THE HIGHEST ORDER SUPERCONVERGENCE FOR BI-\( k \) DEGREE RECTANGULAR ELEMENTS AT NODES: A PROOF OF 2\( k \)-CONJECTURE

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Abstract. We proved the highest order superconvergence \((u - u_h)(z) = O(h^{2k}) \ln h\) at nodes \(z\), based on Element Orthogonality Analysis (EOA), correction techniques and tensor product, where \(u \in W^{2k, \infty}(\Omega)\) is the solution for the Poisson equation \(-\Delta u = f\) in a rectangle \(\Omega\), \(u = 0\) on \(\Gamma\), and \(u_h \in S^k_0\) is its bi-\( k \) degree rectangular finite element approximation. This conclusion is also verified by numerical experiments for \(k = 4, 5\).

1. Introduction

We consider the model boundary value problem to find \(u \in H^1_0\) such that
\[
A(u, v) = (f, v), \quad v \in H^1_0 = \{v : v \in H^1(\Omega), \: v = 0 \text{ on } \Gamma\},
\]
where \(\Omega\) is a rectangular domain with boundary \(\Gamma\) (see Remark 4) and the bilinear form
\[
A(u, v) = \int_{\Omega} (u_x v_x + u_y v_y) dx dy,
\]
is bounded and \(H^1_0\)-coercive, i.e.,
\[
|A(u, v)| \leq C \|u\|_{1,p} \|v\|_{1,p'}, \quad A(v, v) \geq \nu \|v\|_{1,2}, \quad u, v \in H^1_0, \quad 1/p + 1/p' = 1.
\]
Denote by \(W^{k,p}(\Omega)\) the Sobolev space with the norm
\[
\|u\|_{k,p,\Omega} = \left\{ \sum_{i+j \leq k} \int_{\Omega} |D_x^i D_y^j u(x,y)|^p dx dy \right\}^{1/p},
\]
where the subindex \(\Omega\) is often omitted, if there is no confusion. When \(p = 2\), we denote simply \(H^k(\Omega) = W^{k,2}(\Omega)\) and \(\|\\|_k = \|\\|_{k,2}\).

Assume that the domain \(\Omega\) is subdivided into a finite number of the rectangular elements \(K \in J^h\), \(\Omega = \bigcup_{K \in J^h} K\) and the mesh is quasi-uniform. Denote the bi-\( k \) degree finite element space
\[
S^k_0 = S^k_0(\Omega) = \{v : v \in C(\Omega), \quad v|_K \in P_k(x) \otimes P_k(y), \quad v = 0 \text{ on } \partial \Omega \} \subset H^1_0,
\]


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where \( P_k(x) \) is the space of polynomials of degree \( \leq k \) and \( \otimes \) is the tensor product operator (often omitted below). Define the finite element solution \( u_h \in S_0^h \) satisfying

\[
(2) \quad A(u_h, v) = (f, v), \quad v \in S_0^h.
\]

By (1) and (2), we have the following orthogonal relation:

\[
(3) \quad A(u - u_h, v) = 0, \quad v \in S_0^h.
\]

Under certain conditions the optimal error estimates

\[
\|u - u_h\|_{0, p, \Omega} \leq Ch^{k+1}\|u\|_{k+1, p, \Omega}, \quad 2 \leq p < \infty,
\]

\[
\|u - u_h\|_{0, \infty, \Omega} \leq Ch^{k+1}\ln h^{n(k)}\|u\|_{k+1, \infty, \Omega}, \quad p = \infty,
\]

are proved (see [5, 17, 23, 25]), where \( n(1) = 1 \) and \( n(k) = 0 \) if \( k > 1 \).

However, it is found that the finite element solution or its derivatives at some specific points have possibly higher order accuracy, which is called superconvergence by J. Douglas. In 1973, Douglas and Dupont \[14, 15\] proved that for the two-point boundary value problem the \( k \)-degree finite element solution \( u_h \) at each node \( x_j \) is superconvergent, i.e.,

\[
(4) \quad (u - u_h)(x_j) = O(h^{2k})\|u\|_{k+1}, \quad k \geq 2,
\]

where only \( u \in H^{k+1} \) is required. The estimate (4) is optimal and sharp, which is verified by the example in [15].

Soon, Douglas, Dupont, and Wheeler \[16\], considered bi-\( k \) degree rectangular finite element solution \( u_h \) for Poisson equation (1) in a rectangle \( \Omega \) and proved superconvergence estimates \( \|u_h - u_f\| \leq Ch^{k+2}\|u\|_{k+3} \), \( k \geq 3 \), and \( (u - u_h)(z) = O(h^{k+2}) \) at nodes \( z \), where the comparison function \( u_f \) is constructed by quasi-projection and tensor product. Actually, this paper \[16\] contains much richer ideas. Later on, Chen \[7\] proved that bi-\( k \) degree rectangular finite element is superconvergent at \( k+1 \)-order Lobatto points by the element analysis method, in particular, at nodes \( z \) (in \( L^2 \)-norm in \[7\]),

\[
\| (u - u_h)(z) \| \leq Ch^{k+1} \ln h \| u \|_{k+1, \infty}, \quad l = k = 2, 3, \quad l = 3 \text{ if } k \geq 4,
\]

which is further generalized to the case of variable coefficients \[14\]. However, when \( l = k \geq 4 \) the study meets the essential difficulty (because the regularized Green’s function \( g^z \) has the singularity and the optimal regularity is \( ||g^z||_{2, 1} \leq C|\ln h| \), see Section 3).

On the other hand, Bramble and Schatz \[3\] and Thomée \[26\] used the kernel function \( K_h(x, y) \) with small support to construct the convolution \( K_h^k \ast u_h \) (a post-processing or recovery) of finite element solution \( u_h \) and obtained the highest order superconvergence for both function values and derivatives

\[
\| D^l u - K_h^l \ast u_h \|_{0, \infty, \Omega_0} \leq Ch^{2k}\|u\|_{2k, \infty, \Omega_1} + C\|u - u_h\|_{-s, p, \Omega}, \quad s > 0,
\]

where \( \Omega_0 \subset \subset \Omega_1 \subset \subset \Omega \), the mesh is uniform in \( \Omega_1 \), and the negative norm \( \|u - u_h\|_{-s, p, \Omega} \) is of the highest order \( O(h^{2k}) \) under some conditions. Whether the rectangular finite element solution naturally has the highest order superconvergence at nodes is an attractive problem, which is one of six open problems proposed in \[13\] pp. 593-599]. Recently, this was emphasized by Zhou and Lin \[33\], as a conjecture. All of this research can be concluded as the following \( 2k \)-conjecture.
2k-Conjecture of Superconvergence. Under some conditions, bi-\( k \) degree rectangular element \( u_h \in S^k \) for (1) has the highest order superconvergence at nodes \( z \):

\[ (u - u_h)(z) = O(h^{2k}), \quad k \geq 2. \]

The question is so difficult that no definite answer was given for a long time.

We recall that up to now five main superconvergence methods have been developed: quasi-projection method [16], local averaging method [3, 26], element analysis method [6, 7, 8, 13, 22, 31, 32], computer-based method [1, 2] and local symmetrical theory [24, 29, 30]. In the element (orthogonality) analysis method (EOA), a new technique of orthogonality correction [9, 11, 12] is introduced; also see [4, 20, 21]. This paper will give a rigorous proof of this conjecture based on EOA, orthogonality correction and tensor product. Our main result is as follows.

**Theorem 1.** Assume that the rectangular mesh in a rectangle \( \Omega \) is quasi-uniform, and problem (1) has a solution

\[ u \in W^{2k,p}(\Omega) \cap H^1_0(\Omega) \quad 2 \leq p \leq \infty, \quad k \geq 2. \]

Then bi-\( k \) degree rectangular element solution \( u_h \in S^k \) on the nodal set \( T_h \) has the highest order superconvergence in the discrete \( L^p \)-norm

\[ |||u - u_h|||_{0,p,T_h} = \left\{ \sum_{z \in T_h} |(u - u_h)(z)|^p h^{2k} \right\}^{1/p} \leq C h^{2k} ||u||_{2k,p,\Omega}, \quad 2 \leq p < \infty, \]

and the highest order pointwise superconvergence

\[ \max_{z \in T_h} |(u - u_h)(z)| \leq C h^{2k} |\ln h| ||u||_{2k,\infty,\Omega}, \quad p = \infty. \]

This result is quite similar to (4) in the one-dimensional case, where the slowly increasing factor \( |\ln h| \) is not essential and perhaps can be removed (see Remark 4). This conclusion is also verified by our numerical experiments for \( k = 4, 5 \). The proof shows that Theorem 1 is valid in the higher dimensional case and the superconvergence conclusion may be generalized to non-rectangular domains (see Remark 4). Moreover, whether the regularity requirement could be decreased to \( W^{k+1,p}(\Omega) \) as in a one-dimensional case is another difficult question.

In recent years, The Third and Fourth International Conferences on Superconvergence and A Posteriori Error Estimates, held in Changsha, China in 2004, and Prague, Czech in 2008, respectively, and the corresponding special issues [34, 35] were published. Thus, some new superconvergence results on tetrahedral quadratic finite elements and triangular finite elements [36–40] were obtained.

**Remark 1.** Under some conditions the solution of (1) in a non-smooth domain has higher regularity. For example, in a rectangle \( \Omega \), Fufaev [18] and Volkov [28] proved that if \( f \in C^{1,\alpha}(\Omega) \), \( 0 < \alpha < 1 \) and \( f = 0 \) at angular nodes, then \( u \in C^{3,\alpha}(\Omega) \); if \( f \in C^{3,\alpha}(\Omega) \), \( 0 < \alpha < 1 \) and \( f = f_{xx} = f_{yy} = 0 \) at angular nodes, then \( u \in C^{5,\alpha}(\Omega) \).

2. **M-type Projection in a One-dimensional Element**

Subdividing \( J = (0, 1) \) by the nodes \( x_0 = 0 < x_1 < x_2 < \cdots < x_N = 1 \), denote the element \( K_i = (x_i, x_{i+1}) \), its midpoint \( \bar{x}_i = (x_i + x_{i+1})/2 \) and the half-step length \( h_i = (x_{i+1} - x_i)/2 \). Using a linear transform \( x = \bar{x}_i + h_i s, s \in E = (-1, 1) \) in \( K_i \), the function \( f(x) = f(\bar{x}_i + h_is) \) is still denoted by \( f(s) \). Note that \( D^2_s f = h_i^2 D^2_x f = O(h^2) \).
Introduce Legendre’s polynomials in $E$,

\begin{equation}
0 \leq 1, \quad l_1 = s, \quad l_2 = (3s^2 - 1)/2, \quad l_3 = (5s^3 - 3s)/2, \ldots, \quad l_n = \gamma_n D_n^s(s^2 - 1)^n,
\end{equation}

where $\gamma_n = 1/(2n)!!$, the inner product $(l_i, l_j) = 0, i \neq j$, and $c_{j+1} = (l_j, l_j) = 2/(2j + 1)$.

Integrating $l_j(s)$ leads to $M$-type polynomials [6],

\begin{align}
M_0 &= 1, \quad M_1 = s, \quad M_2 = (s^2 - 1)/2, \quad M_3 = (s^3 - s)/2, \ldots, \quad M_{n+1}(s) = \gamma_n D_s^{n-1}(s^2 - 1)^n,
\end{align}

which are quasi-orthogonal: $d_{ij} = (M_i, M_j) \neq 0$ for $j - i = 0, \pm 2$, otherwise $d_{ij} = 0$. So $M_i(s)$ is not orthogonal to at most three bases $M_{i-2}, M_i, M_{i+2}$. Obviously, $M_n(\pm 1) = 0$ for $n \geq 2$.

To construct an $M$-type projection of $u(s)$, expanding $u_s(s)$ as an $L$-type series,

\begin{equation}
u_s(s) = \sum_{j=0}^{\infty} b_{j+1} l_j(s), \quad b_{j+1} = j(u_s, l_j), \quad j_1 = j + 1/2,
\end{equation}

and integrating in $s$, we get an $M$-type series [6] [7] and its partial sum

\begin{equation}
u(s) = \sum_{j=0}^{\infty} b_j M_j(s), \quad u_I = Q^s u = \sum_{j=0}^{k} b_j M_j(s),
\end{equation}

where $b_0 = (u(1) + u(-1))/2, b_1 = (u_s, l_0)/2 = (u(1) - u(-1))/2$. Then $u_I(\pm 1) = u(\pm 1)$ at two endpoints, which guarantees that the $u_I$ constructed in each element is continuous in $J$. Integrating by parts, the coefficient $b_{j+1}$ has the following expression and estimate:

\begin{align}
b_{j+1} &= C_j \int_E D_{s}^{i+1} u(s) D_{s}^{j-i}(s^2 - 1)^j ds, \quad i \leq j, \quad C_j = (-1)^j j! \gamma_j, \quad |b_{j+1}| \leq C h^{i+1-1/p} ||u||_{i+1, p, K}, \quad i \leq j.
\end{align}

Note that if $u$ is a polynomial of degree $k$, then all coefficients $b_j = 0, \quad j > k$. This is the basis to use the Bramble-Hilbert lemma.

It is easy to see that the remainder

\begin{equation}
R^k = E^s u = u - Q^s u = \sum_{j=k+1}^{\infty} b_j M_j(s) = \sum_{j=1}^{\infty} \gamma_{k+j-1} b_{k+j} D_{s}^{k+j-2}(s^2 - 1)^{k+j-1}
\end{equation}

has the orthogonal properties

\begin{equation}
R^k(\pm 1) = 0; \quad D_s R^k \perp P_{k-1}(s), \quad k \geq 1; \quad R^k \perp P_{k-2}(s), \quad k \geq 2,
\end{equation}

and error estimates (by the Bramble-Hilbert lemma)

\[ ||R^k||_{0, q, K} \leq C h^{k+1+1/q-1/p} ||u||_{k+1, p, K}, \quad 1 \leq p, q \leq \infty. \]

Defining an integral operator in $K = (-h, h)$ gives us

\[ S_x w = \int_{-h}^{x} w(x) dx = h S_s w, \quad S_s w = \int_{-1}^{s} w(s) ds, \]

so, obviously,

\[ ||S_x w||_{L^p(K)} \leq \left( \int_{K} |w(x)| dx \right)^{1/p} \leq 2h ||w||_{L^p(K)}. \]
When \( k \geq 2, 1 \leq i \leq k - 1 \), we have

\[
S^i_s R^k = \sum_{j=1}^{\infty} \gamma_{k+j-1} b_{k+j} D_s^k \gamma_{k+j-2-i} (s^2 - 1)^{k+j-1}, \quad S^i_s R(\pm 1) = 0,
\]

and \( S^i_s R^k \perp P_{k-2-i} (s) \), \( 1 \leq i \leq k - 2 \), but \( S^{k-1}_s R^k \) has no orthogonality.

The projection \( u_I = Q^* u \) and the orthogonality of \( R^k = u - Q_s u \) play an important role in studying superconvergence \([8][11][12][13]\). Unfortunately, the argument is invalid for \( k > 3 \) in the multi-dimensional case, because the corresponding regularized Green’s function \( g^\theta(x) \in H^1_0(J) \cap C^k(J) \cap C^k(J_1) \) (i.e., continuous and piecewise smooth) such that

\[
L g^\theta(x) = \delta(x - y), \quad 0 < x, y < 1, \quad w(y) = A(w, g^\theta), \quad w \in H^1_0(J).
\]

If taking the node \( y = x_j \) and \( g_I \in S^h \) of the \( M \)-type projection of \( g^\theta \), we have

\[
||g - g_I||_{0,J_i} \leq C h^{k+1} ||g||_{k+1,J_i} \leq C h^{k+1}, \quad \sum_{K \in J_i} ||g_I||^2_{k,K} \leq C, \quad i = 0, 1,
\]

and (noticing \( (g - g_I)_x \perp \theta_x \))

\[
\theta(x_j) = A(\theta, g) = A(\theta, g_I) + A(\theta, g - g_I) = (R^k, g_I) + (\theta, g - g_I) = O(h^{2k})||u||_{k+1}.
\]

Finally, we derive \( (u - u_h)(x_j) = O(h^{2k}) \) at each node \( x_j \) (as obtained in \([13][15]\)). Unfortunately, the argument is invalid for \( k > 3 \) in the multi-dimensional case, because the corresponding regularized Green’s function \( g^\theta \in H^1_0(\Omega) \) has the singularity and the optimal regularity \( ||g^\theta||_{2,1} \leq C \ln h \). To prove the \( 2k \)-conjecture we have to propose a new correction technique in the element band.

3. ELEMENT ORTHOGONALITY ANALYSIS (EOA)

Assuming that \( u_I \in S^h \) is a comparison function to be defined, we split the error \( u - u_h = R - \theta, R = u - u_I, \theta = u_h - u_I \) and rewrite \( [3] \) as

\[
A(\theta, v) = A(R, v), \quad v \in S^h_0.
\]

The idea behind EOA is to construct \( u_I \) super close to \( u_h \) and prove the basic estimate

\[
A(\theta, v) = A(R, v) = O(h^{k+1+\alpha}||v||_{2,p',\Omega}^\alpha), \quad v \in S^h, \quad v' = p/(p - 1),
\]

where \( ||v||_{2,p',\Omega} = (\sum_K ||v||^2_{p,K})^{1/p} \) is the mesh-norm.

Subdivide a square \( \Omega = J \times J \) into \( N \times N \) rectangular elements \( K_{ij} = (x_i, x_{i+1}) \times (y_j, y_{j+1}) \). For simplicity of notation, we denote the half-step lengths \( h_i = k_i = \frac{1}{2} h \), \( k_i = k_{i+1} \).
$h = 1/(2N)$ in the $x, y$-directions. Then $K_{i l}$ is transformed to a reference element $E = (-1, 1) \times (-1, 1)$ by a linear transform $x = \bar{x}_i + h s, y = \bar{y}_i + h t$, and the function $u(x, y)$ becomes $u(s, t)$. Obviously, $D^p_y D^q_z u(s, t) = h^{p+q} D^p_y D^q_z u(x, y) = O(h^{p+q})$.

Constructing $k$-degree $M$-type projections $Q^x, Q^y$ and their tensor product $u_I = Q^y Q^x u \in S^h$ in $K_{i l}$, the error $R = u - u_I$ has the following important decomposition [16]:

$$R = u - u_I = (I - Q^x)u + (I - Q^y)(I - Q^x)u = E^x u + E^y u - E^y E^x u.$$  

For example, in an element-band $B = J \times (y_l, y_{l+1})$ we have the remainder

$$E^y u = \sum_{j=1}^{\infty} b_{k+j}(x)M_{k+j}(t), \quad x \in J, \quad y = \bar{y}_l + h t,$$

where

$$b_{k+j}(x) = \gamma_{k+j-1} h^{k+1} \int_{-1}^{1} D_y^{k+1} u(x, y) D_t^{j-1} (t^2 - 1)^{k+j-1} dt, \quad j \geq 1.$$

$$||D_x^{k-1} b_{k+j}(x)||_{0,B} \leq C h^{k+1} ||D_x^{k-1} D_y^{k+1} u||_{0,B} \leq C h^{k+1} ||u||_{2k,B}.$$

As mentioned above, there are nice orthogonal properties:

$$(E^x u)_x \perp P_{k-1}(x), \quad E^x u \perp P_{k-2}(x), \quad (E^y u)_y \perp P_{k-1}(y), \quad E^y u \perp P_{k-2}(y).$$

In particular, $(E^x u)_x \perp v_x, (E^y u)_y \perp v_y, v \in S^h_0$. As a result only two terms in $A(R, v)$ remain:

$$A(\theta, v) = A(R, v) = \sum_K \int_K (E^y u_x v_x + E^x u_y v_y) dx dy, \quad v \in S^h.$$

Obviously, $(E^y u)_x = E^y u_x$ is a smooth function in $x$. By Green’s formula,

$$\sum_K \int_K E^y u_x v_x dx dy = \sum_K \left\{ \int_\gamma E^y u_x v dx - \int_K E^y u_{xx} v dx dy \right\},$$

in which all inner linear integrals cancel each other out, while only the integral along $\Gamma$ remains and disappears as $v = 0$ on $\Gamma$ (this is the simplest combination cancellation). We treat $E^x u_x v_y$ in the same way. So we have [7]

$$A(\theta, v) = A(R, v) = - \sum_K \int_K (E^y u_{xx} v + E^x u_{yy} v) dx dy.$$

When $k \geq 3$, integrating by parts twice, we have [7]

$$A(\theta, v) = A(R, v) = - \sum_K \int_K \left( S^2_y E^y u_{xx} v_{yy} + S^2_x E^x u_{yy} v_{xx} \right) dx dy$$

$$\leq C h^{k+3} \max_{(x, y) \in \Omega} |D^{k+3} u(x, y)| \sum_K \int_K |D^2 v| dx dy \leq C h^{k+3} ||u||_{k+3, \infty} ||v||_{2,1}^*.$$  

Using the discrete Green’s function $g_h \in S^h$ and $||g_h||_{2,1}^* \leq C \ln h$ (see Section 6), we get

$$\max \{|\theta(z)| \leq C h^{k+3} \ln h | ||u||_{k+3, \infty}, \quad k \geq 3.$$  

As $R(z) = 0$, $e(z) = -\theta(z)$ at nodes $z$, then Theorem [1] holds for $k = 3$.

This is the best result up to now. For a long time, the remainders $S^2_y E^y u_{xx}, S^2_x E^x u_{yy} = O(h^{k+3})$ could not be cancelled. Although $S^2_y E^y u_{xx} \perp P_{k-4}(y)$ for $k \geq 4$, it cannot be integrated by parts again, because the regularized Green’s
function $g^z$ does not have higher regularity. This is the essential difficulty mentioned in Section 1.

We found that the conjecture is still valid for $k = 4$ using the correction technique in an element. Recently, $k = 4$ was also discussed in [33]. This paper further finds that constructing the correction function in an element-band $B$ (rather than in an element) is more effective.

The basic idea of orthogonality correction [9 10 11 12] is to construct $W \in S^h$ such that $W = 0$ at nodes and

$$A(\theta - W, v) = A(R - W, v) = O(h^{k+1+\alpha})||v||^2_{2,p',\Omega}, \quad v \in S^h, \quad \alpha > 0,$$

is of the higher order. Then by using the discrete Green’s function we get

$$\max_{(x,y) \in \Omega} |(\theta - W)(x, y)| \leq C h^{k+1+\alpha} |\ln h|.$$

Coming back to each element, we have superconvergence at nodes $z$:

$$(u - u_h)(z) = (R - W)(z) - (\theta - W)(z) = 0 + O(h^{k+1+\alpha})|\ln h|.$$

To prove the conjecture, it is necessary to construct the optimal correction function $W \in S^h_0(\Omega)$ in order to get (20) with $\alpha = k - 1$ and guarantee $W(z) = 0$ at all nodes $z$. This is an elaborate work.

4. AN EXPANSION WITH SMALL PARAMETER IN AN ELEMENT-BAND

Following V. Thomee [27], we define a subspace of Sobolev space $W^{L,p}(\Omega)$ by

$$\tilde{W}^{L,p}(\Omega) = \{ u : u \in W^{L,p}(\Omega), \Delta^m u = 0 \text{ on } \Gamma, \quad m < L/2 \}, \quad 2 \leq p \leq \infty.$$

In this section we shall prove the following key result under the stronger constraint (21), which means that the $2k$-conjecture is valid.

**Theorem 2.** Assume that the solution of (11) satisfies

$$u \in \tilde{W}^{k-1,p}(\Omega) \cap W^{2k,p}(\Omega), \quad 2 \leq p \leq \infty, \quad \Omega = J \times J.$$

Then a corresponding correction function $w_h \in S^h$ can be constructed such that $w_h = 0$ at all nodes $z$ and

$$|A(\theta - w_h, v)| \leq C h^{2k}||u||_{2k,p} ||v||^2_{2,p'}, \quad v \in S^h_0, \quad 1/p + 1/p' = 1,$$

where $||v||^2_{2,p'}$ is the mesh-norm.

**Proof.** Under the assumption (21), define a specific index $m = [(k - 2)/2]$, which is the maximum integer such that $m \leq (k - 2)/2$. So $k = 2m + 2$ or $k = 2m + 3$.

As $u = 0$ on $\Gamma$, the tangent derivatives $D^l_y u = 0$, $0 \leq l < 2k$ on the boundary $\Gamma_x$ (see Figure 1). If $\Delta u = 0$ on $\Gamma$, we have $D^2_x u = -D^2_y u = 0$ and $D^l_y D^2_x u = 0$ on $\Gamma_x$, for $0 \leq l < 2k - 2$. In general, we have

$$D^l_y D^j_x u = 0 \quad \text{on } \Gamma_x, \quad \text{for } 2j \leq 2m < k - 1, \quad l + 2j < 2k.$$

Similarly, we also have $D^l_y D^j_y u = 0$ on $\Gamma_y$, for $2j \leq 2m < k - 1, \quad l + 2j < 2k$.

For simplicity, we first consider one term $R_1 = E^y u$ in (19) (similarly for $R_2 = E^x u$). Each element-band $B = J \times (y_l, y_{l+1})$ is transformed to $B' = J \times E$ by $y = y_l + ht$, then

$$A(\theta, v) = \sum_{B'} h \int_{B'} (\theta_x v_x + h^{-2} \theta_t v_t) dx dt = A(R_1, v), \quad v \in S^h_0,$$
where \( A(R_1, v) = \sum_{B'} h \int_{B'} E^y u_{xx} v dx dt, \)
\[
R_1 = E^y u = \sum_{j=k+1}^{\infty} b_j(x) M_j(t) \perp P_{k-2}(t), \quad y = \bar{y}_l + ht, \quad t \in E.
\]

Note that \( M_j(t) \perp P_k(t) \) for \( j > k+2; \) actually, in addition to the two terms \( b_{k+1}(x) \) and \( b_{k+2}(x), \) all other terms in \( A(R_1, v) \) disappear. Whereas the coefficients \( b_{k+i}(x) = \nu_{k+i} h^{k+1} \int_{-1}^{1} D^k M_j(t) D_{i-1}^y(t^2 - 1)^{k+i-1} dt \in \hat{W}^{k-1,p}(J), \quad i = 1, 2, \)
are smooth functions in \( x \in J = (0, 1), \) and by differentiating,
\[
|D^2 b_{k+i}(x)| \leq C h^{k+1} \int_{-1}^{1} |D^2 b_{k+i}(x, \bar{y}_l + ht)| dt, \quad 2j \leq 2m < k - 1, \quad i = 1, 2,
\]
\[
||D^2 b_{k+i}(x)||_{0,p,B} \leq C h^{k+1} ||u||_{2k-1,p,B}, \quad 2j \leq 2m < k - 1, \quad i = 1, 2.
\]

We shall concretely solve the problem with the low order term \( b(x) = b_{k+1}(x): \)
\[
A(\theta, v) = A(R_1, v) = -\sum_{B'} h \int_{B'} b_{xx}(x) M_{k+1}(t) v dx dt, \quad v \in S^h, \quad B' = J \times E.
\]

The main difficulty in constructing the correction function is that the small parameter \( h^2 \) contained in \( A(w, v) \) makes the problem singularly perturbed. However, from the condition (21), \( D^2 b(x) = 0 \) on \( \Gamma_x \) for \( 2j \leq 2m < k - 1, \) that makes it possible to treat the small parameter. For this purpose we shall construct the correction function \( w_h \in S^h_0(B') \) in each \( B' \) as
\[
w_h = \sum_{j=1}^{m} W_j(x, t), \quad W_j = h^{j}Q_h G_j(x) F_j(t), \quad F_j(t) = \sum_{i=2}^{k} a_{ji} M_i(t), \quad F_j(\pm 1) = 0,
\]
where \( G_j = D^2 b(x), \) \( Q_h \) is the one-dimensional \( M \)-type projection operator defined in Section 2 and \( a_{ji} \) are the constants to be defined. We define \( F_j(t) \) one-by-one as follows.
First, when \( k \geq 4 \), rewriting

\[
M_{k+1}(t) = D_t^2 \Phi_1(t) \perp P_{k-2}(t), \quad \Phi_1(t) = cD_t^{k-3}(t^2 - 1)^k \perp P_{k-4}(t),
\]

we construct the first correction function \( W_1 = h^2Q_hG_1(x)F_1(t) \). Taking \( v = \xi(x)M_i(t), i = 0, 1, ..., k \), in (20) and integrating by parts, we can calculate the residue

\[
A(\theta - W_1, v) = -A(W_1, v) + A(R_1, v)
\]

\[
= h \sum_B \int_{B'} \{-h^2(Q_hG_1)_x\xi_xF_1M_i - Q_hG_1\xi F_1t\iota_{i-1} - G_1\xi \Phi_{1t}M_i\}dxdt
\]

\[
= h \sum_B \int_{B'} \{-h^2G_1\xi_xF_1M_i - Q_hG_1\xi F_1t\iota_{i-1} + G_1\xi \Phi_{1t}l_{i-1}\}dxdt
\]

\[
= h \sum_B \int_{B'} \{h^2D_x^2G_1\xi F_1M_i - Q_hG_1\xi (F_1t - \Phi_{1t})l_{i-1}(t)dxdt + r_1,
\]

where \((Q_hG_1 - G_1)_x \perp \xi_x\) is used and the remainder follows:

\[
r_1 = -\sum_B \int_B (G_1 - Q_hG_1)\Phi_1v_{xy}h^2\,dxdy = O(h^2) \sum_B \|G_1 - Q_hG_1\|_{0,p,B}\|v\|_{2,p',B}
\]

\[
= \sum_B O(h^{k-1})\|G_1\|_{k-3,p,B}\|v\|_{2,p',B} = O(h^{2k})\|u\|_{2k,p,\Omega}\|v\|_{2,p',\Omega} = r_h.
\]

To define \( F_1 \), an important idea is to project \( \Phi_1 \) to \( F_1(t) = \sum_{j=0}^k a_{1j}M_j(t) \) such that

\[
\int_E (F_{1t} - \Phi_{1t})l_{i-1}dt = 0, \quad i = 1, 2, ..., k.
\]

Using \( \Phi_{1t} \perp P_{k-3} \), we can determine that

\[
F_1 = a_{1,k-1}M_{k-1}(t) \perp P_{k-4}(t),
\]

where the orthogonality of \( F_1(t) \) decreases the degree by 2. Therefore, we have

\[
A_B(\theta - W_1, v) = h \sum_B \int_{B'} h^2D_x^2G_1(x)F_1(t)\xi(x)M_i(t)dxdt + r_1, \quad v \in S^h,
\]

which still has the form similar to (20). This treatment will be repeatedly used later.

When \( k \geq 6 \), rewriting \( F_1 = a_{1,k-1}M_{k-1}(t) = D_t^2 \Phi_2(t) \), we construct \( W_2 = h^4Q_hG_2(x)F_2(t) \), \( G_2 = D_x^2G_1 \) and calculate the residue

\[
A(\theta - W_1 - W_2, v) = -A(W_2, v) + h \sum_B \int_{B'} h^2G_2(x)\xi(x)D_t^2 \Phi_2(t)M_i(t)dxdt + r_h
\]

\[
= h \sum_B \int_{B'} \{-h^4(Q_hG_2)_x\xi_xF_2M_i - h^2Q_hG_2\xi F_2t\iota_{i-1} - h^2G_2\xi \Phi_{2t}l_{i-1}\}dxdt + r_h
\]

\[
= h \sum_B \int_{B'} \{h^4D_x^2G_2\xi F_2M_i - h^2Q_hG_2\xi (F_2t + \Phi_{2t})l_{i-1}\}dxdt + r_2 + r_h,
\]
where the remainder
\[
\begin{align*}
  r_2 &= h \sum_B \int_B h^2 (Q_h G_2 - G_2) \Phi_2 v_y h^2 dxdy \\
  &= O(h^4) \sum_B ||G_2 - Q_h G_2||_{0,p,B} ||v||_{2,p',B} \\
  &= \sum_B O(h^{k-1}) ||G_2||_{k-5,p,B} ||v||_{2,p',B} = O(h^{2k}) ||u||_{2k,p,\Omega} ||v||_{2,p',\Omega} = r_h.
\end{align*}
\]

We define \( F_2(t) = \sum_{j=0}^k a_{2j} M_j(t) \) such that
\[
\int_E (\Phi_{2t} + F_{2t}) l_{i-1} dt = 0, \quad i = 1, 2, \ldots, k.
\]

Using \( \Phi_{2t} \perp P_{k-5} \), we can determine
\[
F_2 = a_{2,k-1} M_{k-1}(t) + a_{2,k-3} M_{k-3}(t) \perp P_{k-6}(t),
\]
where the orthogonality of \( F_2(t) \) decreases the degree by 2 once again. So we get
\[
A(\theta - W_1 - W_2, v) = h \sum_B \int_{B'} h^4 D_2^2 G_2 \xi F_2 M_x dxdt + r_h, \ v \in S_0^h(\Omega).
\]

In general, when \( 2m + 2 \leq k \), we can define the \( k \)-degree polynomials \( F_j \) one-by-one (\( 1 \leq j \leq m \)),
\[
F_j(t) = a_{j1} M_{k-1}(t) + a_{j3} M_{k-3}(t) + \cdots + a_{j2j-1} M_{k-2j+1}(t) \perp P_{k-2j-2}(t),
\]
\[
F_j(\pm 1) = 0,
\]
and we get \( w_h \) in (27) such that the corresponding residue could be rewritten as
\[
(28) \quad A(\theta - w_h, v) = h^{2m} \sum_B \int_B (G_m)_x F_m(t) v_x dxdy + r_h, \ v \in S_0^h(\Omega).
\]

When \( k = 2m + 2 \) is even, the last polynomial
\[
F_m(t) = a_{m1} M_{k-1}(t) + a_{m3} M_{k-3}(t) + \cdots + a_{m,k-3} M_3(t) \perp P_0, \ F_m(\pm 1) = 0,
\]
is odd. Integrating by parts, we get
\[
\begin{align*}
  A(\theta - w_h, v) &= -h^{2m+1} \sum_B \int_B D_x^{2m+1} b(x) S_t F_m(t) v_{xy} dxdy + r_h \\
  &= O(h^{k-1}) h^{k+1} \sum_B \int_B |D^{2k}u| |v_{xy}| dxdy \\
  &\quad + r_h = r_h, k = 2m + 2.
\end{align*}
\]

When \( k = 2m + 3 \) is odd, the last polynomial
\[
F_m(t) = a_{m1} M_{k-1}(t) + a_{m3} M_{k-3} + \cdots + a_{m,2m-1} M_4(t) \perp P_1(t), \ F_m(\pm 1) = 0,
\]
Theorem 3. Assume that the solution of \( u \) satisfies (1) is even. Integrating by parts twice, we have

\[
A(\theta - w_h, v) = -h^{2m} \sum_B \int_B (G_m)_{xx} F_m(t) v dx dy + r_h
\]

\[
= -h^{2m+2} \sum_B \int_B D_x^{2m+2} b(x) S'_t F_m(t) v_{yy} dx dy + r_h
\]

\[
= O(h^{k-1}) \sum_B \int_B |D_x^{k-1} b(x)| |v_{yy}| dx dy + r_h = r_h, \quad k = 2m + 3.
\]

Therefore, when \( R = R_1 = b_{k+1}(x) M_{k+1}(t) \) for any \( k \geq 4 \), the desired estimate (22) can be derived by (21) and (30).

We can also discuss \( R_1 = b_{k+2}(x) M_{k+2}(t) \) in \( \dot{W}^{k-1,p}(J_x) \) in a similar way. This time the only difference is that \( W_j = h^{2j} Q_h(D_x^{2j} b_{k+2}) F_j(t) \in S^h_0(B') \) with \( F_j(t) = a_{j0} M_k(t) + a_{j2} M_{k-2} + \cdots \).

Now we should study another remainder, \( R_2 = E^x u \) in the corresponding band \( B^* = (x_0, x_{k+1}) \times J_y \) and construct \( w_h \in S^h_0(B^*) \) to satisfy (22). Its analysis is quite parallel to that of \( R_1 = E^y u \).

Finally, summarizing these analyses and estimates, Theorem 2 is proved.

5. THE STUDY OF THE SOLUTION \( u \in W^{2k,p}(\Omega) \cap H^1_0 \)

In this section we shall remove the stronger constraints (21) on \( u \) in Theorem 2.

**Theorem 3.** Assume that the solution of (1) satisfies

\[
u \in W^{2k,p}(\Omega) \cap H^1_0(\Omega),
\]

then the correction function \( w_h + W_h \in S^h_0 \) can be constructed such that \( w_h + W_h = 0 \) at all nodes and

\[
A(\theta - w_h - W_h, v) = r_h, \quad r_h = O(h^{2k}) ||u||_{2k,p} ||v||_{2,p'}, \quad v \in S^h_0,
\]

still holds.

**Proof.** Under the condition (31), the function \( b(x) = b_{k+1}(x) = 0 \) on \( \Gamma_x \), but in general the derivatives \( D_x^{2j} b(x) \neq 0 \) on \( \Gamma_x \). So we shall construct a \( k \)-degree polynomial \( G(x) \) in a whole interval \( J_x = (0, 1) \), \( G(0) = G(1) = 0 \), such that

\[
F(x) = b(x) - G(x), \quad D_x^l F(0) = D_x^l F(1) = 0, \quad l = 2, 4, \ldots, 2m < k - 1.
\]

Thus the problem (13) is decomposed to

\[
A(\theta, v) = \sum_B h \int_{B'} F_x(x) M_{k+1}(t) v_x dx dt + \sum_B h \int_{B'} G_x(x) M_{k+1}(t) v_x dx dt.
\]

Note that \( F(x) \in \dot{W}^{k-1,p}(J_x) \), by Theorem 2 we can construct the corresponding \( w_h \in S^h_0(B) \) such that

\[
A(\theta - w_h, v) = \sum_B h \int_{B'} G_x(x) M_{k+1}(t) v_x dx dt + r_h, \quad v \in S^h_0,
\]

where \( r_h \) is of the highest order (22). Then we will treat \( G \). Because the \( k \)-degree polynomial \( G(x) \) contains only \( k - 1 \) degrees of freedom, we shall directly construct the desired correction polynomial \( W_h \in S^h_0(B') \) such that (32) holds.
1. The construction of $G$. Note that the function
\[ b(x) = b_{k+1}(x) = C_1 h^k \int_{y_l}^{y_{l+1}} D_y^{k+1} u(x, y) (t^2 - 1)^k dy \in W^{k-1, p}(J_x), \]
and the derivatives $D_x^{k-2} b(x) \in C(J)$, as the trace of $D_x^{2k-1} u \in W^{1, p}$. Denoting the parameters by
\[ D_x^l b(0) = \alpha_t, \quad D_x^l b(1) = \beta_l, \quad l = 2, 4, \ldots, 2m < k - 1, \]
and using the imbedding theorem, we have
\[ |\alpha_t| + |\beta_l| \leq C \|b\|_{k-1, p, J} \leq C h^{k+1/\nu} \|u\|_{2k, p, B}, \quad l = 2i \leq 2m < k - 1. \]
We introduce the polynomial series
\[
\begin{align*}
\phi_0 &= 1, \quad \phi_1 = x, \quad \phi_2 = (x^2 - x)/2!, \quad \phi_3 = (x^3 - x)/3!, \\
\phi_4 &= (x^4 - 2x^3 + x)/4!, \quad \phi_5 = (3x^5 - 10x^3 + 7x)/(3 \cdot 5!), \ldots,
\end{align*}
\]
with the following properties:
\[ D_x^j \phi_j(x) = \phi_{j-2}(x), \quad \phi_j(0) = \phi_j(1) = 0, \quad \text{for} \quad j \geq 2, \]
\[ D_x^{2i} \phi_{2i}(x) = 1, \quad D_x^{2i} \phi_{2i+1}(x) = x. \]
Using these bases, $\phi_j$, and the parameters $\alpha_j, \beta_j$, we can define the function
\[ g_j(x) = \alpha_j \phi_j(x) + (\beta_j - \alpha_j) \phi_{j+1}(x), \quad g_j(0) = g_j(1) = 0, \quad 2 \leq j = 2i < k - 1, \]
with the derivatives
\[ D_x^{2i} g_j(0) = D_x^{2i} g_j(1) = 0, \quad D_x^j g_j(0) = \alpha_j, \quad D_x^j g_j(1) = \beta_j, \quad 2i < j = 2l < k - 1. \]
Therefore, the $k$-degree polynomial
\[ G(x) = g_2(x) + g_4(x) + \cdots + g_{2m}(x) \in S_0^h(J), \]
has the desired properties in \[(33).\]

2. The construction of $W_h$. For simplicity, we consider only the term $G(x) = \alpha_l \phi_l(x), \quad l = 2i < k - 1$ (or taking $G = (\beta_l - \alpha_l) \phi_{l+1}(x)$) and construct the corresponding finite element solution
\[ W_h(x, t) = \alpha_t \sum_{p,q=2}^{k} a_{pq} \phi_p(x) M_q(t) \in S_0^h(B'), \]
satisfying
\[ A_B(W_h, v') = h \alpha_t \int_{B'} \phi_{ix}(x) M_{k+1}(t) v'_x dx dt, \quad v' \in S_0^h(B'), \]
where $a_{pq}$ are $(k-1)^2$ parameters to be defined. Reducing the factor $\alpha_t$ in both sides of equation \[(41),\] taking $v' = \phi_i(x) M_j(t), \quad i, j = 2, 3, \ldots, k$ (note that $M_0(t), M_1(t) \notin S_0^h(B')$) and denoting by
\[ K_{ij} = (\phi_{ix}, \phi_{jx})_J, \quad m_{ij} = (\phi_i, \phi_j)_J, \quad c_{j+1} = (l_j, l_j)_E, \quad d_{ij} = (M_i, M_j)_E, \]
the integrals in $J_x$ and $E$, where for $i$ fixed, $d_{ij} \neq 0$ for at most three indices $j - i = 0, \pm 2$, we get a linear system of equations
\[
\begin{align*}
\sum_{p,q=2}^{k} K_{pqi} d_{qj} a_{pq} + \sum_{p=2}^{k} m_{pi} c_{pj} a_{pq} &= h^2 d_{k+1,j} K_{li}, \quad i, j = 2, 3, \ldots, k.
\end{align*}
\]
Denoting the unknowns
\[ X_q = (a_{2q}, a_{3q}, \ldots, a_{kq})^T, \quad X = (X_2, X_3, \ldots, X_k)^T, \]
the vector
\[ H_{l,k-1} = d_{k+1,k-1}(K_{l2}, K_{l3}, \ldots, K_{lk})^T, \]
and the symmetrical positive definite matrix of order \( k-1 \),
\[ K = [K_{pi}] > 0, \quad M = [m_{ip}] > 0, \]
then (42) can be rewritten as a three-diagonal matrix equation:

\[
\begin{bmatrix}
  c_2MX_2 \\
  c_3MX_3 \\
  \vdots \\
  c_jMX_j \\
  \vdots \\
  c_{k-1}MX_{k-1} \\
  c_kMX_k
\end{bmatrix} + \begin{bmatrix} K(d_{22}X_2 + d_{24}X_4) \\
  K(d_{33}X_3 + d_{35}X_5) \\
  \vdots \end{bmatrix} = \begin{bmatrix} 0 \\
  0 \\
  \vdots \end{bmatrix}.
\]

This is an absolutely diagonally dominated linear system. Using Lemma 1 below, we have
(43)
\[ X_k = O(h), \quad X_{k-1} = O(h^2), \quad X_j = O(h^{k-1+j}), \quad X_3 = O(h^{k-2}), \quad X_2 = O(h^{k-1}), \]
where the \( X_2, X_3 \) play an important role in the proof of Theorem 3.

Finally, coming back to (35) and taking a general test function \( v = v' + v^* \in S_0^h(\Omega) \), then (11) is exactly satisfied for the local test function \( v' = \xi M_j(t) \in S_0^h(B') \), \( \xi = \phi_i(x) \), \( i, j = 2, 3, \ldots, k \), and
\[
A(\theta - w_h - W_h, v) = r_h + \sum_B r'_{B}, \quad v = v' + v^* \in S_0^h(\Omega).
\]

Whereas the global test function \( v^* = \xi(x)\eta(t) \), \( \xi = \phi_i, \quad i = 2, 3, \ldots, k, \quad \eta = \gamma_0 + \gamma_1 t \), will bring the following residue in each \( B' \) (note that \( M_{k+1} \perp \eta, \quad k \geq 3 \))
\[
r'_{B} = A_B(W_h, v) - h \int_{B'} G_xM_{k+1}(t)v_xdxdt = A_B(W_h, v^*)
\]
\[
= h \sum_{p,q=2}^{k} \alpha_l \int_{B'} a_{pq}(\phi_{px}M_q(t)\xi_x(\eta t + \gamma_0) + h^{-2}\phi_{lq-1}\xi\eta t + \gamma_0))dxdt,
\]
where \( l_{q-1} \perp \gamma_1 \), the second term disappears automatically, and only \( q = 2, 3 \) in the first term remain. By (37) and (13), \( |\alpha_i| \leq C h^{k+1/p'} |u|_{2p,B}, \quad |X_3|h + |X_2| \leq Ch^{k-1} \), we have
\[
|\sum_B r'_{B}| = h \sum_{B} |\alpha_l \sum_{p,=2}^{k} (\phi_{px}, \xi_x)J(a_{p3d31\gamma_1} + a_{p2d20\gamma_0})|
\]
\[
\leq Ch^{k+1} \sum_B |u|_{2k,p,B}(|X_3|h|\eta| + |X_2| |\eta|)h^{1/p'} ||\xi_x||_{0,p',J}
\]
\[
\leq Ch^{2k} \sum_B |u|_{2k,p,B} ||v||_{2p',B} = r_h.
\]
The other terms of \( G \) can be treated similarly. This completes the proof of Theorem 3. \[ \square \]
In studying the correction function we often encountered a special linear system of \( n \) equations
\[
KX = b, \quad K = [K_{ij}]_{n \times n}, \quad X = (X_1, \ldots, X_n)^T, \quad b = (b_1, \ldots, b_n)^T,
\]
whose coefficients \( K_{ij} \) satisfy
\[
K_{ii} = c_i + O(h), \quad |c_i| \geq c > 0, \quad K_{ij} = O(h^{|i-j|}), \quad i \neq j \leq n,
\]
and is called absolutely diagonally dominated [12, pp. 39-40].

**Lemma 1** ([12]). Assume that for \( h \) suitably small, \( K \) is absolutely diagonally dominated and \( b_j = O(h^{n-j}) \), \( j = 1, 2, \ldots, n \), then the solution \( X = (X_1, X_2, \ldots, X_n)^T \) of (44) satisfies
\[
X_j = O(h^{n-j}), \quad j = 1, 2, \ldots, n.
\]

**Proof.** We give a simplified proof. Using the substitute
\[
Y_j = X_j h^{j-n}, \quad K'_ij = K_{ij} h^{-|i-j|} = O(1), \quad b'_i = b_i h^{i-n} = O(1),
\]
the original \( i \)-th linear equation becomes (deducing a common factor \( h^{n-i} \)
\[
K'_{i1} h^{2i-2} Y_1 + K'_{i2} h^{2i-4} Y_2 + \cdots + K'_{i,i-1} h^2 Y_{i-1} + K'_{i,i} + K'_{i,i+1} Y_{i+1} + \cdots + K'_{i,n} Y_n = b'_i.
\]
This is upper triangularly absolutely dominated. Obviously, \( \det(K') = c + O(h) \), \( c \neq 0 \) for \( h \) suitably small. Then by Cramer’s rule the solution \( Y = O(1) \) is bounded. Lemma 1 is proved.

6. **Proof of Theorem 1**

In the previous two sections, by Theorem 3 for \( u \in W^{2k,p}(\Omega) \cap H_0^1 \), we have constructed a correction function \( W = u_h + W_h \in S_0^h(\Omega), W(z) = 0 \) at all nodes \( z \), and obtained the following basic estimate:
\[
|A(\theta - W, v)| \leq C h^{2k} \|u\|_{2k,p} \|v\|_{2,p'}, \quad v \in S_0^h.
\]
In the following, we shall prove Theorem 1 by the duality argument.

**Proof.** 1. **Discrete \( L^p \)-estimate**, \( 2 \leq p < \infty \). Make a conjugate problem \( g \in H_0^1(\Omega) \) such that \( A(g, v) = (\psi, v) \), \( v \in H_0^1 \). By the theory of partial differential equation in a rectangle, there is the regularity estimate [19]
\[
|g|_{2,p,\Omega} \leq C(p) \|\psi\|_{0,p,\Omega}, \quad 1 < p < \infty.
\]
Let \( g_h \in S_0^h \) be the \( k \)-degree finite element projection of \( g \), \( A(g - g_h, v) = 0, v \in S_h \), so [17] [23]
\[
|g - g_h|_{1,p} \leq C h \|g\|_{2,p} \leq C h \|\psi\|_{0,p}.
\]
Denoting by \( g_I \in S_0^h \) the interpolant of \( g \), and using the inverse estimate we have
\[
|g_I|_{2,p} \leq |g|_{2,p} + |g_h - g_I|_{2,p} \leq C \|g\|_{2,p} + C h^{-1} \|g_h - g_I\|_{1,p} \leq C \|g\|_{2,p}.
\]
For any \( k \) (odd or even), taking \( v = g^2 \) in (46) leads to
\[
|(\theta - W, \psi)| = |A(\theta - W, g_h)| \leq C h^{2k} \|u\|_{2k,p} \|g_h\|_{2,p'} \leq C h^{2k} \|u\|_{2k,p} \|\psi\|_{0,p'}.
\]
Taking \( \psi = |\theta - W|^p \text{sign}(\theta - W) \) and using the inverse estimate, we have
\[
|\theta - W|_{0,p} \leq C h^{2k} \|u\|_{2k,p}, \quad \max_{z \in \Omega} |\theta - W(z)| \leq C h^{-2/p} \|\theta - W\|_{0,p}.
\]
Noting that \( R(z) = W(z) = 0 \) at nodes \( z \in T_h \), we get the discrete \( L^p \)-estimate
\[
|u - u_h|_{0,p,T_h} = |\theta - W|_{0,p,T_h} \leq C |\theta - W|_{0,p} \leq C h^{2k} \|u\|_{2k,p,T_h}.
\]
In practice, superconvergence in the discrete $L^2$-norm is often used.

2. **Pointwise estimate.** The regularity estimate (17) is invalid for $p = 1, \infty$, so one can introduce the regularized Green’s function $g^*(x) \in H^1_0(\Omega)$ and its finite element projection $g_h^*(x) \in S^h_0$ such that

$$A(g^*, v) = (\delta_h^*, v), \quad v \in H^1_0; \quad A(g^*_h, v) = (\delta_h^*, v) = v(z), \quad v \in S^h_0,$$

where $\delta^*_h$ is the discrete $\Delta$ function satisfying $\langle \delta^*_h, v \rangle = v(z)$, $v \in S^h_0$. For the linear element, the following basic estimates are proved by the weighted norm (17),

$$\|g^*\|_{2,1} \leq C |\ln h|, \quad \|g^* - g^*_h\|_{1,1} \leq Ch |\ln h|, \quad \|g^*_h\|_{2,1} \leq C\|g^*\|_{2,1} \leq C |\ln h|,$$

which are also valid for degree $k > 1$ (maybe the factor $|\ln h|$ can be removed; see remark 3).

Taking $v = g^*_h$ and $p = \infty$ in (46), we get the maximum norm estimate

$$\max_{z \in \Omega} |(\theta - W)(z)| \leq Ch^{2k} |\ln h| \|u\|_{2k, \infty}.$$

Recalling $u - u_h = R - \theta$ and $R = W = 0$ at node $z \in T_h$, we have

$$\max_{z \in T_h} |(u - u_h)(z)| \leq Ch^{2k} |\ln h| \|u\|_{2k, \infty}.$$

Summarizing (49) and (51), Theorem 1 is proved. \qed

**Remark 3.** It is possible to remove the logarithm factor $|\ln h|$ in (5) for $k > 1$. Scott [25] studied the Neumann problem $-\Delta u + u = f$ on a smooth convex domain and proved

$$\|g^* - g^*_h\|_{1,1} \leq Ch^k |\ln h|^n(k), \quad n(k) = 1 \text{ if } k = 1, \quad n(k) = 0 \text{ if } k > 1,$$

where no factor $|\ln h|$ appears if $k > 1$. Thus, it directly leads to the optimal maximum estimate

$$\|u - u_h\|_{0, \infty} \leq Ch^{k+1} \|u\|_{k+1, \infty}, \quad k > 1.$$

Note that it is not immediately clear if the analysis of Scott’s results can be extended for convex polygonal domains.

**Remark 4.** In this paper, recalling the proofs of Theorems 2 and 3, we found that when $\Omega$ is a non-convex polygonal domain formed by several rectangles (for example, the L-shaped domain), the error estimate (49) (for example, taking $p = p' = 2$) is still valid. However, when $\Omega$ has the maximum inner angle $\alpha \pi$, $\alpha = 3/2$, the solution $g \in H^1_0$ of problem $A(g, v) = (\psi, v), \quad v \in H^1_0$ has the lower regularity [19]

$$\|g\|_{2, p'} \leq C(p') \|\psi\|_{0, p'}, \quad 1 < p' < 2\alpha/(2\alpha - 1) = 3/2.$$

Because the spaces $W^{2, p'}$ can be embedded into $H^s$ with the index $2 - 2/p' = s - 1$, $\|g\|_{s, 2} \leq C \|g\|_{2, p'}$, using the error estimate $\|g - g_h\|_1 \leq Ch^{s-1} \|g\|_{s, 2}$ and the inverse property

$$\|g_h\|_2^2 \leq Ch^{s-2} \|g_h\|_s^2 \leq Ch^{s-2} (\|g_T\|^*_s + Ch^{s-1} \|g_h - g_T\|_1) \leq Ch^{s-2} \|\psi\|,$$

$\beta = s - 2 = 1 - 2/p' < -1/3$, we get (similar to (48) and (49))

$$\|\theta - W\| \leq Ch^{2k+\beta} \|u\|_{2k}, \quad \|u - u_h\|_{0, \Omega_h} \leq Ch^{2k+\beta} \|u\|_{2k}.$$

Therefore, the concave angle $3\pi/2$ makes the superconvergence decrease by about $h^{1/3}$. 

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7. Numerical experiments for \( k = 4, 5 \)

Consider the Poisson equation in a square \( \Omega = (0, 1)^2 \),

\[-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.\]

The exact solution \( u(x, y) = \sin(3\pi x)e^{(2x-1,1)^2} \sin(2\pi y)e^{(3y-1,4)^2} \) is given. Subdivide \( \Omega \) into \( N \times N \) square meshes \( Z_N \), \( N = 4, 8, 16 \). Denote by \( e^N = u^N - u \) the error of bi-\( k = 4, 5 \) degree finite element solutions \( u^N \). We investigate the error at nine common nodes \( x_i, y_j = 0.25, 0.50, 0.75 \) and their averaging square root value

\[|e^N|_2 = \left( \sum_{i,j=1}^{3} |e^N(x_i, y_j)|^2 / 9 \right)^{1/2}, \quad \text{(theoretical ratio } 2^{2k}).\]

For comparison, the \( L^2 \)-norm error \( ||e^N|| \) is listed in the last line (theoretical ratio \( 2^{k+1} \)).

Table 1. \( N = 4, 8, 16, 32 \), the errors of bi-4 degree elements at nine nodes (error ratio), theoretical ratio \( tr = 256 \).

| \((x_i, y_j)\) | \(|e^4|\) | \(|e^8|(tr = 256)\) | \(|e^{16}|(tr = 256)\) | \(|e^{32}|(tr = 256)\) |
|---------------|---------|----------------|----------------|----------------|
| (0.25, 0.25)  | 8.0760e-6 | 7.2850e-8(111) | 3.2721e-10(223) | 1.4202e-12(230) |
| (0.50, 0.25)  | 1.0131e-5 | 5.0669e-8(200) | 2.0589e-10(246) | 8.1029e-13(254) |
| (0.75, 0.25)  | 1.8425e-5 | 2.2812e-7(81)  | 1.0350e-09(220) | 4.3041e-12(240) |
| (0.25, 0.50)  | 6.1598e-5 | 2.5503e-7(243) | 1.0136e-09(252) | 4.0059e-12(253) |
| (0.50, 0.50)  | 1.5061e-5 | 5.8385e-8(258) | 2.3192e-10(252) | 9.4545e-13(245) |
| (0.75, 0.50)  | 1.5927e-4 | 6.6880e-7(238) | 2.6839e-09(249) | 1.0643e-11(252) |
| (0.25, 0.75)  | 2.2390e-5 | 1.1232e-7(199) | 4.6484e-10(242) | 1.9191e-12(242) |
| (0.50, 0.75)  | 9.4271e-6 | 4.1842e-8(225) | 1.6772e-10(249) | 6.5090e-13(258) |
| (0.75, 0.75)  | 5.7601e-5 | 3.1799e-7(181) | 1.3345e-09(238) | 5.3877e-12(248) |
| \( |e^N|_2 \)   | 6.1327e-5 | 2.7711e-7(221) | 1.1319e-09(245) | 4.5284e-12(213) |
| \( ||e^N|| \) | 6.0608e-3 | 2.9991e-4(20)  | 1.0902e-5(28)   | 3.5604e-7(31)   |

We observe in Table 1 that the ratio of errors \( |e^N|_2 \) on the meshes \( Z_4, Z_8, Z_{16}, Z_{32} \) are 221, 245, 213, respectively. Their accuracy increases 3, 4, 5 digitally respectively compared with \( ||e^N|| \). The error ratios at different nodes have some vibration.

We observe in Table 2 that the ratio of errors \( |e^N|_2 \) on the triple meshes \( Z_4, Z_8, Z_{16} \) are 978, 1113, respectively. Their accuracy increases 3, 4, 5 digitally respectively compared with \( ||e^N|| \). Because the double accuracy is accepted, the error \( |e^{16}| \) is inexact.

The error at a node depends on the local property of derivatives of \( u \), and the ratios of error on different nodes can be off if the solution is very oscillatory (for example, the error at \((0.25, 0.75)\) is less), so the discrete \( L^2 \)-norm \( |e^N|_2 \) will be more stable.
Table 2. \( N = 4,8,16 \), the errors of bi-5 degree elements at nine nodes (error ratio), theoretical ratio \( tr = 1024 \).

<table>
<thead>
<tr>
<th>((x_i, y_j))</th>
<th>(e^1)</th>
<th>(e^3)</th>
<th>(e^5)</th>
<th>(e^{10})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.25, 0.25)</td>
<td>3.5851e - 7</td>
<td>4.7001e - 10</td>
<td>4.1633e - 13</td>
<td>1129</td>
</tr>
<tr>
<td>(0.50, 0.25)</td>
<td>7.1605e - 8</td>
<td>11523</td>
<td>9.7703e - 14</td>
<td>733</td>
</tr>
<tr>
<td>(0.75, 0.25)</td>
<td>1.3176e - 8</td>
<td>9(704)</td>
<td>1.2159e - 12</td>
<td>1132</td>
</tr>
<tr>
<td>(0.25, 0.50)</td>
<td>5.0654e - 7</td>
<td>10(1249)</td>
<td>4.1145e - 13</td>
<td>986</td>
</tr>
<tr>
<td>(0.50, 0.50)</td>
<td>1.9526e - 7</td>
<td>10(1202)</td>
<td>1.4727e - 13</td>
<td>1326</td>
</tr>
<tr>
<td>(0.75, 0.50)</td>
<td>1.2171e - 6</td>
<td>9(1248)</td>
<td>1.1036e - 12</td>
<td>1103</td>
</tr>
<tr>
<td>(0.25, 0.75)</td>
<td>1.0110e - 8</td>
<td>10(225)</td>
<td>8.4821e - 14</td>
<td>1120</td>
</tr>
<tr>
<td>(0.50, 0.75)</td>
<td>1.0028e - 8</td>
<td>10(949)</td>
<td>1.122e - 13</td>
<td>902</td>
</tr>
<tr>
<td>(0.75, 0.75)</td>
<td>2.8026e - 10</td>
<td>503</td>
<td>2.4691e - 13</td>
<td>1135</td>
</tr>
<tr>
<td>(e^{12}_{12})</td>
<td>6.4422e - 7</td>
<td>9(78)</td>
<td>5.9167e - 13</td>
<td>1113</td>
</tr>
<tr>
<td>(|e^N|)</td>
<td>1.0421e - 3</td>
<td>503</td>
<td>2.5270e - 05</td>
<td>56</td>
</tr>
</tbody>
</table>

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References


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