MATHEMATICS OF COMPUTATION
Volume 82, Number 281, January 2013, Pages 25–43
S 0025-5718(2012)02611-1
Article electronically published on June 8, 2012

MINIMAL FINITE ELEMENT SPACES FOR $2m$-TH-ORDER PARTIAL DIFFERENTIAL EQUATIONS IN R^n

MING WANG AND JINCHAO XU

ABSTRACT. This paper is devoted to a canonical construction of a family of piecewise polynomials with the minimal degree capable of providing a consistent approximation of Sobolev spaces H^m in R^n (with $n \geq m \geq 1$) and also a convergent (nonconforming) finite element space for $2m$-th-order elliptic boundary value problems in R^n. For this class of finite element spaces, the geometric shape is n-simplex, the shape function space consists of all polynomials with a degree not greater than m, and the degrees of freedom are given in terms of the integral averages of the normal derivatives of order $m-k$ on all subsimplexes with the dimension $n-k$ for $1 \leq k \leq m$. This sequence of spaces has some natural inclusion properties as in the corresponding Sobolev spaces in the continuous cases.

The finite element spaces constructed in this paper constitute the only class of finite element spaces, whether conforming or nonconforming, that are known and proven to be convergent for the approximation of any $2m$-th-order elliptic problems in any R^n, such that $n \geq m \geq 1$. Finite element spaces in this class recover the nonconforming linear elements for Poisson equations ($m = 1$) and the well-known Morley element for biharmonic equations ($m = 2$).

1. INTRODUCTION

In the study of qualitative and numerical analysis of partial differential equations and, in general, of approximation theory, we are often interested in the approximation of functions in Sobolev spaces by piecewise polynomials (such as finite element spaces) defined on a partition of the domain by, say, a number of simplexes.

For Sobolev space H^1, it is easy to construct approximation subspaces comprising piecewise polynomial subspaces of any degree (that are defined on simplicial partitions of the underlying domain) by conforming finite element discretization (see [10]). It turns out, though, that it is much more difficult to construct conforming finite element spaces, namely piecewise polynomial subspaces, of H^2. The minimal degree of conforming elements is 5 for $n = 2$ (the well-known Argyris elements, see [10]) and 9 for $n = 3$ (see [45]). For a general Sobolev space H^m, constructing piecewise polynomial subspaces becomes increasingly difficult as the differential order m and/or the spatial dimension n increases.

Received by the editor September 30, 2008 and, in revised form, August 30, 2011.
2010 *Mathematics Subject Classification.* Primary 65N30.
Key words and phrases. Finite element space, minimal degree, conforming, nonconforming, consistent approximation, $2m$-th-order elliptic problem.
This work was supported by the National Natural Science Foundation of China (10871011).
This work was supported by the National Science Foundation, DMS 0749202 and DMS 0915153.

In order to use piecewise polynomials with a lower degree, nonconforming finite element spaces, i.e., finite element spaces that are not necessarily subspaces of H^2, have been constructed and used in practice, such as the Morley element [21, 23, 38] and the rectangle Morley element (see [47, 41]), the Veubake elements [36], the new Zienkiewicz-type element [40], the Adini element [2, 20, 41], three-dimensional (or higher) Bogner-Fox-Schmit element [41], the 12- and 15-parameter plate bending elements (see [47]), the cubic element and incomplete cubic element given in [39]. Obviously, not all piecewise polynomial spaces are convergent finite element spaces. In addition, certain "continuity" or consistency conditions must be imposed. Such conditions have been widely studied in the literature; cf. [10, 11, 13, 14], [17]–[19], [22], [24]–[34], [37, 47]. An example of these conditions is the "consistent approximation" condition, in which, the finite element spaces have approximability and are weakly compact (see [32]).

From both theoretical and practical viewpoints, we are particularly interested in consistent approximation spaces for H^m consisting of piecewise polynomials with the smallest possible degree, as denoted by $d_{\min}(m,n)$, in R^n.

It is easy to see that

$$d_{\min}(m,n) \geq m, \quad \forall m \geq 1, n \geq 1. \tag{1.1}$$

Since it is well-known that convergent linear simplicial finite elements can be easily constructed for second-order elliptic boundary value problems in any dimension, we have that

$$d_{\min}(m,n) = m, \quad n \geq m \tag{1.2}$$

is true for $m = 1$. Due to the classic Morley element [21] for biharmonic equations for $n = 2$ and the results in Ruas [23], Wang and Xu [38], (1.2) is also true for $m = 2$. With the new class of consistent approximation spaces to be constructed in this paper, we can conclude that (1.2) is valid for general $m \geq 1$.

A universal construction will be given in this paper for consistent approximation spaces for H^m in R^n (with $n \geq m$) consisting of piecewise polynomials of degree m. This construction can be used as finite element spaces for the discretization of $2m$-th-order elliptic boundary value problems.

Denoted by M_h^m, spaces in this class are given by piecewise polynomials with a degree not greater than m, namely the space P_m. Degrees of freedom for M_h^m in each element are given in terms of the integral averages of the normal derivatives of order $m-k$ on all subsimplexes of dimension $n-k$ for $1 \leq k \leq m$. The total number of these degrees of freedom in each element amounts to the dimension of P_m. One remarkable property of this sequence of spaces M_h^m is that it has certain inclusion properties. The degrees of freedom are just those of the nonconforming linear element when $m = 1$, which are also those of the Morley element [38] when $m = 2$. That is, we recover these two nonconforming elements in a canonical way.

While the construction presented in this paper is mainly motivated by theoretical considerations, the new element can also be applied to practical problems. The modeling for plates in linear elasticity is a classic area wherein fourth-order partial differential equations find their applications in two spatial dimensions. In recent years, modeling in material science has made use of fourth-order equations (see [7, 8, 12, 15, 35]) and also sixth-order equations [5, 42, 43] in three dimensions. Elliptic or parabolic equations of eighth or higher order are rare for practical applications;

however, in the theory of differential geometry (see [9]), elliptic equations of order $m = n/2$ in any dimension n have been used.

In addition to conforming and nonconforming finite element methods, discontinuous Galerkin methods (see [3, 4]), which have been the subject of considerable research interest in recent years, represents another type of discretization method for $2m$-th-order partial differential equations. The discontinuous Galerkin method uses discontinuous piecewise polynomial spaces, and it imposes consistency on these spaces by introducing certain penalty terms on the element interfaces in the discrete variational forms. Thus the study of this type of method focuses exclusively on constructing and analyzing of appropriate discrete variational forms.

The rest of the paper is organized as follows: Section 2 gives a detailed description of our family of minimal degree finite element spaces. Section 3 discusses the convergence and the error estimate for their application to $2m$-th-order elliptic problems. The last section offers some brief concluding remarks.

2. Nonconforming finite element spaces of minimal degree

In this section, we will construct a minimal piecewise polynomial approximation of $H^m(\Omega)$ for $\Omega \subset R^n$ with $n \geq m \geq 1$.

We first introduce some basic notation. Given a nonnegative integer k and a bounded domain $G \subset R^n$ with boundary ∂G, let $H^k(G)$, $H_0^k(G)$, $(\cdot,\cdot)_{k,G}$, $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$ denote the usual Sobolev spaces, inner product, norm, and semi-norm, respectively.

For an n dimensional multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We will use α, β to denote the multi-indexes. Let e_i denote the multi-index with the i-th component 1 and the others 0. For $k \geq 1$, let A_k be the set consisting of all multi-indexes α with $\sum_{i=k+1}^{n} \alpha_i = 0$.

Following the description in [10], a finite element can be represented by a triple (T, P_T, D_T), with T the geometric shape of the element, P_T the shape function space, and D_T the set of the degrees of freedom, such that D_T is P_T-unisolvent.

Let Ω be a bounded polyhedron domain of R^n. Assume that $\{h\}$ is a sequence of positive numbers and $h \to 0$. For each h, let $\mathcal{T}_h$ be a partition of Ω corresponding to a finite element (T, P_T, D_T), and let h be the mesh size, i.e., the maximal diameter of the elements in $\mathcal{T}_h$.

For any element $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest ball containing T, and let ρ_T be the diameter of the largest ball contained in T. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ is quasi-uniform, that is,

$$\max_{x\in T,\ T\in\mathcal{T}_h} h_T \leq \eta \min_{x\in T,\ T\in\mathcal{T}_h} \rho_T, \quad \forall x \in \bar{\Omega}, \tag{2.1}$$

in which η is a positive constant independent of h.

For $|\alpha| \leq m$ and $v_h \in L^2(\Omega)$ with $v_h|_T \in H^m(T)$ $(\forall T \in \mathcal{T}_h)$, we denote $\partial_h^\alpha v_h$ as the partial derivatives of v_h taken piecewise with respect to the partition $\mathcal{T}_h$.

For a subset $B \subset R^n$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials defined on B with a degree not greater than r, and let $Q_r(B)$ be the space of all polynomials with a degree in each variable not greater than r. Define

$$P_{r,h} = \{\, v \in L^2(\Omega) \ :\ v|_T \in P_r(T),\ \forall T \in \mathcal{T}_h\}. \tag{2.2}$$

We will give the description of (T, P_T, D_T) for our new finite element first. Then we will show the P_T-unisolvent property and give the construction and the error estimate of the corresponding interpolation operator. Moreover, we will define the global finite element spaces and show their basic properties, such as the approximation property and the inclusion property.

2.1. **The local degrees of freedom.** For our new element (T, P_T, D_T), T is a simplex and $P_T = P_m(T)$. The set of degrees of freedom, denoted by D_T^m, will be given next.

Given an n-simplex T with vertices a_i, $1 \le i \le n+1$, let $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ be the barycentric coordinates of T. For $1 \le k \le n$, let $\mathcal{F}_{T,k}$ be the set consisting of all $(n-k)$-dimensional subsimplexes of T. For any F in $\mathcal{F}_{T,k}$, let $|F|$ denote its measure, and let $\nu_{F,1}, \cdots, \nu_{F,k}$ be its unit outer normals which are linearly independent.

For $1 \le k \le m$, any $(n-k)$-dimensional subsimplex $F \in \mathcal{F}_{T,k}$ and $\beta \in A_k$ with $|\beta| = m-k$, define

$$d_{T,F,\beta}(v) = \frac{1}{|F|} \int_F \frac{\partial^{|\beta|} v}{\partial \nu_{F,1}^{\beta_1} \cdots \nu_{F,k}^{\beta_k}}, \quad \forall v \in H^m(T). \tag{2.3}$$

By the Sobolev embedding theorems [1], $d_{T,F,\beta}$ is a continuous linear functional on $H^m(T)$. Then the set of the degrees of freedom is given by

$$D_T^m = \{\ d_{T,F,\beta}\ :\ \beta \in A_k \text{ with } |\beta| = m-k,\ F \in \mathcal{F}_{T,k},\ 1 \le k \le m\}. \tag{2.4}$$

That is, the degrees of freedom are the integral averages of the normal derivatives of order $m-k$ on all subsimplexes of dimension $n-k$ for $1 \le k \le m$.

For nonnegative integers i, j with $i \le j$ set

$$C_j^i = \frac{j!}{i!(j-i)!}.$$

Then for each $1 \le k \le m$, T has C_{n+1}^{n-k+1} subsimplexes of dimension $n-k$. For each $(n-k)$-dimensional subsomplex F, the number of all $(m-k)$-th-order direction derivatives with respect to $\nu_{F,1}, \cdots, \nu_{F,k}$ is C_{m-1}^{m-k}. Therefore, by the well-known Vandermonde combinatorial identity, the number of the total degrees of freedom is given by

$$\sum_{k=1}^{m} C_{n+1}^{n-k+1} C_{m-1}^{m-k} = C_{n+m}^{m},$$

which is precisely the dimension of $P_m(T)$.

Let $J = C_{n+m}^m$. We also number all the degrees of freedom by

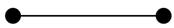
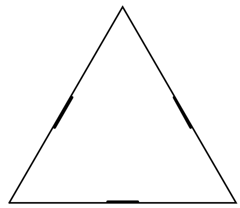
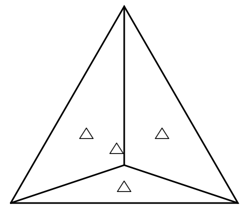
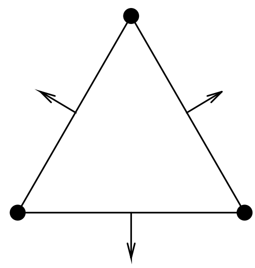
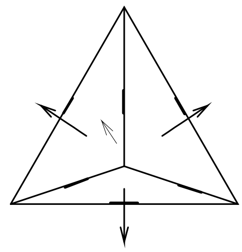
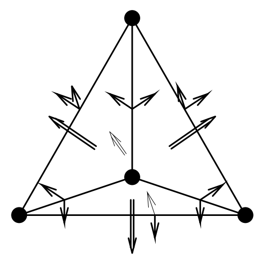
$$d_{T,1}, d_{T,2}, \cdots, d_{T,J}.$$

Then, $D_T^m = \{d_{T,1}, d_{T,2}, \cdots, d_{T,J}\}$.

For $1 \le k \le m$ and an $(n-k)$-dimensional subsimplex F without $a_{j_1}, \cdots, a_{j_k}$ as its vertices, different choices (for $k > 1$) of $\nu_{F,1}, \cdots, \nu_{F,k}$ will lead to equivalent degrees of freedom. The particular and convenient choices of normal directions are as follows:

$$\nu_{F,i} = -\frac{\nabla \lambda_{j_i}}{\|\nabla \lambda_{j_i}\|}, \quad 1 \le i \le k. \tag{2.5}$$

Table 1. Degrees of Freedom

$m \setminus n$	1	2	3
1			
2			
3			

Some special cases: $1 \le m \le 3$**.** We now offer a brief discussion regarding all spaces that corresponding to the three lowest indices $1 \le m \le 3$. The degrees of freedom in these cases are depicted in Table 1 for $m \le n \le 3$. For $m = 1$ and $n = 1$, we obtain the well-known conforming linear element. This is the only conforming element in this family of elements. For $m = 1$ and $n \ge 2$, we obtain the well-known nonconforming linear element. For $m = 2$, we recover the well-known Morley element for $n = 2$ and its generalization to $n \ge 2$ (see Wang and Xu [38]). For $m = 3$ and $n = 3$, we obtain a new cubic element on a simplex that has 20 degrees of freedom.

2.2. **Unisolvent property and canonical nodal interpolation.** We need to show the P_T-unisolvent property of our new finite element. First, we show a crucial property.

Lemma 2.1. *Let $1 \le k \le m$ and $F \in \mathcal{F}_{T,k}$. Then for any $v \in H^m(T)$, the integrals of all its $(m-k)$-th-order derivatives on F,*

$$\int_F \partial^\alpha v, \quad |\alpha| = m - k,$$

are uniquely determined by all $d_{T,F',\beta}(v)$ given in (2.3) with $k \le r \le m$, F' $(n-r)$-dimensional subsimplex of F, $\beta \in A_r$, and $|\beta| = m - r$.

Proof. Let $v \in H^m(T)$. We prove the lemma by induction. When $k = m$,

$$\frac{1}{|F|}\int_F v = d_{T,F,0}(v).$$

The lemma is obviously true.

Assume that the lemma is true for all $k \in \{i+1, \cdots, m\}$ with $1 \le i < m$. We consider the case that $k = i$.

Denote all $(n-k-1)$-dimensional subsimplexes of the $(n-k)$-simplex F by $S_1, S_2, \cdots, S_{n-k+1}$, and the unit outer normal of S_j by $n^{(j)}$, viewed as the boundary of an $(n-k)$-simplex in $(n-k)$-dimensional space. Choose orthogonal unit vectors $\tau_{F,k+1}, \cdots, \tau_{F,n}$ that are tangent to F. Then

$$\nu_{F,1}, \cdots, \nu_{F,k}, \tau_{F,k+1}, \cdots, \tau_{F,n}$$

form a basis of R^n.

Now let $|\alpha| = m-k$. If $\alpha_{k+1} = \cdots = \alpha_n = 0$, then

$$\frac{1}{|F|}\int_F \frac{\partial^{m-k} v}{\partial\nu_{F,1}^{\alpha_1}\cdots\nu_{F,k}^{\alpha_k}\partial\tau_{F,k+1}^{\alpha_{k+1}}\cdots\partial\tau_{F,n}^{\alpha_n}} = d_{T,F,\beta}(v)$$

with $\beta \in A_k$ and $\beta_j = \alpha_j$, $1 \le j \le k$. Otherwise, without loss of generality, let $\alpha_{k+1} > 0$. Green's formula gives

$$\begin{aligned}&\int_F \frac{\partial^{m-k} v}{\partial\nu_{F,1}^{\alpha_1}\cdots\nu_{F,k}^{\alpha_k}\partial\tau_{F,k+1}^{\alpha_{k+1}}\cdots\partial\tau_{F,n}^{\alpha_n}}\\ &\qquad = \sum_{j=1}^{n-k+1} n^{(j)}\cdot\tau_{F,k+1}\int_{S_j}\frac{\partial^{m-k-1} v}{\partial\nu_{F,1}^{\alpha_1}\cdots\nu_{F,k}^{\alpha_k}\partial\tau_{F,k+1}^{\alpha_{k+1}-1}\partial\tau_{F,k+2}^{\alpha_{k+2}}\cdots\partial\tau_{F,n}^{\alpha_n}}.\end{aligned}$$

By the assumption of induction, the right-hand side of the above identity can be expressed in terms of all $d_{T,F',\beta}(v)$ with $k < r \le m$, F' $(n-r)$-dimensional subsimplex of F, $\beta \in A_r$, and $|\beta| = m-r$. Consequently, the lemma is true for $k = i$. □

Lemma 2.2. *For $1 \le i \le J$, there exists a unique polynomial $p_i \in P_m(T)$, such that*

$$d_{T,j}(p_i) = \delta_{ij}, \quad 1 \le j \le J, \tag{2.6}$$

where δ_{ij} is the Kronecker delta.

Proof. As the dimension of $P_m(T)$ is also J, it is sufficient to show that if $p \in P_m(T)$ and

$$d_{T,F,\beta}(p) = 0, \quad \beta \in A_k \text{ with } |\beta| = m-k,\ F \in \mathcal{F}_{T,k},\ 1 \le k \le m, \tag{2.7}$$

then $p \equiv 0$.

By Lemma 2.1 and its proof, we deduce that

$$\int_F \partial^\alpha p = 0, \quad |\alpha| = m-k,\ F \in \mathcal{F}_{T,k},\ 1 \le k \le m. \tag{2.8}$$

By Green's formula and (2.8), we have for all $1 \le k_1 \le k_2 \le \cdots \le k_m \le n$, that,

$$\begin{aligned}\frac{\partial^m p}{\partial x_{k_1}\cdots\partial x_{k_m}} &= \frac{1}{|T|}\int_T \frac{\partial^m p}{\partial x_{k_1}\cdots\partial x_{k_m}}\\ &= \frac{1}{|T|}\sum_{F\in\mathcal{F}_{T,1}}\int_F \frac{\partial^{m-1} p}{\partial x_{k_2}\cdots\partial x_{k_m}}(\nu_{F,1})_{k_1} = 0,\end{aligned}$$

where $|T|$ is the measure of T. That is, $p\in P_{m-1}(T)$.

When $1\le k<m$ and $p\in P_{m-k}(T)$, by (2.8), all $(m-k)$-th-order derivatives of p are zero and p is in $P_{m-k-1}(T)$. Thus, p is a constant. By (2.8), we obtain that $p\equiv 0$. $\square$

Lemma 2.2 shows the P_T-unisolvent property of our new elements, specifically that a polynomial $p\in P_m(T)$ is uniquely determined by $d_{T,j}(p)$, $1\le j\le J$. The polynomial p_i given by (2.6) is the basis function corresponding to degree of freedom $d_{T,i}$. Based on Lemma 2.2, we can define the interpolation operator $\Pi_T: H^m(T)\to P_m(T)$ by

$$\Pi_T v = \sum_{i=1}^{J} p_i d_{T,i}(v),\quad \forall v\in H^m(T). \tag{2.9}$$

We would like to emphasize here that operator Π_T is well-defined for all functions in $H^m(T)$.

By the interpolation theory [10], we obtain the following error estimate of the interpolation operator.

Lemma 2.3. *For $s=0,1$,*

$$|v-\Pi_T v|_{k,T}\le C(\eta)h_T^{m+s-k}|v|_{m+s,T},\ 0\le k\le m+s,\ \forall v\in H^{m+s}(T) \tag{2.10}$$

for all n-simplex T with $h_T\le\eta\rho_T$. Here, $C(\eta)$ is a constant that only depends on η.

2.3. **Global finite element spaces.** We define our piecewise polynomial spaces M_h^m and M_{h0}^m as follows:

(1) M_h^m consists of all functions v_h in $P_{m,h}$, such that for any $k\in\{1,\cdots,m\}$, any $(n-k)$-dimensional subsimplex F of any $T\in\mathcal{T}_h$ and any $\beta\in A_k$ with $|\beta|=m-k$, $d_{T,F,\beta}(v_h)$ is continuous through F.

(2) M_{h0}^m consists of all functions v_h in M_h^m, such that for any $k\in\{1,\cdots,m\}$, any $(n-k)$-dimensional subsimplex F of any $T\in\mathcal{T}_h$ and any $\beta\in A_k$ with $|\beta|=m-k$, if $F\subset\partial\Omega$ then $d_{T,F,\beta}(v_h)=0$.

Define an operator Π_h on $H^m(\Omega)$ as follows:

$$(\Pi_h v)|_T = \Pi_T(v|_T),\quad \forall T\in\mathcal{T}_h,\ \forall v\in H^m(\Omega). \tag{2.11}$$

By the above definition, $\Pi_h v\in M_h^m$ for any $v\in H^m(\Omega)$ and $\Pi_h v\in M_{h0}^m$ for any $v\in H_0^m(\Omega)$.

For convenience, following [44], the symbols $\lesssim$, $\gtrsim$ and $\eqsim$ will be used in the rest of this paper. That $X_1\lesssim Y_1$ and $X_2\gtrsim Y_2$, mean that $X_1\le c_1Y_1$ and $c_2X_2\ge Y_2$ for some positive constants c_1 and c_2 that are independent of mesh size h. That $X_3\eqsim Y_3$ means that $X_3\lesssim Y_3$ and $X_3\gtrsim Y_3$.

We define, for $w\in L^2(\Omega)$ with $w|_T\in H^m(T)$, $\forall T\in\mathcal{T}_h$,

$$\|w\|_{m,h}^2=\sum_{T\in\mathcal{T}_h}\|w\|_{m,T}^2,\quad |w|_{m,h}^2=\sum_{T\in\mathcal{T}_h}|w|_{m,T}^2.$$

Now we consider the approximate property of M_h^m and M_{h0}^m.

Theorem 2.1. *For any $v \in H^{m+1}(\Omega)$,*

$$\|v - \Pi_h v\|_{m,h} \lesssim h|v|_{m+1,\Omega}, \tag{2.12}$$

and for any $v \in H^m(\Omega)$,

$$\lim_{h\to 0} \|v - \Pi_h v\|_{m,h} = 0. \tag{2.13}$$

Proof. First, let $v \in H^{m+1}(\Omega)$. By Lemma 2.3, we obtain (2.12) directly.

Now let $w \in H^m(\Omega)$. Since $H^{m+1}(\Omega)$ is dense in $H^m(\Omega)$, for any $\varepsilon > 0$ there exists $\phi \in H^{m+1}(\Omega)$ such that

$$\|w - \phi\|_{m,\Omega} < \varepsilon.$$

By (2.12), there exists $\tilde{h} > 0$ such that

$$\|\phi - \Pi_h \phi\|_{m,h} < \varepsilon$$

when $h < \tilde{h}$. Therefore by (2.10)

$$\|w - \Pi_h w\|_{m,h} \le \|w - \phi\|_{m,h} + \|\Pi_h (w - \phi)\|_{m,h} + \|\phi - \Pi_h \phi\|_{m,h} \lesssim \varepsilon$$

when $h < \tilde{h}$. This leads to (2.13). □

When $n > 1$, M_h^m is not a subspace of $H^m(\Omega)$ and M_h^m is not a subspace of $C^0(\bar{\Omega})$. Although functions in M_h^m are not continuous on Ω in general, they have some weak continuity. By the definitions of M_h^m and M_{h0}^m, Lemma 2.1 and its proof, the following lemma can be obtained directly.

Lemma 2.4. *Let $k \in \{1, \cdots, m\}$ and F be an $(n-k)$-dimensional subsimplex of $T \in \mathcal{T}_h$. Then, for any $v_h \in M_h^m$ and any $T' \in \mathcal{T}_h$ with $F \subset T'$,*

$$\int_F \partial^\alpha (v_h|_{T'}) = \int_F \partial^\alpha (v_h|_T), \quad |\alpha| = m - k. \tag{2.14}$$

If $F \subset \partial\Omega$, then for any $v_h \in M_{h0}^m$,

$$\int_F \partial^\alpha (v_h|_T) = 0, \quad |\alpha| = m - k. \tag{2.15}$$

An equivalent definition. By Lemma 2.4, we can give an equivalent definition of M_h^m and M_{h0}^m: M_h^m consists of all functions v_h in $P_{m,h}$, such that for any $k \in \{1, \cdots, m\}$, any $(n-k)$-dimensional subsimplex F of any $T \in \mathcal{T}_h$ and any α with $|\alpha| = m - k$, the integral of $\partial_h^\alpha v_h$ over F is continuous through F; M_{h0}^m consists of all functions v_h in M_h^m such that for any $k \in \{1, \cdots, m\}$, any $(n-k)$-dimensional subsimplex F of any $T \in \mathcal{T}_h$ and any α with $|\alpha| = m - k$, if $F \subset \partial\Omega$, then the integral of $\partial_h^\alpha v_h$ over F vanishes.

Lemma 2.5. *Let $|\alpha| < m$ and F be an $(n-1)$-dimensional subsimplex of $T \in \mathcal{T}_h$. Then for any $v_h \in M_h^m$, $\partial_h^\alpha v_h$ is continuous at a point on F at least. If $F \subset \partial\Omega$ and $v_h \in M_{h0}^m$, then $\partial_h^\alpha v_h$ vanishes at a point on F at least.*

Proof. Let $v_h \in M_h^m$ and $T' \in \mathcal{T}_h$ with $F \subset T'$. By Lemma 2.4, there is an $(n - m + |\alpha|)$-dimensional subsimplex F' of F, such that

$$\int_{F'} \partial^\alpha (v_h|_{T'}) = \int_{F'} \partial^\alpha (v_h|_T).$$

Then, $\partial_h^\alpha v_h$ is continuous at a point on F' at least.

If $F \subset \partial\Omega$ and $v_h \in M_{h0}^m$, then there is an $(n - m + |\alpha|)$-dimensional subsimplex F' of F by Lemma 2.4, such that

$$\int_{F'} \partial^\alpha (v_h|_T) = 0.$$

Thus, $\partial_h^\alpha v_h$ vanishes at a point on F' at least. □

2.4. **Inclusion properties.** We will now discuss two simple inclusion properties of our finite element spaces. First, we have the following observation.

Lemma 2.6. *Given any $n > m \geq 1$ and a simplex T, the set of subsimplices of T used to define D_T^m is also used in the definition of D_T^{m+1}. More precisely, the degrees of freedom for D_T^{m+1} can be obtained by taking the integral of one order higher than normal derivatives of functions on the same subsimplexes used for D_T^m, plus the integral average of the function over all the additional $(n - m - 1)$-subsimplexes.*

To obtain a more interesting inclusion property, we define

$$\partial M_h^m = \operatorname{span}\{\partial_h^{e_1} M_h^m, \partial_h^{e_2} M_h^m, \cdots, \partial_h^{e_n} M_h^m\}$$

and

$$\partial M_{h0}^m = \operatorname{span}\{\partial^{e_1} M_{h0}^m, \partial^{e_2} M_{h0}^m, \cdots, \partial^{e_n} M_{h0}^m\}.$$

Theorem 2.2. *Let $n \geq m > 1$, then*

$$\partial M_h^m = M_h^{m-1}, \quad \partial M_{h0}^m = M_{h0}^{m-1}. \tag{2.16}$$

Proof. By the equivalent definition of M_h^m and M_{h0}^m, we obtain directly that

$$\partial M_h^m \subset M_h^{m-1}, \quad \partial M_{h0}^m \subset M_{h0}^{m-1}.$$

For any $k \in \{1, 2, \cdots, m-1\}$, any $T \in \mathcal{T}_h$, any $(n-k)$-dimensional subsimplex F of T, and any $\beta \in A_k$ with $|\beta| = m - 1 - k$, (1) let w be the global basis function of M_h^{m-1} corresponding to degree of freedom $d_{T,F,\beta}$, and (2) let v be the global basis function of M_h^m corresponding to degree of freedom $d_{T,F,\alpha}$ with $\alpha = \beta + (1, 0, \cdots, 0)$. By these definitions, $w = \nu_{F,1}^\top \nabla v$. Then the theorem follows. □

A note on the case $m = 0$. In the above construction, we made the assumption that $m \geq 1$. But we can slightly enlarge this construction to include the trivial case $m = 0$, namely $L^2(\Omega)$ space. Technically, we can just replace the constraint $1 \leq k \leq m$ with $\min(1, m) \leq k \leq m$. In this case, the shape function space is again $P_m(T) = P_0(T)$, namely the constant, and the corresponding degree of freedom is just the volume integral on each simplex. This trivial case of the finite element space, denoted by M_h^0, can be thought of a close relative of M_h^m ($m \geq 1$), but not a direct family member in view of the properties stated in Lemma 2.6.

3. Convergence analysis for the new element

In this section, we will give the convergence analysis for the new class of finite element methods introduced in this paper.

Let b_0 be a nonnegative constant and b_α be positive constants, $|\alpha| = m$. Define

$$a(v, w) = \int_\Omega \Big(\sum_{|\alpha|=m} b_\alpha \partial^\alpha v \, \partial^\alpha w + b_0 v w \Big), \quad \forall v, w \in H^m(\Omega). \tag{3.1}$$

Let W be $H_0^m(\Omega)$ or $H^m(\Omega)$, and let $f \in L^2(\Omega)$. We consider the following variational problem: find $u \in W$, such that

$$a(u,v) = (f,v), \quad \forall v \in W. \tag{3.2}$$

We assume that problem (3.2) has a unique solution for any $f \in L^2(\Omega)$.

The above variational problem corresponds to the following $2m$-th-order partial differential equation:

$$\sum_{|\alpha|=m} (-1)^{|\alpha|}\partial^\alpha(b_\alpha \partial^\alpha u) + b_0 u = f, \quad \text{in } \Omega. \tag{3.3}$$

When $W = H_0^m(\Omega)$, the variational problem (3.2) corresponds to the homogeneous Dirichlet boundary problem of partial differential equation (3.3) with these boundary conditions:

$$\frac{\partial^k u}{\partial \nu^k}\Big|_{\partial\Omega} = 0, \quad 0 \le k \le m-1, \tag{3.4}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^\top$ is the unit outer normal to $\partial\Omega$.

When $W = H^m(\Omega)$, problem (3.2) corresponds to the boundary problem of partial differential equation (3.3) with some natural boundary conditions.

For $v, w \in L^2(\Omega)$ so that $v|_T, w|_T \in H^m(T)$, $\forall T \in \mathcal{T}_h$, we define

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T \Big(\sum_{|\alpha|=m} b_\alpha \partial^\alpha v \, \partial^\alpha w + b_0 v w \Big). \tag{3.5}$$

When $W = H^m(\Omega)$, let U_h be M_h^m; otherwise, let U_h be M_{h0}^m. The nonconforming finite element method for problem (3.2) corresponding to the new element given in Section 2 is to find $u_h \in U_h$, such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in U_h. \tag{3.6}$$

Next, we will discuss the convergence property of solution u_h of problem (3.6).

3.1. **Weak continuity.** Let V_h be a nonconforming finite element space to approximate $H^m(\Omega)$ corresponding to $\mathcal{T}_h$, and let V_{h0} be the corresponding subspace of V_h to approximate $H_0^m(\Omega)$.

We say that V_h has weak continuity (or weak discontinuity) if for any v_h in V_h, any $(n-1)$-dimensional face F of $T \in \mathcal{T}_h$ and any $|\alpha| < m$, $\partial_h^\alpha v_h$ is continuous at a point on F at least. Correspondingly, we say that V_{h0} satisfies the weak zero-boundary condition if for any v_h in V_{h0}, any $(n-1)$-dimensional face F of $T \in \mathcal{T}_h$ with $F \subset \partial\Omega$ and any $|\alpha| < m$, $\partial_h^\alpha v_h$ vanishes at a point on F at least.

By Lemma 2.5, we know that M_h^m has weak continuity and that M_{h0}^m satisfies the weak zero-boundary condition.

With weak continuity, we obtain that v_h is a single polynomial of a degree less than m on whole Ω if $v_h \in V_h$ and $|v_h|_{m,h} = 0$. Moreover, $v_h \equiv 0$ when $v_h \in V_{h0}$ and the weak zero-boundary condition is satisfied; that is, $|\cdot|_{m,h}$ is a norm of V_{h0}. In this sense, the weak continuity and the weak zero-boundary condition are viewed as necessary.

We assume in the rest of the section that a nonnegative integer t exists, such that $V_h \subset P_{t,h}$ for all h. We also assume that $\mathcal{T}_h$ is a partition consisting of n-simplexes or consisting of n-cubes with their sides parallel to some respective coordinate axes. Following the method used in [37], we have the following lemma.

Lemma 3.1. *Let V_h have weak continuity and let V_{h0} satisfy the weak zero-boundary condition. Then, for any $v_h \in V_h$ and any $|\alpha| < m$ there exists a piecewise polynomial $v_\alpha \in H^1(\Omega)$ such that*

$$|\partial_h^\alpha v_h - v_\alpha|_{j,h} \lesssim h^{m-|\alpha|-j}|v_h|_{m,h}, \quad 0 \le j \le m - |\alpha|, \tag{3.7}$$

and v_α can be chosen in $H_0^1(\Omega)$ when $v_h \in V_{h0}$.

Proof. For set $B \subset R^n$, let $\mathcal{T}_h(B) = \{ T \in \mathcal{T}_h \ : \ B \cap T \neq \emptyset \}$ and $N_h(B)$ be the number of the elements in $\mathcal{T}_h(B)$.

Let $v_h \in V_h$, $|\alpha| < m$. For $T \in \mathcal{T}_h$, denote by v_h^T the continuous extension of v_h from the interior of T to T. Given any $(n-1)$-dimensional face F of T, let us define the jump of $\partial_h^\alpha v_h$ across F as follows: $[\partial_h^\alpha v_h]_F = \partial^\alpha v_h^T|_F - \partial^\alpha v_h^{T'}|_F$ if $F = T \cap T'$ for some other $T' \in \mathcal{T}_h$ and $[\partial_h^\alpha v_h]_F = \partial^\alpha v_h^T|_F$ if $F = T \cap \partial\Omega$.

First, we show that if $F \not\subset \partial\Omega$ or $v_h \in V_{h0}$, then

$$[\partial_h^\alpha v_h]_F^2 \lesssim h^{2(m-|\alpha|)-n} \sum_{T' \in \mathcal{T}_h, F \subset T'} |v_h|_{m,T'}^2. \tag{3.8}$$

By the weak continuity and the weak zero-boundary condition there exists $x \in F$ such that $[\partial_h^\alpha v_h]_F$ vanishes at x, this leads to

$$[\partial_h^\alpha v_h]_F^2 \le h^2 \max_{y \in F} \left[\frac{\partial}{\partial \tau} \partial_h^\alpha v_h\right]_F^2 (y) \lesssim h^2 \sum_{|\alpha'|=|\alpha|+1} \max_{y \in F} \left[\partial_h^{\alpha'} v_h\right]_F^2 (y)$$

where τ is a unit tangent of F. Repeating the same argument, we have

$$[\partial_h^\alpha v_h]_F^2 \lesssim h^{2(m-|\alpha|)} \sum_{|\alpha'|=m} \max_{y \in F} \left[\partial_h^{\alpha'} v_h\right]_F^2 (y).$$

By the inverse inequality, we obtain (3.8).

Let $l = m - |\alpha|$ and $0 \le j \le l$. If T is an n-simplex, then we take $S_{l,T} = P_l(T)$ and $\Pi_{l,T}$ the interpolating operator corresponding to the element of n-simplex of type (l), otherwise take $S_{l,T} = Q_l(T)$ and $\Pi_{l,T}$ the interpolating operator corresponding to the element of n-cube of type (l) (see [10], pp. 48 and 57). Let $\Xi_{l,T}$ be the set of nodal points of $\Pi_{l,T}$.

Now we define $v_\alpha \in H^1(\Omega)$ as follows: for all $T \in \mathcal{T}_h$, $v_\alpha|_T \in S_{l,T}$ and for each $x \in \Xi_{l,T}$ if $x \in \partial\Omega$ and $v_h \in V_{h0}$, then $v_\alpha(x) = 0$, otherwise

$$v_\alpha(x) = \frac{1}{N_h(x)} \sum_{T' \in \mathcal{T}_h(x)} \partial^\alpha v_h^{T'}(x). \tag{3.9}$$

Then v_α is well-defined, and $v_\alpha \in H_0^1(\Omega)$ when $v_h \in V_{h0}$.

By the interpolating theory,

$$|\partial_h^\alpha v_h - \Pi_{l,T} \partial_h^\alpha v_h|_{j,T} \lesssim h^{m-|\alpha|-j}|v_h|_{m,T}. \tag{3.10}$$

Using the affine argument, we can show the following inequality:

$$|p|_{j,T}^2 \lesssim h^{n-2j} \sum_{x \in \Xi_{l,T}} |p(x)|^2, \quad \forall p \in S_{l,T}. \tag{3.11}$$

Since $\Pi_{l,T}\partial_h^\alpha v_h - v_\alpha|_T \in S_{l,T}$,

$$|\Pi_{l,T}\partial_h^\alpha v_h - v_\alpha|_{j,T}^2 \lesssim h^{n-2j} \sum_{x \in \Xi_{l,T}} |\Pi_{l,T}\partial^\alpha v_h^T(x) - v_\alpha(x)|^2. \tag{3.12}$$

If $x \in \Xi_{l,T} \cap \Omega$ or $v_h \notin V_{h0}$, then by (3.9) we have

$$|\Pi_{l,T}\partial^\alpha v_h^T(x) - v_\alpha(x)|^2 = \Big|\frac{1}{N_h(x)} \sum_{T' \in \mathcal{T}_h(x)} \Big(\partial^\alpha v_h^T(x) - \partial^\alpha v_h^{T'}(x)\Big)\Big|^2.$$

For $T' \in \mathcal{T}_h(x)$ and $T' \neq T$, there exist $T_1, \cdots, T_L \in \mathcal{T}_h(x)$ such that $T_1 = T$, $T_L = T'$ and $\tilde{F}_j = T_j \cap T_{j+1}$ is a common $(n-1)$-dimensional face of T_j and T_{j+1}, $1 \le j < L$. By (3.8) and the fact that $N_h(x)$ is bounded, we obtain

$$\Big|\partial^\alpha v_h^T(x) - \partial^\alpha v_h^{T'}(x)\Big|^2 \lesssim \sum_{j=1}^{L-1} \max_{y \in F_j} [\partial_h^\alpha v_h]_{F_j}^2(y) \lesssim h^{2(m-|\alpha|)-n} \sum_{T' \in \mathcal{T}_h(x)} |v_h|_{m,T'}^2.$$

If $x \in \Xi_{l,T} \cap \partial\Omega$ and $v_h \in V_{h0}$, then we have by definition of v_α,

$$|\Pi_{l,T}\partial^\alpha v_h^T(x) - v_\alpha(x)|^2 = |\partial^\alpha v_h^T(x)|^2 \lesssim h^{2(m-|\alpha|)-n} \sum_{T' \in \mathcal{T}_h(x)} |v_h|_{m,T'}^2.$$

From (3.12) we derive that

$$|\Pi_{l,T}\partial_h^\alpha v_h - v_\alpha|_{j,T}^2 \lesssim h^{2(m-|\alpha|-j)} \sum_{T' \in \mathcal{T}_h(T)} |v_h|_{m,T'}^2. \tag{3.13}$$

By (3.10), (3.13) and the triangle inequality, we get

$$|\partial_h^\alpha v_h - v_\alpha|_{j,h}^2 \lesssim h^{2(m-|\alpha|-j)} \sum_{T \in \mathcal{T}_h} \sum_{T' \in \mathcal{T}_h(T)} |v_h|_{m,T'}^2. \tag{3.14}$$

Then (3.7) follows. □

Theorem 3.1. *Let V_h have weak continuity, and let V_{h0} satisfy the weak zero-boundary condition. Then, the generalized inequality of Poincaré-Friedrichs,*

$$\|v_h\|_{m,h} \lesssim |v_h|_{m,h}, \quad \forall v_h \in V_{h0}, \tag{3.15}$$

and the generalized Poincaré inequality

$$\|v_h\|_{m,h}^2 \lesssim |v_h|_{m,h}^2 + \sum_{|\alpha|<m} \Big(\int_\Omega \partial_h^\alpha v_h\Big)^2, \quad \forall v_h \in V_h, \tag{3.16}$$

are true.

Proof. The following inequalities are true.

$$\|v\|_{1,\Omega} \lesssim |v|_{1,\Omega}, \quad \forall v \in H_0^1(\Omega), \tag{3.17}$$

$$\|v\|_{1,\Omega}^2 \lesssim |v|_{1,\Omega}^2 + \Big(\int_\Omega v\Big)^2, \quad \forall v \in H^1(\Omega). \tag{3.18}$$

For $v_h \in V_{h0}$, $|\alpha| < m$, let $v_\alpha \in H_0^1(\Omega)$ be as in (3.7). Then from (3.17) and (3.7),

$$\begin{aligned}\|\partial_h^\alpha v_h\|_{0,\Omega}^2 &\lesssim \|\partial_h^\alpha v_h - v_\alpha\|_{0,\Omega}^2 + \|v_\alpha\|_{0,\Omega}^2 \\ &\lesssim |v_h|_{m,h}^2 + |v_\alpha|_{1,\Omega}^2 \lesssim |v_h|_{m,h}^2 + |v_h|_{|\alpha|+1,h}^2.\end{aligned}$$

Consequently,

$$|v_h|_{k,h} \lesssim |v_h|_{m,h} + |v_h|_{k+1,h}, \quad 0 \le k < m. \tag{3.19}$$

This leads to (3.15).

By (3.18) and the same argument, we obtain (3.16). □

By Lemma 2.5 and Theorem 3.1, we have

Corollary 3.1. *The following inequalities are true:*

$$\|v_h\|_{m,h} \lesssim |v_h|_{m,h}, \quad \forall v_h \in M_{h0}^m, \tag{3.20}$$

$$\|v_h\|_{m,h}^2 \lesssim |v_h|_{m,h}^2 + \sum_{|\alpha|<m} \Big(\int_\Omega \partial_h^\alpha v_h\Big)^2, \quad \forall v_h \in M_h^m. \tag{3.21}$$

3.2. **Consistent approximation.** The first condition guaranteeing the convergence property is the approximation condition. When $W = H^m(\Omega)$, let W_h be V_h, otherwise, let W_h be V_{h0}. It is said that $\{W_h, W\}$ satisfies the approximation condition if

$$\lim_{h\to 0} \inf_{v_h \in W_h} \|v - v_h\|_{m,h} = 0, \quad \forall v \in W. \tag{3.22}$$

By means of the interpolation theory (see [10]), the approximation condition can be handled easily.

By the approximation theory, $\{P_{r,h}, H^m(\Omega)\}$ satisfies the approximation condition when $r \geq m$, while $\{P_{m-1,h}, H^m(\Omega)\}$ fails. Then, among the piecewise polynomial approximations to $H^m(\Omega)$, the m-th degree is the lowest.

It is said that $\{W_h, W\}$ satisfies the consistent condition if for any infinite sequence $\{v_{h_k}\}$ with $v_{h_k} \in W_{h_k}$ and $h_k \to 0$ as $k \to \infty$, such that $\{\partial_{h_k}^\alpha v_{h_k}\}$ is weakly convergent, in $L^2(\Omega)$, to v^α for each multi-index α satisfying $|\alpha| \leq m$, it is always true that $v^0 \in W$ and $v^\alpha = \partial^\alpha v^0$ for all $|\alpha| \leq m$.

It is said that $\{W_h\}$ is a consistent approximation of W if $\{W_h, W\}$ satisfies both the approximation condition and the consistent condition.

The bilinear form $a_h(\cdot,\cdot)$ is said to be uniformly W_h-elliptic if

$$\|v_h\|_{m,h}^2 \lesssim a_h(v_h, v_h), \quad \forall v_h \in W_h. \tag{3.23}$$

When $a_h(\cdot,\cdot)$ is uniformly W_h-elliptic and $\{W_h\}$ is a consistent approximation of W, the corresponding nonconforming element method for problem (3.2) is convergent (see [32]).

To check the consistent condition, we can use the generalized patch test proposed in [32]. Other sufficient conditions that are easier to achieve can also be used, such as the patch test (see [6, 16, 36, 37]), the weak patch test [37], the F-E-M test [29], and the IPT test [46].

Theorem 3.2. *Both $\{M_h^m, H^m(\Omega)\}$ and $\{M_{h0}^m, H_0^m(\Omega)\}$ satisfy the approximation condition and consistent condition.*

Proof. Theorem 2.1 leads to that $\{M_h^m, H^m(\Omega)\}$ and $\{M_{h0}^m, H_0^m(\Omega)\}$ satisfy the approximation condition.

Let $\varphi \in C_0^\infty(\Omega)$ (or $C_0^\infty(R^n)$) and $\{v_{h_k}\}$ be an infinite sequence with $v_{h_k} \in M_{h_k}^m$ (or $M_{h_k 0}^m$) and $h_k \to 0$ as $k \to \infty$, such that $\{\partial_{h_k}^\alpha v_{h_k}\}$ is weakly convergent, in $L^2(\Omega)$, to v^α for each multi-index α satisfying $|\alpha| \leq m$.

Now, let $1 \leq i \leq n$ and $|\alpha| < m$. By Lemma 3.1, we have that for each k, a piecewise polynomial $v_{\alpha k} \in H^1(\Omega)$ (or $H_0^1(\Omega)$) exists, such that

$$|\partial_{h_k}^\alpha v_{h_k} - v_{\alpha k}|_{j,h_k} \lesssim h_k^{m-|\alpha|-j} |v_{h_k}|_{m,h_k}, \quad 0 \leq j \leq m - |\alpha|. \tag{3.24}$$

We obtain from (3.24), Green's formula and the Schwarz inequality that

$$\Big|\int_\Omega (\varphi\partial_{h_k}^{\alpha+e_i} v_{h_k} + \partial^{e_i}\varphi\partial_{h_k}^{\alpha} v_{h_k})\Big|$$

$$= \Big|\int_\Omega \Big(\varphi\partial_{h_k}^{e_i}(\partial_{h_k}^{\alpha} v_{h_k} - v_{\alpha k}) + \partial^{e_i}\varphi(\partial_{h_k}^{\alpha} v_{h_k} - v_{\alpha k})\Big)\Big|$$

$$\lesssim h_k^{m-|\alpha|-1}\|\varphi\|_{1,\Omega}|v_{h_k}|_{m,h_k},$$

and this leads to

$$\int_\Omega (\varphi v^{\alpha+e_i} + \partial^{e_i}\varphi v^{\alpha}) = 0, \tag{3.25}$$

when $|\alpha| < m-1$.

Let $|\alpha| = m-1$. Given $T \in \mathcal{T}_h$ and an $(n-1)$-dimensional face F of T, let $P_F^0 : L^2(F) \to P_0(F)$ be the orthogonal projection. By Lemma 2.4 and Green's formula, we have

$$\int_\Omega (\varphi\partial_{h_k}^{\alpha+e_i} v_{h_k} + \partial^{e_i}\varphi\partial_{h_k}^{\alpha} v_{h_k}) = \sum_{T\in\mathcal{T}_{h_k}}\sum_{F\subset\partial T}\int_F \varphi\partial_{h_k}^{\alpha} v_{h_k}\nu_i$$

$$= \sum_{T\in\mathcal{T}_{h_k}}\sum_{F\subset\partial T}\int_F (\varphi - P_F^0\varphi)(\partial_{h_k}^{\alpha} v_{h_k} - P_F^0\partial_{h_k}^{\alpha} v_{h_k})\nu_i.$$

By the Schwarz inequality and the interpolation theory in [10], we obtain that

$$\Big|\int_\Omega (\varphi\partial_{h_k}^{\alpha+e_i} v_{h_k} + \partial^{e_i}\varphi\partial_{h_k}^{\alpha} v_{h_k})\Big|$$

$$\le \sum_{T\in\mathcal{T}_{h_k}}\sum_{F\subset\partial T}\|\varphi - P_F^0\varphi\|_{0,F}\|\partial_{h_k}^{\alpha} v_{h_k} - P_F^0\partial_{h_k}^{\alpha} v_{h_k}\|_{0,F}$$

$$\lesssim h_k\sum_{T\in\mathcal{T}_{h_k}}|\varphi|_{1,T}|v_{h_k}|_{m,T} \lesssim h_k|\varphi|_{1,\Omega}|v_{h_k}|_{m,h_k}.$$

Thus, (3.25) is also true when $|\alpha| = m-1$.

Consequently, $v^\alpha = \partial^\alpha v^0$ for all $|\alpha| \le m$ and $v^0 \in H^m(\Omega)$ (or $H_0^m(\Omega)$).

$\{M_h^m, H^m(\Omega)\}$ and $\{M_{h0}^m, H_0^m(\Omega)\}$ satisfy the consistent condition. □

By Theorem 3.2, we know that M_h^m is a consistent approximation of $H^m(\Omega)$ and that M_{h0}^m is a consistent approximation of $H_0^m(\Omega)$.

By Corollary 3.1, Theorem 3.2 and the result given in [32], we obtain the following theorem directly.

Theorem 3.3. *For any $f \in L^2(\Omega)$, the solution u_h of problem* (3.6) *converges to the solution u of problem* (3.2)*:*

$$\lim_{h\to 0}\|u - u_h\|_{m,h} = 0.$$

3.3. **Error estimate.** Now, we discuss the error estimate of the nonconforming finite element solution of problem (3.6) when $W = H_0^m(\Omega)$ and $U_h = M_{h0}^m$. Let u be the solution to problem (3.2) and let u_h be the solution to problem (3.6).

Lemma 3.2. *Let $r = \max\{m+1, 2m-1\}$. If $u \in H^r(\Omega)$ and $f \in L^2(\Omega)$, then*

$$\sup_{0\neq v_h\in M_{h0}^m,} \frac{\left|a_h(u,v_h)-(f,v_h)\right|}{\|v_h\|_{m,h}} \lesssim \sum_{k=1}^{r-m} h^k|u|_{m+k,\Omega} + h^m\|f\|_{0,\Omega}. \tag{3.26}$$

Proof. Let $v_h \in M_{h0}^m$. Set $f' = f - b_0 u$, then

$$f' = (-1)^m \sum_{|\alpha|=m} \partial^\alpha(b_\alpha \partial^\alpha u), \tag{3.27}$$

$$\|f'\|_{0,\Omega} \lesssim \|f\|_{0,\Omega}, \tag{3.28}$$

and

$$a_h(u,v_h) - (f,v_h) = \sum_{T\in\mathcal{T}_h} \int_T \Big(\sum_{|\alpha|=m} b_\alpha \partial^\alpha u\, \partial^\alpha v_h\Big) - (f', v_h). \tag{3.29}$$

Given $|\alpha| = m$, it can be written as $\alpha = \sum_{i=1}^m e_{j_{\alpha,i}}$. Set

$$\alpha_{(k)} = \sum_{i=1}^k e_{j_{\alpha,i}}, \quad \alpha'_{(k)} = \alpha - \sum_{i=1}^k e_{j_{\alpha,i}}, \quad 0 \le k \le m.$$

Define

$$E_1 = \sum_{|\alpha|=m} \int_\Omega \Big(b_\alpha \partial^\alpha u\, \partial_h^\alpha v_h + b_\alpha \partial^{\alpha+\alpha_{(1)}} u\, \partial_h^{\alpha'_{(1)}} v_h\Big),$$

and

$$E_2 = \sum_{k=1}^{m-2} (-1)^k \sum_{|\alpha|=m} b_\alpha \int_\Omega \Big(\partial^{\alpha+\alpha_{(k)}} u\, \partial_h^{\alpha'_{(k)}} v_h + \partial^{\alpha+\alpha_{(k+1)}} u\, \partial_h^{\alpha'_{(k+1)}} v_h\Big),$$

$$E_3 = \int_\Omega \Big((-1)^{m-1} \sum_{|\alpha|=m} b_\alpha \partial^{\alpha+\alpha_{(m-1)}} u\, \partial_h^{\alpha'_{(m-1)}} v_h - f'\, v_h\Big),$$

when $m > 1$. Then by (3.27) and (3.29),

$$a_h(u,v_h) - (f,v_h) = \begin{cases} E_1 + E_2 + E_3, & m > 1, \\ E_1, & m = 1. \end{cases} \tag{3.30}$$

By Lemma 2.4 and Green's formula, we have

$$\begin{aligned} E_1 &= \sum_{|\alpha|=m} b_\alpha \sum_{T\in\mathcal{T}_h} \int_{\partial T} \partial^\alpha u \partial_h^{\alpha'_{(1)}} v_h \nu_{j_{\alpha,1}} \\ &= \sum_{|\alpha|=m} b_\alpha \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F \Big(\partial^\alpha u - P_F^0 \partial^\alpha u\Big)\Big(\partial_h^{\alpha'_{(1)}} v_h - P_F^0 \partial_h^{\alpha'_{(1)}} v_h\Big) \nu_{j_{\alpha,1}}. \end{aligned}$$

Using the Schwarz inequality and the interpolation theory, we obtain

$$|E_1| \lesssim \sum_{|\alpha|=m} \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \|\partial^\alpha u - P_F^0 \partial^\alpha u\|_{0,F} \left\|\partial_h^{\alpha'_{(1)}} v_h - P_F^0 \partial_h^{\alpha'_{(1)}} v_h\right\|_{0,F}$$

$$\lesssim \sum_{T\in\mathcal{T}_h} h|u|_{m+1,T}|v_h|_{m,T}.$$

That is,

$$|E_1| \lesssim h|u|_{m+1,\Omega}|v_h|_{m,h}. \tag{3.31}$$

When $m > 1$, let $v_\beta \in H_0^1(\Omega)$ be the piecewise polynomial as in (3.7) for $|\beta| < m$. The Green formula leads to that

$$E_2 = \sum_{k=1}^{m-2} (-1)^k \sum_{|\alpha|=m} b_\alpha \int_\Omega \partial^{\alpha+\alpha_{(k)}} u\, \partial_h^{e_{j_{\alpha,k+1}}} \left(\partial_h^{\alpha'_{(k+1)}} v_h - v_{\alpha'_{(k+1)}}\right)$$

$$+ \sum_{k=1}^{m-2} (-1)^k \sum_{|\alpha|=m} b_\alpha \int_\Omega \partial^{\alpha+\alpha_{(k+1)}} u \left(\partial_h^{\alpha'_{(k+1)}} v_h - v_{\alpha'_{(k+1)}}\right),$$

$$E_3 = \int_\Omega \left((-1)^{m-1} \sum_{|\alpha|=m} b_\alpha \partial^{\alpha+\alpha_{(m-1)}} u\, \partial_h^{\alpha'_{(m-1)}} (v_h - v_0) - f'(v_h - v_0)\right).$$

We have by the Schwarz inequality, the triangle inequality, (3.7) and (3.28),

$$|E_2| + |E_3| \lesssim \left(h^m \|f\|_{0,\Omega} + \sum_{k=1}^{m-1} h^k |u|_{m+k,\Omega}\right)|v_h|_{m,h}. \tag{3.32}$$

By (3.30), (3.31), and (3.32), we obtain the desired estimation. □

By Corollary 3.1, Theorem 2.1, Lemma 3.2, and the well-known Strang Lemma (see [31] or [10]), we obtain the following theorem.

Theorem 3.4. *Let u be the solution to problem (3.2) with $W = H_0^m(\Omega)$, and let u_h be the solution to problem (3.6) with $U_h = M_{h0}^m$. Then for any $f \in L^2(\Omega)$,*

$$\|u - u_h\|_{m,h} \lesssim \sum_{k=1}^{r-m} h^k |u|_{m+k,\Omega} + h^m \|f\|_{0,\Omega}, \tag{3.33}$$

when $u \in H^r(\Omega)$ and $r = \max\{m+1, 2m-1\}$.

When $W = H^m(\Omega)$ and the corresponding nonconforming finite element method is used, the same error estimates can also be obtained by similar arguments provided that $b_0 > 0$.

Since the approximate error of M_h^m and M_{h0}^m in norm $\|\cdot\|_{m,h}$ is of order $O(h)$ only, the error estimate given by (3.33) is optimal.

4. Concluding remarks

This construction of the consistent approximation of Sobolev spaces with minimal degree piecewise polynomials is motivated by theoretical considerations and an interest in applications for practical problems. In this paper, a new consistent approximation to m-th order Sobolev spaces of n dimensions with $n \ge m \ge 1$ is proposed in a canonical fashion, and the convergence and the error estimate for application to $2m$-th-order elliptic problems in R^n are shown. The new class of nonconforming elements has several attractive properties. For example, this new class

- provides consistent approximation with minimal degree piecewise polynomials;
- offers degrees of freedom that fit perfectly well;
- recovers the well-known nonconforming linear elements for $m = 1$ and the Morley element for $m = 2$ in a canonical fashion; and
- has the inclusion property.

Though it is of great theoretical interest, this new type of finite element also has the potential to be useful in practice. In encountering high-order partial differential equations, we often try to transform them into a system of lower-order equations. Such a practice is based on the position that higher-order partial differential equations are too difficult to be efficiently discretized by finite element or finite difference methods. One strong message this paper sends is that a direct discretization of high-order partial differential equations is also practical. For example, a sixth-order partial differential equation in 3 dimensions can be discretized by a piecewise cubic polynomial that has 20 degrees of freedom on each element. This is not very difficult to carry out in practice.

References

[1] Adams, R. A., *Sobolev Spaces*, Academic Press, New York, 1975. MR0450957 (56:9247)

[2] Adini, A. and Glough, R. W., Analysis of plate bending by the finite element method, NSF report G, 7337, 1961.

[3] Arnold, D., Brezzi, F., Cockburn, B., and Marini, D., Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J Numer Anal*, **39** (2002), 1749–1779. MR1885715 (2002k:65183)

[4] Baker, G. A., Finite element methods for elliptic equations using nonconforming elements, *Math Comp*, **31** (1977), 45–59. MR0431742 (55:4737)

[5] Barrett, J. W., Langdon, S., and Nürnberg, R., Finite element approximation of a sixth-order nonlinear degenerate parabolic equation, *Numer Math*, **96** (2004), 401–434. MR2028722 (2004k:65168)

[6] Bazeley, G. P., Cheung, Y. K., Irons, B. M., and Zienkiewicz, O. C., Triangular elements in bending–conforming and nonconforming solutions, in *Proc. Conf. Matrix Methods in Structural Mechanics*, Air Force Ins. Tech., Wright-Patterson A. F. Base, Ohio, 1965.

[7] Cahn, J. W., On spinodal decomposition, *Acta Metallurgica*, **9** (1961), 795–801.

[8] Cahn, J. W. and Hilliard, J. E., Free energy of a nonuniform system I. interfacial free energy, *J Chem. Phys.*, **28** (1958), 258–267.

[9] Chang, A. S. and Chen, W., A note on a class of higher order comformally covariant equations, *Discrete and Continuous Dymanical Systems*, **7** (2001), 275–281. MR1808400 (2001m:35101)

[10] Ciarlet, P. G., *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, New York, 1978. MR0520174 (58:25001)

[11] Crouzeix, M. and Raviart, P. A., Conforming and nonconforming finite element methods for solving the stationary Stokes equations, *RAIRO Anal. Numer.*, **7** (1973), 33–76. MR0343661 (49:8401)

[12] Elliott, C. M. and Zheng, S., On the Cahn-Hilliard Equation, *Archive for Rational Mechanics and Analysis*, **96** (1986), 339–357. MR855754 (87k:80007)

[13] Falk, R. S., Nonconforming finite element methods for the equations of linear elasticity, *Math. Comp.*, **57** (1991), 529–550. MR1094947 (92a:65290)

[14] Feng, Kang, On the theory of discontinuous finite elements, *Numerica Mathematica Sinica*, **1** (1979), 378–385. MR657317 (83f:65175)

[15] Fife, P. C., Models for phase separation and their mathematics, *Electronic Journal of Differential Equations*, **2000**, 8 (2000), 1–26. MR1772733 (2001k:35147)

[16] Irons, B. M. and Razzaque, A., *Experience with the patch test for convergence of finite element methods*, in *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, A. R. Aziz, ed., Academic Press, New York, 1972, 557–587. MR0423839 (54:11813)

[17] Lascaux, P. and Lesaint, P., Some nonconforming finite elements for the plate bending problem, *RAIRO Anal. Numer.*, **R-1** (1985), 9–53. MR0423968 (54:11941)

[18] Lesaint, P., On the convergence of Wilson's nonconforming element for solving the elastic problem, *Comput. Meths. Appl. Mech. Eng.*, **7** (1976), 1–6. MR0455479 (56:13717)

[19] Lesaint, P. and Zlamal, M., Convergence of the nonconforming Wilson element for arbitrary quadrilateral meshes, *Numer. Math.*, **36** (1980), 33–52. MR595805 (81m:65171)

[20] Melosh, R. J., Basis of derivation of matrices for direct stiffness method, *AIAA J.*, **1** (1963), 1631–1637.

[21] Morley, L. S. D., The triangular equilibrium element in the solution of plate bending problems, *Aero. Quart.*, **19** (1968), 149–169.

[22] Rannacher, R., On nonconforming and mixed finite element methods for plate bending problems, *RAIRO Numer. Anal.*, **13** (1979), 369–387. MR555385 (80i:65125)

[23] Ruas, V., A quadratic finite element method for solving biharmonic problems in R^n, *Numer. Math.*, **52** (1988), 33–43. MR918315 (89a:65173)

[24] Shi, Z.-C., An explicit analysis of Stummel's patch test examples, *Int. J. Numer. Meth. Eng.*, **20** (1984), 1233–1246. MR751335 (85g:65096)

[25] Shi, Z.-C., A convergence condition for the quadrilateral Wilson element, *Numer. Math.*, **44** (1984), 349–361. MR757491 (86d:65151)

[26] Shi, Z.-C., On the convergence properties of quadrilateral elements of Sander and Beckers, *Math. Comp.*, **42** (1984), 493–504. MR736448 (85h:65245)

[27] Shi, Z.-C., The generalized patch test for Zienkiewicz's triangles, *J. Comput. Math.*, **2** (1984), 279–286. MR815422 (86m:73051)

[28] Shi, Z.-C., Convergence properties of two nonconforming finite elements, *Comput. Meths. Appl. Mech. Eng.*, **48** (1985), 123–137. MR784479 (87c:73073)

[29] Shi, Z.-C., The F-E-M-test for nonconforming finite elements, *Math. Comp.*, **49** (1987), 391–405. MR906178 (88g:65120)

[30] Shi, Z.-C., On the error estimates of Morley element, *Numerica Mathematica Sinica* **12**, 2 (1990), 113–118. MR1070298 (91i:65182)

[31] Strang, G. and Fix, G. J., *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, 1973. MR0443377 (56:1747)

[32] Stummel, F., The generalized patch test, *SIAM J. Numer. Analysis*, **16** (1979), 449–471. MR530481 (80e:65106)

[33] Stummel, F., The limitation of the patch test, *Int. J. Numer. Meth. Eng.*, **15** (1980), 177-188.

[34] Stummel, F., Basic compactness properties of nonconforming and hybrid finite element spaces, *RAIRO Anal. Numer.*, **4**, 1 (1980), 81–115. MR566091 (81h:65058)

[35] Temam, R., *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences 68, (2nd Edition, 1997) Springer-Verlag, New York, 1988. MR953967 (89m:58056)

[36] Veubake, F. D., Variational principles and the patch test, *Int. J. Numer. Meth. Eng.*, **8** (1974), 783–801. MR0375911 (51:12099)

[37] Wang, M., On the necessity and sufficiency of the patch test for convergence of nonconforming finite elements, *SIAM J. Numer. Anal.,* **39** (2002), 363–384. MR1860273 (2002h:65197)

[38] Wang, M. and Xu, J., The Morley element for fourth order elliptic equations in any dimensions, *Numer. Math.*, **103** (2006), 155–169. MR2207619 (2006i:65205)

[39] Wang, M. and Xu, J., Nonconforming tetrahedral finite elements for fourth order elliptic equations, *Math. Comp.*, **76** (2007), 1–18. MR2261009 (2008e:65367)

[40] Wang, M., Shi, Z.-C., and Xu, J., A new class of Zienkiewicz-type nonconforming element in any dimensions, *Numer. Math.*, **106**, 2 (2007), 335-347. MR2291941 (2008f:65227)

[41] Wang, M., Shi, Z.-C., and Xu, J., Some n-rectangle nonconforming elements for fourth order elliptic equations, *J. Comput. Math.*, **25** (2007), 408–420 MR2337403 (2008e:65366)

[42] Wise, S. M., Lowengrub, J. S., Kim, J. S., et al., Efficient phase-field simulation of quantum dot formation on a trained heteroepitaxial film, *Superlattices and Microstructure*, **36** (2004), 293–304.

[43] Wise, S. M., Lowengrub, J. S., Kim, J. S., et al., Quantum dot formation on a train-patterned epitaxial thin film, *Appl. Phys. Lett.*, **87**, 133102 (2005).
[44] Xu, J., Iterative methods by space decomposition and subspace correction, *SIAM Review*, **34**, 4 (1992), 581–613. MR1193013 (93k:65029)
[45] Zenicek, A., Polynomial approximation on tetrahedrons in the finite element method, *J. Approx. Theory*, **7** (1973), 334–351. MR0350260 (50:2753)
[46] Zhang, H. Q. and Wang, M., On the compactness of quasi-conforming element spaces and the convergence of quasi-conforming element method, *Appl. Math. Mech. (English edition)*, **7** (1986), 443–459. MR860997 (88c:65103)
[47] Zhang, H. Q. and Wang, M., *The Mathematical Theory of Finite Elements*, Science Press, Beijing, 1991.

Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing, China
E-mail address: mwang@math.pku.edu.cn

Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing China –and– Department of Mathematics, Pennsylvania State University
E-mail address: xu@math.psu.edu
URL: http://www.math.psu.edu/xu/