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# COMPLETE MONOTONICITY AND RELATED PROPERTIES OF SOME SPECIAL FUNCTIONS

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ABSTRACT. We completely determine the set of $s, t > 0$ for which the function $L_{s,t}(x) := x - \frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t+1}$ is a Bernstein function, that is $L_{s,t}(x)$ is positive with completely monotonic derivative on $(0, \infty)$. The complete monotonicity of several closely related functions is also established.

## 1. INTRODUCTION

Euler's gamma function $\Gamma(x)$ is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$. Its logarithmic derivative $\psi(x) = \Gamma'(x)/\Gamma(x)$ is called the psi or digamma function and the derivatives $\psi^{(n)}(x)$ are called polygamma functions. These functions are perhaps the most important of all special functions.

In this paper we continue the investigations of [10], [11], and [12], and extend and strengthen some of the results obtained there by presenting some sharp inequalities concerning the function

$$L_{s,\,t}(x) := x - \frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t+1}\,,$$

for the best possible range of the parameters $s, t > 0$ and $x > 0$. Such inequalities are of importance in establishing sharp estimates for certain trigonometric sums. For more information and an extensive bibliography concerning inequalities of this type we refer to [10] and [12].

Most of the inequalities of the present work are obtained by verifying the complete monotonicity of certain functions. A function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic if it has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \text{ for all } x > 0 \text{ and } n \geq 0. \tag{1.1}$$

J. Dubourdieu [4] proved that if a nonconstant function $f$ is completely monotonic, then strict inequality holds in (1.1). See also [6] for a simpler proof of this result. A necessary and sufficient condition for complete monotonicity is given by Bernstein's theorem (see [21, p. 161]) which states that $f$ is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} dm(t),$$

where $m$ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

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Some important subclasses of completely monotonic functions have been considered in [14]. Koumandos and Pedersen [14] called a function $f : (0, \infty) \to \mathbb{R}$ *completely monotonic of order* $\alpha$ if $x^\alpha f(x)$ is completely monotonic on $(0, \infty)$.

We recall also that the Riemann-Liouville fractional integral $I_\alpha(m)(t)$ of order $\alpha > 0$, of a Borel measure $m$ on $[0, \infty)$ is defined by

$$I_\alpha(m)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, dm(s).$$

The following characterization of completely monotonic functions of positive order was established in [14, Theorem 1.3].

**Theorem 1.1.** *The function $f : (0, \infty) \to \mathbb{R}$ is completely monotonic of order $\alpha > 0$ if and only if $f$ is the Laplace transform of a fractional integral of order $\alpha$ of a positive Radon measure $m$ on $[0, \infty)$, that is,*

$$f(x) = \int_0^\infty e^{-xt}\, I_\alpha(m)(t)\, dt.$$

The following special case will be used in the sequel.

**Corollary 1.2.** *Let $r$ be an integer $\geq 2$. The function $f(x)$ is completely monotonic of order $r$ on $(0, \infty)$ if and only if*

$$f(x) = \int_0^\infty e^{-xt}\, p(t)\, dt\,,$$

*where the integral converges for all $x > 0$ and $p(t)$ is $r-2$ times continuously differentiable on $[0, \infty)$ with*

$$p^{(r-2)}(t) = \int_0^t m\big([0, s]\big)\, ds$$

*for some Radon measure $m$ and $p^{(k)}(0) = 0$ for $0 \leq k \leq r-2$.*

The class of completely monotonic functions order 1 coincides with the class of strongly completely monotonic functions introduced in [20] (see also [14, Proposition 1.1]).

An important counterpart of completely monotonic functions are the *Bernstein* functions. A function $f : (0, \infty) \to (0, \infty)$ is called *a Bernstein function* if $f$ has derivatives of all orders and $f'$ is completely monotonic on $(0, \infty)$. It is easy to see, for example, that $x \mapsto x\,t/(x+t)$ is a Bernstein function on $(0, \infty)$ for all $t > 0$.

We refer to [3] for additional properties and background information about Bernstein functions.

In this paper, we determine the set of all $(s, t)$ with $s, t > 0$ for which $L_{s,t}(x)$ is a Bernstein function on $(0, \infty)$. We also show that some functions involving the function $L_{s,t}(x)$ are completely monotonic of certain positive order. As applications we obtain sharp estimates of the function $L_{s,t}(x)$ for the best possible set of the parameters $s$, $t$.

We organize the paper as follows. In the next section we recall some basic definitions and facts from [10]. In Section 3 the main results of the paper are stated. We give some additional remarks in Section 4. Section 5 contains several lemmas which are important for the proofs of the main results presented in Section 6.

In the final section we show how our results can be applied to obtain sharp estimates for certain trigonometric sums.

## 2. Preliminaries

We first define some auxiliary functions. For $s, t > 0$ and $x > 0$ let

$$\Delta_{s,\,t}(x) := \frac{\Gamma(s)}{\Gamma(t)} \left[ \frac{1}{x^{s-t}} - \frac{\Gamma(x+t)}{\Gamma(x+s)} \right] \tag{2.1}$$

and

$$M_{s,t}(x) := \frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t}. \tag{2.2}$$

Of course, we have

$$\Delta_{s,\,t}(x) = \frac{\Gamma(s)}{\Gamma(t)}\, \frac{1}{x^{s-t+1}}\, L_{s,\,t}(x). \tag{2.3}$$

It is easily seen that

$$L'_{s,\,t}(x) = 1 - \frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t} \left[ x \left( \psi(x+t) - \psi(x+s) \right) + s - t + 1 \right]. \tag{2.4}$$

We also have

$$L''_{s,\,t}(x) = -\frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t+1}\, \Phi_{s,\,t}(x), \tag{2.5}$$

where

$$\begin{aligned} \Phi_{s,\,t}(x) := &\left( \psi(x+t) - \psi(x+s) + \frac{s-t+1}{x} \right)^2 \\ &+ \left( \psi(x+t) - \psi(x+s) + \frac{s-t+1}{x} \right)' . \end{aligned}$$

An important ingredient of our method is the representation (cf. [10, (3.5)])

$$\Phi_{s,\,t}(x) = \int_0^\infty e^{-xu}\, F_{s,\,t}(u)\, du, \tag{2.6}$$

where

$$F_{s,\,t}(u) := \int_0^u \sigma_{s,\,t}(u-v)\, \sigma_{s,\,t}(v)\, dv - u\, \sigma_{s,\,t}(u)\,,$$

with

$$\varphi_{s,t}(u) := \frac{e^{(1-t)u} - e^{(1-s)u}}{e^u - 1}, \quad u \in \mathbb{R} \setminus \{0\}, \quad \text{and} \quad \varphi_{s,t}(0) := s - t,$$

and $\sigma_{s,t}(u) := s - t + 1 - \varphi_{s,t}(u)$.

It is easy to see that

$$F'_{s,\,t}(u) = \int_0^u \sigma'_{s,\,t}(u-v)\, \sigma_{s,\,t}(v)\, dv - u\, \sigma'_{s,\,t}(u) \tag{2.7}$$

and

$$F''_{s,\,t}(u) = u\, \varphi''_{s,\,t}(u) + \int_0^u \varphi'_{s,\,t}(u-v)\, \varphi'_{s,\,t}(v)\, dv. \tag{2.8}$$

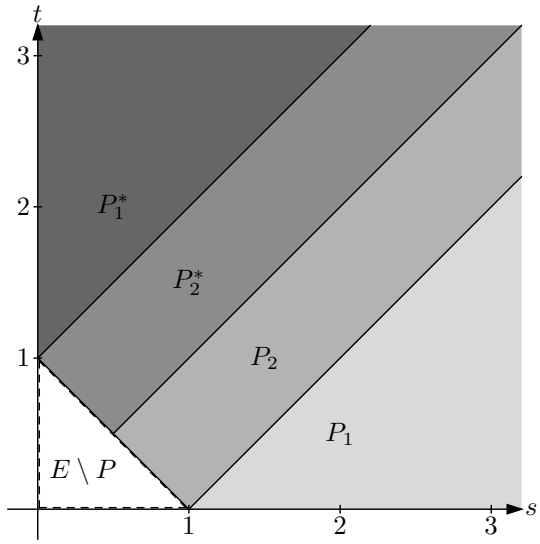


FIGURE 1. The sets $P_1$, $P_2$, $P_1^*$, and $P_2^*$. $H$ is equal to the union of $P_1$ and $P_2$. $H^*$ is equal to the union of $P_1^*$ and $P_2^*$. $P$ is equal to the union of $P_1$, $P_2$, $P_1^*$, and $P_2^*$. $E$ is the first quadrant. The dashed lines describe the boundary of the set $E \setminus P$. The graphs in this paper were created by using the KETpic package for Maple [7].

The following subsets of the first quadrant $E := \{(s,\, t) \in \mathbb{R}^2 \,:\, s, t > 0\}$ will be important in our further considerations (see Figure 1):

$$P := \{(s,t) \in E : s + t \geq 1\}, \quad H := \{(s,t) \in P : s > t\},$$
$$P_1 := \{(s,t) \in E : s - t > 1\}, \quad P_2 := \{(s,t) \in P : 0 < s - t < 1\}.$$

Moreover, if $M$ is any subset of $\mathbb{R}^2$, then we will denote by $M^*$ its reflection with respect to the line $\{(s,\, t) \in \mathbb{R}^2 \,:\, s = t\}$.

It is clear that $L_{s,s}(x) = \Delta_{s,s}(x) = \Phi_{s,s}(x) = 0$ for all $s,\, x > 0$. It is also easy to check that $L_{s,s+1}(x) = -s$, $\Delta_{s,s+1} = -1$, and $\Phi_{s,s+1}(x) = 0$ for all $s,\, x > 0$. In the following we will therefore tacitly assume that for $(s,t) \in E$ we always have $t \neq s$, $s+1$.

The next result gives information about the sign of the functions $L_{s,\,t}(x)$ and $\Delta_{s,\,t}(x)$ on $(0,\, \infty)$.

**Proposition 2.1.** *Let $(s,\, t) \in E$ with $t \neq s$.*

(*i*) *The function $\Delta_{s,\,t}(x)$ is completely monotonic on $(0,\, \infty)$ if and only if $(s,\, t) \in H$.*
(*ii*) *The function $\Delta_{s,\,t}(x)$ changes sign in $(0,\, \infty)$ precisely when $(s,\, t) \in E \setminus P$.*
(*iii*) *We have $\Delta_{s,\,t}(x) < 0$ for all $x > 0$ if and only if $(s,\, t) \in H^*$.*

*Proof.* Using the well-known formulae (see for example [2, p. 615]),

$$\frac{1}{x^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xu}\, u^{\alpha-1}\, du, \quad \alpha > 0$$

and

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} = \frac{1}{\Gamma(s-t)} \int_0^\infty e^{-xu}\, e^{-tu}\, (1-e^{-u})^{s-t-1}\, du, \quad 0 < s-t, \tag{2.9}$$

we see that

$$\Delta_{s,\,t}(x) = \frac{\Gamma(s)}{\Gamma(t)\,\Gamma(s-t)} \int_0^\infty e^{-xu}\, u^{s-t-1} \left[1 - \left(\frac{u}{e^u-1}\right)^{1+t-s} e^{u(1-s)}\right] du. \tag{2.10}$$

In order to prove Statement $(i)$, it will therefore suffice to show that

$$\left(\frac{u}{e^u-1}\right)^{1+t-s} e^{u(1-s)} < 1, \quad \text{for all } \ u > 0, \ \text{ when } \ (s,\,t) \in H,$$

which is equivalent to

$$(1+t-s)\, g(u) - 1 + s > 0, \tag{2.11}$$

where $g(u) := (\log(e^u-1) - \log u)/u$. It is easy to see that $\frac{1}{2} < g(u) < 1$ for all $u > 0$. For $s-t \le 1$ the inequality $g(u) > \frac{1}{2}$ implies $(1+t-s)\, g(u) - 1 + s > 0$ since $(s,\,t) \in P$. For $s-t > 1$ the inequality $g(u) < 1$ implies $(1+t-s)\, g(u) - 1 + s > t > 0$. This establishes (2.11) and thus Statement $(i)$.

Using the asymptotic formula (see [1, 6.1.47])

$$\frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t} = 1 - \frac{(s-t)(s+t-1)}{2x} + \frac{(s-t)(s-t+1)\, p(s,\,t)}{24x^2} + O\left(\frac{1}{x^3}\right), \tag{2.12}$$

as $x \to \infty$, where

$$p(s,t) := 3s^2 + 6st + 3t^2 - 5s - 7t + 2, \tag{2.13}$$

we get $\lim_{x\to\infty} M_{s,t}(x) = 1$. Hence, since by [8, Theorem 2.4] for $(s,\,t) \in H^*$ the function $M_{s,t}(x)$ is completely monotonic (and thus strictly decreasing) on $(0,\,\infty)$, it follows that $M_{s,t}(x) > 1$ for all $x > 0$. This is equivalent to Statement $(iii)$.

By (2.3) and (2.12) we have

$$\frac{\Gamma(t)}{\Gamma(s)} \lim_{x\to\infty} \left(x^{s-t+1} \Delta_{s,\,t}(x)\right) = \lim_{x\to\infty} L_{s,\,t}(x) = \frac{(s-t)(s+t-1)}{2}.$$

On the other hand,

$$\lim_{x\to 0} \Delta_{s,\,t}(x) = \begin{cases} +\infty, & s > t, \\ -1, & s < t \end{cases}$$

The last two formulae imply Statement $(ii)$. □

An immediate consequence of Proposition 2.1 is the following.

**Corollary 2.1.** *Let $(s,t) \in E$ with $t \neq s$. We have $L_{s,\,t}(x) > 0$ for all $x > 0$ if and only if $(s,\,t) \in H$, whereas $L_{s,\,t}(x) < 0$ for all $x > 0$ if and only if $(s,\,t) \in H^*$. The function $L_{s,\,t}(x)$ changes sign in $(0,\,\infty)$ precisely when $(s,\,t) \in E \setminus P$.*

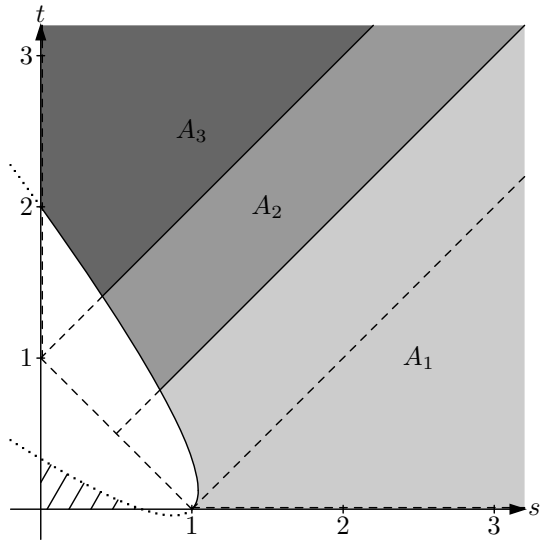


FIGURE 2. The sets $A_1$, $A_2$, and $A_3$. $A$ is equal to the union of $A_1$, $A_2$, and $A_3$. The dashed lines describe the boundaries of the sets $P_1$, $P_2$, $P_1^*$, and $P_2^*$ from Figure 1. The dotted curves describe the parts of the parabola $p(s,t) = 0$ that lie outside $P$. The hatched area is the set of points $(s,t) \in E \setminus P$ with $p(s,t) > 0$.

## 3. Main results

Let

$$A := \{(s,\, t) \in P \,:\, p(s,\, t) \geq 0\}$$

and set

$$A_1 := H \cap A, \quad A_2 := P_2^* \cap A, \quad A_3 := P_1^* \cap A,$$

where $p(s,t)$ is as in (2.13) (see Figure 2).

Our first result concerns complete monotonicity of $\Phi_{s,\,t}(x)$ of positive order.

**Theorem 3.1.**

(*i*) *For* $(s,\, t) \in A_1$ *the function* $\Phi_{s,\,t}(x)$ *is completely monotonic of order* 2 *on* $(0,\, \infty)$.

(*ii*) *For* $(s,\, t) \in A_2$ *the function* $-\Phi_{s,\,t}(x)$ *is completely monotonic of order* 1 *on* $(0,\, \infty)$.

(*iii*) *For* $(s,\, t) \in A_3$ *the function* $\Phi_{s,\,t}(x)$ *is completely monotonic of order* 1 *on* $(0,\, \infty)$.

(*iv*) *For* $(s,\, t) \in E \setminus A$ *with* $t \neq s,\ s+1$, *the function* $\Phi_{s,\,t}(x)$ *changes sign in* $(0,\, \infty)$.

Our next theorem describes the set of all $(s,t) \in E$ for which $L_{s,\,t}(x)$ is a Bernstein function.

**Theorem 3.2.**

(*i*) *For* $(s,t) \in P_1$ *the function* $L_{s,\,t}(x)$ *is strictly increasing and concave on* $(0,\, \infty)$ *but* not *a Bernstein function. In this case we have*

$$0 < L_{s,\,t}(x) < \frac{(s-t)(s+t-1)}{2} \quad \textit{for all } x > 0. \tag{3.1}$$

(*ii*) *For* $(s,t) \in A_1 \setminus P_1$ *the function* $L_{s,t}(x)$ *is a Bernstein function on* $(0, \infty)$ *and*

$$0 < L_{s,t}(x) < \frac{(s-t)(s+t-1)}{2} \quad \textit{for all } x > 0. \tag{3.2}$$

(*iii*) *For* $(s,t) \in A_2$ *the function* $-L_{s,t}(x)$ *is a Bernstein function on* $(0, \infty)$ *and*

$$\frac{(s-t)(s+t-1)}{2} < L_{s,t}(x) < 0 \quad \textit{for all } x > 0. \tag{3.3}$$

(*iv*) *For* $(s,t) \in A_3$ *the function* $-L_{s,t}(x)$ *is completely monotonic on* $(0, \infty)$ *and*

$$-\infty < L_{s,t}(x) < \frac{(s-t)(s+t-1)}{2} \quad \textit{for all } x > 0. \tag{3.4}$$

*All bounds in the inequalities* (3.1)–(3.4) *are sharp. For* $(s, t) \in E \setminus A$ *with* $t \neq s$, $s+1$, *the derivatives* $L'_{s,t}(x)$ *and* $L''_{s,t}(x)$ *change sign in* $(0, \infty)$ *and inequalities* (3.1)–(3.4) *fail to hold for appropriate* $x > 0$.

*Remark* 3.3. The weak version of Theorem 3.2(ii) in the case where $(s,t) \in K \setminus P_1$ has been established in [11, Theorem 1.2 (1)]. $K$ is the set of all $(s,t) \in E$ for which the function $\varphi_{s,t}(u)$ is convex on $(0,\infty)$; see Lemma 5.2 below. In the present work we extend this result to the larger area $(s,t) \in A_1 \setminus P_1$ by employing a totally different method that enables us to cover the case $(s,t) \in K \setminus P_1$ as well. (Compare with Remark 5.8). It is necessary here to correct a misprint that appears on page 1700 of [11], where Theorem 1.2 is stated. The correct definition of the set $S_2$ is $S_2 := S_1 \cap T \cap \{(s,t) \in \mathbb{R}^2 : 0 < s-t < 1\}$. The proof of the whole of [11, Theorem 1.2] is then unaffected.

As a consequence of Theorem 3.2 we derive the following.

**Corollary 3.4.**

(*i*) *For* $(s,t) \in A_1 \setminus P_1$ *the function* $\frac{1}{2}(s-t)(s+t-1) - L_{s,t}(x)$ *is completely monotonic on* $(0, \infty)$.

(*ii*) *For* $(s,t) \in A_2$ *the function* $L_{s,t}(x) - \frac{1}{2}(s-t)(s+t-1)$ *is completely monotonic on* $(0, \infty)$.

(*iii*) *For* $(s,t) \in A_3$ *the function* $\frac{1}{2}(s-t)(s+t-1) - L_{s,t}(x)$ *is completely monotonic on* $(0, \infty)$.

(*iv*) *For* $(s,t) \in A_3$ *the function* $M_{s,t}(x) - 1$ *is completely monotonic of order* 1 *on* $(0, \infty)$.

We shall also prove the following closely related result.

**Theorem 3.5.** *Let* $(s,t) \in E$ *with* $s \neq t$. *The function* $1 - M_{s,t}(x)$ *is completely monotonic on* $(0, \infty)$ *if and only if* $(s,t) \in \overline{P}_2$.

## 4. Remarks

Let

$$z_{s,t}(x) = \left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} - x.$$

The following result is shown in [5] (see also [19]).

**Theorem 4.1.** *Let $s, t > 0$, $r = \min(s, t)$. Then $z_{s,\,t}(x)$ is convex and decreasing on $(-r, \infty)$ for $|s - t| < 1$ and $z_{s,\,t}(x)$ is concave and increasing on $(-r, \infty)$ for $|s - t| > 1$.*

We note that

$$\lim_{x\to\infty} z_{s,\,t}(x) = \frac{s + t - 1}{2}.$$

A weak version of Theorem 3.2 can be obtained by using Theorem 4.1 above.

In particular, we have

*Remark* 4.2.

$$z_{s,\,t}(x) > 0, \quad \text{for } x \in [0, \infty) \quad \text{if and only if} \quad s + t \geq 1,$$

while $z_{s,\,t}(x)$ changes sign in $[0, \infty)$ when $s + t < 1$.

Using Remark 4.2 we easily deduce the following.

*Remark* 4.3.

$$L_{s,\,t}(x) > 0, \quad \text{for } x \in (0, \infty) \quad \text{if and only if} \quad s + t \geq 1 \quad \text{and } s > t,$$
$$L_{s,\,t}(x) < 0, \quad \text{for } x \in (0, \infty) \quad \text{if and only if} \quad s + t \geq 1 \quad \text{and } s < t.$$

$L_{s,\,t}(x)$ changes sign in $(0, \infty)$ when $s+t < 1$. This is another proof of Corollary 2.1.

Using Theorem 4.1 we are able to obtain some estimates for the function $L_{s,\,t}(x)$. First, we prove the following elementary lemma.

**Lemma 4.4.** *For $\alpha > 0$, $\beta \in \mathbb{R}$, $\beta \neq 0, 1$ we define*

$$w_{\alpha,\beta}(x) := x - x^{1-\beta}\,(x + \alpha)^{\beta}, \quad x \in [0, \infty).$$

*If $\beta\,(\beta - 1) > 0$, then $w_{\alpha,\beta}(x)$ is strictly increasing and concave on $[0, \infty)$ and*

$$w_{\alpha,\beta}(0) < w_{\alpha,\beta}(x) < -\alpha\beta, \quad x > 0.$$

*If $\beta\,(\beta - 1) < 0$, then $w_{\alpha,\beta}(x)$ is strictly decreasing and convex on $[0, \infty)$ and*

$$-\alpha\beta < w_{\alpha,\beta}(x) < w_{\alpha,\beta}(0), \quad x > 0.$$

*Proof.* : We first observe that $w_{\alpha,\beta}(0) = 0$ for $\beta < 1$, while

$$\lim_{x\to 0^+} w_{\alpha,\beta}(x) = w_{\alpha,\beta}(0^+) = -\infty$$

for $\beta > 1$. Writing

$$w_{\alpha,\beta}(x) = x\left[1 - \left(1 + \frac{\alpha}{x}\right)^{\beta}\right]$$

we easily obtain

$$\lim_{x\to\infty} w_{\alpha,\beta}(x) = -\alpha\beta \quad \text{and} \quad \lim_{x\to\infty} w'_{\alpha,\beta}(x) = 0.$$

Then we observe that

$$w''_{\alpha,\beta}(x) = -\left(1 + \frac{\alpha}{x}\right)^{\beta} \frac{\beta\,(\beta - 1)\,\alpha^2}{x\,(x + \alpha)^2},$$

whence all our claims in the lemma follow. □

Combination of Lemma 4.4 with Theorem 4.1 yields the following

**Proposition 4.1.** *Let $\omega(s, t) := (\Gamma(t)/\Gamma(s))^{\frac{1}{t-s}}$ and set*

$$\vartheta_{s,\,t}(x) := w_{\omega(s,t),t-s} \quad \textit{and} \quad \eta_{s,\,t}(x) := w_{(s+t-1)/2,t-s}.$$

(*i*) *For* $(s,t) \in P_2^*$ *we have*

$$(s-t)\,\omega(s,t) < \vartheta_{s,\,t}(x) < L_{s,\,t}(x) < \eta_{s,\,t}(x) < 0.$$

(*ii*) *For* $(s,t) \in P_2$ *we have*

$$0 < \eta_{s,\,t}(x) < L_{s,\,t}(x) < \vartheta_{s,\,t}(x) < (s-t)\,\omega(s,t).$$

(*iii*) *For* $(s,t) \in P_1^*$ *we have*

$$-\infty < \eta_{s,\,t}(x) < L_{s,\,t}(x) < \vartheta_{s,\,t}(x) < (s-t)\,\omega(s,t) < 0.$$

(*iv*) *For* $(s,t) \in P_1$ *we have*

$$0 < \vartheta_{s,\,t}(x) < L_{s,\,t}(x) < \eta_{s,\,t}(x) < \frac{(s-t)(s+t-1)}{2}.$$

We observe that only (*iv*) above yields the upper bound

$$L_{s,\,t}(x) < \frac{(s-t)(s+t-1)}{2}, \quad \text{for all } \ x > 0, \tag{4.1}$$

which is sharp, because by (2.12) we have

$$\lim_{x\to\infty} L_{s,\,t}(x) = \frac{(s-t)(s+t-1)}{2}\,. \tag{4.2}$$

Theorem 3.2 is therefore of interest because it establishes (4.1) in all cases through monotonicity of the function $L_{s,\,t}(x)$.

## 5. Lemmas

For the proof of our main results we need the following lemmas.

We begin with monotonicity and convexity properties of the function $\varphi_{s,\,t}(u)$ defined in Section 2.

**Lemma 5.1.**

(*i*) *For* $(s,\,t) \in H$ *the function* $\varphi_{s,\,t}(u)$ *is strictly decreasing on* $(0,\,\infty)$.

(*ii*) *For* $(s,\,t) \in H^*$ *the function* $\varphi_{s,\,t}(u)$ *is strictly increasing on* $(0,\,\infty)$.

(*iii*) *For* $(s,\,t) \in E \setminus P$ *with* $t \neq s$ *the function* $\varphi'_{s,\,t}(u)$ *changes sign in* $(0,\,\infty)$.

*Proof.* See [18] (or [17] for a simpler proof). □

**Lemma 5.2.** *Let* $\varepsilon(s,t) := 2s^2 + 2st + 2t^2 - 3s - 3t + 1$ *and*

$$K := H \cap \{(s,\,t) \in \mathbb{R}^2 \,:\, \varepsilon(s,\,t) \geq 0\}.$$

(*i*) *For* $(s,\,t) \in K$ *we have* $\varphi''_{s,\,t}(u) > 0$ *for all* $u > 0$.

(*ii*) *For* $(s,\,t) \in K^*$ *we have* $\varphi''_{s,\,t}(u) < 0$ *for all* $u > 0$.

(*iii*) *For* $(s,\,t) \in E \setminus (K \cup K^*)$ *with* $t \neq s$ *the function* $\varphi''_{s,\,t}(u)$ *changes sign in* $(0,\,\infty)$.

*Proof.* See [11]. □

This information will be useful when dealing with the function

$$\Xi_{s,t}(u) := \sigma'_{s,t}(u)\sigma'''_{s,t}(u) - (\sigma''_{s,t}(u))^2 \quad s,\ t,\ u \in \mathbb{R}.$$

Observe that $\varphi_{s,t}(u) = -\varphi_{t,s}(u)$ and that therefore, since $\sigma'_{s,t}(u) = -\varphi'_{s,t}(u)$,

$$\Xi_{s,t}(u) = \Xi_{t,s}(u) \quad \text{for all} \quad (s,t) \in \mathbb{R}^2,\ u > 0. \tag{5.1}$$

We will prove the following.

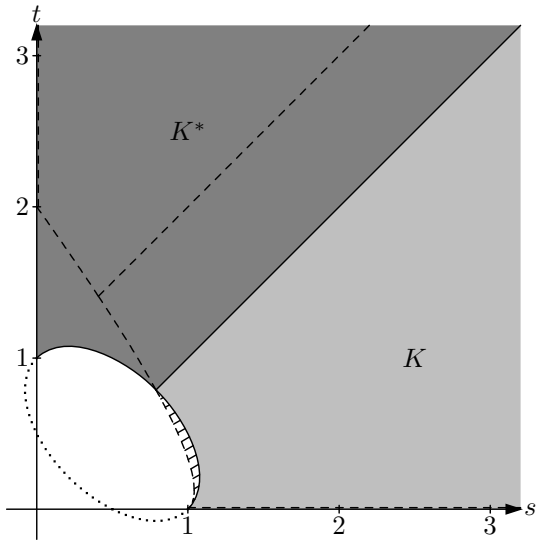


FIGURE 3. The sets $K$ and $K^*$. The dashed curves describe the boundaries of the sets $A_1$, $A_2$, and $A_3$, from Figure 2. The dotted curve describes the part of the ellipse $\varepsilon(s,t) = 0$ that lies outside $P$. The hatched area is $A_1 \setminus K$.

**Lemma 5.3.** *For $(s,t) \in P_2 \cup P_2^*$ we have $\Xi_{s,t}(u) < 0$ for all $u \in (0,\infty)$. For $(s,t) \in A_3$ we have $\Xi_{s,t}(u) > 0$ for all $u \in (0,\infty)$.*

In order to verify this lemma, the change of coordinates

$$T(\alpha,\beta) := (s(\alpha,\beta), t(\alpha,\beta)) := \left(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\right)$$

will prove useful. Observe that the $(\alpha,\beta)$-plane is the $(s,t)$-plane rotated by an angle of $-\frac{\pi}{4}$ and stretched by the factor $\sqrt{2}$. Setting (cf. Figure 4)

$$\begin{aligned} B_1 &:= \{(\alpha,\beta) : 0 < \beta < 1 \le \alpha\}, \\ B_2 &:= \{(\alpha,\beta) : 1 < \alpha,\ 1 < \beta < 2,\ 0 < \mu(\alpha,\beta)\}, \\ B_3 &:= \{(\alpha,\beta) : 2 \le \beta < \alpha\}, \end{aligned}$$

where $\mu(\alpha,\beta) := 3\alpha^2 - \beta^2 - 6\alpha + 4$, it follows that $A_3 \subset T(B_2 \cup B_3)$ and that $P_2^* = T(B_1)$. Hence, if we define

$$\rho(\alpha,\beta,x) := 4x^6(x+1)^{\alpha-2}\Xi_{s(\alpha,\beta),t(\alpha,\beta)}(\log(x+1)),$$

then it is clear that the following lemma implies Lemma 5.3 for $(s,t) \in P_2^* \cup A_3$, and thus, because of (5.1), also for $(s,t) \in P_2$.

**Lemma 5.4.** *For all $x > 0$ we have $\rho(\alpha,\beta,x) < 0$ if $(\alpha,\beta) \in B_1$ and $\rho(\alpha,\beta,x) > 0$ if $(\alpha,\beta) \in B_2 \cup B_3$.*

Observe that, for $\alpha$, $\beta$, $x > 0$,

$$\rho_{\beta,x}(\alpha) := \rho(\alpha,\beta,x) = a(\beta,x)x^2\,\alpha^2 + 2b(\beta,x)x\,\alpha + c(\beta,x),$$

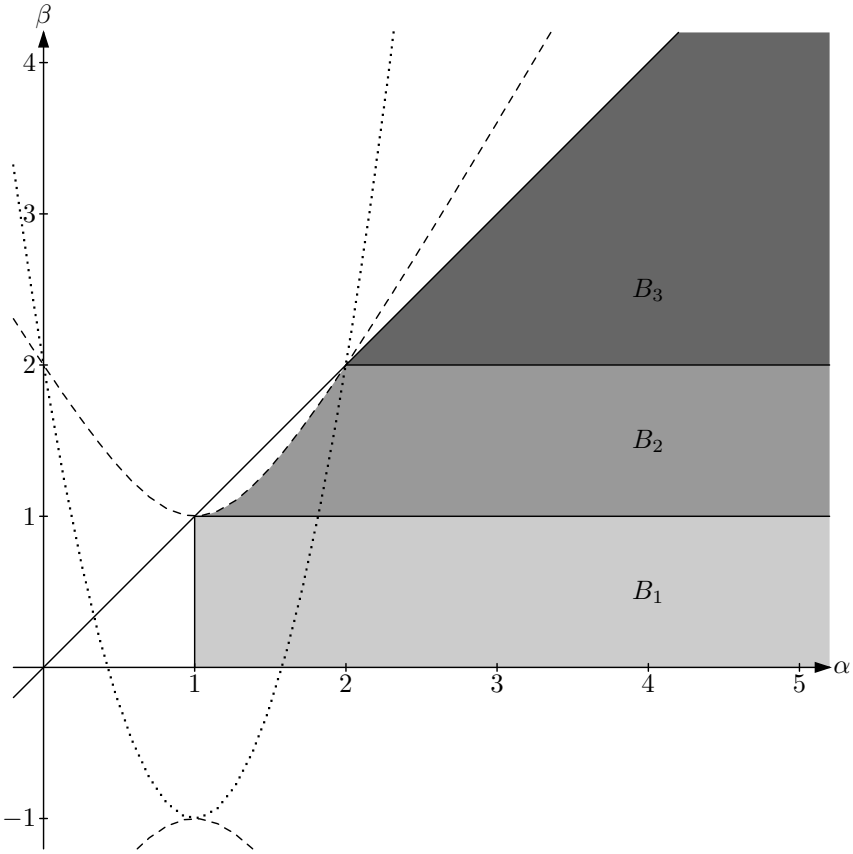


FIGURE 4. The sets $B_1$, $B_2$, $B_3$. The dashed curve describes the set of points $(\alpha, \beta)$ for which $\mu(\alpha, \beta) = 0$, while the dotted curves describe the points $(\alpha, \beta)$ for which $p(s(\alpha, \beta), t(\alpha, \beta)) = 0$.

where

$$\begin{aligned}
a(\beta, x) &:= (x+1)^{1+\beta} + (x+1)^{1-\beta} - \beta^2 x^2 - 2x - 2,\\
b(\beta, x) &:= (x+1)^{1+\beta}\left((1-\beta)x + 4\right) + (x+1)^{1-\beta}\left((1+\beta)x + 4\right)\\
&\quad - 2(\beta^2+1)x^2 - 10x - 8,\\
c(\beta, x) &:= (x+1)^{1+\beta}\left(\beta(\beta-2)x^2 - 8\beta x + 8\right)\\
&\quad + (x+1)^{1-\beta}\left(\beta(\beta+2)x^2 + 8\beta x + 8\right)\\
&\quad + \beta^4 x^4 + 10\beta^2 x^3 + 6\beta^2 x^2 - 16x - 16.
\end{aligned}$$

Hence, $\alpha \mapsto \rho_{\beta,x}(\alpha)$ is a parabola for all $\beta$, $x \in \mathbb{R}$ (this explains the relevance of the coordinate transformation $T$). Before we will present the proof of Lemma 5.4, we will first determine for which $\beta$, $x > 0$ this parabola opens up- or downward.

**Lemma 5.5.** *Let $x > 0$. If $\beta \in (0, 1)$, then $a(\beta, x) < 0$. If $1 < \beta$, then $a(\beta, x) > 0$.*

*Proof.* Set $a_\beta(x) := a(\beta, x)$. It is easy to check that for all $\beta > 0$ we have $a_\beta^{(j)}(0) = 0$, $j = 0, \ldots, 3$, and that

$$a_\beta'''(x) = \beta(\beta^2 - 1)\left((x+1)^{\beta-2} - (x+1)^{-\beta-2}\right)$$

for all $x > 0$. Hence, $a_\beta'''(x)$ is negative for all $x > 0$ if $\beta \in (0,1)$ and positive for all $x > 0$ if $\beta > 1$ and the lemma follows. □

*Proof of Lemma* 5.4. We will prove the lemma separately for the three sets $B_1$, $B_2$, and $B_3$.

*The case* $(\alpha, \beta) \in B_1$. Fix $\beta \in (0,1)$, $x > 0$. We have to prove that for all $\alpha \geq 1$ we have $\rho_{\beta,x}(\alpha) < 0$. Because of Lemma 5.5 this will follow once we show that

$$\rho_{\beta,x}(1) < 0 \quad \text{and} \quad \rho'_{\beta,x}(1) < 0. \tag{5.2}$$

Set $f_\beta(x) := \rho_{\beta,x}(1)$. We calculate

$$f_\beta^{(j)}(0) = 0 \quad \text{for} \quad j = 0, \dots, 5, \tag{5.3}$$

and set

$$\widetilde{f}_\beta(x) := \frac{(x+1)^{4+\beta}}{\beta - \beta^3} f_\beta^{(5)}(x), \qquad 0 < \beta < 1,\ 0 < x. \tag{5.4}$$

For $\beta \in (0,1)$ we then obtain

$$\begin{aligned} \widetilde{f}'_\beta(0) &= 20\beta(\beta^2 - 1) < 0, \\ \widetilde{f}''_\beta(0) &= 20\beta(\beta^2 - 1)(2\beta + 1) < 0, \\ \widetilde{f}'''_\beta(0) &= 20\beta(\beta^2 - 1)(3\beta^2 + 8) < 0, \end{aligned} \tag{5.5}$$

and

$$\widetilde{f}'''_\beta(x) = 4\beta(\beta^2 - 1)(x+1)^{2\beta-3} q_\beta(x), \tag{5.6}$$

where

$$q_\beta(x) := (9 - \beta^2)(2 + \beta)(1 + 2\beta)x^2 + 5(4 - \beta)(2 + \beta)(1 + 2\beta)x + 40 + 15\beta^2$$

is clearly positive for all $x > 0$ and $\beta \in (0,1)$. It thus follows from (5.3)–(5.6) that the first inequality in (5.2) must be true.

Now, set $g_\beta(x) := \rho'_{\beta,x}(1)/(2x)$. Then we have $g_\beta^{(j)}(0) = 0$ for $j = 0, \dots, 4$ and

$$g_\beta^{(4)}(x) = -\beta x(1 - \beta^2)(4 - \beta^2)\left((x+1)^{\beta-3} - (x+1)^{-\beta-3}\right),$$

which is clearly negative for all $\beta \in (0,1)$ and $x > 0$. Hence, also the second inequality in (5.2) is true.

*The case* $(\alpha, \beta) \in B_2$. Fix $\beta \in (1,2)$, $x > 0$. We have to prove that for all $\alpha > 1$ with $\mu(\alpha, \beta) > 0$ or, equivalently, for

$$\alpha > 1 + \sqrt{\frac{\beta^2 - 1}{3}},$$

we have $\rho_{\beta,x}(\alpha) > 0$. Because of Lemma 5.5 this will follow once we show that

$$\rho_{\beta,x}(1 + \sqrt{(\beta^2 - 1)/3}) > 0 \quad \text{for} \quad \beta \in (1,2),\ x > 0 \tag{5.7}$$

and (note that $1 + \sqrt{(\beta^2 - 1)/3} > \beta$ for $\beta \in (1,2)$)

$$\rho'_{\beta,x}(\beta) > 0 \quad \text{for} \quad \beta > 1,\ x > 0. \tag{5.8}$$

Set $f_\beta(x) := 3\rho_{\beta,x}(1 + \sqrt{(\beta^2 - 1)/3})$. Then we have

$$f_\beta^{(j)}(0) = 0, \quad \text{for} \quad j = 0, \dots, 5. \tag{5.9}$$

Setting

$$\widetilde{f}_\beta(x) := \frac{(x+1)^{4+\beta}}{\beta(\beta^2-1)(4-\beta^2)} f_\beta^{(5)}(x), \tag{5.10}$$

it is straightforward to check that for $\beta \in (1,2)$,

$$\begin{aligned} &\widetilde{f}'_\beta(0) = 0, \\ &\widetilde{f}''_\beta(0) = 56\beta\sqrt{3(\beta^2-1)} > 0, \\ &\widetilde{f}'''_\beta(0) = 56\beta(-2\beta^2 + 3\beta\sqrt{3(\beta^2-1)} + 2) > 0. \end{aligned} \tag{5.11}$$

Further, we find

$$\frac{\widetilde{f}'''_\beta(x)}{8\beta(x+1)^{2\beta-3}} = (1+2\beta)(3+\beta)(1+\beta)c_2(\beta)x^2 + (1+2\beta)c_1(\beta)x + c_3(\beta), \tag{5.12}$$

where once again it is straightforward to check that for $\beta \in (1,2)$,

$$\begin{aligned} &c_0(\beta) := 21\beta\sqrt{3(\beta^2-1)} - 14\beta^2 + 14 > 0, \\ &c_1(\beta) := 9\beta\sqrt{3(\beta^2-1)} - 14\beta^2 + 6\sqrt{3(\beta^2-1)} + 14 > 0, \\ &c_2(\beta) := \sqrt{3(\beta^2-1)} - 2\beta + 2 > 0. \end{aligned} \tag{5.13}$$

Relations (5.9)–(5.13) show that (5.7) holds.

Next, set $g_\beta(x) := \rho'_{\beta,x}(\beta)/(2x)$. Then we have

$$g_\beta^{(j)}(0) = 0, \quad \text{for} \quad j = 0, \dots, 4. \tag{5.14}$$

We set

$$\widetilde{g}_\beta(x) := \frac{(x+1)^{3+\beta}}{\beta(\beta^2-1)} g_\beta^{(4)}(x), \tag{5.15}$$

and find that for $\beta > 1$,

$$\begin{aligned} &\widetilde{g}'_\beta(0) = 10\beta(\beta-1) > 0, \\ &\widetilde{g}''_\beta(0) = 4\beta(4\beta^2 - 5\beta + 4) > 0. \end{aligned} \tag{5.16}$$

We further have

$$\widetilde{g}''_\beta(x) = 2\beta(x+1)^{2(\beta-1)} \left((2+\beta)(1+2\beta)x + 2(4\beta^2 - 5\beta + 4)\right). \tag{5.17}$$

Since it is easy to see that $4\beta^2 - 5\beta + 4 > 0$ for $\beta > 1$, (5.8) follows from (5.14)–(5.17).

*The case* $(\alpha, \beta) \in B_3$. Fix $\beta > 2$, $x > 0$. We have to prove that for all $\alpha > \beta$ we have $\rho_{\beta,x}(\alpha) > 0$. Because of Lemma 5.5 and (5.8) this will follow once we show that

$$\rho_{\beta,x}(\beta) > 0 \quad \text{for all} \quad \beta > 2,\ x > 0. \tag{5.18}$$

Let $f_\beta(x) := \rho_{\beta,x}(\beta)$. Then

$$f_\beta^{(j)}(0) = 0, \quad \text{for} \quad j = 0, \dots, 4 \tag{5.19}$$

and we set

$$\widetilde{f}_\beta(x) := \frac{(x+1)^{3-\beta}}{4\beta(\beta^2-1)(\beta-2)} f_\beta^{(4)}(x). \tag{5.20}$$

Then $\widetilde{f}_\beta(0) = 0$ and

$$\widetilde{f}_\beta(x) = 2 + (x+1)^{-2\beta}\left((\beta-3)\beta x^2 - 4\beta x - 2\right),$$

which shows that $\widetilde{f}_\beta(x) \to 2$ as $x \to \infty$. Since

$$\widetilde{f}'_\beta(x) = 2\beta x(\beta-1)(x+1)^{1-2\beta}\left((3-\beta)x+5\right)$$

for $\beta > 2$ has at most one zero in $(0,\infty)$ and is positive for small $x > 0$, we thus see that $\widetilde{f}_\beta(x) > 0$ for all $\beta > 2$, $x > 0$. (5.19) and (5.20) therefore imply (5.18). □

Lemma 5.4 is therefore proven and we can show the following.

**Lemma 5.6.** *For $s$, $t$, $u \in \mathbb{R}$ let*

$$\Theta_{s,t}(u) := \sigma_{s,t}(u)\sigma''_{s,t}(u) - (\sigma'_{s,t}(u))^2.$$

*If $(s,t) \in A_2$, then $\Theta_{s,t}(u) > 0$ for all $u > 0$ and if $(s,t) \in A_3$, then $\Theta_{s,t}(u) < 0$ for all $u > 0$. If $(s,t) \in H^* \setminus (\overline{A_2 \cup A_3})$, then $\Theta_{s,t}(u)$ changes sign in $(0,\infty)$.*

*Proof.* Observe first that

$$\Theta_{s,t}(0) = \sigma''_{s,t}(0) - \left(\sigma'_{s,t}(0)\right)^2 = -\frac{1}{12}(s-t+1)(s-t)p(s,t) \tag{5.21}$$

and, for $t > s$,

$$\lim_{u\to\infty} e^{(s-4)u}(e^u-1)^4\Theta_{s,t}(u) = s^2(s-t+1). \tag{5.22}$$

Hence, for $(s,t) \in H^* \setminus (\overline{A_2 \cup A_3})$ the function $\Theta_{s,t}(u)$ changes sign in $(0,\infty)$ and for every $(s,t) \in A_2 \cup A_3$ there is a $\delta > 0$ such that $\Theta_{s,t}(u)\Theta_{s,t}(v) > 0$ for all $u \in (0,\delta)$ and $v > 1/\delta$. This implies that for every $(s,t) \in A_2 \cup A_3$ the function $\Theta_{s,t}(u)$ has to have an even number of zeros in $(0,\infty)$ (counted according to multiplicity). Suppose now that there are $(s,t) \in A_2 \cup A_3$ and $u^* \in (0,\infty)$ such that

$$0 = \Theta_{s,t}(u^*) = \sigma_{s,t}(u^*)\sigma''_{s,t}(u^*) - (\sigma'_{s,t}(u^*))^2.$$

Because of Lemma 5.2 this is equivalent to $\sigma_{s,t}(u^*) = (\sigma'_{s,t}(u^*))^2/\sigma''_{s,t}(u^*)$ and therefore

$$\Theta'_{s,t}(u^*) = \sigma_{s,t}(u^*)\sigma'''_{s,t}(u^*) - \sigma'_{s,t}(u^*)\sigma''_{s,t}(u^*) = \frac{\sigma'_{s,t}(u^*)}{\sigma''_{s,t}(u^*)}\Xi_{s,t}(u^*)$$

holds. Since $A_2 \subset P_2^*$, Lemmas 5.1, 5.2, and 5.3, thus imply $\Theta'_{s,t}(u)\Theta'_{s,t}(v) > 0$ for every pair $u$, $v \in (0,\infty)$ of zeros of $\Theta_{s,t}(u)$. Hence, $\Theta_{s,t}(u)$ can have at most one zero in $(0,\infty)$ and since it has to have an even number of zeros it follows that $\Theta_{s,t}(u) \neq 0$ for all $(s,t) \in A_2 \cup A_3$ and $u \in (0,\infty)$. Because of (5.21) and (5.22) we obtain $\Theta_{s,t}(u) > 0$ in $(0,\infty)$ if $(s,t) \in A_2$ and $\Theta_{s,t}(u) < 0$ in $(0,\infty)$ if $(s,t) \in A_3$. □

Our next lemma concerns monotonicity and convexity properties of the function $F_{s,t}(u)$ defined in Section 2.

**Lemma 5.7.**

(*i*) *For $(s,t) \in A_1$ we have $F''_{s,t}(u) > 0$ for all $u > 0$.*
(*ii*) *For $(s,t) \in A_2$ we have $F'_{s,t}(u) < 0$ for all $u > 0$.*
(*iii*) *For $(s,t) \in A_3$ we have $F'_{s,t}(u) > 0$ for all $u > 0$.*

*Proof.* First, we prove Statement $(i)$. For $(s,t) \in K$ we immediately obtain $F''_{s,t}(u) > 0$ for all $u > 0$ by employing (2.8) and Lemmas 5.1 and 5.2.

In order to prove $(i)$ also for $(s,t) \in A_1 \cap P_2$ (observe that $A_1 \setminus K \subset A_1 \cap P_2$, cf. Figure 3), we make use of (2.8) and write

$$\int_0^u \sigma'_{s,t}(u-v)\,\sigma'_{s,t}(v)\,dv = \left(\int_0^{u/2} + \int_{u/2}^u\right)\left(\sigma'_{s,t}(u-v)\,\sigma'_{s,t}(v)\right)\,dv.$$

Making the change of variables $\tau = u - v$ in the second integral, we get

$$\int_0^u \sigma'_{s,t}(u-v)\,\sigma'_{s,t}(v)\,dv = 2\int_0^{u/2} \sigma'_{s,t}(u-v)\,\sigma'_{s,t}(v)\,dv. \tag{5.23}$$

Set

$$\delta_{s,t}(v) := \sigma'_{s,t}(u-v)\,\sigma'_{s,t}(v)$$

and

$$\widetilde{r}_{s,t}(v) := \frac{\sigma''_{s,t}(v)}{\sigma'_{s,t}(v)}.$$

By Lemma 5.3 we have $\Xi_{s,t}(v) < 0$ for all $v > 0$ when $(s,t) \in P_2$ and this implies that $\widetilde{r}_{s,t}(v)$ is strictly decreasing. Because of Lemma 5.1 we therefore obtain

$$\begin{aligned}\delta'_{s,t}(v) &= \sigma''_{s,t}(v)\sigma'_{s,t}(u-v) - \sigma'_{s,t}(v)\sigma''_{s,t}(u-v)\\ &= \sigma'_{s,t}(v)\sigma'_{s,t}(u-v)\left(\widetilde{r}_{s,t}(v) - \widetilde{r}_{s,t}(u-v)\right) > 0\end{aligned}$$

for $v < u - v$. Hence $\delta_{s,t}(v) \geq \delta_{s,t}(0) = \sigma'_{s,t}(0)\sigma'_{s,t}(u)$ and thus (2.8), (5.23), and the fact that $v < u - v$ for $v \in (0, \frac{u}{2})$, yield

$$F''_{s,t}(u) \geq u\left(\sigma'_{s,t}(0)\sigma'_{s,t}(u) - \sigma''_{s,t}(u)\right).$$

It remains to show that $\sigma'_{s,t}(0)\sigma'_{s,t}(u) - \sigma''_{s,t}(u) > 0$ for $u > 0$, or, equivalently (because of Lemma 5.1), that

$$\widetilde{r}_{s,t}(u) < \sigma'_{s,t}(0).$$

Since $\widetilde{r}_{s,t}(u)$ is strictly decreasing this reduces to $\widetilde{r}_{s,t}(0) < \sigma'_{s,t}(0)$, or (cf. (5.21))

$$0 < (\sigma'_{s,t}(0))^2 - \sigma''_{s,t}(0) = \frac{1}{12}(s-t+1)(s-t)p(s,t).$$

The last inequality is clearly valid for $(s,t) \in A_1 \cap P_2$ and this completes the proof.

For the proof of $(ii)$ and $(iii)$ we make use of (2.7) and write

$$\int_0^u \sigma'_{s,t}(u-v)\,\sigma_{s,t}(v)\,dv = \left(\int_0^{u/2} + \int_{u/2}^u\right)\left(\sigma'_{s,t}(u-v)\,\sigma_{s,t}(v)\right)\,dv.$$

By the change of variables $\tau = u - v$ in the second integral we get

$$\int_{u/2}^u \sigma'_{s,t}(u-v)\,\sigma_{s,t}(v)\,dv = \int_0^{u/2} \sigma_{s,t}(u-\tau)\,\sigma'_{s,t}(\tau)\,d\tau.$$

Hence

$$\int_0^u \sigma'_{s,t}(u-v)\,\sigma_{s,t}(v)\,dv = \int_0^{u/2} \left[\sigma'_{s,t}(u-v)\,\sigma_{s,t}(v) + \sigma_{s,t}(u-v)\,\sigma'_{s,t}(v)\right]\,dv. \tag{5.24}$$

Recall that $v < u - v$ for $0 < v < u/2$. Define the function of $v$

$$q_{s,t}(v) := \frac{\sigma_{s,t}(v)}{\sigma'_{s,t}(v)}$$

and note that by Lemmas 5.1 and 5.2 we have $\sigma'_{s,t}(v) = -\varphi'_{s,t}(v) < 0$ and $\sigma''_{s,t}(v) = -\varphi''_{s,t}(v) > 0$, for all $v > 0$ when $(s,t) \in A_2 \cup A_3$.

Now, let $(s,t) \in A_2$. Then, by Lemma 5.6, we have $\Theta_{s,t}(v) > 0$ for all $v > 0$ and $(s,t) \in A_2$ and therefore $q_{s,t}(v)$ is strictly decreasing on $[0,\infty)$. Hence,

$$q_{s,t}(u-v) < q_{s,t}(v), \quad \text{for} \quad 0 < v < u/2,$$

and thus

$$\sigma_{s,t}(u-v)\,\sigma'_{s,t}(v) < \sigma'_{s,t}(u-v)\,\sigma_{s,t}(v), \quad \text{for} \quad 0 < v < u/2. \tag{5.25}$$

Combining (5.24) with (5.25) we obtain

$$\int_0^u \sigma'_{s,t}(u-v)\,\sigma_{s,t}(v)\,dv < 2\int_0^{u/2} \sigma'_{s,t}(u-v)\,\sigma_{s,t}(v)\,dv. \tag{5.26}$$

We define

$$r_{s,t}(v) := \frac{\sigma'_{s,t}(v)}{\sigma''_{s,t}(v)}.$$

By Lemma 5.3 we have $\Xi_{s,t}(v) < 0$ for all $v > 0$ and $(s,t) \in A_2$. Therefore the function $r_{s,t}(v)$ is strictly increasing on $[0,\infty)$ so that

$$r_{s,t}(v) < r_{s,t}(u-v) \quad \text{for} \quad 0 < v < u/2. \tag{5.27}$$

Let

$$\gamma_{s,t}(v) := \sigma'_{s,t}(u-v)\,\sigma_{s,t}(v) \quad \text{for} \quad 0 \le v \le u/2.$$

It follows from (5.27) that

$$\begin{aligned} \gamma'_{s,t}(v) &= \sigma'_{s,t}(u-v)\,\sigma'_{s,t}(v) - \sigma''_{s,t}(u-v)\,\sigma_{s,t}(v) \\ &= \sigma'_{s,t}(v)\,\sigma''_{s,t}(u-v)\,[r_{s,t}(u-v) - q_{s,t}(v)] \\ &< \sigma'_{s,t}(v)\,\sigma''_{s,t}(u-v)\,[r_{s,t}(v) - q_{s,t}(v)] \\ &= -\frac{\sigma''_{s,t}(u-v)}{\sigma''_{s,t}(v)}\,\Theta_{s,t}(v) < 0 \quad \text{for} \quad 0 < v < u/2. \end{aligned} \tag{5.28}$$

Therefore the function $\gamma_{s,t}(v)$ is strictly decreasing on $[0,\, u/2]$ and since $\sigma_{s,t}(0) = 1$ we obtain $\gamma_{s,t}(v) < \gamma_{s,t}(0) = \sigma'_{s,t}(u)$ for $0 < v < u/2$. It follows from this and (5.26) that

$$\int_0^u \sigma'_{s,t}(u-v)\,\sigma_{s,t}(v)\,dv < u\,\sigma'_{s,t}(u) \quad \text{for all} \quad u > 0, \tag{5.29}$$

which, because of (2.7), means that $F'_{s,t}(u) < 0$ for all $u > 0$ and $(s,t) \in A_2$. $(ii)$ is thus proven.

In the case where $(s,t) \in A_3$, the reverse inequalities hold in (5.25), (5.26) and (5.27) and therefore the reverse inequalities also hold in (5.28) and (5.29). From this we conclude that $F'_{s,t}(u) > 0$ for all $u > 0$ and $(s,t) \in A_3$. Hence, $(iii)$ is proven and the proof of the lemma is complete. □

*Remark* 5.8. The case $(s,t) \in A_1 \cap K$ of Statement $(i)$ of this lemma has already been shown in [11] (by applying Lemmas 5.1 and 5.2). Thus the only case in which Statement $(i)$ was actually open was the case in which $(s,t) \in A_1 \setminus K$.

The methods of the proof of [11, Theorem 1.1] can also be used in order to prove the following extension of [12, Lemma 2]: For $(s,t) \in A_1 \setminus K$ the function $\varphi''_{s,t}(u)$ has exactly one zero $u_{s,t}$ in $(0,\infty)$ and is strictly increasing in $(0, u_{s,t})$. Employing

this extension of [12, Lemma 2] (instead of [12, Lemma 2] itself) in the proof of [12, Theorem 5], one immediately obtains a second proof of Lemma 5.7 $(i)$.

Nevertheless, given (5.21) and our proof of Lemma 5.7, it is clear that the sign of the function $F_{s,t}(u)$ (and thus also the complete monotonicity of the functions $\Phi_{s,t}(x)$ and $L'_{s,t}(x)$, as we will see below) is naturally explained by the signs of the composite functions $\Theta_{s,t}(u)$ and $\Xi_{s,t}(u)$ and not so much by the signs of the single functions $\varphi^{(j)}_{s,t}(u)$, $j = 0, \dots, 3$.

## 6. Proofs of the main results

*Proof of Theorem* 3.1. From Lemma 5.7 $(i)$ and the expressions (2.7) and (2.8) we infer that for $(s,t) \in A_1$ we have

$$F''_{s,\,t}(u) > 0, \quad F'_{s,\,t}(u) > 0, \quad F_{s,\,t}(u) > 0 = F_{s,\,t}(0), \quad \text{for all} \quad u > 0.$$

It therefore follows from (2.6) and Corollary 1.2 that $x^2\,\Phi_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$, which shows that Statement $(i)$ is true.

In a similar manner, applying $(ii)$ and $(iii)$ of Lemma 5.7 and taking into consideration Theorem 1.1, we infer that if $(s,t) \in A_2$, then $-x\,\Phi_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$, while if $(s,t) \in A_3$, then $x\,\Phi_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$. Hence, Statements $(ii)$ and $(iii)$ are also true.

Statement $(iv)$ of the theorem follows by studying the asymptotic behaviour of the function $\Phi_{s,\,t}(x)$ for $x \to 0^+$ and $x \to \infty$. Indeed, using the expressions given in Section 2, it is easy to see that

$$\lim_{x\to\infty} x^4\,\Phi_{s,\,t}(x) = \frac{1}{12}\,(s-t)\,(s-t+1)\,p(s,t)$$

and

$$\lim_{x\to 0} x^2\Phi_{s,\,t}(x) = (s-t)(s-t+1)\,.$$

Since $(s,t) \in P \setminus A$ implies $p(s,t) < 0$, it follows directly from these formulae that for $(s,\,t) \in P \setminus A$ the function $\Phi_{s,\,t}(x)$ changes sign in $(0,\,\infty)$. On the other hand, for $(s,t) \in E \setminus P$ (note that $E \setminus A = (E \setminus P) \cup (P \setminus A)$) we have $L_{s,t}(0) = 0$ and therefore, as explained in the proof of Theorem 3.2 below (see (6.5)), the positivity (resp. negativity) of the function $\Phi_{s,\,t}(x)$ for all $x > 0$ implies positivity (resp. negativity) of $L_{s,\,t}(x)$ for all $x > 0$. Hence, since by Corollary 2.1 for $(s,t) \in E \setminus P$ the function $L_{s,\,t}(x)$ changes sign in $(0,\,\infty)$, also the function $\Phi_{s,\,t}(x)$ has to change sign in $(0,\infty)$ if $(s,t) \in E \setminus P$. Our result is therefore best possible as far as the sign of the function $\Phi_{s,\,t}(x)$ is concerned.

The proof is complete. □

*Proof of Theorem* 3.2. Using (2.4), (2.5), and (2.12), we obtain results concerning the asymptotic behaviour of $L'_{s,\,t}(x)$ and $L''_{s,\,t}(x)$ for $x \to 0$ and $x \to \infty$. Indeed, we have

$$\lim_{x\to\infty} x^2\,L'_{s,\,t}(x) = \frac{1}{24}\,(s-t)\,(s-t+1)\,p(s,t) \quad \text{for all} \;\; s,t > 0, \tag{6.1}$$

$$\lim_{x\to 0} L'_{s,\,t}(x) = \begin{cases} 1, & s > t, \\ -\infty, & 0 < s-t+1 < 1, \\ +\infty, & s-t+1 < 0, \end{cases} \tag{6.2}$$

and

$$\lim_{x\to\infty} -x^3\, L''_{s,\,t}(x) = \frac{1}{12}\,(s-t)\,(s-t+1)\,p(s,t), \quad \text{for all } \ s,t>0, \tag{6.3}$$

$$\lim_{x\to 0} -\,L''_{s,\,t}(x) = \begin{cases} 0, & s-t>1, \\ +\infty, & 0<s-t<1, \\ -\infty, & 0<s-t+1<1, \\ +\infty, & s-t+1<0. \end{cases} \tag{6.4}$$

Observe that (6.1) in particular implies that $L'_{s,t}(x)\to 0$ as $x\to\infty$ for all $(s,t)\in E$. Because of (2.5) we therefore have

$$\Phi_{s,t}(x)>0,\ \forall x>0, \quad \Rightarrow \quad L''_{s,t}(x)<0,\ \forall x>0, \quad \Rightarrow \quad L'_{s,t}(x)>0,\ \forall x>0, \tag{6.5}$$

for all $(s,t)\in E$ and an analogous chain of implications holds if $\Phi_{s,t}(x)<0$ for all $x>0$.

Relation (6.5), together with Theorem 3.1 $(i)$, shows that for $(s,t)\in P_1$ the function $L_{s,\,t}(x)$ is strictly increasing and concave on $(0,\,\infty)$. Inequality (3.1) therefore follows from (4.2) and the fact that $L_{s,\,t}(0)=0$. If $L'_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$, then so is $-L''_{s,\,t}(x)$. But for $(s,t)\in P_1$ this is impossible because of the first equality in (6.4). We have shown Statement $(i)$.

In order to prove Statement $(ii)$ suppose now that $(s,t)\in A_1\setminus P_1$. Rewriting (2.5) as

$$-L''_{s,\,t}(x) = \frac{\Gamma(x+t)}{\Gamma(x+s)}\,\frac{1}{x^{1-s+t}}\ x^2\,\Phi_{s,\,t}(x)$$

and using Theorem 3.1 $(i)$ and (2.9), we see that $-L''_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$ as a product of completely monotonic functions. Because of (6.5) it follows that also $L'_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$. By Corollary 2.1 we have $L_{s,\,t}(x)>0$ for all $x>0$ when $(s,\,t)\in A_1\setminus P_1$. Therefore, in this case, $L_{s,\,t}(x)$ is a Bernstein function. Inequality (3.2) is obtained as in the case $(i)$.

By Corollary 2.1 we have $L_{s,\,t}(x)<0$ for all $x>0$ when $(s,t)\in A_2\cup A_3$. We write (2.5) as

$$L''_{s,\,t}(x) = -\frac{\Gamma(x+t)}{\Gamma(x+s)}\,x^{s-t}\,x\,\Phi_{s,\,t}(x). \tag{6.6}$$

It is shown in [8, Theorem 2.4] that for $t>s>0$, $s+t\geq 1$, the function $M_{s,t}(x)$ (defined in (2.2)) is completely monotonic on $(0,\,\infty)$. Combining this with (6.6) and Theorem 3.1 $(ii)$, we deduce that for $(s,t)\in A_2$ the function $L''_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$ as a product of completely monotonic functions. Because of (6.5) (applied with the reversed inequalities) this implies that $L'_{s,\,t}(x)<0$ for all $x>0$. Therefore the function $-L_{s,\,t}(x)$ is positive and has a completely monotonic derivative on $(0,\,\infty)$ and is thus a Bernstein function. In particular, the function $L_{s,\,t}(x)$ is strictly decreasing and convex on $(0,\,\infty)$ and therefore inequality (3.3) is valid because of (4.2) and $L_{s,\,t}(0)=0$. Statement $(iii)$ is verified.

In the case where $(s,t)\in A_3$, we use Theorem 3.1 $(iii)$ and (6.6) to deduce that the function $-L''_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$. Because of (6.5) this implies that the function $L'_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$. Hence, since $-L_{s,\,t}(x)>0$ for all $x>0$, also $-L_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$. In particular, the function $L_{s,\,t}(x)$ is strictly increasing and concave on $(0,\,\infty)$ and

the sharp inequality (3.4) is valid because of (4.2). Note that in this case we have $\lim_{x\to 0} L_{s,\,t}(x) = -\infty$. Statement $(iv)$ is thus proven .

Moreover, using (6.1)–(6.4) and the fact that $(s,t) \in P \setminus A$ implies $p(s,t) < 0$, we see that for $(s,\,t) \in P \setminus A$ with the functions $L'_{s,\,t}(x)$ and $L''_{s,\,t}(x)$ change sign in $(0,\,\infty)$. For $(s,t) \in E \setminus P$ (note that $E \setminus A = (E \setminus P) \cup (P \setminus A)$) we have $L_{s,t}(0) = 0$ and therefore it is clear that positivity (resp. negativity) of the function $L'_{s,\,t}(x)$ for all $x > 0$ implies positivity (resp. negativity) of $L_{s,\,t}(x)$ for all $x > 0$. On the other hand, by (6.5) for all $(s,t) \in E$ positivity (resp. negativity) of the function $L''_{s,\,t}(x)$ for all $x > 0$ implies negativity (resp. positivity) of $L'_{s,\,t}(x)$ for all $x > 0$. Hence, since by Corollary 2.1 the function $L_{s,\,t}(x)$ changes sign in $(0,\,\infty)$ if $(s,t) \in E \setminus P$, also the functions $L'_{s,\,t}(x)$ and $L''_{s,\,t}(x)$ must change sign in $(0,\,\infty)$ for $(s,t) \in E \setminus P$.

Finally, by (2.12) the inequality

$$L_{s,\,t}(x) < \frac{(s-t)(s+t-1)}{2} \qquad \text{for all } \ x > 0, \tag{6.7}$$

implies

$$\frac{1}{24}\,(s-t)\,(s-t+1)\,p(s,t) \geq 0, \tag{6.8}$$

and the reverse inequality (6.7) implies the reverse inequality (6.8). Also, for $s - t + 1 > 0$, inequality (6.7) implies

$$\frac{(s-t)(s+t-1)}{2} \geq 0 \tag{6.9}$$

while the reverse inequality (6.7) implies the reverse inequality (6.9). Combining the above, we conclude that inequality (6.7) fails to hold for appropriate $x > 0$ if $(s,\,t) \in \{(s,t) \in E \,:\, s > t\} \setminus A_1$ or $(s,\,t) \in P_1^* \setminus A_3$. Similarly, the reverse inequality (6.7) fails to hold for appropriate $x > 0$ if $(s,\,t) \in P_2^* \setminus A_2$.

The proof of the theorem is complete. □

*Proof of Theorem* 3.5. Suppose that $(s,t) \in P_2$, that is, suppose that $s+t \geq 1$ and $0 < s - t < 1$. We then have $1 - M_{s,t}(x) = L_{s,\,t}(x)/x > 0$ for all $x > 0$.

It will suffice to prove that $M'_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$. Making use of [10, (3.4)], it is easy to see that

$$\begin{aligned} M'_{s,\,t}(x) &= M_{s,\,t}(x)\left[\frac{s-t}{x} + \psi(x+t) - \psi(x+s)\right] \\ &= M_{s,\,t}(x)\int_0^\infty e^{-xu}\left[\varphi_{s,t}(0) - \varphi_{s,t}(u)\right]du, \end{aligned} \tag{6.10}$$

where $\varphi_{s,t}(u)$ is defined as in Section 2. It follows from Lemma 5.1 that in this case we have $-\varphi'_{s,t}(u) > 0$ for all $u > 0$ and therefore it follows from Theorem 1.1 and (6.10) that $x\,M'_{s,\,t}(x)/M_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$. Writing

$$M'_{s,\,t}(x) = x\,\frac{M'_{s,\,t}(x)}{M_{s,\,t}(x)}\,\frac{\Gamma(x+t)}{\Gamma(x+s)}\,x^{s-t-1} \tag{6.11}$$

and taking into consideration (2.9), we conclude that for $(s,t) \in P_2$ the function $M'_{s,\,t}(x)$ is completely monotonic on $(0,\,\infty)$ as a product of completely monotonic functions.

Since for $s+t < 1$ the function $L_{s,\,t}(x)$ changes sign in $(0,\,\infty)$ while for $s+t \geq 1$ and $s < t$ it is negative on $(0,\,\infty)$, $L_{s,\,t}(x)/x$ cannot be completely monotonic on

$(0, \infty)$ in these cases. On the other hand, if $(s, t) \in P_1$, that is, if $s - t > 1$, then it follows from (6.10) and (6.11) that $M'_{s,t}(0) = 0$ and therefore neither $M'_{s,t}(x)$ nor $L_{s,t}(x)/x$ can be completely monotonic on $(0, \infty)$.

This completes the proof of Theorem 3.5. □

*Remark* 6.1. It is known and easy to see [3] that if $f$ is a Bernstein function on $(0, \infty)$, then $f(x)/x$ is completely monotonic on $(0, \infty)$. The converse need not be true. Accordingly, in the case where $(s, t) \in A_1 \setminus P_1$ the result of Theorem 3.5 follows from Theorem 3.2 (*ii*). However, our Theorem 3.5 states that $L_{s,t}(x)/x$ is completely monotonic for $(s, t)$ in a larger set and the result is best possible.

## 7. Further results and applications

Our initial interest in estimates for the functions $L_{s,t}(x)$ and $\Delta_{s,t}(x)$ stems from the following conjecture from [16] concerning the mapping behaviour of the partial sums

$$s_n^\mu(z) := \sum_{k=0}^{n} \frac{(\mu)_k}{k!} z^k, \quad \mu \in (0, 1],\ n \in \mathbb{N},$$

of the function $(1 - z)^{-\mu}$. Here, $(\mu)_k := \mu(\mu + 1) \cdots (\mu + k - 1) = \Gamma(\mu + k)/\Gamma(\mu)$ is the so-called Pochhammer symbol.

**Conjecture 1.** *For $\rho \in (0, 1]$ define $\mu(\rho)$ as the maximal number such that*

$$|\arg(1 - z)^\rho s_n^\mu(z)| \le \frac{\rho\pi}{2}$$

*for all $n \in \mathbb{N}$, $z \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$, and $0 < \mu \le \mu(\rho)$. Then for all $\rho \in (0, 1]$ the number $\mu(\rho)$ is equal to the unique solution $\mu^*(\rho)$ in $(0, 1]$ of the equation*

$$\int_0^{(\rho+1)\pi} \frac{\sin(t - \rho\pi)}{t^{1-\mu}} dt = 0.$$

It is shown in [13] that $\mu^*(\rho)$ is a strictly increasing analytic function of $\rho \in (0, 1)$.

A quite powerful method (called $\Delta$-method from now on) for proving the positivity of trigonometric sums with coefficient sequences of the form $\{\frac{(\mu)_k}{k!}\}$ was presented in [9, 15]. This method was used to prove Conjecture 1 for $\rho = \frac{1}{2}$ and $\rho = \frac{1}{4}$ in [12, 16], where it was shown that for the $\rho$ in question Conjecture 1 is equivalent to the nonnegativity of the trigonometric sums

$$\sigma_n(\rho, \mu, \theta) := \sum_{k=0}^{n} \frac{(\mu)_k}{k!} \sin\left[(2k + \rho)\theta\right]$$

for $n \in \mathbb{N}$, $\theta \in (0, \pi)$, $\rho \in \{\frac{1}{4}, \frac{1}{2}\}$, and $\mu = \mu^*(\rho)$.

As explained in [9, 12, 15, 16] the $\Delta$-method requires sharp bounds for the remainder series of the form

$$\sum_{k=n}^{\infty} \Delta_{1,\mu}(k) e^{ikx} \tag{7.1}$$

with $\mu \in (0, 1)$ (here $\Delta_{s,t}(x)$ is defined as in (2.1)). Observe that for $k \in \mathbb{N}$,

$$\Delta_{1,\mu}(k) = \frac{1}{\Gamma(\mu)} \frac{1}{k^{1-\mu}} - \frac{(\mu)_k}{k!}.$$

Using partial summation and Proposition 2.1 $(i)$, it is easy to see that Theorem 3.2 implies the following.

**Corollary 7.1.** *Suppose that* $(s,t) \in A_1$. *Then*

$$\left|\sum_{k=n}^{\infty} \Delta_{s,t}(k) e^{ikx}\right| \leq \frac{1}{\sin\frac{x}{2}} \frac{1}{n^{s-t+1}} \frac{\Gamma(s)}{\Gamma(t)} \frac{(s-t)(s+t-1)}{2}$$

*for all* $x \in (0, 2\pi)$ *and* $n \in \mathbb{N}$ *with* $n \geq 2$.

This is a substantial extension of [10, Proposition 1].

Since $(1,\mu) \in A_1$ if, and only if, $\frac{1}{3} \leq \mu < 1$, Corollary 7.1 gives the required estimate of the remainder series (7.1) in order to apply the $\Delta$-method for proving the positivity of $\sigma_n(\rho, \mu^*(\rho), \theta)$ not only for $\rho \in \{\frac{1}{4}, \frac{1}{2}\}$, but for all $\rho \in [(\mu^*)^{-1}(\frac{1}{3}), 1]$. Note that $(\mu^*)^{-1}(\frac{1}{3}) = 0.21\ldots$.

The following conjecture thus seems reasonable.

**Conjecture 2.** *Let* $\rho \in (0,1]$. *Then*

$$\sigma_n(\rho, \mu^*(\rho), \theta) > 0$$

*for all* $n \in \mathbb{N}$ *and* $\theta \in (0,\pi)$ *and this trigonometric inequality is sharp. That is, for all* $\rho \in (0,1)$ *and* $\mu \in (\mu^*(\rho), 1]$ *there are* $n \in \mathbb{N}$ *and* $\theta \in (0,\pi)$ *such that* $\sigma_n(\rho, \mu, \theta) < 0$.

Note that the sharpness part of this conjecture is already known (cf. [16]).

In order to make the $\Delta$-method applicable to Conjecture 1 for all $\rho \in (0,1]$, we will now show how to obtain a sharp estimate for the remainder series (7.1) when $\mu \in (0, \frac{1}{3})$.

**Lemma 7.2.** *Let* $\mu \in (0, \frac{1}{3})$. *Then*

$$\Delta_{1,\mu}(x) \leq \frac{x^{\mu-2}}{\Gamma(\mu)} \left( \frac{\mu(1-\mu)}{2} + \frac{\mu(1-\mu)(2-\mu)(1-3\mu)}{24x} \right). \tag{7.2}$$

*Proof.* Making use of (2.9), we have

$$\Delta_{1,\mu}(x) = \frac{1}{\Gamma(\mu)\Gamma(1-\mu)} \int_0^\infty e^{-xu} u^{-\mu} \left(1 - \left(\frac{u}{e^u - 1}\right)^\mu\right) dt.$$

Hence, if we can show that for $\mu \in (0, \frac{1}{3})$ and $u > 0$,

$$1 - \left(\frac{u}{e^u - 1}\right)^\mu < \frac{\mu u}{2} + \frac{\mu(1-3\mu)u^2}{24}, \tag{7.3}$$

then it is straightforward to check that (7.2) holds as required.

Now, in order to prove (7.3) observe that it will suffice to show that

$$u^{-\mu} \left(1 - \frac{\mu u}{2} - \frac{\mu(1-3\mu)u^2}{24}\right) < \frac{1}{(e^u - 1)^\mu} \tag{7.4}$$

for the $\mu$ and $u$ in question. It is easy to check that for $\mu \in (0, \frac{1}{3})$ the parabola

$$p_\mu(u) := 1 - \frac{\mu u}{2} - \frac{\mu(1-3\mu)u^2}{24}$$

has exactly one zero $u_0$ in $(0,\infty)$ and is negative for $u > u_0$. (7.3) is therefore clear for $u \geq u_0$, and in order to prove the inequality also for $u \in (0, u_0)$, it will be enough to show that, for $\mu \in (0, \frac{1}{3})$ and $u \in (0, u_0)$,

$$\left(\log\left(u^{-\mu} p_\mu(u)\right)\right)' < -\mu \left(\log\left(e^u - 1\right)\right)'.$$

This inequality is equivalent to

$$h_\mu(u) := \frac{(3\mu-1)(\mu-2)u^2 + 12(\mu-1)u - 24}{24 p_\mu(u)} < \frac{e^u}{1-e^u}. \tag{7.5}$$

Since it is straightforward to check that this holds for $\mu = 0$ and $u \in (0, u_0)$, our assertion will follow once we show that $\frac{d}{d\mu} h_\mu(u) < 0$ for $\mu \in (0, \frac{1}{3})$ and $u \in (0, u_0)$. We calculate

$$\frac{d}{d\mu} h_\mu(u) = -\frac{2u^2\left((3\mu-1)^2 u - 36\mu + 18\right)}{p_\mu^2(u)}.$$

Since it is easy to see that the right-hand side of this equation is negative for $\mu \in (0, \frac{1}{3})$ and $u \in (0, u_0)$, we obtain (7.3) also for $u \in (u, u_0)$. □

Using partial summation and Proposition 2.1 $(i)$, we see that this lemma implies the following.

**Corollary 7.3.** *Suppose that $\mu \in (0, \frac{1}{3})$. Then*

$$\left|\sum_{k=n}^{\infty} \Delta_{1,\mu}(k) e^{ikx}\right| \leq \frac{\mu(1-\mu)}{2n^{2-\mu}\Gamma(\mu)\sin\frac{x}{2}} \left(1 + \frac{(2-\mu)(1-3\mu)}{12n}\right)$$

*for all $x \in (0, 2\pi)$ and $n \in \mathbb{N}$ with $n \geq 2$.*

Corollaries 7.1 and 7.3 give the necessary means in order to tackle Conjecture 2 for all $\rho \in (0, 1]$.

At this point, however, we should note that while the $\Delta$-method works quite well in order to verify Conjecture 2 for any concretely given $\rho \in (0, 1]$ (e.g. $\rho = \frac{1}{4}$, $\rho = \frac{1}{2}$), the application of the $\Delta$-method in order to prove Conjecture 2 for all $\rho$ in a concretely given interval (e.g. $\rho \in [\frac{1}{2} - 0.01, \frac{1}{2} + 0.01]$) seems to be technically quite hard.

Finally, let us show that Conjecture 2 implies Conjecture 1 for all $\rho \in (0, 1)$.

**Lemma 7.4.** *Let $\mu$, $\rho \in (0, 1]$ and $n \in \mathbb{N}$. Then*

$$|\arg(1-z)^\rho s_n^\mu(z)| \leq \frac{\rho\pi}{2} \tag{7.6}$$

*holds for all $n \in \mathbb{N}$ and $z \in \overline{\mathbb{D}}$ if, and only if,*

$$\sigma_n(\rho, \mu, \theta) \geq 0 \quad \textit{for all} \quad \theta \in [0, \pi]. \tag{7.7}$$

*Proof.* It is clear that (7.6) is equivalent to the two inequalities

$$\operatorname{Im}\left[e^{i\rho\pi/2}(1-z)^\rho s_n^\mu(z)\right] \geq 0, \qquad z \in \overline{\mathbb{D}} \tag{7.8}$$

and

$$\operatorname{Im}\left[e^{-i\rho\pi/2}(1-z)^\rho s_n^\mu(z)\right] \leq 0, \qquad z \in \overline{\mathbb{D}}. \tag{7.9}$$

For $z = e^{2i\theta}$, $\theta \in [0, \pi]$, we have

$$(1-z)^\rho s_n^\mu(z) = (2\sin\theta)^\rho e^{-i\rho\pi/2} \sum_{k=0}^{n} \frac{(\mu)_k}{k!} e^{i(2k+\rho)\theta},$$

and therefore it follows from the minimum principle of harmonic functions that (7.8) and (7.9) are equivalent to

$$\sigma_n(\rho, \mu, \theta) = \sum_{k=0}^{n} \frac{(\mu)_k}{k!} \sin[(2k+\rho)\theta] \geq 0, \quad \theta \in [0, \pi]$$

and

$$-\sigma_n(\rho, \mu, \pi - \theta) = \sum_{k=0}^{n} \frac{(\mu)_k}{k!} \sin\left[(2k+\rho)\theta - \rho\pi\right] \leq 0, \quad \theta \in [0, \pi],$$

respectively. □

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