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THE HIGHEST ORDER SUPERCONVERGENCE FOR BI-k DEGREE RECTANGULAR ELEMENTS AT NODES: A PROOF OF $2k$-CONJECTURE

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ABSTRACT. We proved the highest order superconvergence $(u - u_h)(z) = O(h^{2k})|\ln h|$ at nodes z, based on Element Orthogonality Analysis (EOA), correction techniques and tensor product, where $u \in W^{2k,\infty}(\Omega)$ is the solution for the Poisson equation $-\Delta u = f$ in a rectangle Ω, $u = 0$ on Γ, and $u_h \in S_0^h$ is its bi-k degree rectangular finite element approximation. This conclusion is also verified by numerical experiments for $k = 4, 5$.

1. INTRODUCTION

We consider the model boundary value problem to find $u \in H_0^1$ such that

$$(1) \qquad A(u, v) = (f, v), \quad v \in H_0^1 = \{v \ : \ v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma\},$$

where Ω is a rectangular domain with boundary Γ (see Remark 4) and the bilinear form

$$A(u, v) = \int_\Omega (u_x v_x + u_y v_y) dxdy,$$

is bounded and H_0^1-coercive, i.e.,

$$|A(u, v)| \le C||u||_{1,p}||v||_{1,p'}, \quad A(v, v) \ge \nu||v||_{1,2},\ u, v \in H_0^1, \quad 1/p + 1/p' = 1.$$

Denote by $W^{k,p}(\Omega)$ the Sobolev space with the norm

$$||u||_{k,p,\Omega} = \{\sum_{i+j\le k} \int_\Omega |D_x^i D_y^j u(x, y)|^p dxdy\}^{1/p},$$

where the subindex Ω is often omitted, if there is no confusion. When $p = 2$, we denote simply $H^k(\Omega) = W^{k,2}(\Omega)$ and $||.||_k = ||.||_{k,2}$.

Assume that the domain Ω is subdivided into a finite number of the rectangular elements $K \in J^h$, $\Omega = \bigcup_{K\in J^h} K$ and the mesh is quasi-uniform. Denote the bi-k degree finite element space

$$S_0^h = S_0^h(\Omega) = \{v : \quad v \in C(\Omega), \quad v|_K \in P_k(x) \otimes P_k(y), \quad v = 0 \text{ on } \partial\Omega\} \subset H_0^1,$$

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where $P_k(x)$ is the space of polynomials of degree $\leq k$ and $\otimes$ is the tensor product operator (often omitted below). Define the finite element solution $u_h \in S_0^h$ satisfying

$$A(u_h, v) = (f, v), \quad v \in S_0^h. \tag{2}$$

By (1) and (2), we have the following orthogonal relation:

$$A(u - u_h, v) = 0, \quad v \in S_0^h. \tag{3}$$

Under certain conditions the optimal error estimates

$$||u - u_h||_{0,p,\Omega} \leq Ch^{k+1}||u||_{k+1,p,\Omega}, \quad 2 \leq p < \infty,$$
$$||u - u_h||_{0,\infty,\Omega} \leq Ch^{k+1}|\ln h|^{n(k)}\ ||u||_{k+1,\infty,\Omega}, \quad p = \infty,$$

are proved (see [5, 17, 23, 25]), where $n(1) = 1$ and $n(k) = 0$ if $k > 1$.

However, it is found that the finite element solution or its derivatives at some specific points have possibly higher order accuracy, which is called superconvergence by J. Douglas. In 1973, Douglas and Dupont [14, 15] proved that for the two-point boundary value problem the k-degree finite element solution u_h at each node x_j is superconvergent, i.e.,

$$(u - u_h)(x_j) = O(h^{2k})||u||_{k+1}, \quad k \geq 2, \tag{4}$$

where only $u \in H^{k+1}$ is required. The estimate (4) is optimal and sharp, which is verified by the example in [15].

Soon, Douglas, Dupont, and Wheeler [16], considered bi-k degree rectangular finite element solution u_h for Poisson equation (1) in a rectangle Ω and proved superconvergence estimates $||u_h - u_I||_1 \leq Ch^{k+2}||u||_{k+3}, \ \ k \geq 3$, and $(u - u_h)(z) = O(h^{k+2})$ at nodes z, where the comparison function u_I is constructed by quasi-projection and tensor product. Actually, this paper [16] contains much richer ideas. Later on, Chen [7] proved that bi-k degree rectangular finite element is superconvergent at $k+1$-order Lobatto points by the element analysis method, in particular, at nodes z (in L^2-norm in [7]),

$$|(u - u_h)(z)| \leq Ch^{k+l}|\ln h|\ ||u||_{k+l,\infty}, \quad l = k = 2, 3, \quad l = 3 \ \ if\ k \geq 4,$$

which is further generalized to the case of variable coefficients [14]. However, when $l = k \geq 4$ the study meets the essential difficulty (because the regularized Green's function g^z has the singularity and the optimal regularity is $||g^z||_{2,1} \leq C|\ln h|$, see Section 3).

On the other hand, Bramble and Schatz [3] and Thomee [26] used the kernel function $K_h(x, y)$ with small support to construct the convolution $K_h^k * u_h$ (a post-processing or recovery) of finite element solution u_h and obtained the highest order superconvergence for both function values and derivatives

$$||D^l u - K_h^l * u_h||_{0,\infty,\Omega_0} \leq Ch^{2k}||u||_{2k,\infty,\Omega_1} + C||u - u_h||_{-s,p,\Omega}, \quad s > 0, \quad l = 0, 1,$$

where $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, the mesh is uniform in Ω_1, and the negative norm $||u - u_h||_{-s,p,\Omega}$ is of the highest order $O(h^{2k})$ under some conditions. Whether the rectangular finite element solution naturally has the highest order superconvergence at nodes is an attractive problem, which is one of six open problems proposed in [13, pp. 593-599]. Recently, this was emphasized by Zhou and Lin [33], as a conjecture. All of this research can be concluded as the following $2k$-conjecture.

2k-Conjecture of Superconvergence. Under some conditions, bi-k degree rectangular element $u_h \in S^h$ for (1) has the highest order superconvergence at nodes z:

$$(u - u_h)(z) = O(h^{2k}), \quad k \geq 2.$$

The question is so difficult that no definite answer was given for a long time.

We recall that up to now five main superconvergence methods have been developed: quasi-projection method [16], local averaging method [3, 26], element analysis method [6, 7, 8, 13, 22, 31, 32], computer-based method [1, 2] and local symmetrical theory [24, 29, 30]. In the element (orthogonality) analysis method (EOA), a new technique of orthogonality correction [9, 11, 12] is introduced; also see [4, 20, 21]. This paper will give a rigorous proof of this conjecture based on EOA, orthogonality correction and tensor product. Our main result is as follows.

Theorem 1. *Assume that the rectangular mesh in a rectangle* Ω *is quasi-uniform, and problem* (1) *has a solution*

$$u \in W^{2k,p}(\Omega) \cap H_0^1(\Omega) \quad 2 \leq p \leq \infty, \quad k \geq 2. \tag{5}$$

Then bi-k degree rectangular element solution $u_h \in S^h$ *on the nodal set* T_h *has the highest order superconvergence in the discrete* L^p*-norm*

$$|||u - u_h|||_{0,p,T_h} \equiv \{\sum_{z \in T_h} |(u - u_h)(z)|^p h^2\}^{1/p} \leq Ch^{2k}||u||_{2k,p,\Omega}, \ 2 \leq p < \infty, \tag{6}$$

and the highest order pointwise superconvergence

$$\max_{z \in T_h} |(u - u_h)(z)| \leq Ch^{2k}|\ln h| \ ||u||_{2k,\infty,\Omega}, \quad p = \infty. \tag{7}$$

This result is quite similar to (4) in the one-dimensional case, where the slowly increasing factor $|\ln h|$ is not essential and perhaps can be removed (see Remark 3). This conclusion is also verified by our numerical experiments for $k = 4, 5$. The proof shows that Theorem 1 is valid in the higher dimensional case and the superconvergence conclusion may be generalized to non-rectangular domains (see Remark 4). Moreover, whether the regularity requirement could be decreased to $W^{k+1,p}(\Omega)$ as in a one-dimensional case is another difficult question.

In recent years, The Third and Fourth International Conferences on Superconvergence and A Posteriori Error Estimates, held in Changsha, China in 2004, and Prague, Czech in 2008, respectively, and the corresponding special issues [34, 35] were published. Thus, some new superconvergence results on tetrahedral quadratic finite elements and triangular finite elements [36]–[40] were obtained.

Remark 1. Under some conditions the solution of (1) in a non-smooth domain has higher regularity. For example, in a rectangle Ω, Fufaev [18] and Volkov [28] proved that if $f \in C^{1,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$ and $f = 0$ at angular nodes, then $u \in C^{3,\alpha}(\bar{\Omega})$; if $f \in C^{3,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$ and $f = f_{xx} = f_{yy} = 0$ at angular nodes, then $u \in C^{5,\alpha}(\bar{\Omega})$.

2. M-type projection in a one-dimensional element

Subdividing $J = (0, 1)$ by the nodes $x_0 = 0 < x_1 < x_2 < \cdots < x_N = 1$, denote the element $K_i = (x_i, x_{i+1})$, its midpoint $\bar{x}_i = (x_i + x_{i+1})/2$ and the half-step length $h_i = (x_{i+1} - x_i)/2$. Using a linear transform $x = \bar{x}_i + h_i s$, $s \in E = (-1, 1)$ in K_i, the function $f(x) = f(\bar{x}_i + h_i s)$ is still denoted by $f(s)$. Note that $D_s^j f = h_i^j D_x^j f = O(h^j)$.

Introduce Legendre's polynomials in E,

(8) $l_0 = 1,\ l_1 = s,\ l_2 = (3s^2-1)/2,\ l_3 = (5s^3-3s)/2, ...,\ l_n = \gamma_n D_s^n (s^2-1)^n,$

where $\gamma_n = 1/(2n)!!$, the inner product $(l_i, l_j) = 0, i \neq j$, and $c_{j+1} = (l_j, l_j) = 2/(2j+1)$.

Integrating $l_j(s)$ leads to M-type polynomials [6],

$$M_0 = 1,\ M_1 = s,\ M_2 = (s^2-1)/2,\ M_3 = (s^3-s)/2, ...,\ M_{n+1}(s) = \gamma_n D_s^{n-1}(s^2-1)^n, \tag{9}$$

which are quasi-orthogonal: $d_{ij} = (M_i, M_j) \neq 0$ for $j - i = 0, \pm 2$, otherwise $d_{ij} = 0$. So $M_i(s)$ is not orthogonal to at most three bases M_{i-2}, M_i, M_{i+2}. Obviously, $M_n(\pm 1) = 0$ for $n \geq 2$.

To construct an M-type projection of $u(s)$, expanding $u_s(s)$ as an L-type series,

$$u_s(s) = \sum_{j=0}^{\infty} b_{j+1} l_j(s), \quad b_{j+1} = j_1(u_s, l_j),\ j_1 = j + 1/2,$$

and integrating in s, we get an M-type series [6, 7] and its partial sum

$$u(s) = \sum_{j=0}^{\infty} b_j M_j(s), \quad u_I = Q^s u = \sum_{j=0}^{k} b_j M_j(s), \tag{10}$$

where $b_0 = (u(1) + u(-1))/2, b_1 = (u_s, l_0)/2 = (u(1) - u(-1))/2$. Then $u_I(\pm 1) = u(\pm 1)$ at two endpoints, which guarantees that the u_I constructed in each element is continuous in J. Integrating by parts, the coefficient b_{j+1} has the following expression and estimate:

$$b_{j+1} = C_j \int_E D_s^{i+1} u(s) D_s^{j-i}(s^2-1)^j ds, \quad i \leq j, \quad C_j = (-1)^i j_1 \gamma_j, \tag{11}$$

$$|b_{j+1}| \leq C h^{i+1-1/p} ||u||_{i+1,p,K}, \quad i \leq j. \tag{12}$$

Note that if u is a polynomial of degree k, then all coefficients $b_j = 0,\ j > k$. This is the basis to use the Bramble-Hilbert lemma.

It is easy to see that the remainder

$$R^k = E^s u = u - Q^s u = \sum_{j=k+1}^{\infty} b_j M_j(s) = \sum_{j=1}^{\infty} \gamma_{k+j-1} b_{k+j} D_s^{k+j-2}(s^2-1)^{k+j-1} \tag{13}$$

has the orthogonal properties

$$R^k(\pm 1) = 0; \quad D_s R^k \perp P_{k-1}(s),\ k \geq 1; \quad R^k \perp P_{k-2}(s),\ k \geq 2, \tag{14}$$

and error estimates (by the Bramble-Hilbert lemma)

$$||R^k||_{0,q,K} \leq C h^{k+1+1/q-1/p} ||u||_{k+1,p,K}, \quad 1 \leq p, q \leq \infty.$$

Defining an integral operator in $K = (-h, h)$ gives us

$$S_x w = \int_{-h}^{x} w(x) dx = h S_s w, \quad S_s w = \int_{-1}^{s} w(s) ds,$$

so, obviously,

$$||S_x w||_{L^p(K)} \leq (\int_K | \int_K |w(x)| dx|^p dx)^{1/p} \leq 2h ||w||_{L^p(K)}.$$

When $k \geq 2$, $1 \leq i \leq k-1$, we have

$$S_s^i R^k = \sum_{j=1}^{\infty} \gamma_{k+j-1} b_{k+j} D_s^{k+j-2-i} (s^2-1)^{k+j-1}, \quad S_s^i R(\pm 1) = 0,$$

and $S_s^i R^k \perp P_{k-2-i}(s)$, $1 \leq i \leq k-2$, but $S_s^{k-1} R^k$ has no orthogonality.

The projection $u_I = Q^s u$ and the orthogonality of $R^k = u - Q_s u$ play an important role in studying superconvergence [8, 11, 12, 13].

Remark 2. As an example, we consider the one-dimensional problem $Lu = -u_{xx} + u = f, u(0) = u(1) = 0$ and its k-degree finite element $u_h \in S_0^h$, and the error $e = u - u_h = R^k - \theta$ satisfies $A(e, v) = (e_x, v_x) + (e, v) = 0,\ v \in S_0^h$. As $(u - Q_s u)_x \perp v_x$, $\theta = u_h - u_I$ satisfies $A(\theta, v) = A(R^k, v) = (R^k, v)$, and $||\theta||_1 \leq C||R^k|| \leq Ch^{k+1}||u||_{k+1}$. Further, integrating by parts for $k \geq 2$, there is the higher order estimate

$$A(\theta, v) = (-1)^{k-1}(S_x^{k-1} R^k, D_x^{k-1} v) \leq Ch^{2k}||u||_{k+1,J}(\sum_{K \in J} ||v||_{k-1,K}^2)^{1/2}, \quad v \in S_0^h.$$

On the other hand, denote $J_0 = (0, y), J_1 = (y, 1),\ 0 < y < 1$ and construct the Green's function $g^y(x) \in H_0^1(J) \cup C^k(J_0) \cup C^k(J_1)$ (i.e., continuous and piecewise smooth) such that

$$Lg^y(x) = \delta(x - y), \quad 0 < x, y < 1, \quad w(y) = A(w, g^y),\ w \in H_0^1(J).$$

If taking the node $y = x_j$ and $g_I \in S^h$ of the M-type projection of g^y, we have

$$||g - g_I||_{0,J_i} \leq Ch^{k+1}||g||_{k+1,J_i} \leq Ch^{k+1}, \quad \sum_{K \in J_i} ||g_I||_{k,K}^2 \leq C, \quad i = 0, 1,$$

and (noticing $(g - g_I)_x \perp \theta_x$)

$$\theta(x_j) = A(\theta, g) = A(\theta, g_I) + A(\theta, g - g_I) = (R^k, g_I) + (\theta, g - g_I)$$

$$= O(h^{2k})||u||_{k+1}(\sum_{K \in J_0} ||g_I||_{k-1,K}^2 + \sum_{K \in J_1} ||g_I||_{k-1,K}^2 + C)^{1/2} = O(h^{2k})||u||_{k+1}.$$

Finally, we derive $(u - u_h)(x_j) = O(h^{2k})$ at each node x_j (as obtained in [14, 15]). Unfortunately, the argument is invalid for $k > 3$ in the multi-dimensional case, because the corresponding regularized Green's function $g^z \in H_0^1(\Omega)$ has the singularity and the optimal regularity $||g^z||_{2,1} \leq C|\ln h|$. To prove the $2k$-conjecture we have to propose a new correction technique in the element band.

3. Element Orthogonality Analysis (EOA)

Assuming that $u_I \in S^h$ is a comparison function to be defined, we split the error $u - u_h = R - \theta$, $R = u - u_I$, $\theta = u_h - u_I$ and rewrite (3) as

$$A(\theta, v) = A(R, v), \quad v \in S_0^h. \tag{15}$$

The idea behind EOA is to construct u_I super close to u_h and prove the basic estimate

$$A(\theta, v) = A(R, v) = O(h^{k+1+\alpha})||v||_{2,p',\Omega}^*, \quad v \in S^h, \quad \alpha > 0, \quad p' = p/(p-1), \tag{16}$$

where $||v||_{2,p,\Omega}^* = (\sum_K ||v||_{2,p,K}^p)^{1/p}$ is the mesh-norm.

Subdivide a square $\Omega = J \times J$ into $N \times N$ rectangular elements $K_{il} = (x_i, x_{i+1}) \times (y_l, y_{l+1})$. For simplicity of notation, we denote the half-step lengths $h_i = k_l =$

$h = 1/(2N)$ in the x, y-directions. Then K_{il} is transformed to a reference element $E = (-1,1)\times(-1,1)$ by a linear transform $x = \bar{x}_i + hs, y = \bar{y}_l + ht$, and the function $u(x,y)$ becomes $u(s,t)$. Obviously, $D_s^p D_t^q u(s,t) = h^{p+q} D_x^p D_y^q u(x,y) = O(h^{p+q})$.

Constructing k-degree M-type projections Q^x, Q^y and their tensor product $u_I = Q^y Q^x u \in S^h$ in K_{il}, the error $R = u - u_I$ has the following important decomposition [16]:

$$R = u - u_I = (I - Q^x)u + (I - Q^y)u - (I - Q^y)(I - Q^x)u = E^x u + E^y u - E^y E^x u.$$

For example, in an element-band $B = J \times (y_l, y_{l+1})$ we have the remainder

$$E^y u = \sum_{j=1}^{\infty} b_{k+j}(x) M_{k+j}(t), \quad x \in J, \quad y = \bar{y}_l + ht,$$

where

$$b_{k+j}(x) = \gamma_{k+j-1} h^{k+1} \int_{-1}^{1} D_y^{k+1} u(x,y) D_t^{j-1} (t^2-1)^{k+j-1} dt, \quad j \geq 1. \tag{17}$$

$$||D_x^{k-1} b_{k+j}(x)||_{0,B} \leq C h^{k+1} ||D_x^{k-1} D_y^{k+1} u||_{0,B} \leq C h^{k+1} ||u||_{2k,B}.$$

As mentioned above, there are nice orthogonal properties:

$$(E^x u)_x \perp P_{k-1}(x),\ E^x u \perp P_{k-2}(x), \quad (E^y u)_y \perp P_{k-1}(y), \quad E^y u \perp P_{k-2}(y).$$

In particular, $(E^x u)_x \perp v_x, (E^y u)_y \perp v_y, v \in S_0^h$. As a result only two terms in $A(R,v)$ remain:

$$A(\theta, v) = A(R, v) = \sum_K \int_K (E^y u_x v_x + E^x u_y v_y) dxdy,\ v \in S^h. \tag{18}$$

Obviously, $(E^y u)_x = E^y u_x$ is a smooth function in x. By Green's formula,

$$\sum_K \int_K E^y u_x v_x dxdy = \sum_K \{ \oint_\gamma E^y u_x v dy - \int_K E^y u_{xx} v dxdy \},$$

in which all inner linear integrals cancel each other out, while only the integral along Γ remains and disappears as $v = 0$ on Γ (this is the simplest combination cancellation). We treat $E^x u_y v_y$ in the same way. So we have [7]

$$A(\theta, v) = A(R, v) = -\sum_K \int_K (E^y u_{xx} v + E^x u_{yy} v) dxdy. \tag{19}$$

When $k \geq 3$, integrating by parts twice, we have [7]

$$A(\theta, v) = A(R, v) = -\sum_K \int_K (S_y^2 E^y u_{xx} v_{yy} + S_x^2 E^x u_{yy} v_{xx}) dxdy$$

$$\leq C h^{k+3} \max_{(x,y)\in\Omega} |D^{k+3} u(x,y)| \sum_K \int_K |D^2 v| dxdy \leq C h^{k+3} ||u||_{k+3,\infty} ||v||_{2,1}^*.$$

Using the discrete Green's function $g_h \in S^h$ and $||g_h||_{2,1}^* \leq C|\ln h|$ (see Section 6), we get

$$\max |\theta(z)| \leq C h^{k+3} |\ln h|\ ||u||_{k+3,\infty}, \quad k \geq 3.$$

As $R(z) = 0$, $e(z) = -\theta(z)$ at nodes z, then Theorem 1 holds for $k = 3$.

This is the best result up to now. For a long time, the remainders $S_y^2 E^y u_{xx}$, $S_x^2 E^x u_{yy} = O(h^{k+3})$ could not be cancelled. Although $S_y^2 E^y u_{xx} \perp P_{k-4}(y)$ for $k \geq 4$, it cannot be integrated by parts again, because the regularized Green's function g^z does not have higher regularity. This is the essential difficulty mentioned in Section 1.

We found that the conjecture is still valid for $k = 4$ using the correction technique in an element. Recently, $k = 4$ was also discussed in [33]. This paper further finds that constructing the correction function in an element-band B (rather than in an element) is more effective.

The basic idea of orthogonality correction [9, 10, 11, 12] is to construct $W \in S^h$ such that $W = 0$ at nodes and

$$A(\theta - W, v) = A(R - W, v) = O(h^{k+1+\alpha})||v||^*_{2,p',\Omega}, \quad v \in S^h, \quad \alpha > 0, \tag{20}$$

is of the higher order. Then by using the discrete Green's function we get

$$\max_{(x,y)\in\Omega} |(\theta - W)(x, y)| \le Ch^{k+1+\alpha}|\ln h|.$$

Coming back to each element, we have superconvergence at nodes z:

$$(u - u_h)(z) = (R - W)(z) - (\theta - W)(z) = 0 + O(h^{k+1+\alpha})|\ln h|.$$

To prove the conjecture, it is necessary to construct the optimal correction function $W \in S^h_0(\Omega)$ in order to get (20) with $\alpha = k - 1$ and guarantee $W(z) = 0$ at all nodes z. This is an elaborate work.

4. An expansion with small parameter in an element-band

Following V. Thomee [27], we define a subspace of Sobolev space $W^{L,p}(\Omega)$ by

$$\dot{W}^{L,p}(\Omega) = \{u : \quad u \in W^{L,p}(\Omega), \quad \Delta^m u = 0 \text{ on } \Gamma, \quad m < L/2\}, \quad 2 \le p \le \infty.$$

In this section we shall prove the following key result under the stronger constraint (21), which means that the $2k$-conjecture is valid.

Theorem 2. *Assume that the solution of* (1) *satisfies*

$$u \in \dot{W}^{k-1,p}(\Omega) \cap W^{2k,p}(\Omega), \quad 2 \le p \le \infty, \quad \Omega = J \times J. \tag{21}$$

Then a corresponding correction function $w_h \in S^h$ can be constructed such that $w_h = 0$ at all nodes z and

$$|A(\theta - w_h, v)| \le Ch^{2k}||u||_{2k,p}||v||^*_{2,p'}, \ v \in S^h_0, \quad 1/p + 1/p' = 1, \tag{22}$$

where $||v||^*_{2,p'}$ is the mesh-norm.

Proof. Under the assumption (21), define a specific index $m = [(k-2)/2]$, which is the maximum integer such that $m \le (k-2)/2$. So $k = 2m + 2$ or $k = 2m + 3$.

As $u = 0$ on Γ, the tangent derivatives $D^l_y u = 0, \ 0 \le l < 2k$ on the boundary Γ_x (see Figure 1). If $\Delta u = 0$ on Γ, we have $D^2_x u = -D^2_y u = 0$ and $D^l_y D^2_x u = 0$ on Γ_x, for $0 \le l < 2k - 2$. In general, we have

$$D^l_y D^{2j}_x u = 0 \quad \text{on} \quad \Gamma_x, \quad \text{for} \quad 2j \le 2m < k - 1, \quad l + 2j < 2k.$$

Similarly, we also have $D^l_x D^{2j}_y u = 0$ on Γ_y, for $2j \le 2m < k - 1, \ l + 2j < 2k$.

For simplicity, we first consider one term $R_1 = E^y u$ in (19) (similarly for $R_2 = E^x u$). Each element-band $B = J \times (y_l, y_{l+1})$ is transformed to $B' = J \times E$ by $y = \bar{y}_l + ht$, then

$$A(\theta, v) \equiv \sum_{B'} h \int_{B'} (\theta_x v_x + h^{-2}\theta_t v_t)dxdt = A(R_1, v), \ v \in S^h_0, \tag{23}$$

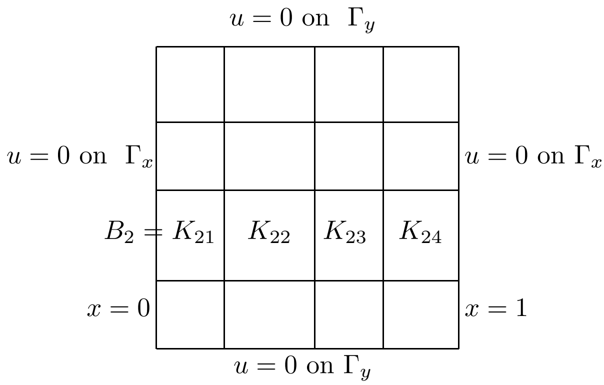

FIGURE 1. The element-band B_2 over 4×4 meshes, $J = (0, 1)$.

where $A(R_1, v) = \sum_{B'} h \int_{B'} E^y u_{xx} v dxdt$,

$$R_1 = E^y u = \sum_{j=k+1}^{\infty} b_j(x) M_j(t) \perp P_{k-2}(t), \quad y = \bar{y}_l + ht, \quad t \in E.$$

Note that $M_j(t) \perp P_k(t)$ for $j > k+2$; actually, in addition to the two terms $b_{k+1}(x)$ and $b_{k+2}(x)$, all other terms in $A(R_1, v)$ disappear. Whereas the coefficients

(24)
$$b_{k+i}(x) = \nu_{k+i} h^{k+1} \int_{-1}^{1} D_y^{k+1} u(x, y) D_t^{i-1} (t^2 - 1)^{k+i-1} dt \in \dot{W}^{k-1,p}(J), \qquad i = 1, 2,$$

are smooth functions in $x \in J = (0, 1)$, and by differentiating,

(25)
$$|D_x^{2j} b_{k+i}(x)| \leq C h^{k+1} \int_{-1}^{1} |D_x^{2j} D_y^{k+1} u(x, \bar{y}_l + ht)| dt, \qquad 2j \leq 2m < k-1, \quad i = 1, 2,$$
$$||D_x^{2j} b_{k+i}(x)||_{0,p,B} \leq C h^{k+1} ||u||_{2k,p,B}, \quad 2j \leq 2m < k-1, \quad i = 1, 2.$$

We shall concretely solve the problem with the low order term $b(x) = b_{k+1}(x)$:

$$(26) \quad A(\theta, v) = A(R_1, v) = -\sum_B h \int_{B'} b_{xx}(x) M_{k+1}(t) v dxdt, \quad v \in S^h, \quad B' = J \times E.$$

The main difficulty in constructing the correction function is that the small parameter h^2 contained in $A(w, v)$ makes the problem singularly perturbed. However, from the condition (21), $D_x^{2j} b(x) = 0$ on Γ_x for $2j \leq 2m < k-1$, that makes it possible to treat the small parameter. For this purpose we shall construct the correction function $w_h \in S_0^h(B')$ in each B' as

(27)
$$w_h = \sum_{j=1}^{m} W_j(x, t), \quad W_j = h^{2j} Q_h G_j(x) F_j(t), \quad F_j(t) = \sum_{i=2}^{k} a_{ji} M_i(t), \quad F_j(\pm 1) = 0,$$

where $G_j = D_x^{2j} b(x)$, Q_h is the one-dimensional M-type projection operator defined in Section 2 and a_{ji} are the constants to be defined. We define $F_j(t)$ one-by-one as follows.

First, when $k \geq 4$, rewriting

$$M_{k+1}(t) = D_t^2\Phi_1(t) \perp P_{k-2}(t), \quad \Phi_1(t) = cD_t^{k-3}(t^2-1)^k \perp P_{k-4}(t),$$

we construct the first correction function $W_1 = h^2 Q_h G_1(x) F_1(t)$. Taking $v = \xi(x)M_i(t), i = 0, 1, ..., k$, in (26) and integrating by parts, we can calculate the residue

$$\begin{aligned}
A(\theta - W_1, v) &= -A(W_1, v) + A(R_1, v)\\
&= h\sum_B \int_{B'} \{-h^2(Q_hG_1)_x\xi_x F_1 M_i - Q_hG_1\xi F_{1t} l_{i-1} - G_1\xi\Phi_{1tt}M_i\}dxdt\\
&= h\sum \int_{B'} \{-h^2 G_{1x}\xi_x F_1 M_i - Q_hG_1\xi F_{1t} l_{i-1} + G_1\xi\Phi_{1t} l_{i-1}\}dxdt\\
&= h\sum \int_{B'} \{h^2 D_x^2 G_1\xi F_1 M_i - Q_hG_1\xi(F_{1t} - \Phi_{1t}) l_{i-1}(t)dxdt + r_1,
\end{aligned}$$

where $(Q_hG_1 - G_1)_x \perp \xi_x$ is used and the remainder follows:

$$\begin{aligned}
r_1 &= -\sum_B \int_B (G_1 - Q_hG_1)\Phi_1 v_{yy} h^2 dxdy = O(h^2)\sum_B ||G_1 - Q_hG_1||_{0,p,B}||v||_{2,p',B}\\
&= \sum_B O(h^{k-1})||G_1||_{k-3,p,B}||v||_{2,p',B} = O(h^{2k})||u||_{2k,p,\Omega}||v||^*_{2,p',\Omega} = r_h.
\end{aligned}$$

To define F_1, an important idea is to project Φ_1 to $F_1(t) = \sum_{j=0}^k a_{1j}M_j(t)$ such that

$$\int_E (F_{1t} - \Phi_{1t}) l_{i-1} dt = 0, \quad i = 1, 2, ..., k.$$

Using $\Phi_{1t} \perp P_{k-3}$, we can determine that

$$F_1 = a_{1,k-1}M_{k-1}(t) \perp P_{k-4}(t),$$

where the orthogonality of $F_1(t)$ decreases the degree by 2. Therefore, we have

$$A_B(\theta - W_1, v) = h\sum_B \int_{B'} h^2 D_x^2 G_1(x) F_1(t)\xi(x) M_i(t) dxdt + r_1, \quad v \in S^h,$$

which still has the form similar to (26). This treatment will be repeatedly used later.

When $k \geq 6$, rewriting $F_1 = a_{1,k-1}M_{k-1}(t) = D_t^2\Phi_2(t)$, we construct $W_2 = h^4 Q_h G_2(x) F_2(t)$, $G_2 = D_x^2 G_1$ and calculate the residue

$$\begin{aligned}
&A(\theta - W_1 - W_2, v) = -A(W_2, v) + h\sum_B \int_{B'} h^2 G_2(x)\xi(x) D_t^2\Phi_2(t) M_i(t) dxdt + r_h\\
&= h\sum \int_B \{-h^4(Q_hG_2)_x\xi_x F_2 M_i - h^2 Q_hG_2\xi F_{2t} l_{i-1} - h^2 G_2\xi\Phi_{2t} l_{i-1}\}dxdt + r_h\\
&= h\sum \int_{B'} \{h^4 D_x^2 G_2\xi F_2 M_i - h^2 Q_hG_2\xi(F_{2t} + \Phi_{2t}) l_{i-1}\}dxdt + r_2 + r_h,
\end{aligned}$$

where the remainder

$$\begin{aligned} r_2 &= h\sum_B \int_B h^2(Q_hG_2 - G_2)\Phi_2 v_{yy} h^2 dxdy \\ &= O(h^4)\sum_B ||G_2 - Q_hG_2||_{0,p,B}||v||_{2,p',B} \\ &= \sum_B O(h^{k-1})||G_2||_{k-5,p,B}||v||_{2,p',B} = O(h^{2k})||u||_{2k,p,\Omega}||v||^*_{2,p',\Omega} = r_h. \end{aligned}$$

We define $F_2(t) = \sum_{j=0}^k a_{2j}M_j(t)$ such that

$$\int_E (\Phi_{2t} + F_{2t})l_{i-1}dt = 0, \quad i = 1, 2, ..., k.$$

Using $\Phi_{2t} \perp P_{k-5}$, we can determine

$$F_2 = a_{2,k-1}M_{k-1}(t) + a_{2,k-3}M_{k-3}(t) \perp P_{k-6}(t),$$

where the orthogonality of $F_2(t)$ decreases the degree by 2 once again. So we get

$$A(\theta - W_1 - W_2, v) = h\sum_B \int_{B'} h^4 D_x^2 G_2 \xi F_2 M_i dxdt + r_h, v \in S_0^h(\Omega).$$

In general, when $2m+2 \le k$, we can define the k-degree polynomials F_j one-by-one $(1 \le j \le m)$,

$$F_j(t) = a_{j1}M_{k-1}(t) + a_{j3}M_{k-3}(t) + \cdots + a_{j,2j-1}M_{k-2j+1}(t) \perp P_{k-2j-2}(t),$$
$$F_j(\pm 1) = 0,$$

and we get w_h in (27) such that the corresponding residue could be rewritten as

$$A(\theta - w_h, v) = h^{2m}\sum_B \int_B (G_m)_x F_m(t) v_x dxdy + r_h, \quad v \in S_0^h(\Omega). \tag{28}$$

When $k = 2m+2$ is even, the last polynomial

$$F_m(t) = a_{m1}M_{k-1}(t) + a_{m3}M_{k-3}(t) + \cdots + a_{m,k-3}M_3(t) \perp P_0, \quad F_n(\pm 1) = 0,$$

is odd. Integrating by parts, we get

$$\begin{aligned} A(\theta - w_h, v) &= -h^{2m+1}\sum_B \int_B D_x^{2m+1} b(x) S_t F_m(t) v_{xy} dxdy + r_h \\ &= O(h^{k-1})h^{k+1}\sum_B \int_B |D^{2k}u|\ |v_{xy}|dxdy \\ &\quad + r_h = r_h, k = 2m+2. \end{aligned} \tag{29}$$

When $k = 2m+3$ is odd, the last polynomial

$$F_m(t) = a_{m1}M_{k-1}(t) + a_{m3}M_{k-3} + \cdots + a_{m,2m-1}M_4(t) \perp P_1(t), \quad F_m(\pm 1) = 0,$$

is even. Integrating by parts twice, we have

$$
\begin{aligned}
A(\theta - w_h, v) &= -h^{2m}\sum_B \int_B (G_m)_{xx}F_m(t)v dxdy + r_h \\
&= -h^{2m+2}\sum_B \int_B D_x^{2m+2}b(x)S_t^2F_m(t)v_{yy}dxdy + r_h \qquad (30)\\
&= O(h^{k-1})\sum_B \int_B |D_x^{k-1}b(x)|\,|v_{yy}|dxdy + r_h = r_h, k = 2m+3.
\end{aligned}
$$

Therefore, when $R = R_1 = b_{k+1}(x)M_{k+1}(t)$ for any $k \geq 4$, the desired estimate (22) can be derived by (29) and (30).

We can also discuss $R_1 = b_{k+2}(x)M_{k+2}(t) \in \dot{W}^{k-1,p}(J_x)$ in a similar way. This time the only difference is that $W_j = h^{2j}Q_h(D_x^{2j}b_{k+2})F_j(t) \in S_0^h(B')$ with $F_j(t) = a_{j0}M_k(t) + a_{j2}M_{k-2} + \cdots$.

Now we should study another remainder, $R_2 = E^x u$ in the corresponding band $B^* = (x_i, x_{i+1}) \times J_y$ and construct $w_h \in S_0^h(B^*)$ to satisfy (22). Its analysis is quite parallel to that of $R_1 = E^y u$.

Finally, summarizing these analyses and estimates, Theorem 2 is proved. □

5. The study of the solution $u \in W^{2k,p}(\Omega) \cap H_0^1$

In this section we shall remove the stronger constraints (21) on u in Theorem 2.

Theorem 3. *Assume that the solution of* (1) *satisfies*

$$u \in W^{2k,p}(\Omega) \cap H_0^1(\Omega), \qquad (31)$$

then the correction function $w_h + W_h \in S_0^h$ can be constructed such that $w_h + W_h = 0$ at all nodes and

$$A(\theta - w_h - W_h, v) = r_h, \quad r_h = O(h^{2k})||u||_{2k,p}||v||_{2,p'}^*, \quad v \in S_0^h, \qquad (32)$$

still holds.

Proof. Under the condition (31), the function $b(x) = b_{k+1}(x) = 0$ on Γ_x, but in general the derivatives $D_x^{2j}b(x) \neq 0$ on Γ_x. So we shall construct a k-degree polynomial $G(x)$ in a whole interval $J_x = (0,1)$, $G(0) = G(1) = 0$, such that

$$F(x) = b(x) - G(x), \quad D_x^l F(0) = D_x^l F(1) = 0, \quad l = 2, 4, ..., 2m < k-1. \qquad (33)$$

Thus the problem (18) is decomposed to

$$A(\theta, v) = \sum_B h \int_{B'} F_x(x)M_{k+1}(t)v_x dxdt + \sum_B h \int_{B'} G_x(x)M_{k+1}(t)v_x dxdt. \qquad (34)$$

Note that $F(x) \in \dot{W}^{k-1,p}(J_x)$, by Theorem 2 we can construct the corresponding $w_h \in S_0^h(B)$ such that

$$A(\theta - w_h, v) = \sum_B h \int_{B'} G_x(x)M_{k+1}(t)v_x dxdt + r_h, \quad v \in S_0^h, \qquad (35)$$

where r_h is of the highest order (32). Then we will treat G. Because the k-degree polynomial $G(x)$ contains only $k-1$ degrees of freedom, we shall directly construct the desired correction polynomial $W_h \in S_0^h(B')$ such that (32) holds.

1. The construction of G. Note that the function

$$b(x) = b_{k+1}(x) = C_1 h^k \int_{y_l}^{y_{l+1}} D_y^{k+1} u(x,y)(t^2-1)^k dy \in W^{k-1,p}(J_x),$$
$$b(0) = b(1) = 0,$$

and the derivatives $D_x^{k-2} b(x) \in C(J)$, as the trace of $D^{2k-1}u \in W^{1,p}(\Omega)$. Denoting the parameters by

$$D_x^l b(0) = \alpha_l, \quad D_x^l b(1) = \beta_l, \quad l = 2, 4, ..., 2m < k-1, \tag{36}$$

and using the imbedding theorem, we have

$$|\alpha_l| + |\beta_l| \le C||b||_{k-1,p,J} \le Ch^{k+1/p'}||u||_{2k,p,B}, \quad l = 2i \le 2m < k-1. \tag{37}$$

We introduce the polynomial series

$$\begin{aligned} &\phi_0 = 1,\ \phi_1 = x,\ \phi_2 = (x^2-x)/2!, \quad \phi_3 = (x^3-x)/3!, \\ &\phi_4 = (x^4 - 2x^3 + x)/4!, \quad \phi_5 = (3x^5 - 10x^3 + 7x)/(3*5!), \ldots, \end{aligned} \tag{38}$$

with the following properties:

$$D_x^2 \phi_j(x) = \phi_{j-2}(x), \quad \phi_j(0) = \phi_j(1) = 0, \quad \text{for} \quad j \ge 2, \tag{39}$$

$$D_x^{2i} \phi_{2i}(x) = 1, \quad D_x^{2i} \phi_{2i+1}(x) = x. \tag{40}$$

Using these bases, ϕ_j, and the parameters α_j, β_j, we can define the function

$$g_j(x) = \alpha_j \phi_j(x) + (\beta_j - \alpha_j)\phi_{j+1}(x), \quad g_j(0) = g_j(1) = 0, \quad 2 \le j = 2i < k-1,$$

with the derivatives

$$D_x^{2i} g_j(0) = D_x^{2i} g_j(1) = 0,\ D_x^j g_j(0) = \alpha_j, \quad D_x^j g_j(1) = \beta_j, \quad 2i < j = 2l < k-1.$$

Therefore, the k-degree polynomial

$$G(x) = g_2(x) + g_4(x) + \cdots + g_{2m}(x) \in S_0^h(J),$$

has the desired properties in (33).

2. The construction of W_h. For simplicity, we consider only the term $G(x) = \alpha_l \phi_l(x)$, $l = 2i < k-1$ (or taking $G = (\beta_l - \alpha_l)\phi_{l+1}(x)$) and construct the corresponding finite element solution

$$W_h(x,t) = \alpha_l \sum_{p,q=2}^{k} a_{pq} \phi_p(x) M_q(t) \in S_0^h(B'),$$

satisfying

$$A_B(W_h, v') = h\alpha_l \int_{B'} \phi_{lx}(x) M_{k+1}(t) v'_x dxdt, \quad v' \in S_0^h(B'), \tag{41}$$

where a_{pq} are $(k-1)^2$ parameters to be defined. Reducing the factor α_l in both sides of equation (41), taking $v' = \phi_i(x)M_j(t), i, j = 2, 3, ..., k$ (note that $M_0(t), M_1(t) \notin S_0^h(B')$) and denoting by

$$K_{ij} = (\phi_{ix}, \phi_{jx})_J, \quad m_{ij} = (\phi_i, \phi_j)_J, \quad c_{j+1} = (l_j, l_j)_E, \quad d_{ij} = (M_i, M_j)_E,$$

the integrals in J_x and E, where for i fixed, $d_{ij} \ne 0$ for at most three indices $j - i = 0, \pm 2$, we get a linear system of equations

$$h^2 \sum_{p,q=2}^{k} K_{pi} d_{qj} a_{pq} + \sum_{p=2}^{k} m_{pi} c_j a_{pj} = h^2 d_{k+1,j} K_{li}, \quad i, j = 2, 3, ..., k. \tag{42}$$

Denoting the unknowns

$$X_q = (a_{2q}, a_{3q}, ..., a_{kq})^T, \quad X = (X_2, X_3, ..., X_k)^T,$$

the vector

$$H_{l,k-1} = d_{k+1,k-1}(K_{l2}, K_{l3}, ..., K_{lk})^T,$$

and the symmetrical positive definite matrix of order $k-1$,

$$K = [K_{pi}] > 0, \quad M = [m_{ip}] > 0,$$

then (42) can be rewritten as a three-diagonal matrix equation:

$$\begin{bmatrix} c_2 M X_2 \\ c_3 M X_3 \\ .... \\ c_j M X_j \\ .... \\ c_{k-1} M X_{k-1} \\ c_k M X_k \end{bmatrix} + h^2 \begin{bmatrix} K(d_{22}X_2 + d_{24}X_4) \\ K(d_{33}X_3 + d_{35}X_5) \\ .... \\ K(d_{j,j-2}X_{j-2} + d_{jj}X_j + d_{j,j+2}X_{j+2}) \\ .... \\ K(d_{k-1,k-3}X_{k-3} + d_{k-1,k-1}X_{k-1}) \\ K(d_{k,k-2}X_{k-2} + d_{kk}X_k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ .... \\ 0 \\ .... \\ h^2 H_{l,k-1} \\ 0 \end{bmatrix}.$$

This is an absolutely diagonally dominated linear system. Using Lemma 1 below, we have
(43)

$$X_k = O(h), \quad X_{k-1} = O(h^2), \quad X_j = O(h^{k-1+j}), \quad X_3 = O(h^{k-2}), \quad X_2 = O(h^{k-1}),$$

where the X_2, X_3 will play an important role in the proof of Theorem 3.

Finally, coming back to (35) and taking a general test function $v = v' + v^* \in S_0^h(\Omega)$, then (41) is exactly satisfied for the local test function $v' = \xi M_j(t) \in S_0^h(B')$, $\xi = \phi_i(x)$, $i, j = 2, 3, ..., k$, and

$$A(\theta - w_h - W_h, v) = r_h + \sum_B r'_B, \quad v = v' + v^* \in S_0^h(\Omega).$$

Whereas the global test function $v^* = \xi(x)\eta(t)$, $\xi = \phi_i$, $i = 2, 3, ..., k$, $\eta = \gamma_0 + \gamma_1 t$, will bring the following residue in each B' (note that $M_{k+1} \perp \eta$, $k \geq 3$)

$$\begin{aligned} r'_B &= A_B(W_h, v) - h\int_{B'} G_x M_{k+1}(t) v_x dxdt = A_B(W_h, v^*) \\ &= h\sum_{p,q=2}^{k} \alpha_l \int_{B'} a_{pq}(\phi_{px} M_q(t)\xi_x(\eta_t t + \gamma_0) + h^{-2}\phi_p l_{q-1}\xi\gamma_1) dxdt, \end{aligned}$$

where $l_{q-1} \perp \gamma_1$, the second term disappears automatically, and only $q = 2, 3$ in the first term remain. By (37) and (43), $|\alpha_l| \leq Ch^{k+1/p'}||u||_{2k,p,B}$, $|X_3|h + |X_2| \leq Ch^{k-1}$, we have

$$\begin{aligned} |\sum_B r'_B| &= h\sum_B |\alpha_l \sum_{p=2}^{k} (\phi_{px}, \xi_x)_J (a_{p3}d_{31}\gamma_1 + a_{p2}d_{20}\gamma_0)| \\ &\leq Ch^{k+1}\sum_B ||u||_{2k,p,B}(|X_3|h|\eta_y| + |X_2|\ |\eta|)h^{1/p'}\ ||\xi_x||_{0,p',J} \\ &\leq Ch^{2k}\sum_B ||u||_{2k,p,B}||v||^*_{2,p',B} = r_h. \end{aligned}$$

The other terms of G can be treated similarly. This completes the proof of Theorem 3. □

In studying the correction function we often encountered a special linear system of n equations

$$KX = b, \quad K = [K_{ij}]_{n\times n}, \ X = (X_1, ..., X_n)^T, \ b = (b_1, ..., b_n)^T, \tag{44}$$

whose coefficients K_{ij} satisfy

$$K_{ii} = c_i + O(h), \quad |c_i| \geq c > 0, \quad K_{ij} = O(h^{|i-j|}), \quad i \neq j \leq n, \tag{45}$$

and is called absolutely diagonally dominated [12, pp. 39-40].

Lemma 1 ([12]). *Assume that for h suitably small, K is absolutely diagonally dominated and $b_j = O(h^{n-j})$, $j = 1, 2, ..., n$, then the solution $X = (X_1, X_2, ..., X_n)^T$ of* (44) *satisfies*

$$X_j = O(h^{n-j}), \quad j = 1, 2, ..., n.$$

Proof. We give a simplified proof. Using the substitute

$$Y_j = X_j h^{j-n}, \quad K'_{ij} = K_{ij} h^{-|i-j|} = O(1), \quad b'_i = b_i h^{i-n} = O(1),$$

the original i-th linear equation becomes (deducing a common factor h^{n-i})

$$K'_{i1}h^{2i-2}Y_1 + K'_{i2}h^{2i-4}Y_2 + \cdots + K'_{i,i-1}h^2 Y_{i-1} + K'_{ii}Y_i + K'_{i,i+1}Y_{i+1} + \cdots + K'_{i,n}Y_n = b'_i.$$

This is upper triangularly absolutely dominated. Obviously, $\det(K') = c + O(h)$, $c \neq 0$ for h suitably small. Then by Cramer's rule the solution $Y = O(1)$ is bounded. Lemma 1 is proved. ☐

6. Proof of Theorem 1

In the previous two sections, by Theorem 3, for $u \in W^{2k,p}(\Omega) \cap H_0^1$, we have constructed a correction function $W = w_h + W_h \in S_0^h(\Omega)$, $W(z) = 0$ at all nodes z, and obtained the following basic estimate:

$$|A(\theta - W, v)| \leq Ch^{2k}||u||_{2k,p}||v||^*_{2,p'}, \ v \in S_0^h. \tag{46}$$

In the following, we shall prove Theorem 1 by the duality argument.

Proof. **1. Discrete L^p-estimate,** $2 \leq p < \infty$. Make a conjugate problem $g \in H_0^1(\Omega)$ such that $A(g, v) = (\psi, v)$, $v \in H_0^1$. By the theory of partial differential equation in a rectangle, there is the regularity estimate [19]

$$||g||_{2,p,\Omega} \leq C(p)||\psi||_{0,p,\Omega}, \quad 1 < p < \infty. \tag{47}$$

Let $g_h \in S_0^h$ be the k-degree finite element projection of g, $A(g - g_h, v) = 0, v \in S^h$, so [17, 23]

$$||g - g_h||_{1,p} \leq Ch||g||_{2,p} \leq Ch||\psi||_{0,p}.$$

Denoting by $g_I \in S_0^h$ the interpolant of g, and using the inverse estimate we have

$$||g_h||^*_{2,p} \leq ||g_I||^*_{2,p} + ||g_h - g_I||^*_{2,p} \leq C||g||_{2,p} + Ch^{-1}||g_h - g_I||_{1,p} \leq C||g||_{2,p}.$$

For any k (odd or even), taking $v = g^z$ in (46) leads to

$$|(\theta - W, \psi)| = |A(\theta - W, g_h)| \leq Ch^{2k}||u||_{2n,p}||g_h||^*_{2,p'} \leq Ch^{2k}||u||_{2k,p}||\psi||_{0,p'}.$$

Taking $\psi = |\theta - W|^{p-1}\text{sign}(\theta - W)$ and using the inverse estimate, we have

$$||\theta - W||_{0,p} \leq Ch^{2k}||u||_{2k,p}, \quad \max_{z\in\Omega}|(\theta - W)(z)| \leq Ch^{-2/p}||\theta - W||_{0,p}. \tag{48}$$

Noting that $R(z) = W(z) = 0$ at nodes $z \in T_h$, we get the discrete L^p-estimate

$$|||u - u_h|||_{0,p,T_h} = |||\theta - W|||_{0,p,T_h} \leq C||\theta - W||_{0,p} \leq Ch^{2k}||u||_{2k,p,\Omega}. \tag{49}$$

In practice, superconvergence in the discrete L^2-norm is often used.

2. Pointwise estimate. The regularity estimate (47) is invalid for $p = 1, \infty$, so one can introduce the regularized Green's function [17] $g^z(x) \in H_0^1(\Omega)$ and its finite element projection $g_h^z(x) \in S_0^h$ such that

$$A(g^z, v) = (\delta_h^z, v), \quad v \in H_0^1; \qquad A(g_h^z, v) = (\delta_h^z, v) = v(z), \quad v \in S_0^h,$$

where δ_h^z is the discrete *Delta* function satisfying $(\delta_h^z, v) = v(z), \quad v \in S_0^h$. For the linear element, the following basic estimates are proved by the weighted norm [17],

$$(50)\quad ||g^z||_{2,1} \le C|\ln h|, \quad ||g^z - g_h^z||_{1,1} \le Ch|\ln h|, \quad ||g_h^z||_{2,1}^* \le C||g^z||_{2,1} \le C|\ln h|,$$

which are also valid for degree $k > 1$ (maybe the factor $|\ln h|$ can be removed; see remark 3).

Taking $v = g_h^z$ and $p = \infty$ in (46), we get the maximum norm estimate

$$\max_{z\in\Omega} |(\theta - W)(z)| \le Ch^{2k}|\ln h| \; ||u||_{2k,\infty}.$$

Recalling $u - u_h = R - \theta$ and $R = W = 0$ at node $z \in T_h$, we have

$$(51)\qquad \max_{z\in T_h} |(u - u_h)(z)| \le Ch^{2k}|\ln h| \; ||u||_{2k,\infty}.$$

Summarizing (49) and (51), Theorem 1 is proved. □

Remark 3. It is possible to remove the logarithm factor $|\ln h|$ in (5) for $k > 1$. Scott [25] studied the Neumann problem $-\Delta u + u = f$ on a smooth convex domain and proved

$$||g^z - g_h^z||_{1,1} \le Ch|\ln h|^{n(k)}, \quad n(k) = 1 \text{ if } k = 1, \quad n(k) = 0 \text{ if } k > 1,$$

where no factor $|\ln h|$ appears if $k > 1$. Thus, it directly leads to the optimal maximum estimate

$$||u - u_h||_{0,\infty} \le Ch^{k+1}||u||_{k+1,\infty}, \quad k > 1.$$

Note that it is not immediately clear if the analysis of Scott's results can be extended for convex polygonal domains.

Remark 4. In this paper, recalling the proofs of Theorems 2 and 3, we found that when Ω is a non-convex polygonal domain formed by several rectangles (for example, the L-shaped domain), the the error estimate (46) (for example, taking $p = p' = 2$) is still valid. However, when Ω has the maximum inner angle $\alpha\pi$, $\alpha = 3/2$, the solution $g \in H_0^1$ of problem $A(g, v) = (\psi, v)$, $v \in H_0^1$ has the lower regularity [19]

$$||g||_{2,p'} \le C(p')||\psi||_{0,p'}, \quad 1 < p' < 2\alpha/(2\alpha - 1) = 3/2.$$

Because the spaces $W^{2,p'}$ can be embedded into H^s with the index $2 - 2/p' = s - 1$, $||g||_{s,2} \le C||g||_{2,p'}$, using the error estimate $||g - g_h||_1 \le Ch^{s-1}||g||_{s,2}$ and the inverse property

$$||g_h||_2^* \le Ch^{s-2}||g_h||_s^* \le Ch^{s-2}(||g_I||_s^* + Ch^{s-1}||g_h - g_I||_1) \le Ch^{s-2}||\psi||,$$

$\beta = s - 2 = 1 - 2/p' < -1/3$, we get (similar to (48) and (49))

$$||\theta - W|| \le Ch^{2k+\beta}||u||_{2k}, \quad |||u - u_h|||_{0,T_h} \le Ch^{2k+\beta}||u||_{2k}.$$

Therefore, the concave angle $3\pi/2$ makes the superconvergence decrease by about $h^{1/3}$.

7. Numerical experiments for $k = 4, 5$

Consider the Poisson equation in a square $\Omega = (0,1)^2$,

$$-\Delta u = f \quad in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega.$$

The exact solution $u(x,y) = \sin(3\pi x)e^{(2x-1.1)^2}\sin(2\pi y)e^{(3y-1.4)^2}$ is given. Subdivide Ω into $N \times N$ square meshes Z_N, $N = 4, 8, 16$. Denote by $e^N = u^N - u$ the error of bi-$k = 4, 5$ degree finite element solutions u^N. We investigate the error at nine common nodes $x_i, y_j = 0.25, 0.50, 0.75$ and their averaging square root value

$$|e^N|_2 = (\sum_{i,j=1}^{3} |e^N(x_i, y_j)|^2/9)^{1/2}, \quad \text{(theoretical ratio } 2^{2k}).$$

For comparison, the L^2-norm error $||e^N||$ is listed in the last line (theoretical ratio 2^{k+1}).

Table 1. $N = 4, 8, 16, 32$, the errors of bi-4 degree elements at nine nodes (error ratio), theoretical ration $tr = 256$.

(x_i, y_j)	$\|e^4\|$	$\|e^8\|(tr = 256)$	$\|e^{16}\|(tr = 256)$	$\|e^{32}\|(tr = 256)$
$(0.25, 0.25)$	$8.0760e-6$	$7.2850e-8(111)$	$3.2721e-10(223)$	$1.4202e-12(230)$
$(0.50, 0.25)$	$1.0131e-5$	$5.0669e-8(200)$	$2.0589e-10(246)$	$8.1029e-13(254)$
$(0.75, 0.25)$	$1.8425e-5$	$2.2812e-7(81)$	$1.0350e-09(220)$	$4.3041e-12(240)$
$(0.25, 0.50)$	$6.1958e-5$	$2.5503e-7(243)$	$1.0136e-09(252)$	$4.0059e-12(253)$
$(0.50, 0.50)$	$1.5061e-5$	$5.8385e-8(258)$	$2.3192e-10(252)$	$9.4545e-13(245)$
$(0.75, 0.50)$	$1.5927e-4$	$6.6880e-7(238)$	$2.6839e-09(249)$	$1.0643e-11(252)$
$(0.25, 0.75)$	$2.2390e-5$	$1.1228e-7(199)$	$4.6484e-10(242)$	$1.9191e-12(242)$
$(0.50, 0.75)$	$9.4271e-6$	$4.1842e-8(225)$	$1.6772e-10(249)$	$6.5090e-13(258)$
$(0.75, 0.75)$	$5.7601e-5$	$3.1799e-7(181)$	$1.3345e-09(238)$	$5.3877e-12(248)$
$\|e^N\|_2$	$6.1327e-5$	$2.7711e-7(221)$	$1.1319e-09(245)$	$4.5284e-12(213)$
$\|\|e^N\|\|$	$6.0608e-3$	$2.9991e-4(20)$	$1.0902e-5(28)$	$3.5604e-7(31)$

We observe in Table 1 that the ratio of errors $|e^N|_2$ on the meshes Z_4, Z_8, Z_{16}, Z_{32} are $221, 245, 213$, respectively. Their accuracy increases 3, 4, 5 digitally respectively compared with $||e_N||$. The error ratios at different nodes have some vibration.

We observe in Table 2 that the ratio of errors $|e^N|_2$ on the triple meshes Z_4, Z_8, Z_{16} are $978, 1113$, respectively. Their accuracy increases 3, 4, 5 digitally respectively compared with $||e_N||$. Because the double accuracy is accepted, the error $|e^{16}|$ is inexact.

The error at a node depends on the local property of derivatives of u, and the ratios of error on different nodes can be off if the solution is very oscillatory (for example, the error at $(0.25, 0.75)$ is less), so the discrete L^2-norm $|e^N|_2$ will be more stable.

TABLE 2. $N = 4, 8, 16$, the errors of bi-5 degree elements at nine nodes (error ratio), theoretical ratio $tr = 1024$.

(x_i, y_j)	$\|e^4\|$	$\|e^8\|(tr = 1024)$	$\|e^{16}\|(tr = 1024)$
$(0.25, 0.25)$	$3.8581e-7$	$4.7001e-10(821)$	$4.1633e-13(1129)$
$(0.50, 0.25)$	$3.7499e-8$	$7.1605e-11(523)$	$9.7703e-14(733)$
$(0.75, 0.25)$	$9.6951e-7$	$1.3764e-09(704)$	$1.2159e-12(1132)$
$(0.25, 0.50)$	$5.0654e-7$	$4.0570e-10(1249)$	$4.1145e-13(986)$
$(0.50, 0.50)$	$2.3470e-7$	$1.9526e-10(1202)$	$1.4727e-13(1326)$
$(0.75, 0.50)$	$1.5178e-6$	$1.2171e-09(1248)$	$1.1036e-12(1103)$
$(0.25, 0.75)$	$2.2784e-8$	$1.0110e-10(225)$	$8.4821e-14(1120)$
$(0.50, 0.75)$	$9.5206e-8$	$1.0028e-10(949)$	$1.1122e-13(902)$
$(0.75, 0.75)$	$1.4115e-7$	$2.8026e-10(503)$	$2.4691e-13(1135)$
$\|e^N\|_2$	$6.4422e-7$	$6.5858e-10(978)$	$5.9167e-13(1113)$
$\|\|e^N\|\|$	$1.0421e-3$	$2.5270e-05(41)$	$4.5200e-07(56)$

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