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GENERATORS OF RATIONAL SPECTRAL TRANSFORMATIONS FOR NONTRIVIAL $\mathcal{C}$-FUNCTIONS

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Abstract. In this paper we consider transformations of sequences of orthogonal polynomials associated with a Hermitian linear functional $\mathcal{L}$ using spectral transformations of the corresponding $\mathcal{C}$-function $F_{\mathcal{L}}$. We show that a rational spectral transformation of $F_{\mathcal{L}}$ with polynomial coefficients is a finite composition of four canonical spectral transformations.

1. Introduction and main results

Let $\mathcal{M}$ be a quasi-definite linear functional in the linear space $\mathbb{P}$ of polynomials with complex coefficients. We denote by $\mu_n = \langle \mathcal{M}, x^n \rangle$, $n \geq 0$, the sequence of moments associated with $\mathcal{M}$. The sequence of monic polynomials orthogonal with respect to $\mathcal{M}$ satisfies a three term recurrence relation (see [6])

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + u_n p_{n-1}(x), \qquad n \geq 1,$$

with initial conditions $p_0(x) = 1$ and $p_1(x) = x - b_0$. The recurrence coefficients are given by

$$b_n = \frac{\langle \mathcal{M}, xp_n^2 \rangle}{\langle \mathcal{M}, p_n^2 \rangle} \text{ and } \quad u_n = \frac{\langle \mathcal{M}, p_n^2 \rangle}{\langle \mathcal{M}, p_{n-1}^2 \rangle}.$$

It has a wide field of applications and has been extensively studied over the years taking into account their connections with spectral theory of Jacobi matrices. The study of perturbations of such a linear functional and their effects on the corresponding Stieltjes function

$$S_{\mathcal{M}}(x) = \left\langle \mathcal{M}, \frac{1}{x - \varsigma} \right\rangle, \tag{1.1}$$

where the functional $\mathcal{M}$ acts on the variable ς, has a significant relevance in this theory. $S_{\mathcal{M}}$ admits the following equivalent representation as a series expansion at infinity

$$S_{\mathcal{M}}(x) = \frac{\mu_0}{x} + \frac{\mu_1}{x^2} + \cdots + \frac{\mu_n}{x^{n+1}} + \cdots ,$$

i.e., it is as a generating function of the sequence of moments for the linear functional $\mathcal{M}$.

If we are interested in characterizing the sequence of monic polynomials $\{p_n\}_{n\geqslant 0}$ orthogonal with respect to $\mathcal{M}$ we could consider the k-th associated sequence of

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monic polynomials $\left\{p_n^{(k)}\right\}_{n\geqslant 0}$ satisfying the three term recurrence relation

$$xp_n^{(k)}(x) = p_{n+1}^{(k)}(x) + b_{n+k}p_n^{(k)}(x) + u_{n+k}p_{n-1}^{(k)}(x), \qquad n \geq 1, \tag{1.2}$$

with initial conditions $p_0^{(k)}(x) = 1$ and $p_1^{(k)}(x) = x - b_k$. Thus we have the following asymptotic expansion around infinity

$$\left(p_n S_{\mathcal{M}} - p_{n-1}^{(1)}\right)(x) = \mathcal{O}(x^{-n-1}).$$

It plays an important role in the theory of Padé rational approximation and continued fractions (see [6]). A rational spectral transformation of the Stieltjes function $S_{\mathcal{M}}$ is a new Stieltjes function defined by

$$S_{\widetilde{\mathcal{M}}}(x) = \left(\frac{aS_{\mathcal{M}} + b}{cS_{\mathcal{M}} + d}\right)(x) \tag{1.3}$$

where a, b, c, and d are coprime polynomials. When $c \equiv 0$, the spectral transformation is said to be linear.

The problem of classifying all possible transformations of orthogonal polynomials corresponding to a generic rational transformation (1.3), in other words, the description of a generator system of the set of rational spectral transformations was raised by Marcellán, Dehesa and Ronveaux [18] in 1990. Two years later, Peherstorfer [22] analyzed a particular class of rational spectral transformations. Indeed, he deduced the relation between the corresponding linear functionals. In 1997 Zhedanov [28] proved that a generic linear spectral transformation of the Stieltjes function can be represented as a finite composition of Christoffel and Geronimus transformations [12] and also that any rational spectral transformation can be obtained as a composition of forward and backward associated elementary transformations [12], Christoffel and Geronimus transformations.

The theory of orthogonal polynomials with respect to nontrivial probability measures supported on the unit circle was initiated by G. Szegő in the beginnings of the twentieth century (see [27]). In [11, 12, 13], an analogue theory for Hermitian linear functionals defined in the linear space of Laurent polynomials with complex coefficients was outlined. Notice that there is a one-to-one correspondence between positive definite Hermitian linear functionals and nontrivial probability measures supported on the unit circle. Many concepts developed on orthogonal polynomials with respect to $\mathcal{M}$ have an analogous in this theory. The counterpart for Hermitian linear functionals of the canonical spectral transformations (1.3) has been introduced by Marcellán and co-workers; see [7, 8, 14, 15] and Peherstorfer, also see [21]. Furthermore, Garza, Hernéndez and Marcellán [8] show that Christoffel and Geronimus transformations for nontrivial probability measures supported on $[-1, 1]$ yield the same kind of transformations for the corresponding measure supported on the unit circle, when the Szegő transformation [27] is applied. For this reason it was conjectured that the generator system of linear spectral transformations of a generic $\mathcal{C}$-function $F_{\mathcal{L}}$ (LST) given by

$$F_{\widetilde{\mathcal{L}}}(z) = \left(\frac{AF_{\mathcal{L}} + B}{D}\right)(z), \tag{1.4}$$

where A, B, and D are polynomials is the same as for Stieltjes functions, i.e. it is constituted by Christoffel and Geronimus transformations. Unfortunately, this does not happen.

The aim of our contribution is to obtain the generators of the set of rational spectral transformations for a generic nontrivial $\mathcal{C}$-function $F_{\mathcal{L}}$ (RST) given by

$$F_{\widetilde{\mathcal{L}}}(z) = \left(\frac{AF_{\mathcal{L}} + B}{CF_{\mathcal{L}} + D}\right)(z),$$

where A, B, C, and D are polynomials, or, equivalently, for quasi-definite Hermitian linear functionals.

In the sequel we will assume that the spectral transformation is irreducible, i.e., A, B, C, and D are coprime up to monomial factors. The main results of our contribution are the following ones.

Theorem 1.1. *A generic LST can be obtained as a finite composition of spectral transformations associated with a modification of the functional by the real part of a polynomial and its inverse.*

Theorem 1.2. *A generic RST can be obtained as a finite composition of linear and associated canonical spectral transformations.*

These results yield, in particular, that we can always split a RST using four elementary spectral transformations.

Our paper is organized as follows. In Section 2 we introduce the basic background to be needed in the sequel. In Section 3 we classify the spectral transformations of a $\mathcal{C}$-function in terms of the moments associated with the corresponding Hermitian linear functional. We also characterize the polynomial coefficients of a generic RST. In Section 4, we show two remarkable examples of RST due to Peherstorfer [21]. In Section 5, some special cases of spectral transformations are studied. Finally, in Section 6 the proofs of Theorems 1.1 and 1.2 are presented.

2. Notation and preliminary results

In the linear space of the Laurent polynomials with complex coefficients $\Lambda = \operatorname{span}\{z^k;\ k \in \mathbb{Z}\}$, we introduce a linear functional $\mathcal{L} : \Lambda \to \mathbb{C}$, satisfying $c_j = \left\langle \mathcal{L}, z^j \right\rangle = \overline{\langle \mathcal{L}, z^{-j} \rangle} = \overline{c}_{-j}$, $j \in \mathbb{Z}$, i.e., $\mathcal{L}$ is a Hermitian linear functional. Then, a bilinear functional associated with $\mathcal{L}$ can be introduced in $\mathbb{P}$, the linear space of polynomials with complex coefficients as follows (see [13]),

$$(p, q)_{\mathcal{L}} = \left\langle \mathcal{L}, p(z)\overline{q}(z^{-1}) \right\rangle$$

with $p, q \in \mathbb{P}$. The complex numbers $(c_j)_{j\in\mathbb{Z}}$ are said to be the moments of the linear functional $\mathcal{L}$ and the infinite matrix

$$\mathbf{T} = \begin{pmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}$$

is the Gram matrix of the above bilinear functional in terms of the canonical basis $\{z^n\}_{n\in\mathbb{Z}_+}$ of $\mathbb{P}$ associated with $\mathcal{L}$. It is known in the literature as a Toeplitz matrix [16].

If $\mathbf{T}_n$, the $(n+1)\times(n+1)$ principal leading submatrix of $\mathbf{T}$, is nonsingular for every $n \geqslant 0$, then $\mathcal{L}$ is said to be quasi-definite. Then there exists a sequence of monic polynomials, orthogonal with respect to $\mathcal{L}$, i.e., there exists a unique sequence of monic polynomials $\{\phi_n\}_{n\geqslant 0}$, $\deg \phi_n = n$ (MOPS, in short), such that

$$(\phi_n, \phi_m)_{\mathcal{L}} = \kappa_n \delta_{n,m},$$

where $\kappa_n \neq 0$ for every $n \geq 0$. This polynomial sequence satisfies two equivalent recurrence relations (see [16, 13, 25, 27])

$$\begin{aligned} \phi_{n+1}(z) &= z\phi_n(z) + \phi_{n+1}(0)\phi_n^*(z), \quad n \geqslant 0, \\ \phi_{n+1}(z) &= \left(1 - |\phi_{n+1}(0)|^2\right) z\, \phi_n(z) + \phi_{n+1}(0)\phi_{n+1}^*(z), \quad n \geqslant 0. \end{aligned}$$

They are known in the literature as forward and backward recurrence relations, respectively, where $\phi_n^*(z) = z^n \overline{\phi}_n(z^{-1}) = z^n (\phi_n)_*(z)$ is the reversed polynomial. The complex numbers $(\phi_n(0))_{n\geqslant 1}$ are known as either Verblunsky, Schur or reflection parameters.

Let $F : A \subset \mathbb{C} \to \mathbb{C}$ be a complex function associated with the linear functional $\mathcal{L}$ defined as follows:

$$F_{\mathcal{L}}(z) = \left\langle \mathcal{L}, \frac{\zeta + z}{\zeta - z} \right\rangle. \tag{2.1}$$

Here $\mathcal{L}$ acts on ζ. $F_{\mathcal{L}}$ is said to be a C-function associated with the linear functional $\mathcal{L}$. Assuming that $\sum_{j=0}^{\infty} c_{-j} z^j$ converges on $|z| \leq \rho$ with $\rho > 0$, $F_{\mathcal{L}}$ is analytic in a neighborhood of $z = 0$ and we get the following representation of $F_{\mathcal{L}}$ as a series expansion at $z = 0$,

$$F_{\mathcal{L}}(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k, \quad |z| \leq \rho, \tag{2.2}$$

where $c_0 \in \mathbb{R}$ and $c_k \in \mathbb{C}$. By a spectral transformation of $F_{\mathcal{L}}$ we mean a new $\mathcal{C}$-function associated with the Hermitian linear functional $\widetilde{\mathcal{L}}$, a modification of $\mathcal{L}$, such that

$$F_{\widetilde{\mathcal{L}}}(z) = \tilde{c}_0 + 2\sum_{k=1}^{\infty} \tilde{c}_{-k} z^k, \quad |z| \leq \rho, \tag{2.3}$$

where $\tilde{c}_k$ are the transformed moments.

If $\Re F_{\mathcal{L}}(z) > 0$ for $|z| < 1$, then (2.2) is the function introduced by Carathéodory in [2] and thus it can be represented in terms of a nontrivial probability measure σ supported on the unit circle as (see [17, 24])

$$F_{\mathcal{L}}(z) = \int \frac{\zeta + z}{\zeta - z} d\sigma(\zeta).$$

For a given polynomial q with leading coefficient η the polynomial of the second kind of q with respect to $\mathcal{L}$ is defined by

$$Q(z) = \begin{cases} \mathcal{L}\left(\dfrac{\zeta + z}{\zeta - z}(q(\zeta) - q(z))\right), & \deg q \in \mathbb{N}\backslash\{0\}, \\ \eta\mathcal{L}1, & \deg q = 0, \end{cases}$$

where $\mathcal{L}$ acts on ζ.

If $H(z) = \sum_{k=0}^{\infty} h_k z^k$ is an analytic function in a neighborhood of $z = 0$, then, as usual, we write

$$H(z) = \mathcal{O}(z^n), \text{ if } h_0 = \cdots = h_{n-1} = 0 \text{ and } H(z) = \dot{\mathcal{O}}(z^n) \text{ if, in addition, } h_n \neq 0.$$

3. Local and global spectral transformations

In terms of the moments we can classify the spectral transformations of a $\mathcal{C}$-function as follows.

(i) *Local spectral transformations*: spectral transformations under the modification of a finite number of moments.

(ii) *Global spectral transformations*: spectral transformations under the modification of a infinite number of moments.

It is clear that the local spectral transformations are given by

$$F_{\tilde{\mathcal{L}}}(z) = (F_{\mathcal{L}} + E)(z),$$

where $E(z) = \sum_{j\in S} m_j z^j$ for $m_j \in \mathbb{C}$ and S a finite subset of nonnegative integer numbers. Hence, a generator system of (i) follows immediately from [5].

Theorem 3.1. *A local spectral transformation can be obtained as a finite composition of spectral transformation associated with the Hermitian linear functional*

$$\mathcal{L}_j = \mathcal{L} + 2\Re(m_j z^j)\frac{dz}{2\pi i z},$$

for $j \in \mathbb{N}$ and $m_j \in \mathbb{C}$.

In general (ii) can be represented by the following rational expression,

$$F_{\tilde{\mathcal{L}}}(z) = \left(\frac{AF_{\mathcal{L}} + B}{CF_{\mathcal{L}} + D}\right)(z), \tag{3.1}$$

where A, B, C, and D are coprime polynomials. This spectral transformation is denoted by RST. If $C \equiv 0$, we have the subclass of linear spectral transformations that will be denoted by LST.

To study a generic RST we will use the following theorem.

Theorem 3.2 ([23])**.** *Let $\mathcal{L}$ be a Hermitian linear functional and $F_{\mathcal{L}}$ the corresponding $\mathcal{C}$-function. Then the following statement holds.*

$$(\phi_n(z), z^j)_{\mathcal{L}} = 0, \qquad \textit{for} \qquad j = 0, \dots, n-1,$$

if and only if,

$$(\phi_n F_{\mathcal{L}} + \omega_n)(z) = \dot{\mathcal{O}}(z^n) \qquad \textit{and} \qquad (\phi_n^* F_{\mathcal{L}} - \omega_n^*)(z) = \dot{\mathcal{O}}(z^{n+1}),$$

where ω_n is the polynomial of the second kind of ϕ_n with respect to $\mathcal{L}$.

In order to prove our main results in the following lemma we will characterize the polynomial coefficients of (3.1).

Lemma 3.3. *Only one of the following two statements holds.*

(i) *The polynomial coefficients in* (3.1) *are Laurent Hermitian polynomials of the same degree such that*

$$A_* = A, \qquad B_* = -B, \qquad C_* = -C, \qquad \textit{and} \qquad D_* = D.$$

(ii) *The polynomial coefficients in* (3.1) *are self-reciprocal polynomials of the same degree such that*

$$A^* = A, \qquad B^* = -B, \qquad C^* = -C, \qquad \textit{and} \qquad D^* = D.$$

Proof. Let $\{\phi_n\}_{n\geqslant 0}$ be the MOPS with respect to a Hermitian linear functional $\mathcal{L}$, with associated $\mathcal{C}$-function (2.1). Let $\{\omega_n\}_{n\geqslant 0}$ be the sequence of polynomials of the second kind for $\mathcal{L}$ and let (3.1) be a RST. Thus, $F_{\mathcal{L}}$ is the RST associated with (3.1) given by

$$F_{\mathcal{L}}(z) = \left(\frac{-DF_{\widetilde{\mathcal{L}}} + B}{CF_{\widetilde{\mathcal{L}}} - A}\right)(z). \tag{3.2}$$

Multiplying and dividing (3.2) for z^ℓ where ℓ is the minimum nonnegative integer number such that $z^\ell A$, $z^\ell B$, $z^\ell C$, and $z^\ell D$ are polynomials and using Theorem 3.2, we immediately get

$$\begin{aligned} z^\ell(\phi_n D - \omega_n C)(z)F_{\widetilde{\mathcal{L}}}(z) + z^\ell(\omega_n A - \phi_n B)(z) &= \dot{\mathcal{O}}(z^{n+\nu}), \\ z^\ell(\phi_n^* D + \omega_n^* z^\ell C)(z)F_{\widetilde{\mathcal{L}}}(z) - z^\ell(\omega_n^* A + \phi_n^* B)(z) &= \dot{\mathcal{O}}(z^{n+1+\nu}), \end{aligned} \tag{3.3}$$

where ν is a positive integer such that

$$z^\ell(A - CF_{\widetilde{\mathcal{L}}})(z) = \dot{\mathcal{O}}(z^\nu).$$

Therefore, we can deduce that the new polynomial coefficients $z^\ell A$, $z^\ell B$, $z^\ell C$, and $z^\ell D$ satisfy the following relations

$$(z^\ell A(z))^* = z^\ell A(z),\ (z^\ell B(z))^* = -z^\ell B(z),\ (z^\ell C(z))^* = -z^\ell C(z), \\ \text{and } (z^\ell D(z))^* = z^\ell D(z).$$

Suppose that $\ell \neq 0$. Then A, B, C, and D are Hermitian Laurent polynomials of the same degree ℓ proving (i). On the other hand, if we suppose that $\ell = 0$, then A, B, C, and D are self-reciprocal polynomials of the same degree. Thus (ii) follows. □

Remark 3.4. Notice that, if the coefficients A, B, C, and D of (3.1) are constant, then, A, D are real numbers and B, C are pure imaginary numbers. An example of this is the Aleksandrov transformation [25].

A generic RST with Hermitian Laurent polynomial coefficients can be transformed into a equivalent RST with self-reciprocal polynomial coefficients of even degree and the converse is also true. Otherwise, if we have a generic RST with self-reciprocal polynomial coefficients of odd degree, then it cannot be transformed into an equivalent RST with Hermitian Laurent polynomial coefficients.

Theorem 3.5. *A generic RST with self-reciprocal polynomial coefficients of odd degree have symmetric generators to the corresponding symmetric RST which has Hermitian Laurent polynomial coefficients.*

Proof. Let $\{\phi_n\}_{n\geqslant 0}$ and $\left\{\tilde{\phi}_n\right\}_{n\geqslant 0}$ be the MOPS associated with the $\mathcal{C}$-functions, $F_{\mathcal{L}}$ and $F_{\widetilde{\mathcal{L}}}$ respectively, where $F_{\widetilde{\mathcal{L}}}$ is a RST of $F_{\mathcal{L}}$ with A, B, C, and D polynomial coefficients. We will assume that they are self-reciprocal polynomials of odd degree. From [20], there is only a unique MOPS sequence $(\psi_n)_{n\geq 0}$ (respectively $(\tilde{\psi}_n)_{n\geq 0}$) such that

$$\tilde{\psi}_{2n}(z) = \tilde{\phi}_n(z^2), \qquad \psi_{2n}(z) = \phi_n(z^2), \qquad n \geq 0,$$

with respect to the symmetric functional $\mathcal{D}$ (respectively $\widetilde{\mathcal{D}}$).

On the other hand, the corresponding symmetric $\mathcal{C}$-functions associated with the symmetric functionals $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are $F_{\mathcal{D}}(z) = F_{\mathcal{L}}(z^2)$ and $F_{\widetilde{\mathcal{D}}}(z) = F_{\widetilde{\mathcal{L}}}(z^2)$, respectively. Therefore,

$$F_{\widetilde{\mathcal{D}}}(z) = \frac{A(z^2)F_{\mathcal{D}}(z) + B(z^2)}{C(z^2)F_{\mathcal{D}}(z) + D(z^2)}$$

is a RST associated with $F_{\mathcal{D}}$ with self-reciprocal polynomial coefficients of even degree. Thus, $F_{\widetilde{\mathcal{D}}}$ is equivalent to a RST with Hermitian Laurent polynomial coefficients. □

According to the previous proposition, throughout this paper we will consider only Hermitian Laurent polynomial coefficients.

4. Rational spectral transformations

Two remarkable examples of RST are due to Peherstorfer [21]. We denote by $\{\phi_n^{(j)}\}_{n\geqslant 0}$ the j-th associated sequence of polynomials of order $j \in \mathbb{N}$ for $\{\phi_n\}_{n\geqslant 0}$ that constitutes analogous of the associated polynomials satisfying (1.2). In this case they are generated by the recurrence relation

$$\phi_{n+1}^{(j)}(z) = z\phi_n^{(j)}(z) + \phi_{n+j+1}(0)(\phi_n^{(j)})^*(z), \qquad n \geq 0. \tag{4.1}$$

Notice that $\{\phi_n^{(j)}\}_{n\geqslant 0}$ is again a sequence of orthogonal polynomials with respect to a new Hermitian linear functional that we will denote by $\mathcal{L}^{(j)}$ (Favard's Theorem, see [13]).

We will assume that j is an even nonnegative integer number in order to obtain Hermitian Laurent polynomial coefficients satisfying Lemma 3.3.

The first example is the forward associated transformation (FT).

Example 4.1. Denoting by $\mathcal{F}_{\mathcal{L}^{(2k)}}$ the FT of $F_{\mathcal{L}}$, i.e., $\mathcal{F}_{\mathcal{L}^{(2k)}}[F_{\mathcal{L}}] = F_{\mathcal{L}^{(2k)}}$, the corresponding $\mathcal{C}$-function is a RST given by

$$F_{\mathcal{L}^{(2k)}}(z) = \frac{(z^{-k}\phi_{2k}(z) + z^k(\phi_{2k})_*(z))F_{\mathcal{L}}(z) + z^{-k}\omega_{2k}(z) - z^k(\omega_{2k})_*(z)}{(z^{-k}\phi_{2k}(z) - z^k(\phi_{2k})_*(z))F_{\mathcal{L}}(z) + z^{-k}\omega_{2k}(z) + z^k(\omega_{2k})_*(z)}.$$

In other words, we remove the first $2k$ Verblunsky coefficients from the original family of Verblunsky coefficients.

On the other hand, if we add complex numbers $\xi_1, \xi_2, \ldots, \xi_{2k}$ with $|\xi_i| \neq 1$, $1 \leqslant i \leqslant 2k$ to the original family of Verblunsky coefficients we have the backward associated sequence of polynomials $(\phi_n^{(-2k)})_{n\geqslant 0}$, as MOPS generated by the Verblunsky parameters $(\xi_i)_{i=1}^{2k} \bigcup (\phi_n(0))_{n\geq 1}$.

The second example is the backward associated transformation (BT).

Example 4.2. Denoting by $\mathcal{F}_{\mathcal{L}^{(-2k)}}$ the BT of $F_{\mathcal{L}}$, i.e., $\mathcal{F}_{\mathcal{L}^{(-2k)}}[F_{\mathcal{L}}] = F_{\mathcal{L}^{(-2k)}}$, the corresponding $\mathcal{C}$-function is a RST given by

$$F_{\mathcal{L}^{(-2k)}}(z) = \frac{(z^{-k}\tilde{\omega}_{2k}(z) + z^k(\tilde{\omega}_{2k})_*(z))F_{\mathcal{L}}(z) + z^k(\tilde{\omega}_{2k})_*(z) - z^k\tilde{\omega}_k(z)}{(z^k(\tilde{\phi}_{2k})_*(z) - z^{-k}\tilde{\phi}_{2k}(z))F_{\mathcal{L}}(z) + z^{-k}\tilde{\phi}_{2k}(z) + z^k(\tilde{\phi}_{2k})_*(z)},$$

where $\tilde{\phi}_{2k}$ (respectively $\tilde{\omega}_{2k}$) is the $2k$-th degree polynomial generated using the complex numbers $\xi_1, \xi_2, \ldots, \xi_{2k}$ (respectively $-\xi_1, -\xi_2, \ldots, -\xi_{2k}$) through the recurrence relation (4.1), i.e, $\tilde{\omega}_{2k}$ is the polynomial of second kind associated with $\tilde{\phi}_{2k}$.

5. Linear spectral transformations

If $\mathcal{L}$ is a Hermitian linear functional and $f \in \Lambda$, the modified functional $f\mathcal{L}$ is defined by

$$\langle f\mathcal{L}, g\rangle = \langle \mathcal{L}, fg\rangle\,, \qquad f, g \in \Lambda.$$

This polynomial modification is Hermitian if and only if f is a Hermitian Laurent polynomial (see [1, 26]). Moreover, $f\mathcal{L}$ is positive definite for a positive definite linear functional $\mathcal{L}$ if f also satisfy the Féjer condition [25]. Furthermore, $f\mathcal{L}$ can be factorized as a product of elementary ones of degree one since the set of zeros of a self-reciprocal polynomial lie on the unit circle or appear in symmetric pairs. In the sequel we will assume for simplicity that the polynomial modification f is a monic Hermitian Laurent polynomial from which we can deduce immediately the more general case.

We denote by $\mathcal{L}_r$ and $\mathcal{L}_{r^{-1}}$ the modification of the Hermitian linear functional $\mathcal{L}$ by the real part of a Hermitian Laurent polynomial of degree one (RT) and its inverse (IT), i.e.,

$$\mathcal{L}_r = (z + z^{-1} - (\alpha + \overline{\alpha}))\mathcal{L}, \qquad (z + z^{-1} - (\alpha + \overline{\alpha}))\mathcal{L}_{r^{-1}} = \mathcal{L}, \qquad \alpha \in \mathbb{C}$$

and by $F_{\mathcal{L}_r}$ and $F_{\mathcal{L}_{r^{-1}}}$ the corresponding $\mathcal{C}$-functions.

These perturbations have been introduced in [4] and [26], and recently studied in [1] with a new approach in the framework of inverse problems for sequences of monic orthogonal polynomials. The authors and Garza (see [10] and [4]) focused the attention on their effect for corresponding $\mathcal{C}$-function. We will show that these transformations play an important role in the representation of a generic LST.

Remark 5.1. For all values of α such that $|\Re(\alpha)| > 1$, the Laurent polynomial $z + z^{-1} - (\alpha + \overline{\alpha})$ can be represented as a polynomial of the form $-\frac{1}{\beta}|z - \beta|^2$, where $\beta \in \mathbb{R} \setminus \{0\}$, i.e., Christoffel and Geronimus transformations are particular cases of these transformations.

From the relation between the moments we get that the $\mathcal{C}$-function associated with $\mathcal{L}_r$ is given by

$$F_{\mathcal{L}_r}(z) = \frac{1}{2}\left((z + z^{-1} - (\alpha + \overline{\alpha}))F_{\mathcal{L}}(z) + c_0(z - z^{-1}) + c_1 - c_{-1}\right). \tag{5.1}$$

whose Laurent polynomial coefficients $z + z^{-1} - (\alpha + \overline{\alpha})$ and $c_0(z - z^{-1}) + c_1 - c_{-1}$ are Hermitian and satisfy Lemma 3.3. In a more general situation, if we consider a finite composition, with order $k \in \mathbb{N}$, of RT defined by

$$\mathcal{L}_{r^{(k)}} = \Re\left(\prod_{i=1}^{k}(z - \alpha_i)\right)\mathcal{L}, \qquad \alpha_i \in \mathbb{C}, \tag{5.2}$$

then we can prove

Theorem 5.2. *A generic spectral transformation with $C \equiv 0$ and $D \equiv 1$ is equivalent to a finite composition of RT. Furthermore, A and B are given by*

$$A = \frac{1}{2}\Re p, \qquad B = P - P_* \tag{5.3}$$

where P is the polynomial of second kind of

$$p(z) = \prod_{i=1}^{k}(z - \alpha_i)$$

with respect to the linear functional (5.2)*.*

Proof. From (5.1) we get that RT is a spectral transformation with $C \equiv 0$ and $D \equiv 1$. On the other hand, if we start with a generic spectral transformation where $C \equiv 0$, B is a Hermitian Laurent polynomial of degree one, and $D \equiv 1$, then, according to (3.3), A should be a Hermitian Laurent polynomial of degree one, and the only choice for this spectral transformation is (5.1) containing one free parameter α.

Now, if we consider (5.2) then, from (5.1) we have

$$F_{\widetilde{\mathcal{L}}}(z) = (AF_{\mathcal{L}} + B)(z) \tag{5.4}$$

where

$$A = (z + z^{-1} - (\alpha_1 + \overline{\alpha}_1)) \cdots (z + z^{-1} - (\alpha_k + \overline{\alpha}_k))$$

is a Hermitian Laurent polynomial of degree k. This transformation contains k free parameters α_k and B is a polynomial of the same type and degree as A.

Conversely, if we start with a generic spectral transformation with

$$A = (z + z^{-1} - (\alpha_1 + \overline{\alpha}_1)) \cdots (z + z^{-1} - (\alpha_k + \overline{\alpha}_k))$$

a Hermitian Laurent polynomial of degree k, $C \equiv 0$, and $D \equiv 1$, then B should be a Hermitian Laurent polynomial of the same degree as A in order to satisfy (3.3). Moreover, it is easy to see that for (5.2) the polynomial B is uniquely determined by means of the sequence of moments $(c_j)_{j\in\mathbb{Z}}$ associated with the linear functional $\mathcal{L}$. Furthermore, (5.3) follows immediately from the definition of (5.2). ☐

Using the same procedure as above, from the relationship between the moments, we obtain the LST associated with $\mathcal{L}_{r^{-1}}$ which is given by

$$F_{\mathcal{L}_{r^{-1}}}(z) = \frac{F_{\mathcal{L}}(z) - \tilde{c}_0(z - z^{-1}) + \tilde{c}_{-1} - \tilde{c}_1}{z + z^{-1} - (\alpha + \overline{\alpha})}, \tag{5.5}$$

whose Laurent polynomial coefficients $-\tilde{c}_0(z-z^{-1}) + \tilde{c}_{-1} - \tilde{c}_1$ and $z + z^{-1} - (\alpha + \overline{\alpha})$ are Hermitian and satisfy Lemma 3.3 as the RT case.

In an analogous way as in the RT case, given a finite composition of order $k \in \mathbb{N}$ of IT,

$$\mathcal{L} = \Re\left(\prod_{i=1}^{k}(z - \alpha_i)\right)\mathcal{L}_{r^{(-k)}}, \qquad \alpha_i \in \mathbb{C}, \tag{5.6}$$

we can deduce

Theorem 5.3. *A generic spectral transformation with $A \equiv 1$ and $C \equiv 0$ is equivalent to a finite composition of IT. Furthermore, B and D are given by*

$$B = \widetilde{P} - \widetilde{P}_*, \qquad D = \Re p, \tag{5.7}$$

where $\widetilde{P}$ is the polynomial of second kind of

$$p(z) = \prod_{i=1}^{k}(z - \alpha_i)$$

with respect to the linear functional (5.6)*.*

Proof. From the above proposition it is straightforward to show that the $\mathcal{C}$-function $F_{\mathcal{L}_{r^{-1}}}$ is the reciprocal of $F_{\mathcal{L}_r}$. However, in contrast to RT, $F_{\mathcal{L}_r}$ contains three free parameters α, c_0 and $\Im c_1$, while $\Re c_1$ is determined by c_0 and the choice of α.

Conversely, if we start with a spectral transformation with $A \equiv 1$, B is a Hermitian Laurent polynomial of degree one and $C \equiv 0$, then according to (3.3), D is a Hermitian Laurent polynomial of degree one with three restrictions for their coefficients. Hence we get just (5.5). Using an analogue of Theorem 5.3 we can complete the the proof. □

Remark 5.4. Notice that for different values $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $\beta_1, \beta_2, \ldots, \beta_k$, $\mathcal{F}_{\mathcal{L}_{r(k)}}[F_{\mathcal{L}}] = F_{\mathcal{L}_{r(k)}}$ and $\mathcal{F}_{\mathcal{L}_{r(-k)}}[F_{\mathcal{L}}] = F_{\mathcal{L}_{r(-k)}}$ we get

$$\begin{aligned}&\mathcal{F}_{\mathcal{L}_{r(k)}}(\alpha_1, \alpha_2, \ldots, \alpha_k) \circ \mathcal{F}_{\mathcal{L}_{r(-k)}}(\beta_1, \beta_2, \ldots, \beta_k)\\ &\quad = \mathcal{F}_{\mathcal{L}_{r(-k)}}(\beta_1, \beta_2, \ldots, \beta_k) \circ \mathcal{F}_{\mathcal{L}_{r(k)}}(\alpha_1, \alpha_2, \ldots, \alpha_k).\end{aligned}$$

Moreover, for the same parameters we have the following relations

$$\begin{aligned}\left(\mathcal{F}_{\mathcal{L}_{r(-k)}} \circ \mathcal{F}_{\mathcal{L}_{r(k)}}\right)(\alpha_1, \alpha_2, \ldots, \alpha_k) &= \mathcal{I},\\ \left(\mathcal{F}_{\mathcal{L}_{r(k)}} \circ \mathcal{F}_{\mathcal{L}_{r(-k)}}\right)(\alpha_1, \alpha_2, \ldots, \alpha_k) &= \mathcal{F}_U(\alpha_1, \ldots, \alpha_k) \circ \mathcal{F}_U(\bar{\alpha}_1, \ldots, \bar{\alpha}_k),\end{aligned}$$

where $\mathcal{F}_U(\alpha_1, \ldots, \alpha_k)$ is the so-called general Uvarov perturbation as the result of the addition of masses at the points $z = \alpha_1, z = \alpha_2, \ldots, z = \alpha_k$; see [3, 4].

6. Proof of the main results

Proof of Theorem 2. Assuming Hermitian Laurent polynomial coefficients, the application of BT with even degree $2k$ to the RST (3.1) of degree k yields a new RST where the transformed Laurent polynomial $\widetilde{C}$ is Hermitian and it is given by

$$\widetilde{C}(z) = z^k (A + C)(z)(\tilde{\phi}_{2k})_*(z) + z^{-k} (C - A)(z)\tilde{\phi}_{2k}(z).$$

Notice that the polynomial $\tilde{\phi}_{2k}$ can be chosen in an arbitrary way. Indeed, instead of choosing arbitrary $2k$ Verblunsky parameters we can choose the polynomial $\tilde{\phi}_{2k}$ satisfying $|\tilde{\phi}_{2k}(0)| \neq 1$ and from the Schur-Cohn-Jury criterion [19] we obtain a sequence of complex numbers $\tilde{\phi}_{2k}(0), \ldots, \tilde{\phi}_1(0)$ with modulus different of 1. Let $(\phi_n(0))_{n\geq 1}$ be the Verblunsky coefficients of the Hermitian linear functional associated with (3.1). Then, by Favard's theorem [13] $(\tilde{\phi}_i(0))_{i=1}^{2k} \cup (\phi_n(0))_{n\geq 1}$ arises as a new sequence of Verblunsky coefficients of a Hermitian linear functional. Notice that it is unique.

On the other hand, in order to preserve the Hermitian character of $\widetilde{C}$, the principal leading coefficients of A and C are different and not symmetric with respect to the origin. Thus

$$\deg\,(A + C)(z) = \deg(C - A)(z) = k.$$

Moreover, without loss of generality, the polynomial $z^k(A+C)(z)$ evaluated at $z = 0$ can be chosen in such a way that its modulus is different from one. Therefore, we can choose the polynomial $\tilde{\phi}_{2k}$ such that

$$\tilde{\phi}_{2k}(z) = -z^k(C + A)(z).$$

Using Lemma 3.3 we obtain the reciprocal polynomial

$$(\tilde{\phi}_{2k})_*(z) = z^{-k}(C - A)(z),$$

leading to $\widetilde{C} \equiv 0$. But this means that we can reduce our RST to a LST and the result follows. □

Proof of Theorem 1. Let $F = \mathcal{F}[F_{\mathcal{L}}]$ be the $\mathcal{C}$-function obtained from (2.2) after a generic LST with Hermitian Laurent polynomial coefficients. Hence, we can apply to F the finite composition of RT given in (5.7). Thus, we get a new $\mathcal{C}$-function as a result of the composition

$$\mathcal{F}_{\mathcal{L}_{r(k)}}\left(\mathcal{F}[F_{\mathcal{L}}(z)]\right)(z) = \left(\frac{\frac{1}{2}(\Re p)AF_{\mathcal{L}} + \frac{1}{2}(\Re p)B(z) + (P - P_*)D}{D}\right)(z)$$

with some polynomials such that the zeros of the polynomial p can be chosen as the parameters of the spectral transformation. Hence, we can always choose $\Re p = \tau D$, where τ is a constant. Thus we obtain

$$\mathcal{F}_{\mathcal{L}_{r(k)}}[F](z) = \left(\tau AF_{\mathcal{L}} + \tau B + P - P_*\right)(z). \tag{6.1}$$

This transformation is reduced to the case considered in (5.4). Therefore, from Theorem 5.3 the $\mathcal{C}$-function $\mathcal{F}_{\mathcal{L}_{r(k)}}[F]$ is obtained from (2.2) by means of a finite composition of RT.

On the other hand, from (6.1) and using a finite composition of IT, we get the $\mathcal{C}$-function

$$F = \mathcal{F}_{\mathcal{L}_{r(-k)}}\left(\mathcal{F}_{\mathcal{L}_{r(k)}}\left(\mathcal{F}[F_{\mathcal{L}}]\right)\right),$$

and so our statement holds. □

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