NEW RESULTS ON REVERSE ORDER LAW FOR \{1, 2, 3\}- AND
\{1, 2, 4\}-INVERSSES OF BOUNDED OPERATORS

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Abstract. In this paper, using some block-operator matrix techniques, we give necessary and sufficient conditions for the reverse order law to hold for \{1, 2, 3\}- and \{1, 2, 4\}-inverses of bounded operators on Hilbert spaces. Furthermore, we present some new equivalents of the reverse order law for the Moore-Penrose inverse.

1. Introduction

Let \(\mathcal{H}\) and \(\mathcal{K}\) be complex Hilbert spaces and let \(\mathcal{B}(\mathcal{H}, \mathcal{K})\) denote the set of all bounded linear operators from \(\mathcal{H}\) to \(\mathcal{K}\). For a given \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K})\), the symbols \(N(A)\) and \(\mathcal{R}(A)\) denote the null space and the range of \(A\), respectively. For given subsets \(M, N\) of \(\mathcal{B}(\mathcal{H}, \mathcal{K})\), by \(MN\) we denote the set consisting of all products \(XY\), where \(X \in M\) and \(Y \in N\).

Recall that \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K})\) has the Moore-Penrose inverse if there exists an operator \(X \in \mathcal{B}(\mathcal{K}, \mathcal{H})\) such that
\[
\begin{align*}
&1. AXA = A, \\
&2. XAX = X, \\
&3. (AX)^* = AX, \\
&4. (XA)^* = XA.
\end{align*}
\]

The Moore-Penrose inverse of an operator \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K})\) exists if and only if \(A\) has closed range and in this case it is unique. It is denoted by \(A^\dagger\).

For any \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K})\), let \(A\{i, j, \ldots, k\}\) denote the set of all operators \(X \in \mathcal{B}(\mathcal{K}, \mathcal{H})\) which satisfy equations (1), (2), ..., (4) of (1.1). In this case \(X\) is a \(\{i, j, \ldots, k\}\)-inverse of \(A\) and is denoted by \(A^{(i,j,\ldots,k)}\). Evidently, \(A\{1, 2, 3, 4\} = \{A^\dagger\}\) when \(A\) has closed range.

The reverse order law for generalized inverses plays an important role in theoretic research and numerical computations in many areas, including the singular matrix problem, ill-posed problems, optimization problems, and statics problems (see, for instance, [1, 9, 13, 14, 18]). These problems have attracted considerable attention since the mid-1960s, and many interesting results for generalized inverses of products of matrices or operators have been obtained (see [5], [6], [10], [11], [15]–[17]). T.N.E. Greville [8] first proved that \((AB)^\dagger = B^\dagger A^\dagger\) if and only if \(\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)\) and \(\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)\), for matrices \(A\) and \(B\). This result was extended to linear bounded operators on Hilbert spaces in [11]. Later, the reverse order law for the Moore-Penrose inverse was considered in rings with involution (see [12]).
Xiong and Zheng [20] considered the reverse order law for \( \{1, 2, 3\} \)- and \( \{1, 2, 4\} \)-generalized inverses of products of two matrices. Their techniques involved expressions for maximal and minimal ranks of the generalized Schur complement. In [2] the authors considered the reverse order law for \( K \)-inverses in the cases \( K \in \{\{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\} \) for elements of a C*-algebra.

In this paper, using block-operator matrix techniques, we consider the reverse order law for \( \{1, 2, 3\} \)- and \( \{1, 2, 4\} \)-inverses of bounded linear operators on Hilbert spaces. We give necessary and sufficient conditions for

\[
B\{1, 2, 3\} \cdot A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}
\]

and

\[
B\{1, 2, 4\} \cdot A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}.
\]

We generalized the results from [2] to the case of bounded linear operators on Hilbert spaces. This is of particular importance especially in statistics where bounded linear operators play a very important role. Furthermore, we present new equivalent conditions for the reverse order law for the Moore-Penrose inverse. It should be pointed out that when restricted to the set of matrices, our results for the reverse order law for the Moore-Penrose inverse yield facts not previously known.

2. Preliminaries

Let \( \mathcal{H}, \mathcal{K} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) have closed range. The operator \( A \) has the following matrix decomposition (see [4], [7])

\[
A = \begin{bmatrix}
A_1 & 0 \\
0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{R}(A^*) \\
\mathcal{N}(A)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A^*)
\end{bmatrix},
\]

where \( A_1 \) is invertible. Also, \( A^\dagger \) has the form

\[
A^\dagger = \begin{bmatrix}
A_1^{-1} & 0 \\
0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A^*)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{R}(A^*) \\
\mathcal{N}(A)
\end{bmatrix}.
\]

If \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) has closed range, then we can explicitly describe the sets \( A\{1, 2, 3\} \) and \( A\{1, 2, 4\} \) using the representation of \( A \) given by (2.1).

**Lemma 2.1.** Let \( \mathcal{H}, \mathcal{K} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) have closed range. Then

\[
A\{1, 2, 3\} = \left\{ \begin{bmatrix}
A_1^{-1} & 0 \\
X_3 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A^*)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{R}(A^*) \\
\mathcal{N}(A)
\end{bmatrix} : X_3 \in \mathcal{B}(\mathcal{R}(A), \mathcal{N}(A)) \right\}
\]

and

\[
A\{1, 2, 4\} = \left\{ \begin{bmatrix}
A_1^{-1} & X_2 \\
0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A^*)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{R}(A^*) \\
\mathcal{N}(A)
\end{bmatrix} : X_2 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{R}(A^*)) \right\}.
\]

**Proof.** Suppose that \( A \) and \( A^\dagger \) are given by (2.1) and (2.2), respectively. Since

\[
A\{1, 2, 3\} = \{ A^\dagger + (I - A^\dagger A)X A A^\dagger : X \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \},
\]
we have that $A^{(1,2,3)} \in A\{1,2,3\}$ if and only if

$$A^{(1,2,3)} = A^\dagger + (I - A^\dagger)XXA^\dagger$$

for some $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$.

If $N = XU$, we get that

$$N = XU$$

Remark where

First, let us note that

Proof. First, let us note that

$$(\Rightarrow)$$ Suppose that $R(B^*) \cap N(AB) = \{0\}$ and let $x \in R(B) \cap N(A)$. By (2.3), there exists $y \in R(B^*)$ such that $By = x$. Now, $y \in R(B^*) \cap N(AB)$, i.e., $y = 0$, so $x = By = 0$.

$$(\Leftarrow)$$ If we suppose that $R(B) \cap N(A) = \{0\}$ and take $u \in R(B^*) \cap N(AB)$, we get that $Bu \in R(B) \cap N(A)$, i.e., $Bu = 0$. Using (2.3) it follows that $u = 0$.

Let us introduce the following notation: if a Hilbert space $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$ where $\mathcal{U}_i \perp \mathcal{U}_j$ for $i \neq j$, then we shall write $\mathcal{H} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$. If $\mathcal{U}$ is a complement space of a Hilbert space $\mathcal{H}$ we shall denote by $\mathcal{H} \oplus \perp \mathcal{U}$ the unique subspace $\mathcal{V}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{U} \oplus \perp \mathcal{V}$.

Remark 2.1. Let $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $R(A), R(B)$ and $R(AB)$ are closed. Denote by

\[
\begin{align*}
\mathcal{H}_1 &= R(B) \cap N(A), \\
\mathcal{H}_2 &= R(B) \ominus \perp \mathcal{H}_1, \\
\mathcal{H}_3 &= N(B^*) \cap N(A), \\
\mathcal{H}_4 &= N(B^*) \ominus \perp \mathcal{H}_3,
\end{align*}
\]

Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{L}$ can be decomposed as

$\mathcal{H} = R(B) \oplus \perp N(B^*), \quad \mathcal{K} = R(A) \oplus \perp N(A^*), \quad \mathcal{L} = R(B^*) \ominus \perp N(B),$

where

$R(B) = \mathcal{H}_1 \oplus \perp \mathcal{H}_2, \quad N(B^*) = \mathcal{H}_3 \ominus \perp \mathcal{H}_4, \quad R(A) = \mathcal{K}_1 \oplus \perp \mathcal{K}_2, \quad R(B^*) = \mathcal{L}_1 \ominus \perp \mathcal{L}_2.$

We can prove that $B^\dagger (R(B) \cap N(A)) = R(B^*) \cap N(AB).$ Indeed let $x \in R(B^*) \cap N(AB).$ Then $x \in R(B^\dagger) = R(B^\dagger B) = B^\dagger R(B)$ and $ABx = 0$, i.e., $Bx \in N(A).$ So we have $x = B^\dagger Bx \in B^\dagger N(A).$ Finally, $x \in B^\dagger (R(B) \cap N(A))$. On the other hand, let $y \in B^\dagger (R(B) \cap N(A))$, i.e., $y \in R(B^\dagger B) = R(B^*)$, then for some $z \in R(B) \cap N(AB)$ we have $y = B^\dagger z$. So $ABy = ABB^\dagger z = Az = 0$, i.e., $y \in N(AB)$. Thus, by Lemma 2.2, we get

$\mathcal{H}_2 = R(B) \Leftrightarrow \mathcal{H}_1 = \{0\} \Leftrightarrow N(AB) = N(B) \Leftrightarrow \mathcal{L}_1 = \{0\} \Leftrightarrow \mathcal{L}_2 = R(B^*)$. 

Let us introduce the following notation: if a Hilbert space $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$ where $\mathcal{U}_i \perp \mathcal{U}_j$ for $i \neq j$, then we shall write $\mathcal{H} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$. If $\mathcal{U}$ is a complement space of a Hilbert space $\mathcal{H}$ we shall denote by $\mathcal{H} \oplus \perp \mathcal{U}$ the unique subspace $\mathcal{V}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{U} \oplus \perp \mathcal{V}$.
Furthermore,
\[ \mathcal{H}_2 = \{0\} \iff \mathcal{H}_1 = \mathcal{R}(B) \iff \mathcal{R}(B) \subset \mathcal{N}(A) \iff \mathcal{L}_1 = \mathcal{R}(B^*) \iff \mathcal{L}_2 = \mathcal{R}(A). \]

Throughout the paper we will use the notation from the above remark.

A similar result to the following one but for the case of two Hilbert spaces has been presented in [17]. Now we give a different proof in the case of three Hilbert spaces.

**Lemma 2.3.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B) \) and \( \mathcal{R}(AB) \) are closed.

1. If \( AB \neq 0 \) and \( \mathcal{N}(AB) \neq \mathcal{N}(B) \), then \( A \) and \( B \) have the following operator matrix forms:

   \[
   A = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix},
   \]

   and

   \[
   B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},
   \]

   where \( A_{12}, B_{11}, B_{22} \) are invertible operators and \( A_{24} \) is a surjection.

2. If \( AB \neq 0 \) and \( \mathcal{N}(AB) = \mathcal{N}(B) \), then \( A \) and \( B \) have the following operator matrix forms:

   \[
   A = \begin{bmatrix} A_{12} & 0 & A_{14} \\ 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix},
   \]

   and

   \[
   B = \begin{bmatrix} B_{22} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},
   \]

   where \( A_{12}, B_{22} \) are invertible operators and \( A_{24} \) is a surjection.

3. If \( AB = 0 \) and \( \mathcal{N}(AB) \neq \mathcal{N}(B) \), then \( A \) and \( B \) have the following operator matrix forms:

   \[
   A = \begin{bmatrix} 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},
   \]

   and

   \[
   B = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},
   \]

   where \( B_{11} \) and \( A_{24} \) are invertible.
Proof. We will assume that the spaces \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) are decomposed as in Remark 2.1, so that the conclusions of the remark also hold. 

(1) Suppose that \( AB \neq 0 \) and \( N(AB) \neq N(B) \). We have that \( B \) can be represented by

\[
B = \begin{bmatrix}
B_{11} & B_{12} & 0 \\
B_{21} & B_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
L_1 \\
L_2 \\
N(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
\]

where \( \widehat{B} = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix} : \begin{bmatrix}
L_1 \\
L_2
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2
\end{bmatrix} \) is invertible.

Since \( B\mathcal{L}_1 \subset \mathcal{H}_1 \), we get that \( B_{21} = 0 \). Now from the invertibility of \( \widehat{B} \), we get that \( B_{11} : \mathcal{L}_1 \rightarrow \mathcal{H}_1 \) and \( B_{22} : \mathcal{L}_2 \rightarrow \mathcal{K}_2 \) are invertible.

Now, we will prove that \( A \) has a matrix form given by (2.4). Suppose that

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34}
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
N(A^*)
\end{bmatrix}.
\]

The reductions \( A_{1i} \) and \( A_{i3} \) for \( i = 1, 2, 3 \) are null operators because \( \mathcal{K}_1, \mathcal{K}_3 \subset N(A) \). The range of the reductions \( A_{3j}, (j = 1, 2, 3, 4) \) is \( N(A^*) \), so \( A_{3j} = 0 \).

Now we will prove that \( A_{22} = 0 \): for any \( x \in \mathcal{K}_2 \subset \mathcal{R}(B) \), there exists \( y \in \mathcal{K} \) such that \( By = x \). Now, \( Ax = ABy \in \mathcal{K}_1 \) and \( Ax = A_{12}x + A_{22}x \). Since \( A_{12}x \in \mathcal{K}_1 \), we get that \( A_{22}x = 0 \).

In order to prove that \( A_{12} \) is bijective, first we will prove that \( N(A_{12}) = \{0\} \): let \( u \in \mathcal{K}_2 \) be such that \( A_{12}u = 0 \). Then \( u \in N(A) \) which implies that \( u \in \mathcal{K}_1 \cap \mathcal{K}_2 = \{0\} \).

To prove that \( A_{12} : \mathcal{K}_2 \rightarrow \mathcal{K}_1 \) is surjective take any \( k \in \mathcal{K}_1 = \mathcal{R}(AB) \). There exists \( k' \in \mathcal{K} \) such that \( ABk' = k \). Since \( Bk' \in \mathcal{R}(B) = \mathcal{K}_1 \oplus \mathcal{H}_2 \), there exist \( h_1 \in \mathcal{H}_1 \) and \( h_2 \in \mathcal{K}_2 \) such that \( Bk' = h_1 + h_2 \). Now, \( Ah_2 = A(Bk' - h_1) = k \), i.e., \( A_{12}h_2 = k \).

The surjective property of \( A_{24} : \mathcal{H}_4 \rightarrow \mathcal{K}_2 \) follows from the fact that for any \( u \in \mathcal{K}_2 \), there exists \( v \in \mathcal{K} \) such that \( Av = u \). Let us decompose \( v = \sum_{i=1}^{4} v_i \), where \( v_i \in \mathcal{H}_i \). It is evident that \( A_{24}v_4 = u \).

The proof of (2) and (3) is analogous. \( \square \)

3. Main results

Z. Xiong and B. Zheng [20] presented necessary and sufficient conditions for

\[
B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\},
\]

in the case when \( A \) and \( B \) are matrices. Here, we give another characterization of (3.1) for linear bounded operators on Hilbert spaces using techniques which are completely different from those used in [20]. First, we will give the following remark:

Remark 3.1. Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B) \) and \( \mathcal{R}(AB) \) are closed, \( AB \neq 0 \) and \( N(AB) \neq N(B) \). Then we can suppose that the operators \( A \) and \( B \) are represented by (2.4) and (2.5),
respectively. By Lemma 2.1, $X \in B\{1, 2, 3\}$ if and only if there exist operators $F_{11}$ and $F_{12}$ such that

$$
(3.2) \quad X = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1} B_{12} B_{22}^{-1} & 0 & 0 \\ 0 & B_{22}^{-1} & 0 & 0 \\ F_{11} & F_{12} & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \mathcal{K}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ N(B) \end{bmatrix}.
$$

To describe the set $A\{1, 2, 3\}$, suppose that an arbitrary $Y \in A\{1, 2, 3\}$ is given by

$$
(3.3) \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \\ Y_{41} & Y_{42} & Y_{42} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \mathcal{K}_4 \end{bmatrix}.
$$

Since $AY$ is hermitian, we get that

$$
AY = \begin{bmatrix} A_{12} Y_{21} + A_{14} Y_{41} & A_{12} Y_{22} + A_{14} A_{42} & 0 \\ A_{24} Y_{41} & A_{24} Y_{42} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ N(A^*) \end{bmatrix},
$$

where $A_{12} Y_{22} + A_{14} A_{42} = (A_{24} Y_{41})^*$ and $A_{12} Y_{21} + A_{14} Y_{41}$, $A_{24} Y_{42}$ are hermitian. Since $AY$ is an orthogonal projection on $\mathbb{R}(A)$, from the definition of the subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ we can conclude that $A_{12} Y_{21} + A_{14} Y_{41} = I$, $A_{24} Y_{42} = I$, $A_{12} Y_{22} + A_{14} A_{42} = 0$ and $A_{24} Y_{41} = 0$. Now, from $YAY = 0$, we get that $Y_{i3} = 0$, for $i = 1, 4$. Hence, $Y \in A\{1, 2, 3\}$ if and only if

$$
(3.4) \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ Y_{31} & Y_{32} & 0 \\ Y_{41} & Y_{42} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \mathcal{K}_4 \end{bmatrix},
$$

where the $Y_{ij}$ satisfy the following equalities:

$$
(3.5) \quad \begin{cases} Y_{2i} A_{24} Y_{42} = Y_{i2}, & i = 1, 4, \\
A_{12} Y_{21} + A_{14} Y_{41} = I \mathcal{K}_1, \\
A_{12} Y_{22} + A_{14} A_{42} = 0, \\
A_{24} Y_{42} = I \mathcal{K}_2, A_{24} Y_{41} = 0. \end{cases}
$$

Since

$$
(3.6) \quad AB = \begin{bmatrix} 0 & A_{12} B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ N(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ N(A^*) \end{bmatrix},
$$

we get that $Z \in (AB)\{1, 2, 3\}$ if and only if there exist operators $N_1$ and $N_2$ such that

$$
(3.7) \quad Z = \begin{bmatrix} N_1 & 0 & 0 \\ B_{22}^{-1} A_{12}^{-1} & 0 & 0 \\ N_2 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ N(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ N(B) \end{bmatrix}.
Theorem 3.1. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:

(i) $B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$.

(ii) $\mathcal{R}(B) = \mathcal{R}(A^*AB) \oplus [\mathcal{R}(B) \cap \mathcal{N}(A)]$, $\mathcal{R}(AB) = \mathcal{R}(A)$.

Proof. We use the decompositions of the spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ and the matrix decompositions of operators $A$, $B$ given in Lemma 2.3. We distinguish two cases:

(1) First, suppose that $\mathcal{N}(AB) \neq \mathcal{N}(B)$. We have that the operators $A$ and $B$ are represented by (2.4) and (2.5), respectively. Also, in Remark 3.1, we have given characterizations of the sets $A\{1, 2, 3\}$, $B\{1, 2, 3\}$ and $(AB)\{1, 2, 3\}$ which we will use in this proof.

(i) $\Rightarrow$ (ii): Let arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ be given by (3.2) and (3.4), respectively. Then

\[
XY = \begin{bmatrix}
M_1 & M_2 & 0 \\
B_{22}^{-1}Y_{22} & B_{22}^{-1}Y_{22} & 0 \\
F_{11}Y_{11} + F_{12}Y_{21} & F_{11}Y_{12} + F_{12}Y_{22} & 0
\end{bmatrix}
\]

(3.8)

where $M_1 = B_{11}^{-1}Y_{11} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{21}$, $M_2 = B_{11}^{-1}Y_{12} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{22}$.

Since $XY \in (AB)\{1, 2, 3\}$, we conclude that $XY$ must be of the form $Z$ given by (3.7) for some operators $N_1$ and $N_2$. If we compare (3.8) and (3.7), we get that $Y_{12} = 0$, $Y_{22} = 0$ and $Y_{21} = A_{12}^{-1}$. Hence, it follows that the system of the operator equations (3.5) is such that $Y_{12}$, $Y_{22}$ and $Y_{21}$ are uniquely determined. Since $A_{24}$ is surjective and $A_{24}Y_{42} = I\mathbb{K}_2$ we get that $Y_{12} = 0$ if and only if $\mathbb{K}_2 = \{0\}$, i.e., $A_{24} = 0$. If this were not true, then $Y_{12}$ could be taken to be an arbitrary operator on an appropriate subspace, which is not the case. Now, from the first equation of (3.5), we get that $Y_{12} = 0$, $i = 1, 4$. Since $Y_{41}$ can be arbitrary, to avoid that $Y_{21} = A_{12}^{-1}$ it must be that $A_{14} = 0$. It is evident that $A_{24} = 0$ is equivalent to $\mathcal{R}(AB) = \mathcal{R}(A)$.

Now, simple computation shows that

\[
A^*AB = \begin{bmatrix}
0 & 0 & 0 \\
0 & A_{12}^*A_{12}B_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
\mathcal{N}(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathbb{K}_1 \\
\mathbb{K}_2 \\
\mathbb{K}_3 \\
\mathbb{K}_4
\end{bmatrix},
\]

and finally we get $\mathcal{R}(A^*AB) = \mathcal{R}(B) \oplus [\mathcal{R}(B) \cap \mathcal{N}(A)]$.

(ii) $\Rightarrow$ (i): Suppose $\mathcal{R}(A^*AB) = \mathcal{R}(B) \oplus [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and $\mathcal{R}(AB) = \mathcal{R}(A)$. We must show that for arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ there exists $Z \in (AB)\{1, 2, 3\}$ such that $XY = Z$.

From $\mathcal{R}(AB) = \mathcal{R}(A)$, we get that $\mathbb{K}_2 = \{0\}$, i.e., $A_{24} = 0$. Also by $\mathcal{R}(A^*AB) = \mathcal{R}(B) \oplus [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and the fact that

\[
A^*AB = \begin{bmatrix}
0 & 0 & 0 \\
0 & A_{12}^*A_{12}B_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
\mathcal{N}(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathbb{K}_1 \\
\mathbb{K}_2 \\
\mathbb{K}_3 \\
\mathbb{K}_4
\end{bmatrix},
\]
where $A_{12}$ and $B_{22}$ are invertible, we have $A_{14} = 0$.

Now, we get that $Y \in A\{1, 2, 3\}$ if and only if

\begin{equation}
Y = \begin{bmatrix}
Y_{11} & 0 & 0 \\
A_{12} & 0 & 0 \\
Y_{31} & 0 & 0 \\
Y_{41} & 0 & 0
\end{bmatrix} \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
N(A^*) \\
\mathcal{K}_3 \\
\mathcal{K}_4
\end{bmatrix} = \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
\end{equation}

where $Y_{11}, Y_{31}, Y_{14}$ are arbitrary. It is evident that for arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ there exists $Z \in (AB)\{1, 2, 3\}$ such that $XY = Z$, i.e.,

$B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}.$

(2) When $N(AB) = N(B)$, the operators $A$ and $B$ are represented by (2.6) and (2.7), respectively, and the proof is analogous to case (1).

**Remark 3.2.** 1° If $AB = 0$, then $(AB)\{1, 2, 3\} = \{0\}$. In the case when $A = 0$ or $B = 0$, evidently $B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$. If it is not the case, we have that $AB = 0 \iff \mathcal{H}_2 = \{0\} \iff \mathcal{K}_1 = \mathcal{K}(B) \iff \mathcal{L}_2 = \{0\} \iff \mathcal{L}_1 = \mathcal{K}(B^*) \iff \mathcal{K}_1 = \{0\} \iff \mathcal{X}_2 = \mathcal{K}(A)$. Also, $A$ and $B$ are represented by (2.8) and (2.9), respectively, so arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ are represented by

\[
X = \begin{bmatrix}
B_{11}^{-1} & 0 & 0 \\
F_1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_3 \\
N(A^*) \\
\mathcal{K}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_4
\end{bmatrix},
\]

and

\[
Y = \begin{bmatrix}
F_2 & 0 \\
F_3 & 0 \\
A_{24}^{-1} & 0
\end{bmatrix} \begin{bmatrix}
\mathcal{K}_2 \\
N(A^*) \\
\mathcal{K}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
\]

for some operators $F_1, F_2$ and $F_3$.

By simple computation, we observe that

\[
XY = \begin{bmatrix}
B_{11}^{-1}F_2 & 0 \\
F_1F_2 & 0
\end{bmatrix} \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_2 \\
\mathcal{K}_4
\end{bmatrix} \neq 0,
\]

i.e., $B\{1, 2, 3\}A\{1, 2, 3\} \neq \{0\}$.

Hence,

$AB = 0, A \neq 0, B \neq 0 \Rightarrow B\{1, 2, 3\}A\{1, 2, 3\} \not\subseteq (AB)\{1, 2, 3\}.$

2° From Theorem 3.1 we conclude that the condition

$$(ABB^\dagger)AABB^\dagger = BB^\dagger \text{ or } (AB)(AB)^\dagger = AA^\dagger$$

from [2] Theorem 3.3] can be replaced by the sole condition $(AB)(AB)^\dagger = AA^\dagger$, i.e., $\mathcal{K}(AB) = \mathcal{K}(A)$.

A similar result in the case $K = \{1, 2, 4\}$ follows from Theorem 3.1 by reversal of products:

**Theorem 3.2.** Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:

(i) $B\{1, 2, 4\}A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}$.

(ii) $\mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus_{\perp} [\mathcal{R}(A^*) \cap N(B^*)]$, $N(AB) = N(B)$. 

Remark 3.3. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed, $AB \neq 0$ and $N(AB) \neq N(B)$. We have that operator $B$ is represented by (2.5), so

\begin{equation}
(3.10) \quad B^\dagger = \begin{bmatrix}
B^{-1}_{11} & -B^{-1}_{11}B_{12}B^{-1}_{22} & 0 & 0 \\
0 & B^{-1}_{22} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
N(B)
\end{bmatrix}.
\end{equation}

Also, if we suppose that the operator $A$ is represented by (2.5), by Remark 3.1 we get that $A^\dagger = Y$ is represented by (3.4), where $Y_{ij}$ satisfy (3.5). Now, since $YA$ is an orthogonal projection on $N(A)^\perp$, we get that

$$YA = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & I
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},$$

which implies that $Y_{21}A_{12} = I_{3\mathcal{H}_2}, Y_{41}A_{12} = 0, Y_{21}A_{14} + Y_{22}A_{24} = 0, Y_{41}A_{14} + Y_{42}A_{24} = I_{3\mathcal{H}_4}$. Now by $YAY = 0$ we get that $Y_{11} = 0, Y_{12} = 0, Y_{31} = 0$ and $Y_{32} = 0$. Hence,

\begin{equation}
(3.11) \quad A^\dagger = \begin{bmatrix}
0 & Y_{12} & 0 \\
Y_{21} & Y_{22} & 0 \\
0 & Y_{32} & 0 \\
Y_{41} & Y_{42} & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
N(A^*)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
\end{equation}

where

\begin{equation}
(3.12) \begin{cases}
Y_{i2}A_{24}Y_{42} = Y_{i2}, i = 1, 4, \\
A_{12}Y_{21} + A_{14}Y_{41} = I_{\mathcal{H}_2}, \\
A_{12}Y_{22} + A_{14}Y_{42} = 0, \\
A_{24}Y_{42} = I_{\mathcal{H}_2}, A_{24}Y_{41} = 0, \\
Y_{21}A_{14} + Y_{22}A_{24} = 0, \\
Y_{41}A_{14} + Y_{42}A_{24} = I_{3\mathcal{H}_4}, \\
Y_{21}A_{12} = I_{3\mathcal{H}_2}, Y_{41}A_{12} = 0.
\end{cases}
\end{equation}

Simple computation shows that

\begin{equation}
(3.13) \quad B^\dagger A^\dagger = \begin{bmatrix}
-B^{-1}_{11}B_{12}B^{-1}_{22}Y_{21} & M_3 & 0 \\
B^{-1}_{22}Y_{21} & B^{-1}_{22}Y_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
N(A^*)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
N(B)
\end{bmatrix},
\end{equation}

where $M_3 = B^{-1}_{11}Y_{12} - B^{-1}_{11}B_{12}B^{-1}_{22}Y_{22}$.

Using the previous remark, we obtain the following result:

**Theorem 3.3.** Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:

(i) $B^\dagger A^\dagger \in (AB)\{1, 2, 3\}$.

(ii) $B\{1, 2, 3\}A^\dagger \subseteq (AB)\{1, 2, 3\}$.

(iii) $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus [\mathcal{R}(B) \cap N(A)]$. 

3.1. We distinguish two cases:

(1) Let \( N(AB) \neq N(B) \).

(i) \( \Rightarrow \) (iii) If \( B^\dagger A^\dagger \in (AB)\{1,2,3\} \), then there exists an operator \( Z \in (AB)\{1,2,3\} \) such that \( B^\dagger A^\dagger = Z \), where \( Z \) is represented by (3.7). Comparing (3.7) with (3.13), we obtain \( Y_{21} = A_{12}^{-1}, Y_{22} = 0, Y_{12} = 0 \).

We have that (3.12) implies \( Y_{21} = A_{12}^{-1} \) only if \( A_{14} = 0 \) which implies the invertibility of \( A_{24} \). Hence

\[
A = \begin{bmatrix}
0 & A_{12} & 0 & 0 \\
0 & 0 & 0 & A_{24} \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

It is easy to get \( R(A^*AB) = R(B) \ominus [R(B) \cap N(A)] \).

(iii) \( \Rightarrow \) (i) Since \( R(A^*AB) = R(B) \ominus [R(B) \cap N(A)] \) is equivalent to \( A_{14} = 0 \), we obtain from (3.12) that \( Y_{21} = A_{12}^{-1}, Y_{22} = 0, Y_{12} = 0 \). Hence, \( B^\dagger A^\dagger \in (AB)\{1,2,3\} \).

(ii) Using the representation of an arbitrary \( X \in B\{1,2,3\} \) given by (3.2), we get that \( XA^\dagger \in (AB)\{1,2,3\} \) if and only if \( B^\dagger A^\dagger \in (AB)\{1,2,3\} \).

(2) If \( N(AB) = N(B) \), the proof is analogous to case (1).

The case \( K = \{1,2,4\} \) is treated completely analogously, and the corresponding result follows by taking adjoints, or by reversal of products.

**Theorem 3.4.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H},\mathcal{K}) \), \( B \in \mathcal{B}(\mathcal{L},\mathcal{H}) \) be such that \( R(A), R(B), R(AB) \) are closed and \( AB \neq 0 \). Then the following statements are equivalent:

(i) \( B^\dagger A^\dagger \in (AB)\{1,2,4\} \).

(ii) \( B^\dagger A\{1,2,4\} \subseteq (AB)\{1,2,4\} \).

(iii) \( R(BB^*A^*) = R(A^*) \ominus [R(A^*) \cap N(B^*)] \).

From the above two theorems, we get the following equivalent condition for the reverse order law for the Moore-Penrose inverse.

**Theorem 3.5.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H},\mathcal{K}) \), \( B \in \mathcal{B}(\mathcal{L},\mathcal{H}) \) be such that \( R(A), R(B), R(AB) \) are closed and \( AB \neq 0 \). Then the following statements are equivalent:

(i) \( (AB)^\dagger = B^\dagger A^\dagger \).

(ii) \( R(A^*AB) = R(B) \ominus [R(B) \cap N(A)] \) and \( R(BB^*A^*) = R(A^*) \ominus [R(A^*) \cap N(B^*)] \).

**Remark 3.4.** The conditions (ii) from Theorem 3.5 are equivalent to the conditions \( R(A^*AB) \subseteq R(B) \) and \( R(BB^*A^*) \subseteq R(A^*) \) given in the paper by Greville [8] for matrices. Also, they are equivalent to those given in [5] Theorem 2.2 (c) in the case of bounded linear operators on Hilbert space.

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