cap K_{P'}) = 0, \quad \text{if} \quad P \neq P'.$$

The union of all control volumes constitutes another partition T_n^* of Ω which is called the dual partition of T_n. In general, we require T_n^* to be perfect and quasi-uniform. There are different constructions of control volumes and dual partitions. For instance, when $\Omega \subset \mathbf{R}^2$, we have the barycenter dual partition and the circumcenter dual partition (Voronoi mesh) (cf. [1, 3, 4, 27, 30]). Recently, the dual partition was constructed in [5] using only the control volumes corresponding to the nodes in a true subset of Ω_n.

We next describe the trial space associated with the primary partition T_n and the test space associated with the dual partition T_n^*. Generally speaking, the trial space U_n can be a finite element space on the grid T_n and the test space V_n is an appropriate piecewise polynomial space on the dual grid T_n^*. For example, when the trial space has a Lagrange type basis, the corresponding test space may be spanned by the characteristic functions of the control volumes on the dual partition T_n^*. When the trial space has a Hermite type basis, the corresponding test space may be spanned by the "generalized" characteristic functions of the control volumes on the dual partition T_n^*, where the generalized characteristic function is zero outside the control volume and within it the polynomial in the term of the Taylor expansion for which the corresponding derivative is used in the construction of the basis function for the trial space. A general construction of the trial space and the test space of the FVM using higher order polynomials was described recently in [9], where geometric conditions on the elements and equivalent morn conditions were given to ensure the resulting FVM bilinear form is bounded and positive definite.

Specifically, the trial space U_n can be a conforming or nonconforming Lagrange type or Hermite type finite element space. For simplicity, in this paper we restrict U_n to a conforming finite element space defined by

$$U_n := \left\{u_n \in C^{k-1}(\overline{\Omega}) : u_n|_K \in \mathbf{P}_r, \quad \text{for all } K \in T_n, \quad u_n|_{\partial\Omega} = 0\right\} \tag{2.7}$$

where for a positive integer r, $\mathbf{P}_r$ is the space of polynomials of total degree at most r. That is, we choose U_n as the space of piecewise polynomials of total degree at most r over T_n, which globally belong to the Sobolev space $H^k(\Omega)$ and satisfy the homogeneous boundary condition on $\partial\Omega$. When $k = 1$, U_n is a Lagrange finite element space which has the basis $\Phi_n := \{\phi_P : P \in \Omega_n\}$ where $\phi_P|_K \in \mathbf{P}_r$, for $K \in T_n$ and $\phi_P(P') = \delta_{P,P'}$, for $P, P' \in \Omega_n$. When $k \geq 2$, or equivalently some directional derivatives of the functions in U_n are continuous, U_n is a Hermite finite element space. For simplicity we use the same notation $\Phi_n = \{\phi_P : P \in \Omega_n\}$ for a basis of U_n even when U_n is a Hermite finite element space, although in this case ϕ_P is not necessarily a nodal basis function or Ω_n is not necessarily a set of the nodes in T_n, but an abstract index set.

The test space V_n is defined associated with the dual partition T_n^* by

$$V_n := \left\{v_n \in L^2(\overline{\Omega}) : v_n|_{K^*} \in \mathbf{P}_{r'}, \text{for all } K^* \in T_n^*, v_n|_{\partial\Omega} = 0\right\}. \tag{2.8}$$

Usually, we choose $0 \leq r' < r$ and require $\dim(U_n) = \dim(V_n) = D(n)$. Note that we allow a test function to have the regularity order lower than that for a trial function. Clearly, V_n contains piecewise polynomials of total degree r' associated with the dual partition T_n^* and $V_n \subset L^2(\overline{\Omega}) = W^{0,2}(\Omega)$. Since spaces U_n and V_n have the same dimension, we denote by $\tilde{\Phi}_n := \{\psi_P : P \in \Omega_n\}$ a basis for V_n.

When the basis of the trial space U_n is chosen as a Lagrange type, we often let $r' = 0$ and choose the basis of V_n as the characteristic functions on the dual element. Namely, we choose $\tilde{\Phi}_n = \{\chi_P : P \in \Omega_n\}$ where for each $P \in \Omega_n$, χ_P denotes the characteristic function on the control volume K_P^*. When U_n is a Hermite type finite element space, the basis of V_n might be chosen as the "generalized" characteristic functions of the dual elements which is zero outside the dual element and within the dual element is the polynomials in the terms of the Taylor expansions for which the corresponding derivatives are used in the construction of the basis function for the trial space.(cf., [9, 27, 30]). For instance, when $\Omega \subset \mathbf{R}^2$ and U_n is the Zienkiewicz finite element space where the function values and the first partial derivatives at the nodes are used in the construction of the basis, the dual basis is chosen as

$$\tilde{\Phi}_n = \{\chi_P, (x_i - P_i)\chi_P : P := (P_1, P_2) \in \Omega_n, i = 1, 2\},$$

because the corresponding Taylor polynomial of degree one consists of the constant term and the first degree terms, where $x := (x_1, x_2) \in \mathbf{R}^2$. The functions in the set $\tilde{\Phi}_n$ are the typical examples of the generalized functions.

To describe the FVM, we introduce the bilinear form for equations (2.2) and (2.3)

$$a_n(u, v_n) := \sum_{K^* \in T_n^*} \int_{K^*} (A \bigtriangledown u) \cdot \bigtriangledown v_n dx - \sum_{K^* \in T_n^*} \int_{\partial K^*} (A \bigtriangledown u) \cdot \mathbf{n}\, v_n ds, \tag{2.9}$$

where $u \in H_0^1(\Omega)$ and $v_n \in V_n$ and $\mathbf{n}$ is the normal vector outward to K^*, (cf., [30]). We shall call $a_n(\cdot, \cdot)$ the FVM bilinear form. Note that unlike the FEM bilinear form which is independent of the partition, the FVM bilinear form a_n depends on the the dual partition T_n^*. Moreover, when $r' = 0$, the test space reduces to the usual piecewise constant space and a_n reduces to

$$a_n(u, v_n) = -\sum_{K^* \in T_n^*} \int_{\partial K^*} (A \bigtriangledown u) \cdot \mathbf{n}\, v_n ds.$$

By letting

$$(f, v_n) := \sum_{K^* \in T_n^*} \int_{K^*} f(x) v_n(x) dx, \quad \text{for} \quad v_n \in V_n,$$

the FVM is a numerical scheme to find $u_n \in U_n$ such that

$$a_n(u_n, v_n) = (f, v_n), \quad \text{for all } v_n \in V_n. \tag{2.10}$$

The operator and the matrix resulted from the bilinear form $a_n(u_n, v_n)$ are both ill-conditioned. The main purpose of this paper is to study multilevel preconditioning of the resulting operator and the corresponding matrix.

We find it convenient to use an interpolation projector Π_n from the trial space U_n onto the test space V_n by

$$\Pi_n u_n := \sum_{P \in \Omega_n} \alpha_P \psi_P, \quad \text{for} \quad u_n = \sum_{P \in \Omega_n} \alpha_P \phi_P \in U_n, \tag{2.11}$$

which was originally introduced in [27] (see also, [30]). With this projector, we have that $\Pi_n \phi_P = \psi_P$, for all $P \in \Omega_n$. Because spaces U_n and V_n have the same dimension, we observe that Π_n is invertible. In terms of this operator, the system (2.10) is equivalent to

$$a_n(u_n, \Pi_n w_n) = (f, \Pi_n w_n), \quad \text{for all} \quad w_n \in U_n. \tag{2.12}$$

We remark that unlike the finite element method, the bilinear form $a_n(\cdot, \Pi_n \cdot)$ is *not* symmetric in general, that is,

$$a_n(u_n, \Pi_n w_n) \neq a_n(w_n, \Pi_n u_n), \quad u_n, w_n \in U_n.$$

It is this lack of symmetry that requires special attention for the analysis of multilevel preconditioning.

It is well known (cf. [9, 20, 30, 43]) that the convergence of FVM depends on the boundedness and positive definiteness of its bilinear form. In this paper we assume that the bilinear form is bounded and positive definite. Specifically, we assume that there exist positive constants N and M_0 such that for all $n \geq N$,

$$|a_n(u_n, \Pi_n w_n)| \leq M_0 \|u_n\|_1 \|w_n\|_1, \quad \text{for all} \quad u_n, w_n \in U_n. \tag{2.13}$$

The bound (2.13) of the bilinear form for various FVM schemes appeared in [9, 20, 30, 43]. We also suppose that there exist positive constants γ_0 and N such that for all $n \geq N$,

$$a_n(u_n, \Pi_n u_n) \geq \gamma_0 \|u_n\|_1^2, \quad \text{for all} \quad u_n \in U_n. \tag{2.14}$$

For the linear FVM, (2.14) is a well-known result (see [1, 3, 27, 29, 30, 48]). Proving (2.14) for the higher order or high-dimensional FVM schemes is a challenging task. It was established in [30] for quadratic and cubic FVMs under some restrictions on the primary triangulation. For the quadratic FVM case, (2.14) was proved in [43] under rather weak conditions on the geometry of the triangles in the partition. General geometric conditions which ensure the validity of (2.13) and (2.14) for higher order FVM schemes for the two-dimensional second order equations was presented recently in [9]. The geometric conditions and norm equivalence arguments presented in [9] improve the existing results for the known FVM schemes and provide new results for a large class of FVM schemes which are not known in the literature. Note that the boundedness condition (2.13) and the positive definiteness (2.14) of the bilinear form lead to the error estimate of the FVM solution. For more information on convergence of the FVM, the readers are referred to [9, 43].

We close this section by providing a two-dimensional quadratic FVM scheme (cf., [9, 30, 43]). Suppose that $\Omega \subset \mathbf{R}^2$ is a polygonal domain and let T_n denote a regular and quasi-uniform triangulation of Ω. That is, T_n consists of a finite number of triangle elements K_Q, where Q is the barycenter of the triangle. We denote by $\overline{\Omega}_n$, $\overline{M}_n$ and Ω_n^*, respectively, the set of the vertices, the set of the midpoints of the common side of two adjacent triangle elements, and the set of the barycenters of the triangle elements in T_n. Moreover, we set $\overset{\circ}{\Omega}_n = \overline{\Omega}_n \setminus \partial\Omega$ and $\overset{\circ}{M}_n = \overline{M}_n \setminus \partial\Omega$. The dual partition T_n^* consists of the polygons $K_{P_0}^*$ surrounding $P_0 \in \overline{\Omega}_n$ and K_M^* surrounding $M \in \overline{M}_n$. The polygons $K_{P_0}^*$ and K_M^* are called the dual elements. Figures 1 and 2 illustrate these constructions. Figure 1 gives an example for the construction of $K_{P_0}^*$, where $P_0 \in \overset{\circ}{\Omega}_n$, P_i, $i = 1, 2, \ldots, 7$, are its adjacent vertices, and that P_{0i} is a point on the line segment joining P_0 and P_i such that $\overline{P_0 P_{0i}} = \frac{1}{3}\, \overline{P_0 P_i}$, where $\overline{PQ}$ denotes the length of the line segment joining points P and Q. We connect P_{0i}, $i = 1, 2, \ldots, 7$, successively to form the polygon $K_{P_0}^*$ which surrounds P_0. Figure 2 is to illustrate the construction of K_M^*, where

$M \in \overline{M}_n$ is the midpoint of the common side of two adjacent triangle elements $K_{Q_1} := \Delta P_0P_1P_2$ and $K_{Q_2} := \Delta P_0P_1P_3$. We denote by $Q_{12}, Q_{13}, Q_{02}, Q_{03}$ the midpoints of $\overline{P_{01}P_{02}}$, $\overline{P_{01}P_{03}}$, $\overline{P_{10}P_{12}}$ and $\overline{P_{10}P_{13}}$, respectively. The polygon K^*_M surrounding M is obtained by connecting P_{10}, Q_{03}, Q_2, Q_{13}, P_{01}, Q_{12}, Q_1, Q_{02} and P_{10} successively.

In this special case, the trial space U_n is chosen as the space spanned by the Lagrange quadratic elements with respect to the triangulation T_n. Specifically, for each $P_0 \in \mathring{\Omega}_n$ and $M_0 \in \mathring{M}_n$, the corresponding basis functions are the piecewise quadratic polynomials satisfying the interpolation conditions

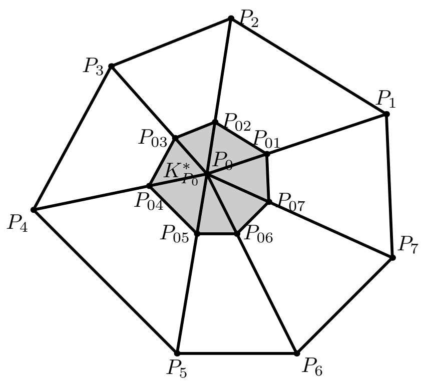

FIGURE 1. The Dual Element$K^*_{P_0}$

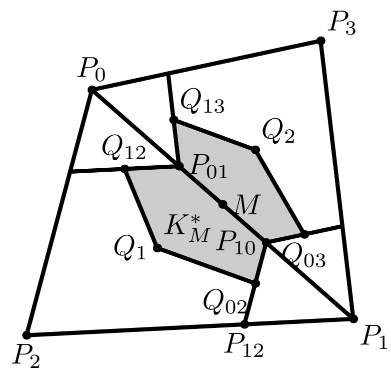

FIGURE 2. The Dual ElementK^*_M

$$\varphi_{P_0}(P) = \begin{cases} 1, & P = P_0, \\ 0, & P \in (\overline{\Omega}_n \cup \overline{M}_n) \setminus \{P_0\} \end{cases}$$

and

$$\varphi_{M_0}(P) = \begin{cases} 1, & P = M_0, \\ 0, & P \in (\overline{\Omega}_n \cup \overline{M}_n) \setminus \{M_0\}, \end{cases}$$

respectively. Thus, we have that

$$U_n := \operatorname{span}\{\varphi_{P_0}, \varphi_{M_0} : P_0 \in \mathring{\Omega}_n, M_0 \in \mathring{M}_n\}.$$

While the test space V_n is the piecewise constant function space with respect to the dual partition T^*_n. Namely, for each $P_0 \in \mathring{\Omega}_n$ and $M_0 \in \mathring{M}_n$, the corresponding basis functions are the characteristic functions of $K^*_{P_0}$ and $K^*_{M_0}$, respectively; that is, $\psi_{P_0}(P) = \chi_{K^*_{P_0}}(P)$ and $\psi_{M_0}(P) = \chi_{K^*_{M_0}}(P)$, $P \in \Omega$. Hence, we obtain that

$$V_n = \operatorname{span}\{\psi_{P_0}, \psi_{M_0} : P_0 \in \mathring{\Omega}_n, M_0 \in \mathring{M}_n\}.$$

The corresponding FVM bilinear form is given by

$$\begin{aligned} a_n(u_n, v_n) = & - \sum_{P_0 \in \mathring{\Omega}_n} v_n(P_0) \int_{\partial K^*_{P_0}} (A \nabla u_n \cdot \mathbf{n}) \psi_{P_0} ds \\ & - \sum_{M \in \mathring{M}_n} v_n(M) \int_{\partial K^*_M} (A \nabla u_n \cdot \mathbf{n}) \psi_M ds \end{aligned}$$

and the quadratic finite volume scheme for (2.2) and (2.3) is to find $u_n \in U_n$ such that

$$a_n(u_n, \psi_{P_0}) = (f, \psi_{P_0}), \quad P_0 \in \overset{\circ}{\Omega}_n, \quad a_n(u_n, \psi_M) = (f, \psi_M), \quad M \in \overset{\circ}{M}_n\ .$$

3. Preconditioning of the FVM operator

This section is devoted to the development of a multilevel preconditioning for the operator induced by the FVM bilinear form a_n. We will study both two-sided and one-sided preconditioning.

We first introduce the operator corresponding to the bilinear form $a_n(\cdot, \Pi_n \cdot)$. The continuity (2.13) of the bilinear form $a_n(\cdot, \Pi_n \cdot)$ and the Riesz representation theorem imply that there exists a unique bounded operator $\mathcal{A}_n : U_n \to U_n$ such that for all $u_n, w_n \in U_n$,

$$(\mathcal{A}_n u_n, w_n) = a_n(u_n, \Pi_n w_n). \tag{3.1}$$

We call $\mathcal{A}_n$ the FVM operator. Note that, in general, the FVM operator is *not* self-adjoint due to the nonsymmetry of the bilinear form $a_n(\cdot, \Pi_n \cdot)$. It is also clear that when inequality (2.14) holds, the operator $\mathcal{A}_n$ is positive definite. Now, by the Riesz representation theorem, we also have $F_n \in U_n$, such that

$$(F_n, w_n) = (f, \Pi_n w_n), \quad \text{for all} \quad w_n \in U_n. \tag{3.2}$$

By (3.1) and (3.2), equation (2.12) has an equivalent operator form

$$\mathcal{A}_n u_n = F_n. \tag{3.3}$$

The main purpose of this section is to precondition the nonself-adjoint operator $\mathcal{A}_n$ in the L_2 norm. To this end, we first present a property of operator $\mathcal{A}_n$.

Lemma 3.1. *If bilinear form $a_n(\cdot, \cdot)$ satisfies hypotheses* (2.13) *and* (2.14)*, then, for each $n \in \mathbf{N}$, $\mathcal{A}_n$ is bounded from L_2 to L_2 and $\mathcal{A}_n^{-1}$ exists as a bounded operator from L_2 to L_2.*

Proof. Since U_n is a finite dimensional space, there exist two positive constants α_n, β_n, such that

$$\alpha_n \|v_n\|_0 \le \|v_n\|_1 \le \beta_n \|v_n\|_0, \quad \text{for all } v_n \in U_n. \tag{3.4}$$

Thus, the L_2 boundedness of $\mathcal{A}_n$ follows directly from inequalities (2.13) and (3.4). The existence of $\mathcal{A}_n^{-1}$ in L_2 is obtained from the inequality (2.14). Moreover, inequalities (2.14) and (3.4) imply the boundedness of $\mathcal{A}_n^{-1}$ in L_2. □

We define the condition number of an operator. The operator norm of $\mathcal{B} : U_n \to U_n$ induced by the L_2 norm $\|\cdot\|_0$ and the finite dimensional subspace U_n is defined by

$$\|\mathcal{B}\|_0 := \sup_{g \in U_n} \frac{(\mathcal{B}g, g)}{(g, g)}.$$

When $\mathcal{B}^{-1} : U_n \to U_n$ exists as a bounded linear operator, we let $\kappa(\mathcal{B})$ denote the condition number of operator $\mathcal{B}$, i.e., $\kappa(\mathcal{B}) := \|\mathcal{B}\|_0 \|\mathcal{B}^{-1}\|_0$. Notice that constants α_n and β_n in the proof of Lemma 3.1 may depend on n, the condition number of $\mathcal{A}_n$ are not necessarily uniformly bounded. This motivates preconditioning $\mathcal{A}_n$.

Two sequences of operators, $\mathcal{B}_n^1$ and $\mathcal{B}_n^2$, $n \in \mathbf{N}$, are equivalent, if there exist two positive constants c_1, c_2 such that for all $n \in \mathbf{N}$,

$$c_1(\mathcal{B}_n^1 g, g) \le (\mathcal{B}_n^2 g, g) \le c_2(\mathcal{B}_n^1 g, g), \quad \text{for all} \quad g \in U_n. \tag{3.5}$$

When $\mathcal{B}_n^1$ and $\mathcal{B}_n^2$, $n \in \mathbf{N}$, are equivalent, we write $\mathcal{B}_n^1 \sim \mathcal{B}_n^2$. A common approach to precondition a sequence of operators $\mathcal{A}_n$ is to find a sequence of operators $\mathcal{C}_n$ such that $\mathcal{C}_n^{-1} \sim \mathcal{A}_n$.

Lemma 3.2. *If bilinear form $a_n(\cdot,\cdot)$ satisfies hypotheses* (2.13) *and* (2.14)*, then the following statements are equivalent.*

(i) *There exist two positive constants $\gamma \leq \Gamma$ and a sequence of positive definite operators $\mathcal{C}_n, n \in \mathbf{N}$, such that*

$$\gamma(\mathcal{C}_n^{-1}g,g) \leq \|g\|_1^2 \leq \Gamma(\mathcal{C}_n^{-1}g,g), \quad \text{for all} \quad g \in U_n. \tag{3.6}$$

(ii) *The sequence $\mathcal{C}_n^{-1} \sim \mathcal{A}_n$.*

Proof. It is easy to conclude from inequalities (2.13), (2.14) and definition (3.1) that

$$\gamma_0\|g\|_1^2 \leq (\mathcal{A}_n g,g) \leq M_0\|g\|_1^2, \quad \text{for all} \quad g \in U_n, \quad n \in \mathbf{N}. \tag{3.7}$$

Therefore, when (3.6) holds, we have that $\mathcal{C}_n^{-1} \sim \mathcal{A}_n$ with $c_1 := \gamma\gamma_0$ and $c_2 := M_0\Gamma$.

Conversely, we suppose that $\mathcal{C}_n^{-1} \sim \mathcal{A}_n$. Then, there exist two positive constants c_1, c_2 such that

$$c_1(\mathcal{C}_n^{-1}g,g) \leq (\mathcal{A}_n g,g) \leq c_2(\mathcal{C}_n^{-1}g,g), \quad \text{for all} \quad g \in U_n, \quad n \in \mathbf{N}. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\frac{c_1}{M_0}(\mathcal{C}_n^{-1}g,g) \leq \|g\|_1^2 \leq \frac{c_2}{\gamma_0}(\mathcal{C}_n^{-1}g,g), \quad \text{for all} \quad g \in U_n, \quad n \in \mathbf{N},$$

which leads to (3.6) with $\gamma := \frac{c_1}{M_0}$ and $\Gamma := \frac{c_2}{\gamma_0}$. □

As a consequence of the last lemma, we have the following uniform boundedness of the condition numbers of preconditioned operators $\mathcal{C}_n^{\frac{1}{2}}\mathcal{A}_n\mathcal{C}_n^{\frac{1}{2}}$.

Theorem 3.3. *Suppose that there exist two positive constants $\gamma \leq \Gamma$ and a sequence of self-adjoint positive definite operators $\mathcal{C}_n$, $n \in N$, such that* (3.6) *holds. If bilinear form $a_n(\cdot,\cdot)$ satisfies hypotheses* (2.13) *and* (2.14)*, then*

$$\kappa(\mathcal{C}_n^{\frac{1}{2}}\mathcal{A}_n\mathcal{C}_n^{\frac{1}{2}}) \leq \frac{M_0\Gamma}{\gamma_0\gamma}, \tag{3.9}$$

where γ_0 and M_0 are the positive constants appearing in (2.13) *and* (2.14)*, respectively.*

Proof. By Lemma 3.2, we have that $C_n^{-1} \sim \mathcal{A}_n$. That is,

$$(\mathcal{C}_n^{-1}h,h) \sim (\mathcal{A}_n h,h), \quad \text{for all} \quad h \in U_n.$$

Let $h := \mathcal{C}_n^{\frac{1}{2}}g$. Since $\mathcal{C}_n$ are self-adjoint, we observe that

$$(g,g) \sim (\mathcal{C}_n^{\frac{1}{2}}\mathcal{A}_n\mathcal{C}_n^{\frac{1}{2}}g,g), \quad \text{for all} \quad g \in U_n.$$

Thus, the condition numbers $\kappa(\mathcal{C}_n^{\frac{1}{2}}\mathcal{A}_n\mathcal{C}_n^{\frac{1}{2}})$ are uniformly bounded. By a detailed analysis of the constants in inequalities (2.13), (2.14) and (3.6), we conclude the desired result. □

A result similar to that in Theorem 3.3 for the FEM may be found in [15]; see also [14]. In fact, it is the wavelet preconditioning for the FEM that motivates this work.

The preconditioners presented in Theorem 3.3 are two-sided. We next present a one-sided preconditioner for operator $\mathcal{A}_n$. To this end, we define the inner product with respect to operator $\mathcal{A}_n$. Let $\mathcal{A}_n^* : U_n \to U_n$ denote the adjoint operator of $\mathcal{A}_n$ defined by

$$(\mathcal{A}_n v, w) = (v, \mathcal{A}_n^* w), \quad \text{for all} \quad v, w \in U_n.$$

We introduce a self-adjoint operator by setting $\hat{\mathcal{A}}_n := \frac{\mathcal{A}_n + \mathcal{A}_n^*}{2}$. It follows from (2.14) and (2.13) that $\hat{\mathcal{A}}_n$, $n \in \mathbf{N}$, are uniformly bounded and positive definite. We then define the inner product related to $\mathcal{A}_n$ by $(v, w)_{\mathcal{A}_n} := (\hat{\mathcal{A}}_n v, w)$. The corresponding norm is defined for all $w \in U_n$ by $\|w\|_{\mathcal{A}_n} := (\hat{\mathcal{A}}_n w, w)^{\frac{1}{2}}$ and the corresponding operator norm is defined for $\mathcal{B} : U_n \to U_n$ by

$$\|\mathcal{B}\|_{\mathcal{A}_n} := \sup \left\{ \frac{(\mathcal{B}w, w)_{\mathcal{A}_n}}{(w, w)_{\mathcal{A}_n}} : w \in U_n \right\}. \tag{3.10}$$

Note that $\|w\|_{\mathcal{A}_n} = (\mathcal{A}_n w, w)^{\frac{1}{2}}$, since $(\mathcal{A}_n w, w) = (\hat{\mathcal{A}}_n w, w)$. Correspondingly, we define the condition number associated with $\mathcal{A}_n$ by

$$\kappa_{\mathcal{A}_n}(\mathcal{B}) := \|\mathcal{B}\|_{\mathcal{A}_n} \|\mathcal{B}^{-1}\|_{\mathcal{A}_n}. \tag{3.11}$$

By (2.13), (2.14) and (3.1), we conclude that $\|w\|_{\mathcal{A}_n}$ is equivalent to $\|w\|_1$, that is, $\|w\|_{\mathcal{A}_n} \sim \|w\|_1$.

Lemma 3.4. *Let $\mathcal{A}_n : U_n \to U_n$ be a sequence of positive definite operators. If $\mathcal{C}_n : U_n \to U_n$ is a sequence of self-adjoint and positive definite operators with $\mathcal{C}_n^{-1} \sim \mathcal{A}_n$, then*

$$\kappa(\mathcal{C}_n^{\frac{1}{2}} \mathcal{A}_n \mathcal{C}_n^{\frac{1}{2}}) \sim \kappa_{\mathcal{A}_n}(\mathcal{C}_n \mathcal{A}_n). \tag{3.12}$$

Proof. It suffices to show that

$$\|\mathcal{C}_n^{\frac{1}{2}} \mathcal{A}_n \mathcal{C}_n^{\frac{1}{2}}\|_0 \sim \|\mathcal{C}_n \mathcal{A}_n\|_{\mathcal{A}_n} \quad \text{and} \quad \left\| \left(\mathcal{C}_n^{\frac{1}{2}} \mathcal{A}_n \mathcal{C}_n^{\frac{1}{2}} \right)^{-1} \right\|_0 \sim \|(\mathcal{C}_n \mathcal{A}_n)^{-1}\|_{\mathcal{A}_n}. \tag{3.13}$$

We present only a proof for the first relation in (3.13) since the second is proved similarly. Note that $\mathcal{C}_n$ and $\mathcal{C}_n^{-1}$ are self-adjoint and positive definite. For any $v \in U_n$ with $v \neq 0$, we let $w := \mathcal{C}_n^{\frac{1}{2}} v$ and observe that $w \neq 0$. Hence, we have that

$$\frac{\|\mathcal{C}_n^{\frac{1}{2}} \mathcal{A}_n \mathcal{C}_n^{\frac{1}{2}} v\|_0}{\|v\|_0} = \frac{\|\mathcal{C}_n^{-\frac{1}{2}} \mathcal{C}_n \mathcal{A}_n w\|_0}{\|\mathcal{C}_n^{-\frac{1}{2}} w\|_0}. \tag{3.14}$$

Observing that $\mathcal{C}_n^{\frac{1}{2}}$ and $\mathcal{C}_n^{-\frac{1}{2}}$ are also self-adjoint, we have that

$$\frac{\|\mathcal{C}_n^{-\frac{1}{2}} \mathcal{C}_n \mathcal{A}_n w\|_0}{\|\mathcal{C}_n^{-\frac{1}{2}} w\|_0} = \frac{\left(\mathcal{C}_n^{-1} \mathcal{C}_n \mathcal{A}_n w, \mathcal{C}_n \mathcal{A}_n w\right)^{\frac{1}{2}}}{\left(\mathcal{C}_n^{-1} w, w\right)^{\frac{1}{2}}}. \tag{3.15}$$

By the equivalence $\mathcal{C}_n^{-1} \sim \mathcal{A}_n$, we conclude that

$$\frac{\left(\mathcal{C}_n^{-1} \mathcal{C}_n \mathcal{A}_n w, \mathcal{C}_n \mathcal{A}_n w\right)^{\frac{1}{2}}}{\left(\mathcal{C}_n^{-1} w, w\right)^{\frac{1}{2}}} \sim \frac{(\mathcal{A}_n \mathcal{C}_n \mathcal{A}_n w, \mathcal{C}_n \mathcal{A}_n w)^{\frac{1}{2}}}{(\mathcal{A}_n w, w)^{\frac{1}{2}}} = \frac{\|\mathcal{C}_n \mathcal{A}_n w\|_{\mathcal{A}_n}}{\|w\|_{\mathcal{A}_n}}, \tag{3.16}$$

where we have used the fact that $\|w\|^2_{\mathcal{A}_n} = (\hat{\mathcal{A}}_n w, w) = (\mathcal{A}_n w, w)$. From (3.14), (3.15) and (3.16), we obtain that

$$\frac{\|\mathcal{C}_n^{\frac{1}{2}}\mathcal{A}_n\mathcal{C}_n^{\frac{1}{2}}v\|_0}{\|v\|_0} \sim \frac{\|\mathcal{C}_n\mathcal{A}_n w\|_{\mathcal{A}_n}}{\|w\|_{\mathcal{A}_n}}.$$

Therefore, by the definition of the operator norm we conclude the first equation of (3.13). □

As a direct consequence of Lemmas 3.2 and 3.4, we have the following result.

Theorem 3.5. *If bilinear form $a_n(\cdot,\cdot)$ satisfies* (2.13) *and* (2.14) *and if there exist positive constants $\gamma \le \Gamma$ such that the self-adjoint positive definite operators $\mathcal{C}_n$, $n \in \mathbf{N}$ satisfy* (3.6)*, then*

$$\kappa_{\mathcal{A}_n}(\mathcal{C}_n\mathcal{A}_n) = O(1), \quad n \to \infty. \tag{3.17}$$

We now describe a construction of self-adjoint positive definite operator $\mathcal{C}_n$ (cf., [15]) which ensures equivalence norm condition (3.6). To this end, we require the primary partition is constructed so that the corresponding trial spaces U_n, $n \in N_0$, are nested in the sense that

$$U_n \subset U_{n+1}, \quad \text{for all} \quad n.$$

We also require that

$$\overline{\bigcup_{i=0}^{\infty} U_i} = L^2(\Omega).$$

However, we do not require test spaces V_n to be nested in order to include the commonly used FVM schemes in our setting. Suppose that $Q_j, j \in \mathbf{N}_0$ is a sequence of linear projectors that maps any U_n, $n \ge j$, onto U_j. For each $j \in \mathbf{N}_0$, we define

$$W_{j+1} := (Q_{j+1} - Q_j)U_{j+1}, \tag{3.18}$$

which yields the direct sum decomposition

$$U_{j+1} = U_j \oplus W_{j+1}. \tag{3.19}$$

Write $W_0 := U_0$ so that $U_n = \bigoplus_{j=0}^n W_j, n \in N_0$. The corresponding multilevel representation of a $g \in U_n$ is given by

$$g = Q_0 g + \sum_{j=1}^{n}(Q_j - Q_{j-1})g. \tag{3.20}$$

Define operator $\mathcal{C}_n^{-1} : U_n \to U_n$ by

(3.21)

$$(\mathcal{C}_n^{-1}g, \tilde{g}) = (Q_0 g, Q_0\tilde{g}) + \sum_{j=1}^{n} 2^{2j}\big((Q_j - Q_{j-1})g, (Q_j - Q_{j-1})\tilde{g}\big), \quad \text{for all} \quad g, \tilde{g} \in U_n.$$

Note that in this case $\mathcal{C}_n^{-1}$ is self-adjoint.

We next present a generalization of preconditioner $\mathcal{C}_n$. For all $j = 0, 1, \ldots, n$, let $\mathcal{R}_j : U_j \to U_j$ be a self-adjoint positive definitive operator satisfying

$$(\mathcal{R}_j v_j, w_j) \sim 2^{-2j}\,(v_j, w_j)\,, \quad \text{for all} \quad v_j, w_j \in U_j. \tag{3.22}$$

We define operator $(\tilde{\mathcal{C}}_n)^{-1} : U_n \to U_n$ by

$$((\tilde{\mathcal{C}}_n)^{-1}g, \tilde{g}) = (\mathcal{R}_0^{-1}Q_0 g, Q_0\tilde{g}) + \sum_{j=1}^{n} \left(\mathcal{R}_j^{-1}(Q_j - Q_{j-1})g, (Q_j - Q_{j-1})\tilde{g}\right) \quad \text{for all} \quad g, \tilde{g} \in U_n. \tag{3.23}$$

It can be seen that $(\tilde{\mathcal{C}}_n)^{-1} \sim \mathcal{C}_n^{-1} \sim \mathcal{A}_n$ and thus, $\tilde{\mathcal{C}}_n$ can be used to precondition operator $\mathcal{A}_n$. Moreover, the results in Theorems 3.3 and 3.5 hold when $\mathcal{C}_n$ is replaced by $\tilde{\mathcal{C}}_n$.

In particular, when Q_j are orthogonal, which ensures (cf. [2, 38]) that

$$\mathcal{C}_n = \sum_{j=0}^{n} 2^{-2j}(Q_j - Q_{j-1}), \tag{3.24}$$

(the operators $\mathcal{C}_n$ are the BPX preconditioner introduced in [2]), we choose

$$\mathcal{C}_n^0 := \sum_{j=0}^{n} \mathcal{R}_j Q_j. \tag{3.25}$$

It is easily shown that $\mathcal{C}_n^0 \sim \mathcal{C}_n$ and thus the results in Theorems 3.3 and 3.5 also hold when $\mathcal{C}_n$ is replaced by $\mathcal{C}_n^0$. The operators $\mathcal{C}_n^0$ are called *parallel subspace correction preconditioners* and were used in [2, 38, 40] to precondition the operators induced by the finite element method.

To close this section, we recall the conditions presented in [19] that guarantee hypothesis (3.6) of Theorems 3.3 and 3.5. Let $\omega_2(g, t, \Omega)_2$ denote the usual second order L_2 modulus of smoothness of a function g for $l \geq 1$, we also introduce the semi-norm $|\cdot|_l$ defined for each $v \in H^l(\Omega)$ by

$$|v|_l := \left(\sum_{|j|=l} \int_\Omega |D^j v|^2\right)^{\frac{1}{2}}.$$

Lemma 3.6. *Suppose that there exists a positive constant c and a $\tilde{\gamma} > 1$ such that for all $n \in N_0$ and $g \in U_n$,*

$$\omega_2(g, t, \Omega)_2 \leq c\sigma_n(t)^{\tilde{\gamma}}\|g\|_0, \quad g \in U_n \tag{3.26}$$

where $\sigma_n(t) := \min\{1, t2^n\}$. Suppose that Q_j, $j \in \mathbf{N}_0$ are uniformly bounded on $L_2(\Omega)$. If there exists a positive constant c such that for all $n \in \mathbf{N}_0, v \in H^l(\Omega)$,

$$\inf_{g \in U_n} \|v - g\|_0 \leq c2^{-nl}|v|_l, \tag{3.27}$$

for some $l > 1$, and if $\mathcal{C}_n^{-1}$ is defined by (3.21)*, then there exist positive constants γ, Γ such that for all $g \in U_n, n \in \mathbf{N}_0$, inequalities* (3.6) *hold.*

Inequalities (3.26) and (3.27) are called the Bernstein (inverse) estimate and the Jackson (direct) estimate, respectively. In the following, we present conditions on partitions of two-dimensional domains which guarantee the Bernstein and Jackson estimates. For $\Omega \subset \mathbf{R}^2$, a triangular partition (or rectangular partition) T_n of Ω is called γ-quasi-uniform, if there exists a positive constant γ such that

$$\max\left\{\frac{h_K}{\rho_{K'}} : K, K' \in T_n\right\} \leq \gamma. \tag{3.28}$$

For the triangular partitions, (3.28) is equivalent to (2.4) and (2.5) and for the rectangular partitions, (3.28) is equivalent to (2.4) and the first assumption in (2.6). It is shown in [32] that if for each $n \in \mathbf{N}_0$, T_n is γ-quasi-uniform for some fixed constant γ, if there exist two positive constants c_1, c_2, such that $c_1 2^{-n} \leq h_n \leq c_2 2^{-n}$, for all $n \in \mathbf{N}_0$ and if the trial space $U_n, n \in \mathbf{N}_0$ are nested, then the Bernstein estimate (3.26) holds with $\tilde{\gamma} = 3/2$ and the Jackson estimate holds (3.27) with $l := r + 1$.

If the trial spaces $U_n, n \in \mathbf{N}_0$ have the above described properties, then by Lemma 3.6, Theorems 3.3 and 3.5 hold.

4. Preconditioning of the FVM matrix

We discuss in this section the preconditioning of the coefficient matrix of the FVM.

For convenience, we let $\Lambda(n) := \{1, 2, \ldots, D(n)\}$ and order the points $P \in \Omega_n$ so that the basis Φ_n and $\widetilde{\Phi}_n$ defined in Section 2 can be written as $\Phi_n = \{\phi_i : i \in \Lambda(n)\}$ and $\widetilde{\Phi}_n = \{\chi_i : i \in \Lambda(n)\}$. We introduce the FVM matrix

$$\mathbf{A}_n := [a_n(\phi_i, \chi_j) : i, j \in \Lambda(n)]$$

and let $\mathbf{f}_n := [(f, \chi_j) : j \in \Lambda(n)]$. With these notations, the variational equation (2.10) becomes the linear system

$$\mathbf{A}_n \mathbf{u}_n = \mathbf{f}_n, \tag{4.1}$$

where $\mathbf{u}_n \in R^{D(n)}$.

Note that the matrix $\mathbf{A}_n$ induced by the FVM is often nonsymmetric. It is known (cf. [22, 34]) that the general minimal residual method (GMRES) is an efficient scheme for solving a linear system with a nonsymmetric coefficient matrix. Consider the nonsymmetric linear system

$$\mathbf{G}\mathbf{x} = \mathbf{g}. \tag{4.2}$$

For the m-th approximate solution $\mathbf{x}_m$ of (4.2) we define the residual of (4.2) at $\mathbf{x}_m$ by

$$\mathbf{r}_m := \mathbf{g} - \mathbf{G}\mathbf{x}_m.$$

We let $\mu_1 := \inf \frac{(\mathbf{G}\mathbf{x}, \mathbf{x})}{(\mathbf{x}, \mathbf{x})}$ and $\mu_2 := \sup \frac{\|\mathbf{G}\mathbf{x}\|_0}{\|\mathbf{x}\|_0}$. The following result is due to (cf. [22, 41]).

Lemma 4.1. *If $\mu_1 > 0$, then the GMRES method converges and*

$$\|\mathbf{r}_m\|_0^2 \leq \left(1 - \frac{\mu_1}{\mu_2}\right)^m \|\mathbf{r}_0\|_0^2. \tag{4.3}$$

While using the GMRES method to solve equation (4.1), we have that $\mathbf{G} := \mathbf{A}_n$ and $\mathbf{g} := \mathbf{f}_n$. Thus, the parameters μ_1 and μ_2 may depend on n and it might happen that $\frac{\mu_1}{\mu_2} \to 0$ as $n \to +\infty$. In this case, the GMRES method either does not converge or converges slowly. For the GMRES method to converge fast while applying it to (4.1) it is necessary to precondition (4.1). We will show that the preconditioners for the finite element matrix also precondition the finite volume matrix for the same boundary value problems.

We first study the preconditioners based on wavelets. For all $j = 0, 1, \ldots, n$, let W_{j+1} be an orthogonal complement of U_j in U_{j+1} which is defined by (3.18). For the index set $\widetilde{\Lambda}(j)$ associated with space W_j, we let $\Psi_j = \{\psi_{j,l} : l \in \widetilde{\Lambda}(j)\}$ denote the wavelet basis for W_j. Then, $\Psi^n = \bigcup_{j=0}^n \Psi_j$ is the wavelet basis for U_n. Let $\mathbf{L}$ denote the change of basis matrix from the basis Φ_n to the basis Ψ^n. That is,

$$(\Psi^n)^T = {\Phi_n}^T \mathbf{L}.$$

By introducing

$$\widetilde{\mathbf{A}}_n := \mathbf{L}^T \mathbf{A}_n \mathbf{L}, \quad \widetilde{\mathbf{u}}_n := \mathbf{L}^{-1} \mathbf{u}_n, \quad \widetilde{\mathbf{f}}_n := \mathbf{L}^T \mathbf{f}_n, \tag{4.4}$$

the discrete form (4.1) of the FVM is equivalent to the linear system

$$\widetilde{\mathbf{A}}_n \widetilde{\mathbf{u}}_n = \widetilde{\mathbf{f}}_n. \tag{4.5}$$

In the next theorem we present a preconditioner for the FVM matrix $\mathbf{A}_n$ in terms of matrix $\mathbf{L}$.

Theorem 4.2. *If bilinear form $a_n(\cdot,\cdot)$ satisfied* (2.13) *and* (2.14) *with the constants M_0, γ_0 and there exist positive constants γ, Γ and $d_{j,l}$, $l \in \widetilde{\Lambda}(j)$, $j = 0, 1, \ldots, n$, such that for each $g \in U_n$ with $g := \sum_{j=0}^{n} \sum_{l \in \widetilde{\Lambda}(j)} y_{j,l}\psi_{j,l}$,*

$$\gamma \sum_{j=0}^{n} \sum_{l \in \widetilde{\Lambda}(j)} |d_{j,l} y_{j,l}|^2 \le \|g\|_1^2 \le \Gamma \sum_{j=0}^{n} \sum_{l \in \widetilde{\Lambda}(j)} |d_{j,l} y_{j,l}|^2, \tag{4.6}$$

then, for all $n \in N$,

$$\kappa(\mathbf{D}^{-1}\mathbf{L}^T \mathbf{A}_n \mathbf{L} \mathbf{D}^{-1}) \le \frac{M_0 \Gamma}{\gamma_0 \gamma} \tag{4.7}$$

where the diagonal matrix $\mathbf{D} := [d_{j,l} : l \in \widetilde{\Lambda}(j), j = 0, 1, \ldots, n]$.

Proof. For each $g, \tilde{g} \in U_n$ with $\tilde{g} := \sum_{j=0}^{n} \sum_{l \in \widetilde{\Lambda}(j)} \tilde{y}_{j,l}\psi_{j,l}$, we have that

$$a_n(g, \Pi_n \tilde{g}) = \mathbf{y}_n^T \widetilde{\mathbf{A}}_n \widetilde{\mathbf{y}}_n.$$

By (2.13) and the second inequality in (4.6), we have that

$$|\mathbf{y}_n^T \widetilde{\mathbf{A}}_n \widetilde{\mathbf{y}}_n| = |a_n(g, \Pi_n \tilde{g})| \le M_0 \|g\|_1 \|\tilde{g}\|_1 \le M_0 \Gamma \|\mathbf{D}\mathbf{y}_n\|_0 \|\mathbf{D}\widetilde{\mathbf{y}}_n\|_0.$$

In other words, we observe that

$$|(\mathbf{D}\mathbf{y}_n)^T (\mathbf{D}^{-1} \widetilde{\mathbf{A}}_n \mathbf{D}^{-1})(\mathbf{D}\widetilde{\mathbf{y}}_n)| \le M_0 \Gamma \|\mathbf{D}\mathbf{y}_n\|_0 \|\mathbf{D}\widetilde{\mathbf{y}}_n\|_0.$$

This ensures that

$$\|\mathbf{D}^{-1} \widetilde{\mathbf{A}}_n \mathbf{D}^{-1}\|_0 \le M_0 \Gamma. \tag{4.8}$$

On the other hand, from (2.14) and the first inequality in (4.6), we obtain that

$$\mathbf{y}_n^T \widetilde{\mathbf{A}}_n \mathbf{y}_n = a_n(g, \Pi_n g) \ge \gamma_0 \|g\|_1^2 \ge \gamma_0 \gamma \|\mathbf{D}\mathbf{y}_n\|_0^2,$$

that is,

$$(\mathbf{D}\mathbf{y}_n)^T (\mathbf{D}^{-1} \widetilde{\mathbf{A}}_n \mathbf{D}^{-1})(\mathbf{D}\mathbf{y}_n) \ge \gamma_0 \gamma \|\mathbf{D}\mathbf{y}\|_0^2.$$

This yields that

$$\|(\mathbf{D}^{-1} \widetilde{\mathbf{A}}_n \mathbf{D}^{-1})^{-1}\|_0 \le \frac{1}{\gamma_0 \gamma}. \tag{4.9}$$

From (4.8), (4.9) and (4.4), we complete the proof of (4.7). ◻

We now apply the GMRES method to the preconditioned equation

$$\mathbf{D}^{-1}\widetilde{\mathbf{A}}_n\mathbf{D}^{-1}\mathbf{x} = \mathbf{D}^{-1}\mathbf{f}_n. \tag{4.10}$$

The next corollary gives a uniform rate of convergence of the method.

Corollary 4.3. *If $\mathbf{r}_m$ is the residual of equation* (4.10) *at the m-th approximate solution $\mathbf{x}_m$, then*

$$\|\mathbf{r}_m\|_0^2 \le \left(1 - \frac{\gamma_0\gamma}{M_0\Gamma}\right)^m \|\mathbf{r}_0\|_0^2. \tag{4.11}$$

Proof. By (4.8) and (4.9) in the proof of Theorem 4.2 and the definition of μ_1 and μ_2, we obtain that $\mu_1 \ge \gamma_0\gamma$ and $\mu_2 \le M_0\Gamma$. Estimate (4.11) follows directly from these bounds and Lemma 4.1. □

We remark that condition (4.6) is fulfilled if an additional stability hypothesis for the wavelet basis is satisfied. For each $j \in \mathbf{N}_0$, we say that the basis $\Psi_j := \{\psi_{j,i} : k \in \widetilde{\Lambda}(j)\}$ is stable, if there exist a constant $c_j > 0$ such that

$$\left\| \sum_{i\in\widetilde{\Lambda}(j)} y_{j,i}\psi_{j,i} \right\|_0^2 \sim c_j^2 \sum_{i\in\widetilde{\Lambda}(j)} |y_{j,i}|^2. \tag{4.12}$$

It was shown in [15] that if Ψ_j is stable for all $j \in \mathbf{N}_0$ and (3.6) holds for $\mathcal{C}_n$ defined in (3.21), then inequality (4.6) holds with $d_{j,i} := 2^j c_j$, $j = 0, 1, \dots, n$, $i \in \widetilde{\Lambda}_j$.

We now turn our attention to a one-sided matrix preconditioner associated with the operator $\mathcal{C}_n^0$. For $k \le n$ we let $\mathbf{E}_k$ denote the representation matrix of the nodal basis $\Phi_k = [\phi_{k,1}, \dots, \phi_{k,D(k)}]$ of U_k in terms of the nodal basis $\Phi_n = [\phi_{n,1}, \dots, \phi_{n,D(n)}]$ of U_n, that is,

$$\Phi_k = \Phi_n\mathbf{E}_k. \tag{4.13}$$

Moreover, let $\mathbf{R}_k$ denote the the matrix representation of operator $\mathcal{R}_k$ (cf., [38]), and define

$$\mathbf{C}_n^0 := \sum_{k=0}^{n} \mathbf{E}_k\mathbf{R}_k\mathbf{E}_k^T. \tag{4.14}$$

Since $\mathbf{C}_n^0$ is nonsingular, the discrete form (4.1) of the FVM is equivalent to the linear system

$$\mathbf{C}_n^0\mathbf{A}_n\mathbf{u}_n = \mathbf{C}_n^0\mathbf{f}_n. \tag{4.15}$$

We define an inner product with respect to $\mathbf{A}_n$ by

$$(\mathbf{v}, \mathbf{w})_{\mathbf{A_n}} := (\hat{\mathbf{A}}_\mathbf{n}\mathbf{v}, \mathbf{w}), \quad \text{for all } \ \mathbf{v}, \mathbf{w} \in R^{D(n)},$$

where $\hat{\mathbf{A}}_\mathbf{n} := \frac{1}{2}(\mathbf{A}_n + \mathbf{A}_n^t)$ and $\mathbf{A}_n^t$ is the transpose of the matrix $\mathbf{A}_n$. We also define an induced norm of vector $\mathbf{u} \in R^{D(n)}$ by

$$\|\mathbf{u}\|_{\mathbf{A}_n} := (\mathbf{u}, \mathbf{u})_{\mathbf{A_n}}^{\frac{1}{2}}. \tag{4.16}$$

For a $D(n) \times D(n)$ matrix $\mathbf{A}$, we let $\|\mathbf{A}\|_{\mathbf{A}_n}$ denote the matrix norm of $\mathbf{A}$ induced by the vector norm $\|\mathbf{u}\|_{\mathbf{A}_n}$. Thus, $\mathbf{A}_n$-condition number $\kappa_{\mathbf{A}_n}$ is defined accordingly.

We next consider the two-sided preconditioned matrix

$$\tilde{\mathbf{A}}_n := (\mathbf{C}_n^0)^{\frac{1}{2}}\mathbf{A}_n(\mathbf{C}_n^0)^{\frac{1}{2}}.$$

Theorem 4.4. *If $\mathbf{C}_n^0$ is the matrix defined by* (4.14)*, then* $\kappa(\tilde{\mathbf{A}}_n) = O(1)$.

Proof. For a $\mathbf{u} \in R^{D(n)}$, we let $u_n := \Phi_n \mathbf{u}$. By (3.1), we have that

$$\mathbf{u}^T \mathbf{A}_n \mathbf{u} = a(u_n, \Pi_n u_n) = (\mathcal{A}_n u_n, u_n). \tag{4.17}$$

Let $\mathbf{v} := (\mathbf{C}_n^0)^{-\frac{1}{2}} \mathbf{u}$ and $v_n := \Phi_n \mathbf{v}$. Then, $v_n = (\mathcal{C}_n^0)^{-\frac{1}{2}} u_n$, where $\mathcal{C}_n^0$ is the operator defined by (3.25). Thus, by (4.17), we obtain that

$$\mathbf{v}^T \tilde{\mathbf{A}}_n \mathbf{v} = (\mathcal{A}_n (\mathcal{C}_n^0)^{\frac{1}{2}} v_n, (\mathcal{C}_n^0)^{\frac{1}{2}} v_n) = ((\mathcal{C}_n^0)^{\frac{1}{2}} \mathcal{A}_n (\mathcal{C}_n^0)^{\frac{1}{2}} v_n, v_n), \tag{4.18}$$

where we have used the fact that $\mathcal{C}_n^0$ is self-adjoint. It follows from (4.18) that

$$\kappa(\tilde{\mathbf{A}}_n) = \kappa\left((\mathcal{C}_n^0)^{\frac{1}{2}} \mathcal{A}_n (\mathcal{C}_n^0)^{\frac{1}{2}} \right).$$

Since $\mathcal{C}_n^0 \sim \mathcal{C}_n$, Theorem 3.3 is valid when $\mathcal{C}_n$ is replaced by $\mathcal{C}_n^0$. Thus, we have the desired result. ☐

The next lemma shows that the condition number of $\tilde{\mathbf{A}}_n$ is equivalent to the $\mathbf{A}_n$-condition number of $\mathbf{C}_n^0 \mathbf{A}_n$.

Lemma 4.5. *If $\mathbf{C}_n^0$ is the matrix defined by* (4.14)*, then* $\kappa_{\mathbf{A}_n}(\mathbf{C}_n^0 \mathbf{A}_n) \sim \kappa(\tilde{\mathbf{A}}_n)$.

Proof. By the same arguments used in the proof of Lemma 3.4, we show by a change of variables that

$$\|\tilde{\mathbf{A}}_n\|_0 \sim \|\mathbf{C}_n \mathbf{A}_n\|_{\mathbf{A_n}} \quad \text{and} \quad \|\tilde{\mathbf{A}}_n^{-1}\|_0 \sim \|\left(\mathbf{C}_n \mathbf{A}_n\right)^{-1}\|_{\mathbf{A_n}}.$$

Thus, the desired equivalence relation follows. ☐

As a direct consequence of Theorem 4.4 and Lemma 4.5, we have the next result regarding the one-sided preconditioning.

Theorem 4.6. *If $\mathbf{C}_n^0$ is the matrix defined by* (4.14)*, then* $\kappa_{\mathbf{A}_n}(\mathbf{C}_n^0 \mathbf{A}_n) = O(1)$.

Now we consider solving the preconditioned linear system (4.15) by using the GMRES method. In each iteration step of the GMRES method, we use the inner product $(\cdot, \cdot)_{\mathbf{A_n}}$ instead of the l_2-inner product. Let $\mathbf{r}_m$ denote the residual for equation (4.15) at the m-th approximate solution and define

$$\mu_1(n) := \inf \frac{(\mathbf{C}_n^0 \mathbf{A}_n, \mathbf{x})_{\mathbf{A_n}}}{(\mathbf{x}, \mathbf{x})_{\mathbf{A_n}}} \quad \text{and} \quad \mu_2(n) := \sup \frac{\|\mathbf{C}_n^0 \mathbf{A}_n \mathbf{x}\|_{\mathbf{A_n}}}{\|\mathbf{x}\|_{\mathbf{A_n}}}.$$

The estimate (4.3) can be modified according to the inner product $(\cdot, \cdot)_{\mathbf{A_n}}$ as

$$\|\mathbf{r}_m\|_{\mathbf{A_n}}^2 \le \left(1 - \frac{\mu_1(n)}{\mu_2(n)}\right)^m \|\mathbf{r}_0\|_{\mathbf{A_n}}^2. \tag{4.19}$$

The next corollary shows the uniform convergence rate of the method.

Corollary 4.7. *There exists a constant $\rho \in (0, 1)$ such that for all n,*

$$\|\mathbf{r}_m\|_{\mathbf{A_n}}^2 \le (1 - \rho)^m \|\mathbf{r}_0\|_{\mathbf{A_n}}^2. \tag{4.20}$$

Proof. By Lemma 4.5 and Theorem 4.4, we have that

$$\|\mathbf{C}_n^0 \mathbf{A}_n\|_{\mathbf{A}_n} \sim \|(\mathbf{C}_n^0 \mathbf{A}_n)^{-1}\|_{\mathbf{A}_n} = O(1).$$

Therefore, there exist two constants ρ_1 and ρ_2 such that for all positive integer n, $\mu_1(n) \ge \rho_1$ and $\mu_2(n) \le \rho_2$. Letting $\rho := \rho_1/\rho_2$, the estimate (4.20) is a direct consequence of the above inequalities and (4.19). ☐

Noticing that when n is large, the GMRES may entail a high computational effort. For this reason, in practice, we use a variant of the GMRES, the GMRES(k) which restarts the iterations after k steps, to reduce the computational cost.

5. A linear FVM scheme

In this section, we consider solving (2.1) in the two-dimensional space with discontinuous coefficients. Specifically, we suppose that $\Omega \subset \mathbf{R}^2$ and $A(x) = \alpha(x)I$, where I is the 2×2 identity matrix, $\alpha \in L^\infty(\Omega)$ is a piecewise smooth function and there exists a positive $\alpha_0 > 0$ such that $\alpha(x) \geq \alpha_0$, for all $x \in \Omega$. We shall prove that the FVM bilinear form using *linear* elements for the trial space for solving the equation can be expressed as the linear FEM bilinear form for solving the same equation with a *small* perturbation. As a result, the corresponding FVM matrix can be preconditioned by an efficient preconditioner for the FEM matrix.

We now describe the linear FVM for solving the equation. Let $\mathcal{T}_n$ be a quasi-uniform, shape regular triangulation of Ω and U_n the associated linear finite element space. We construct the dual partition $\mathcal{T}_n^*$ with the barycentric mesh described below (see, for example, [9, 30, 43]). The control volume K_P^* associated with each node P is obtained by connecting successively with lines the midpoints of the adjacent edges of P and the barycenters of the adjacent triangles of P. The test space V_n consists of the piecewise constant functions with respect to the dual mesh $\mathcal{T}_n^*$. The corresponding FVM is to find a solution $u_n \in U_n$ such that equation (2.10) is satisfied with the FVM bilinear form

$$a_n(u,v) = -\sum_{P\in\Omega_n} v(P)\int_{\partial K_P^*} (\alpha\nabla u\cdot \mathbf{n})\psi_P ds, \quad \text{for all} \;\; u\in U_n, v\in V_n,$$

and Ω_n being the set of all vertices in $\mathcal{T}_n$ and ψ_P, $P\in\Omega_n$, are a basis for the test space V_n.

The finite element bilinear form for the same boundary value problem has the form

$$a_e(u,v) = (\alpha\nabla u, \nabla v), \quad \text{for all} \;\; u,v\in U_n. \tag{5.1}$$

It is well known (see e.g., [1, 30, 43]) that when the coefficient α is piecewise constant with respect to $\mathcal{T}_n$, we have the identity

$$a_n(u,v) = a_e(u,v), \quad \text{for all} \;\; u,v\in U_n.$$

We shall study the more general case when α is piecewise continuous. For this purpose, we introduce the modulus of piecewise continuity

$$m(\alpha,\mathcal{T}_n) := \sup_{K\in\mathcal{T}_n} m(\alpha,K)$$

where

$$m(\alpha,K) := \sup_{x,y\in K} |\alpha(x)-\alpha(y)|.$$

Note that the quantity $m(\alpha,\mathcal{T}_n)$ presents only the variance of α in the interior of elements of $\mathcal{T}_n$. It does not depend on the jump of α across edges of $\mathcal{T}_n$. Since α is piecewise continuous, we have that

$$\lim_{n\to\infty} m(\alpha,\mathcal{T}_n) = 0.$$

We also introduce the relative modulus of piecewise continuity by setting

$$m_r(\alpha, \mathcal{T}_n) := \sup_{K \in \mathcal{T}_n} \frac{m(\alpha, K)}{\alpha_K}$$

where α_K is the average of α in K defined by

$$\alpha_K := \frac{1}{|K|} \int_K \alpha(x) dx.$$

Since $\alpha(x) \geq \alpha_0$ for all $x \in \Omega$, we also have

$$\lim_{n \to \infty} m_r(\alpha, \mathcal{T}_n) = 0. \tag{5.2}$$

The next theorem shows that the FVM bilinear form is a perturbation of the FEM bilinear form.

Theorem 5.1. *There exists a positive constant c such that for all positive integers n and for all $v \in U_n$,*

$$\left| a_n(v, \Pi_n v) - a_e(v, v) \right| \leq c m_r(\alpha, \mathcal{T}_n) a_e(v, v). \tag{5.3}$$

Thus, there exists an integer $N_0 \in \mathbf{N}$ such that for all $n \geq N_0$ and for all $v \in U_n$,

$$\frac{1}{2} a_e(v, v) \leq a_n(v, \Pi_n v) \leq \frac{3}{2} a_e(v, v). \tag{5.4}$$

Proof. For each $K \in \mathcal{T}_n$ we define the elementwise FVM bilinear form for $w \in U_n, v \in V_n$ by

$$a_{n,K}(w, v) := -\sum_{P \in \Omega_n \cap K} v(P) \int_{\partial K_P^* \cap K} (\alpha \nabla w \cdot \mathbf{n}) \psi_P ds,$$

and the elementwise FEM bilinear form for $u, v \in U_n$ by

$$a_{e,K}(w, v) := \int_K \alpha(x) \nabla w \cdot \nabla v \ dx.$$

Because ∇v is constant in each triangle $K \in \mathcal{T}_n$, we have that

$$a_{e,K}(v, v) = \int_K \alpha_K \nabla v \cdot \nabla v \ dx.$$

By the identity (cf. [1, 43])

$$\int_K \nabla v \cdot \nabla v dx = -\sum_{P \in \Omega_n \cap K} v(P) \int_{\partial K_P^* \cap K} (\nabla v \cdot \mathbf{n}) \psi_P ds,$$

we have that

$$a_{e,K}(v, v) = -\sum_{P \in \Omega_n \cap K} v(P) \int_{\partial K_P^* \cap K} (\alpha_K \nabla v \cdot \mathbf{n}) \psi_P ds.$$

Therefore, for all $v \in U_n$,

$$\begin{aligned} a_{n,K}(v, \Pi_n v) - a_{e,K}(v, v) &= -\sum_{P \in \Omega_n \cap K} v(P) \int_{\partial K_P^* \cap K} (\alpha - \alpha_K)(\nabla v \cdot \mathbf{n}) \psi_P ds \\ &= -\sum_{E \in \mathcal{E}_n^*} \int_{E \cap K} (\alpha - \alpha_K) \frac{\partial v}{\partial \mathbf{n}} [\Pi_n v] ds \end{aligned}$$

where $\mathcal{E}_n^*$ is the set of interior edges of the dual partition $\mathcal{T}_n^*$ and $[\Pi_n v]$ is the jump of $\Pi_n v$ across the edge E. The equation above leads to the inequality

$$|a_{n,K}(v, \Pi_n v) - a_{e,K}(v,v)| \leq cm(\alpha, K)|v|_{1,K}^2$$

where the constant c depends only on the minimal angle of $\mathcal{T}_n$. Noting that

$$a_{e,K}(v,v) = \alpha_K |v|_{1,K}^2, \quad \text{for all} \quad v \in U_n$$

we conclude that

$$|a_{n,K}(v, \Pi_n v) - a_{e,K}(v,v)| \leq Cm_r(\alpha, K) a_{e,K}(v,v).$$

Summing the above inequality for all $K \in \mathcal{T}_n$, we obtain the desired inequality (5.3).

Finally, using the equation (5.2), we observe that there exists an $N_0 > 0$ such that for all $n \geq N_0$,

$$cm_r(\alpha, \mathcal{T}_n) \leq \frac{1}{2}.$$

Thus, the inequality (5.4) is a direct consequence of the above estimate and (5.3). □

We remark that the constant c which appeared in the previous theorem depends only on the minimal angle of the triangulation T_n and is independent of the jump in the coefficient α.

We next discuss the preconditioning of the FVM matrix by making use of the result in the last theorem. It follows from Theorem 5.1 that for sufficiently large n, the FVM bilinear form $a_n(\cdot,\cdot)$ is equivalent to the FEM bilinear form $a_e(\cdot,\cdot)$. This implies that the resulting FEM matrix $\mathbf{A}_{e,n}$ is a good preconditioner for the corresponding FVM matrix $\mathbf{A}_n$, and an efficient preconditioner for the FEM matrix $\mathbf{A}_{e,n}$ will be efficient for the precondition of the FVM matrix $\mathbf{A}_n$. To this end, we recall an effective preconditioner, proposed in [42], for the FEM matrix $\mathbf{A}_{e,n}$ in the case when the coefficient α has *large* jumps with respect to $\mathcal{T}_n$. Define the mapping $Q_j^\alpha : H_0^1 \to U_n$ by

$$\sum_{K \in \mathcal{T}_n} \int_K \alpha (Q_j^\alpha u) v dx = \sum_{K \in \mathcal{T}_n} \int_K \alpha_K u v dx, \quad \text{for all} \quad v \in U_n.$$

The preconditioner is defined by

$$\mathcal{B}_n^{-1} = \sum_{j=0}^{n} \mathcal{R}_j Q_j^\alpha \tag{5.5}$$

where R_j is the same self-adjoint operator defined by (3.22). In the next theorem, we demonstrate that this preconditioner which was shown in [42] efficient for the FEM matrix $\mathbf{A}_{e,n}$ can also be used for the FVM matrix $\mathbf{A}_n$. A similar preconditioner was discussed in [43].

Theorem 5.2. *If $\mathbf{A}_n$ and $\mathbf{A}_{e,n}$ denote the stiffness matrices induced by the FVM bilinear form $a_n(\cdot,\cdot)$ and the FEM form $a_e(\cdot,\cdot)$, respectively, then there exists N_0 such that for all $n \geq N_0$*

$$\kappa(\mathbf{A}_{e,n}^{-\frac{1}{2}} \mathbf{A}_n \mathbf{A}_{e,n}^{-\frac{1}{2}}) \lesssim 1, \quad \kappa_{\mathbf{A}_n}(\mathbf{A}_{e,n}^{-1}\mathbf{A}_n) \lesssim 1. \tag{5.6}$$

Moreover, if $\mathbf{B}_n^{-1}$ *denotes the matrix for the precondition operator* B_n^{-1} *defined by* (5.5)*, then*

$$\kappa(\mathbf{B}_n^{\frac{1}{2}}\mathbf{A}_n\mathbf{B}_n^{\frac{1}{2}}) \lesssim |\log h_n|^2, \quad \kappa_{\mathbf{A}_n}(\mathbf{B}_n\mathbf{A}_n) \lesssim |\log h_n|^2, \tag{5.7}$$

where h_n *is the mesh size of* $\mathcal{T}_n$*.*

Proof. The estimates (5.6) are direct consequences of (5.4). It remains to show the estimate (5.7). By Lemma 4.2 in [42], for all $v \in U_n$, there holds

$$\frac{1}{|\log h_n|^2}(\mathcal{B}_n^{-1}v, v) \lesssim a_e(v, v) \lesssim (\mathcal{B}_n^{-1}v, v).$$

These inequalities together with (5.4) ensure that for all $v \in U_n$,

$$\frac{1}{|\log h_n|^2}(\mathcal{B}_n^{-1}v, v) \lesssim a_n(v, \Pi_n v) \lesssim (\mathcal{B}_n^{-1}v, v).$$

Therefore, by the same arguments used in Section 4, we conclude that the desired estimates (5.7) hold. □

6. Special cases and numerical examples

We consider in this section seven special cases on preconditioning the FVM matrices or their GMRES iteration algorithms and present numerical examples to confirm the estimates on condition numbers of the FVM matrices. In the first three cases, we use wavelet preconditioners (in which cases the wavelet bases of the complement space can be easily constructed) and in the last four cases we consider BPX preconditioners. Since Lemma 4.5 ensures that the $\mathbf{A}_n$-condition number of the one-sided preconditioned matrix is equivalent to that of its corresponding two-sided preconditioned matrix, in this section we shall only consider the two-sided preconditioning. The numerical results presented in this section are all obtained by using Matlab.

6.1. **Wavelet preconditioning.**

Case 1: The Linear FVM Schemes for Two Point Boundary Value Problems.

In this case and the next two cases, we consider the wavelet precondition. In these cases, the preconditioned matrix is defined as

$$\tilde{\mathbf{A}}_n := \mathbf{D}^{-1}\mathbf{L}^T\mathbf{A}_n\mathbf{L}\mathbf{D}^{-1}, \tag{6.1}$$

where $\mathbf{D}$ is the diagonal matrix which will be defined in the specific context and $\mathbf{L}$ is the change of basis matrix from the nodal basis to the wavelet basis, specifically, in this example, we have that

$$\mathbf{D} := \text{diag }(d_{ji} : j = 0, 1, \dots, n, i \in \tilde{\Lambda}(j)), \quad \text{with} \quad d_{ji} := 2^j.$$

Let $I := [a, b]$ and $p \in C^1(I)$ be a given function with $p_1 \geq p(x) \geq p_0$, for all $x \in I$ and for some positive constants p_0, p_1. Set $Lu := (pu')'$ and for a given function $f \in L^2(I)$, we consider the two point boundary value problems

$$Lu = f \quad \text{in} \quad I, \tag{6.2}$$

$$u(a) = u(b) = 0. \tag{6.3}$$

The multilevel augmentation method for solving this problem was introduced in [8] and a wavelet preconditioner for the finite element method for this problem was developed in [24].

For each $n \geq 0$ we introduce a partition T_n for I by $a = x_0 < x_1 < \ldots < x_{N(n)} = b$. Writing $I_i := [x_{i-1}, x_i]$, the trial space U_n is the space of continuous piecewise linear functions on the partition T_n satisfying the boundary condition (6.3). Let $\phi_{n,j}$ be a piecewise linear function with knots x_i, $i = 1, 2, \ldots, D(n) := N(n) - 1$, satisfying $\phi_{n,j}(x_i) = \delta_{j,i}$, for all $i, j = 0, 1, \ldots, N(n)$. Then $\Phi_n := \{\phi_{n,j} : j \in \Lambda(n)\}$ forms a nodal basis for space U_n, where $\Lambda(n) := \{1, 2, \ldots, D(n)\}$. Let T_n^* be the dual partition of I with nodes given by $a = x_0 < x_{\frac{1}{2}} < \ldots < x_{N(n)-\frac{1}{2}} < x_{N(n)} = b$, where $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$, $i = 0, 1, \ldots, D(n)$. The corresponding dual elements are $I_0^* = [x_0, x_{\frac{1}{2}}]$, $I_i^* = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i \in \Lambda(n)$, $I_{N(n)}^* = [x_{N(n)-\frac{1}{2}}, x_{N(n)}]$. For all $j \in \Lambda(n)$, let $\chi_{n,j}$ be the characteristic function of I_j^*. The set of functions $\{\chi_{n,j} : j \in \Lambda(n)\}$ constitutes a basis for the test space V_n. In this case, the bilinear form is given by

$$a(u_n, \chi_{n,j}) := -[p(x_{j+\frac{1}{2}})u_n'(x_{j+\frac{1}{2}}) - p(x_{j-\frac{1}{2}})u_n'(x_{j-\frac{1}{2}})], \quad j \in \Lambda(n) \tag{6.4}$$

and the finite volume matrix is given by $\mathbf{A}_n := [a(\phi_{n,i}, \chi_{n,j}) : i, j \in \Lambda(n)]$.

To precondition the matrix $\mathbf{A}_n$, we first describe a wavelet basis for each of the wavelet spaces W_j, for $j \geq 1$. For each $j \in \mathbf{N}_0$, we let $\widetilde{D}(j) := \dim W_j$ and $\widetilde{\Lambda}(j) := \{1, 2, \ldots, \widetilde{D}(j)\}$. A pre-wavelet basis of W_1 is given by (cf., [14, 23])

$$\psi_{1,1} = \frac{9}{10}\phi_{1,1} - \frac{3}{5}\phi_{1,2} + \frac{1}{10}\phi_{1,3},$$

$$\psi_{1,l} = \frac{1}{10}\phi_{1,2l-3} - \frac{3}{5}\phi_{1,2l-2} + \phi_{1,2l-1} - \frac{3}{5}\phi_{1,2l} + \frac{1}{10}\phi_{1,2l+1}, \quad l = 2, 3, \ldots, N(0) - 1,$$

$$\psi_{1,N(0)} = \frac{1}{10}\phi_{1,2N(0)-3} - \frac{3}{5}\phi_{1,2N(0)-2} + \frac{9}{10}\phi_{1,2N(0)-1}.$$

For each $j > 1$, a basis of W_j is obtained by the dyadic-dilation and integer-shift of the basis of W_1. We denote by $\Psi_j := \{\psi_{j,l} : l \in \widetilde{\Lambda}(j)\}$ the basis of $\bigcup_{l \leq j} W_l$.

TABLE 1

n	N	$\kappa(\mathbf{A}_n)$	$\kappa(\tilde{\mathbf{A}}_n)$
0	3	6.1	6.1
1	7	27.5	18.5
2	15	117.1	21.8
3	31	486.7	23.5
4	63	1992.4	24.7
5	127	8084.1	26.8
6	255	32623.0	28.3
7	511	131210.0	29.6

Theorem 4.2 ensures that the condition numbers of $\tilde{\mathbf{A}}_n$ are uniformly bounded. This result is confirmed by the numerical example with $I = [0, 1]$ and $p(x) := 1 + \sqrt{x}$. Table 1 compares the condition number of $\mathbf{A}_n$ with that of $\tilde{\mathbf{A}}_n$ at different levels n, where $N := D(n)$ denotes the size of the corresponding matrix.

Case 2: The Bilinear FVM Schemes on Rectangular Grids for 2-D Laplace Equations.

We consider (2.2) and (2.3) in a rectangular region $\Omega = [a,b] \times [c,d]$. We choose the trial space U_n and the test space V_n, respectively, as the tensor product space of the univariate trial space and test space on intervals $[a,b]$ and $[c,d]$ described in Example 1. We denote $\Lambda_1(n) := \{1, 2, \ldots, N_1(n) - 1\}$ and $\Lambda_2(n) := \{1, 2, \ldots, N_2(n) - 1\}$ and set $\Lambda(n) := \Lambda_1(n) \times \Lambda_2(n)$. Let $\phi^1_{n,j}, j \in \Lambda_1(n)$ and $\phi^2_{n,j}, j \in \Lambda_2(n)$ be the nodal basis on $[a,b]$ and $[c,d]$, respectively (see Example 1 for the details). For all $(i,j) \in \Lambda(n)$, we define $\phi_{n,i,j}(x,y) := \phi^1_{n,i}(x)\phi^2_{n,j}(y)$. Then, $\Phi_n := \{\phi_{n,i,j} : (i,j) \in \Lambda(n)\}$ constitutes a nodal basis for U_n. For all $(i,j) \in \Lambda(n)$, let $\chi_{n,i,j}$ be the tensor product characteristic functions. Then, $\tilde{\Psi}_n := \{\chi_{n,i,j} : (i,j) \in \Lambda(n)\}$ constitutes a basis for V_n. The finite volume matrix is identified with $\mathbf{A}_n := [a(\phi_{n,k,l}, \chi_{n,i,j}), (k,l), (i,j) \in \Lambda(n)]$, where

$$a(\phi_{n,k,l}, \chi_{n,i,j}) = -\int_{\partial K^*_{P_{i,j}}} (A\nabla\phi_{n,k,l}) \cdot \mathbf{n}\chi_{n,i,j} ds.$$

Next we describe the pre-wavelets of U_n. For each $j \geq 1$, let W_j be the orthogonal complement of U_{j-1} in U_j. The basis of W_1 is obtained again by the tensor product of the univariate pre-wavelets, that is,

$$\psi^1_{ij}(x,y) := \phi^1_{0,i}(x)\psi^2_{1,j}(y), \quad \psi^2_{ij}(x,y) := \psi^1_{1,i}(x)\phi^2_{0,j}(y), \quad \psi^3_{ij}(x,y) := \psi^1_{1,i}(x)\psi^2_{1,j}(y).$$

The pre-wavelet basis of W_j, for each $j > 1$, is obtained by dyadic-dilations and integer-translations of the basis functions for W_1. Let $\widetilde{\Lambda}(j)$ denote the index set associated with the pre-wavelets basis Ψ_j for W_j so that $\Psi_j := \{\psi_{j,k,l} : (k,l) \in \widetilde{\Lambda}(j)\}$. It is known in [23] that $\widetilde{\Psi}_j, j \in \mathbf{N}_0$ is a sequence of stable basis. Also, the sequence of orthogonal projectors $\{Q_i : i \in \mathbf{N}_0\}$ is uniformly bounded on $L_2(\Omega)$. Thus, we have the norm equivalence (4.6) with $d_{j,\mathbf{l}} := 2^{-j}$ for all $\mathbf{l} = (l_1, l_2) \in \widetilde{\Lambda}(j)$. With the pre-wavelet basis $\Psi^{(n)} := \bigcup_{j=0}^n \Psi_j$ of U_n, the linear system (4.1) is written in the form (4.5). Let $\mathbf{D} := \mathrm{diag}(d_{j,\mathbf{l}} : \mathbf{1} \in \widetilde{\Lambda}(n))$. Moreover, since the norm equivalence (4.6) holds, by Theorem 4.2, the condition numbers of the preconditioned matrices $\tilde{\mathbf{A}}_n$ defined by (6.1) are uniformly bounded. This result is confirmed by the numerical example presented below.

Consider a boundary value problem of the Laplace equation

$$\begin{cases} -\triangle u(x,y) = f(x,y), & (x,y) \in \Omega := [0,1] \times [0,1], \\ u(x,y) = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{6.5}$$

In Table 2 we compare the condition numbers of matrices $\mathbf{A}_n$ and $\tilde{\mathbf{A}}_n$ at different level n, where $N := D(n)^2$ denotes the size of the matrix $\mathbf{A}_n$.

TABLE 2

n	N	$\kappa(\mathbf{A}_n)$	$\kappa(\tilde{\mathbf{A}}_n)$
1	9	3.6	3.6
2	49	13.4	25.1
3	225	52.3	31.8
4	961	207.9	35.4
5	3969	830.4	36.4

Case 3: The Linear FVM Schemes on Triangular Grids for 2-D Laplace Equations.

We consider (2.2) and (2.3) in a convex polygonal domain $\Omega \subset \mathbf{R}^2$. Let T_n be a triangular partition of Ω, and T_n^* the corresponding barycenter dual partition (or circumcenter dual partition) of T_n. We denote by $\overline{\Omega}_n$ the set of nodes of T_n, by $\overset{\circ}{\Omega}_n = \overline{\Omega}_n \setminus \partial\Omega$ the set of the inner nodes and by Ω_n^* the set of nodes of the dual partition. Let K_Q be the triangular element with barycenter (or circumcenter) $Q \in \Omega_n^*$ and K_P^* the dual element surrounding $P \in \overline{\Omega}_n$.

The trial space U_n is the continuous piecewise linear functions with respect to the partition T_n on Ω satisfying the boundary condition and the test space is the space of piecewise constant functions with respect to the partition T_n^*. Specifically, let $\phi_{n,P}$ be the two-dimensional tent functions satisfying $\phi_{n,P}(P') = \delta_{P,P'}$ for all $P, P' \in \overset{\circ}{\Omega}_n$ and thus, $\Phi_n := \{\phi_{n,P} : P \in \overset{\circ}{\Omega}_n\}$ form the nodal basis for U_n. Moreover, for each $P \in \overset{\circ}{\Omega}_n$, we denote by $\chi_{n,P}$ the characteristic function of K_P^*. Then, $\tilde{\Psi}_n := \{\chi_{n,P} : P \in \overset{\circ}{\Omega}_n\}$ constitutes a basis for V_n. The finite volume matrix has the form $\mathbf{A}_n := [a(\phi_{n,P}, \chi_{n,P'}) : P, P' \in \overset{\circ}{\Omega}_n]$, where

$$a(\phi_{n,P}, \chi_{n,P'}) = -\int_{\partial K_{P'}^*} (A\nabla\phi_{n,P}) \cdot \mathbf{n}\chi_{n,P'} ds.$$

To precondition $\mathbf{A}_n$, we use the two-dimensional pre-wavelets constructed in [35]. Other constructions of two-dimensional pre-wavelets can be found in [18, 25]. We now recall the construction. The basis of W_0 consists of the nodal basis functions of V_0, i.e., $\psi_{0,P} := \phi_{0,P}$, for $P \in \overset{\circ}{\Omega}_0$. For each $0 < j \leq n$, we add pre-wavelets corresponding to the new nodal points in the following way. For each $P \in \overset{\circ}{\Omega}_j$, let $\phi_{j,P}$ denote the nodal basis function on level j corresponding to the point P. For each $P \in \overset{\circ}{\Omega}_{j-1}$, we define the auxiliary function $\theta_{j,P} \in U_j$ by $\theta_{j,P} := \phi_{j,P} - \frac{1}{8}\phi_{j-1,P}$. For a coarse-grid point P, $\theta_{j,P}$ is a linear combination of a finer grid and coarse-grid nodal basis functions. For $P \in \overset{\circ}{\Omega}_j \setminus \overset{\circ}{\Omega}_{j-1}$, we define

$$\psi_{j,P} := \phi_{j,P} - \sum_{P' \in I_{j-1}} \frac{(\phi_{j,P}, \phi_{j-1,P'})_{L_2}}{(\theta_{j,P}, \phi_{j-1,P'})_{L_2}} \theta_{j,P'}.$$

It is known (cf. [35]) that $\Psi^{(n)} := \{\psi_{j,P} : 0 \leq j \leq n, P \in \overset{\circ}{\Omega}_j \setminus \overset{\circ}{\Omega}_{j-1}\}$ constitutes a Riesz basis (with respect to the L_2 norm) for U_n. Let $\mathbf{L}$ denote the change basis matrix between the nodal basis and the pre-wavelet basis and let $\mathbf{D}$ be the diagonal matrix defined as in the last example. By Theorem 4.2, the condition numbers of the matrices $\tilde{\mathbf{A}}_n$ defined by equation (6.1) are uniformly bounded. We confirm this result by the numerical example, where we consider the boundary value problem (6.5). In Table 3, we compare the condition number of $\mathbf{A}_n$ and $\tilde{\mathbf{A}}_n$ at a different level n. Again in Table 3, N denotes the size of $\mathbf{A}_n$.

TABLE 3

n	N	$\kappa(\mathbf{A}_n)$	$\kappa(\tilde{\mathbf{A}}_n)$
1	9	5.8	12.2
2	49	25.3	33.2
3	225	103.1	46.2
4	961	414.3	51.8
5	3969	1659.4	54.2

6.2. **BPX-preconditioning.**

Case 4: The Quadratic FVM Schemes on Triangular Grids for 2-D Laplace Equations.

In the next four cases, we use the two-sided BPX-preconditioner

$$\mathbf{C}_n = \sum_{k=0}^{n} \mathbf{E}_k \mathbf{D}_k^{-1} \mathbf{E}_k^{\mathbf{T}}, \tag{6.6}$$

where $\mathbf{E}_k$ is defined in Section 4 and $\mathbf{D}_k$ is the diagonal matrix with the diagonal entries equal to the diagonal entries of $\mathbf{A}_k$. The resulting preconditioned matrix has the form

$$\tilde{\mathbf{A}}_n := \mathbf{C}_n^{1/2} \mathbf{A}_n \mathbf{C}_n^{1/2}. \tag{6.7}$$

In this example, we consider the Laplace equation (2.2) and (2.3) in a square region $\Omega := [0,1] \times [0,1]$ and precondition the FVM schemes described at the end of Section 2 for solving the problem. We choose the trial space U_n as the piecewise quadratic finite element space on T_n and the test space V_n as the piecewise constant space corresponding to the dual partition T_n^*.

In Table 4 we present the computed condition numbers of matrices $\mathbf{A}_n$ and $\tilde{\mathbf{A}}_n$: at different levels. The numerical results confirm the uniform boundedness of the condition numbers of the preconditioned matrices, ensured by Theorem 4.4. We solve the corresponding linear algebraic systems by using the GMRES(10). That is, we restart the process after 10 steps in the GMRES iteration. In Table 4, in addition to the condition numbers of the matrices, we also present the iteration numbers for both the unpreconditioned method and the preconditioned method that allow us to fulfill the error tolerance 10^{-6}. In this section, we use iter0 and iter1 to denote the iteration numbers for the unpreconditioned and preconditioned FVM schemes, respectively. We observe that the numbers of iterations are bounded above by 37 for different matrix sizes for the preconditioned schemes. This implies that for a given error tolerance, the computational cost for solving the preconditioned linear system is of $\mathcal{O}(n)$.

TABLE 4

n	N	$\kappa(\mathbf{A}_n)$	$\kappa(\tilde{\mathbf{A}}_n)$	iter0	iter1
1	9	5.8	5.5	13	13
2	49	25.3	9.1	26	24
3	225	103.1	12.1	65	29
4	961	414.3	15.0	292	33
5	3969	659.4	17.6	1121	36
6	16129	6639.5	19.7	4468	37
7	65025	26560.1	21.5	17754	37

Case 5: The linear FVM Schemes for Equations with Discontinuous Coefficients.

We consider the equation

$$\begin{cases} -\nabla\cdot(A(x,y)\nabla u(x,y)) = f(x,y), & (x,y)\in\Omega, \\ u(x,y)=0, & (x,y)\in\partial\Omega, \end{cases}$$

where $\Omega := [0,1]\times[0,1]$ and the coefficient function A is defined as

$$A(x,y) := \begin{cases} 1, & 0\le x\le 1/2, \\ \epsilon, & 1/2 < x \le 1. \end{cases}$$

The linear FVM is used to discretize this equation. In our numerical experiment, the value of ϵ is chosen to be $0.5, 0.1, 0.05, 0.01, 0.005, 0.001$. In all these cases, we use the same preconditioner $\mathbf{C}_n$ as in Example 4 for different ϵ.

We present in Tables 5a and 5b, respectively, the condition numbers of the unpreconditioned matrices $\mathbf{A}$ and the two-sided preconditioned matrices $\tilde{\mathbf{A}}_n$ at different levels n for different values of ϵ. Noting that in this example $\mathbf{A}_n$ is a symmetric matrix and thus $\kappa_{\mathbf{A}_n}(\mathbf{C}_n\mathbf{A}_n)\sim\kappa(\tilde{\mathbf{A}}_n)$, the numerical results presented in Table 5b also give the condition numbers of the one-sided preconditioning. We observe that for unpreconditioned matrices, the condition numbers increase significantly as the n increases (to infinity) or as the parameter ϵ decreases (to zero). However, the condition numbers of the preconditioned matrices are uniformly bounded with respect to n and ϵ. These results support our theoretical estimates presented in Section 5.

TABLE 5a: The condition numbers of the unpreconditioned matrices $\mathbf{A}_n$

n	N	$\epsilon=0.5$	$\epsilon=0.1$	$\epsilon=0.05$	$\epsilon=0.01$	$\epsilon=0.005$	$\epsilon=0.001$
1	9	7.2	23.5	45.6	223.2	445.3	2222.3
2	49	34.3	108.8	207.1	1000.4	1992.9	9932.7
3	225	144.7	454.2	860.4	4137.8	8237.8	41040.9
4	961	587.0	1838.4	3477.6	16704.1	33250.9	165636.1
5	3969	2357.0	7376.4	13948.0	66978.0	133320.0	664100.0

TABLE 5b: The condition numbers of the preconditioned matrices $\tilde{\mathbf{A}}_n$

n	N	$\epsilon = 0.5$	$\epsilon = 0.1$	$\epsilon = 0.05$	$\epsilon = 0.01$	$\epsilon = 0.005$	$\epsilon = 0.001$
1	9	3.0	3.2	3.2	3.3	3.3	3.3
2	49	5.4	5.6	5.6	5.7	5.7	5.7
3	225	7.1	7.3	7.4	7.4	7.4	7.4
4	961	8.4	8.6	8.7	8.7	8.7	8.7
5	3969	9.3	9.6	9.7	9.7	9.7	9.7

Case 6: The Hermite Element FVM for Two Point Boundary Value Problems.

We consider solving the two-point boundary value problem

$$Lu = f \quad \text{in} \quad I, \tag{6.8}$$

$$u(a) = 0, u'(b) = 0, \tag{6.9}$$

by using the Hermite cubic element FVM (cf. [30]). In this case, the trial space U_n is chosen as the cubic finite element space of the Hermite type. The test space V_n is chosen as the piecewise linear function space with basis functions given by

$$\psi_{0,j}(x) = \begin{cases} 1, & x_{j-1/2} \le x \le x_{j+1/2}, \\ 0, & \text{elsewhere} \end{cases}$$

and

$$\psi_{1,j}(x) = \begin{cases} x - x_j, & x_{j-1/2} \le x \le x_{j+1/2}, \\ 0, & \text{elsewhere.} \end{cases}$$

The cubic FVM for solving (6.8) and (6.9) is to find $u_n \in U_n$ such that

$$\begin{cases} a(u_n, \psi_{0,j}) = (f, \psi_{0,j}), j = 1, 2, \ldots, n, \\ a(u_n, \psi_{1,j}) = (f, \psi_{1,j}), j = 0, 1, \ldots, n. \end{cases}$$

We use the two-sided BPX-preconditioners to precondition the linear system above. In Table 6 we present the computed condition numbers of matrices $\mathbf{A}_n$ and $\tilde{\mathbf{A}}_n$ at different levels, where $\tilde{\mathbf{A}}_n$ is defined by (6.7). We report in Table 6 the computed condition numbers of the un-preconditioned and preconditioned matrices, and as well as the iteration numbers for solving the linear system to the accuracy within the tolerance 10^{-6} when the GMRES is used. We observe that both the condition numbers of preconditioned matrices and the iteration numbers for preconditioned systems are uniformly bounded.

TABLE 6

n	N	$\kappa(\mathbf{A}_n)$	$\kappa(\hat{\mathbf{A}}_n)$	iter0	iter1
1	2	58.0	2.4	4	3
2	4	410.8	3.0	8	7
3	8	1852.2	3.5	14	10
4	16	7619.1	4.0	22	11
5	32	30673.7	4.5	38	13
6	64	122863.2	4.8	70	14
7	128	491562.3	5.0	133	14
8	256	1966240.5	5.3	261	14
9	512	7864716.8	5.4	517	15
10	1024	31581495.2	5.5	1028	15

Case 7: The Quadratic FVM Schemes.

We precondition the quadratic FVM schemes studied in [5]. We again consider the Laplace equation (2.3) and (2.4) in $\Omega := [0,1] \times [0,1]$. The primal partition T_n of the domain is constructed with isosceles right triangles and the corresponding dual partition T_n^* is the standard barycenter mesh for the linear FVM schemes. The trial space U_n is the standard quadratic finite element space on T_n. The test space V_n constitutes the piecewise constant functions on T_n^* and the quadratic bubble functions at the edges of T_n. We precondition the resulting linear system with the two-sided BPX-preconditionners $\mathbf{C}_n$.

In Table 7 we present the condition numbers of the unpreconditioned matrices $\mathbf{A}_n$ and the preconditioned matrices $\tilde{\mathbf{A}}_n$ at different levels. We confirm that the condition numbers of the preconditioned matrices are uniformly bounded. Again, the corresponding linear algebraic systems is solved by using the GMRES(10) to accuracy within the error tolerance 10^{-6}. The iteration numbers are also listed in Table 7.

TABLE 7

n	N	$\kappa(\mathbf{A}_n)$	$\kappa(\tilde{\mathbf{A}}_n)$	iter0	iter1
1	9	4.8	4.6	14	14
2	49	11.2	11.2	28	27
3	225	38.7	16.4	41	33
4	961	150.3	20.8	97	35
5	3969	596.7	24.5	416	35
6	16129	2382.4	27.5	1524	36
7	65025	9525.2	30.0	5895	35

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