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NEW RESULTS ON REVERSE ORDER LAW FOR $\{1,2,3\}$- AND $\{1,2,4\}$-INVERSES OF BOUNDED OPERATORS

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ABSTRACT. In this paper, using some block-operator matrix techniques, we give necessary and sufficient conditions for the reverse order law to hold for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of bounded operators on Hilbert spaces. Furthermore, we present some new equivalents of the reverse order law for the Moore-Penrose inverse.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a given $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. For given subsets M, N of $\mathcal{B}(\mathcal{H},\mathcal{K})$, by MN we denote the set consisting of all products XY, where $X \in M$ and $Y \in N$.

Recall that $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ has the Moore-Penrose inverse if there exists an operator $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that

$$\begin{array}{ll} (1)\ AXA = A, & (2)\ XAX = X, \\ (3)\ (AX)^* = AX, & (4)\ (XA)^* = XA. \end{array} \tag{1.1}$$

The Moore-Penrose inverse of an operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ exists if and only if A has closed range and in this case it is unique. It is denoted by $A^\dagger$.

For any $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, let $A\{i,j,\dots,k\}$ denote the set of all operators $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ which satisfy equations (i), (j), $\dots$, (k) of (1.1). In this case X is a $\{i,j,\dots,k\}$-inverse of A and is denoted by $A^{(i,j,\dots,k)}$. Evidently, $A\{1,2,3,4\} = \{A^\dagger\}$, when A has closed range.

The reverse order law for generalized inverses plays an important role in theoretic research and numerical computations in many areas, including the singular matrix problem, ill-posed problems, optimization problems, and statics problems (see, for instance, [1, 9, 13, 14, 18]). These problems have attracted considerable attention since the mid-1960s, and many interesting results for generalized inverses of products of matrices or operators have been obtained (see [5], [6], [10], [11], [15]–[17]). T.N.E. Greville [8] first proved that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$, for matrices A and B. This result was extended to linear bounded operators on Hilbert spaces in [11]. Later, the reverse order law for the Moore-Penrose inverse was considered in rings with involution (see [12]).

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Xiong and Zheng [20] considered the reverse order law for $\{1,2,3\}$- and $\{1,2,4\}$-generalized inverses of products of two matrices. Their techniques involved expressions for maximal and minimal ranks of the generalized Schur complement. In [2] the authors considered the reverse order law for K-inverses in the cases $K \in \{\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\}\}$ for elements of a C*-algebra.

In this paper, using block-operator matrix techniques, we consider the reverse order law for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of bounded linear operators on Hilbert spaces. We give necessary and sufficient conditions for

$$B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,2,3\}$$

and

$$B\{1,2,4\} \cdot A\{1,2,4\} \subseteq (AB)\{1,2,4\}.$$

We generalized the results from [2] to the case of bounded linear operators on Hilbert spaces. This is of particular importance especially in statistics where bounded linear operators play a very important role. Furthermore, we present new equivalent conditions for the reverse order law for the Moore-Penrose inverse. It should be pointed out that when restricted to the set of matrices, our results for the reverse order law for the Moore-Penrose inverse yield facts not previously known.

2. Preliminaries

Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ have closed range. The operator A has the following matrix decomposition (see [4], [7])

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{2.1}$$

where A_1 is invertible. Also, $A^\dagger$ has the form

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}. \tag{2.2}$$

If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ has closed range, then we can explicitly describe the sets $A\{1,2,3\}$ and $A\{1,2,4\}$ using the representation of A given by (2.1).

Lemma 2.1. *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ have closed range. Then*

$$A\{1,2,3\} = \left\{ \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} : X_3 \in \mathcal{B}(\mathcal{R}(A),\mathcal{N}(A)) \right\}$$

and

$$A\{1,2,4\} = \left\{ \begin{bmatrix} A_1^{-1} & X_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} : X_2 \in \mathcal{B}(\mathcal{N}(A^*),\mathcal{R}(A^*)) \right\}.$$

Proof. Suppose that A and $A^\dagger$ are given by (2.1) and (2.2), respectively. Since

$$A\{1,2,3\} = \{A^\dagger + (I - A^\dagger A)XAA^\dagger : X \in \mathcal{B}(\mathcal{K},\mathcal{H})\},$$

we have that $A^{(1,2,3)} \in A\{1,2,3\}$ if and only if

$$\begin{aligned} A^{(1,2,3)} &= A^\dagger + (I - A^\dagger A)XAA^\dagger \\ &= \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & 0 \end{bmatrix}, \end{aligned}$$

for some $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}$. The proof for the case of $\{1,2,4\}$-inverses follows similarly. ☐

Lemma 2.2. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. Then $\mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}$ if and only if $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$.*

Proof. First, let us note that

$$B|_{\mathcal{R}(B^*)} : \mathcal{R}(B^*) \to \mathcal{R}(B) \text{ is an invertible operator.} \tag{2.3}$$

$(\Rightarrow)$: Suppose that $\mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}$ and let $x \in \mathcal{R}(B) \cap \mathcal{N}(A)$. By (2.3), there exists $y \in \mathcal{R}(B^*)$ such that $By = x$. Now, $y \in \mathcal{R}(B^*) \cap \mathcal{N}(AB)$, i.e., $y = 0$, so $x = By = 0$.

$(\Leftarrow)$: If we suppose that $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$ and take $u \in \mathcal{R}(B^*) \cap \mathcal{N}(AB)$, we get that $Bu \in \mathcal{R}(B) \cap \mathcal{N}(A)$, i.e., $Bu = 0$. Using (2.3) it follows that $u = 0$. ☐

Let us introduce the following notation: if a Hilbert space $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$ where $\mathcal{U}_i \perp \mathcal{U}_j$ for $i \neq j$, then we shall write $\mathcal{H} = \mathcal{U}_1 \oplus^\perp \cdots \oplus^\perp \mathcal{U}_k$. If $\mathcal{U}$ is a complement space of a Hilbert space $\mathcal{H}$ we shall denote by $\mathcal{H} \ominus^\perp \mathcal{U}$ the unique subspace $\mathcal{V}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{U} \oplus^\perp \mathcal{V}$.

Remark 2.1. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. Denote by

$$\begin{cases} \mathcal{H}_1 = \mathcal{R}(B) \cap \mathcal{N}(A), \\ \mathcal{H}_2 = \mathcal{R}(B) \ominus^\perp \mathcal{H}_1, \\ \mathcal{H}_3 = \mathcal{N}(B^*) \cap \mathcal{N}(A), \\ \mathcal{H}_4 = \mathcal{N}(B^*) \ominus^\perp \mathcal{H}_3, \end{cases} \qquad \begin{cases} \mathcal{K}_1 = \mathcal{R}(AB), \\ \mathcal{K}_2 = \mathcal{R}(A) \ominus^\perp \mathcal{R}(AB), \end{cases} \qquad \begin{cases} \mathcal{L}_1 = B^\dagger \mathcal{H}_1, \\ \mathcal{L}_2 = \mathcal{R}(B^*) \ominus^\perp B^\dagger \mathcal{H}_1. \end{cases}$$

Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ can be decomposed as

$$\mathcal{H} = \mathcal{R}(B) \oplus^\perp \mathcal{N}(B^*), \quad \mathcal{K} = \mathcal{R}(A) \oplus^\perp \mathcal{N}(A^*), \quad \mathcal{L} = \mathcal{R}(B^*) \oplus^\perp \mathcal{N}(B),$$

where

$$\mathcal{R}(B) = \mathcal{H}_1 \oplus^\perp \mathcal{H}_2, \quad \mathcal{N}(B^*) = \mathcal{H}_3 \oplus^\perp \mathcal{H}_4, \quad \mathcal{R}(A) = \mathcal{K}_1 \oplus^\perp \mathcal{K}_2, \quad \mathcal{R}(B^*) = \mathcal{L}_1 \oplus^\perp \mathcal{L}_2.$$

We can prove that $B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A)) = \mathcal{R}(B^*) \cap \mathcal{N}(AB)$. Indeed let $x \in \mathcal{R}(B^*) \cap \mathcal{N}(AB)$. Then $x \in \mathcal{R}(B^\dagger) = \mathcal{R}(B^\dagger B) = B^\dagger \mathcal{R}(B)$ and $ABx = 0$, i.e., $Bx \in \mathcal{N}(A)$. So we have $x = B^\dagger Bx \in B^\dagger \mathcal{N}(A)$. Finally, $x \in B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A)$. On the other hand, let $y \in B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A)\ )$, i.e., $y \in \mathcal{R}(B^\dagger B) = \mathcal{R}(B^*)$, then for some $z \in \mathcal{R}(B) \cap \mathcal{N}(A)$ we have $y = B^\dagger z$. So $ABy = ABB^\dagger z = Az = 0$, i.e., $y \in \mathcal{N}(AB)$. Thus, by Lemma 2.2, we get

$$\mathcal{H}_2 = \mathcal{R}(B) \Leftrightarrow \mathcal{H}_1 = \{0\} \Leftrightarrow \mathcal{N}(AB) = \mathcal{N}(B) \Leftrightarrow \mathcal{L}_1 = \{0\} \Leftrightarrow \mathcal{L}_2 = \mathcal{R}(B^*).$$

Furthermore,

$$\mathcal{H}_2 = \{0\} \Leftrightarrow \mathcal{H}_1 = \mathcal{R}(B) \Leftrightarrow \mathcal{R}(B) \subset \mathcal{N}(A) \Leftrightarrow \mathcal{L}_1 = \mathcal{R}(B^*) \Leftrightarrow \mathcal{L}_2 = \{0\} \Leftrightarrow \mathcal{K}_1 = \{0\} \Leftrightarrow \mathcal{K}_2 = \mathcal{R}(A).$$

Throughout the paper we will use the notation from the above remark.

A similar result to the following one but for the case of two Hilbert spaces has been presented in [17]. Now we give a different proof in the case of three Hilbert spaces.

Lemma 2.3. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed.*

(1) *If $AB \neq 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$, then A and B have the following operator matrix forms:*

$$A = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \tag{2.4}$$

and

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}, \tag{2.5}$$

where A_{12}, B_{11}, B_{22} are invertible operators and A_{24} is a surjection.

(2) *If $AB \neq 0$ and $\mathcal{N}(AB) = \mathcal{N}(B)$, then A and B have the following operator matrix forms:*

$$A = \begin{bmatrix} A_{12} & 0 & A_{14} \\ 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \tag{2.6}$$

and

$$B = \begin{bmatrix} B_{22} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}, \tag{2.7}$$

where A_{12}, B_{22} are invertible operators and A_{24} is a surjection.

(3) *If $AB = 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$, then A and B have the following operator matrix forms:*

$$A = \begin{bmatrix} 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \tag{2.8}$$

and

$$B = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}, \tag{2.9}$$

where B_{11} and A_{24} are invertible.

Proof. We will assume that the spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are decomposed as in Remark 2.1, so that the conclusions of the remark also hold.

(1) Suppose that $AB \neq 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$. We have that B can be represented by

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where $\widehat{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}$ is invertible.

Since $B\mathcal{L}_1 \subset \mathcal{H}_1$, we get that $B_{21} = 0$. Now from the invertibility of $\widehat{B}$, we get that $B_{11} : \mathcal{L}_1 \to \mathcal{H}_1$ and $B_{22} : \mathcal{L}_2 \to \mathcal{K}_2$ are invertible.

Now, we will prove that A has a matrix form given by (2.4). Suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix}.$$

The reductions A_{i1} and A_{i3} for $i = 1, 2, 3$ are null operators because $\mathcal{H}_1, \mathcal{H}_3 \subset \mathcal{N}(A)$. The range of the reductions $A_{3j}, (j = 1, 2, 3, 4)$ is $\mathcal{N}(A^*)$, so $A_{3j} = 0$.

Now we will prove that $A_{22} = 0$: for any $x \in \mathcal{H}_2 \subset \mathcal{R}(B)$, there exists $y \in \mathcal{K}$ such that $By = x$. Now, $Ax = ABy \in \mathcal{K}_1$ and $Ax = A_{12}x + A_{22}x$. Since $A_{12}x \in \mathcal{K}_1$, we get that $A_{22}x = 0$.

In order to prove that A_{12} is bijective, first we will prove that $\mathcal{N}(A_{12}) = \{0\}$: let $u \in \mathcal{H}_2$ be such that $A_{12}u = 0$. Then $u \in \mathcal{N}(A)$ which implies that $u \in \mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$.

To prove that $A_{12} : \mathcal{H}_2 \to \mathcal{K}_1$ is surjective take any $k \in \mathcal{K}_1 = \mathcal{R}(AB)$. There exists $k' \in \mathcal{K}$ such that $ABk' = k$. Since $Bk' \in \mathcal{R}(B) = \mathcal{H}_1 \oplus^{\perp} \mathcal{H}_2$, there exist $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$ such that $Bk' = h_1 + h_2$. Now, $Ah_2 = A(Bk' - h_1) = k$, i.e., $A_{12}h_2 = k$.

The surjective property of $A_{24} : \mathcal{H}_4 \to \mathcal{K}_2$ follows from the fact that for any $u \in \mathcal{K}_2$, there exists $v \in \mathcal{H}$ such that $Av = u$. Let us decompose $v = \sum_{i=1}^{4} v_i$, where $v_i \in \mathcal{H}_i$. It is evident that $A_{24}v_4 = u$.

The proof of (2) and (3) is analogous. □

3. Main results

Z. Xiong and B. Zheng [20] presented necessary and sufficient conditions for

$$B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}, \tag{3.1}$$

in the case when A and B are matrices. Here, we give another characterization of (3.1) for linear bounded operators on Hilbert spaces using techniques which are completely different from those used in [20]. First, we will give the following remark:

Remark 3.1. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed, $AB \neq 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$. Then we can suppose that the operators A and B are represented by (2.4) and (2.5), respectively. By Lemma 2.1, $X \in B\{1,2,3\}$ if and only if there exist operators F_{11} and F_{12} such that

$$X = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & B_{22}^{-1} & 0 & 0 \\ F_{11} & F_{12} & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}. \tag{3.2}$$

To describe the set $A\{1,2,3\}$, suppose that an arbitrary $Y \in A\{1,2,3\}$ is given by

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \\ Y_{41} & Y_{42} & Y_{42} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}. \tag{3.3}$$

Since AY is hermitian, we get that

$$AY = \begin{bmatrix} A_{12}Y_{21} + A_{14}Y_{41} & A_{12}Y_{22} + A_{14}A_{42} & 0 \\ A_{24}Y_{41} & A_{24}Y_{42} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $A_{12}Y_{22} + A_{14}A_{42} = (A_{24}Y_{41})^*$ and $A_{12}Y_{21} + A_{14}Y_{41}$, $A_{24}Y_{42}$ are hermitian. Since AY is an orthogonal projection on $\mathcal{R}(A)$, from the definition of the subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ we can conclude that $A_{12}Y_{21} + A_{14}Y_{41} = I$, $A_{24}Y_{42} = I$, $A_{12}Y_{22} + A_{14}A_{42} = 0$ and $A_{24}Y_{41} = 0$. Now, from $YAY = 0$, we get that $Y_{i3} = 0$, for $i = \overline{1,4}$. Hence, $Y \in A\{1,2,3\}$ if and only if

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ Y_{31} & Y_{32} & 0 \\ Y_{41} & Y_{42} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}, \tag{3.4}$$

where the Y_{ij} satisfy the following equalities:

$$\begin{cases} Y_{i2}A_{24}Y_{42} = Y_{i2}, \ i = \overline{1,4}, \\ A_{12}Y_{21} + A_{14}Y_{41} = I_{\mathcal{K}_1}, \\ A_{12}Y_{22} + A_{14}Y_{42} = 0, \\ A_{24}Y_{42} = I_{\mathcal{K}_2}, A_{24}Y_{41} = 0. \end{cases} \tag{3.5}$$

Since

$$AB = \begin{bmatrix} 0 & A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix}. \tag{3.6}$$

we get that $Z \in (AB)\{1,2,3\}$ if and only if there exist operators N_1 and N_2 such that

$$Z = \begin{bmatrix} N_1 & 0 & 0 \\ B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ N_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}. \tag{3.7}$$

Theorem 3.1. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

(i) $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$.
(ii) $\mathcal{R}(B) = \mathcal{R}(A^*AB) \oplus^{\perp} [\mathcal{R}(B) \cap \mathcal{N}(A)]$, $\mathcal{R}(AB) = \mathcal{R}(A)$.

Proof. We use the decompositions of the spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ and the matrix decompositions of operators A, B given in Lemma 2.3. We distinguish two cases:

(1) First, suppose that $\mathcal{N}(AB) \neq \mathcal{N}(B)$. We have that the operators A and B are represented by (2.4) and (2.5), respectively. Also, in Remark 3.1 we have given characterizations of the sets $A\{1,2,3\}$, $B\{1,2,3\}$ and $(AB)\{1,2,3\}$ which we will use in this proof.

(i) $\Rightarrow$ (ii): Let arbitrary $X \in B\{1,2,3\}$ and $Y \in A\{1,2,3\}$ be given by (3.2) and (3.4), respectively. Then

$$
\begin{aligned}
XY &= \begin{bmatrix} M_1 & M_2 & 0 \\ B_{22}^{-1}Y_{21} & B_{22}^{-1}Y_{22} & 0 \\ F_{11}Y_{11} + F_{12}Y_{21} & F_{11}Y_{12} + F_{12}Y_{22} & 0 \end{bmatrix} \\
&: \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix},
\end{aligned} \tag{3.8}
$$

where $M_1 = B_{11}^{-1}Y_{11} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{21}$, $M_2 = B_{11}^{-1}Y_{12} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{22}$.

Since $XY \in (AB)\{1,2,3\}$, we conclude that XY must be of the form Z given by (3.7) for some operators N_1 and N_2. If we compare (3.8) and (3.7), we get that $Y_{12} = 0$, $Y_{22} = 0$ and $Y_{21} = A_{12}^{-1}$. Hence, it follows that the system of the operator equations (3.5) is such that Y_{12}, Y_{22} and Y_{21} are uniquely determined. Since A_{24} is surjective and $A_{24}Y_{42} = I_{\mathcal{K}_2}$ we get that $Y_{12} = 0$ if and only if $\mathcal{K}_2 = \{0\}$, i.e., $A_{24} = 0$. If this were not true, then Y_{12} could be taken to be an arbitrary operator on an appropriate subspace, which is not the case. Now, from the first equation of (3.5), we get that $Y_{i2} = 0$, $i = \overline{1,4}$. Since Y_{41} can be arbitrary, to avoid that $Y_{21} = A_{12}^{-1}$ it must be that $A_{14} = 0$. It is evident that $A_{24} = 0$ is equivalent to $\mathcal{R}(AB) = \mathcal{R}(A)$.

Now, simple computation shows that

$$
A^*AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{12}^*A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},
$$

and finally we get $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^{\perp} [\mathcal{R}(B) \cap \mathcal{N}(A)]$.

(ii) $\Rightarrow$ (i): Suppose $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^{\perp} [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and $\mathcal{R}(AB) = \mathcal{R}(A)$. We must show that for arbitrary $X \in B\{1,2,3\}$ and $Y \in A\{1,2,3\}$ there exists $Z \in (AB)\{1,2,3\}$ such that $XY = Z$.

From $\mathcal{R}(AB) = \mathcal{R}(A)$, we get that $\mathcal{K}_2 = \{0\}$, i.e., $A_{24} = 0$. Also by $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^{\perp} [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and the fact that

$$
A^*AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{12}^*A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & A_{14}^*A_{12}B_{22} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},
$$

where A_{12} and B_{22} are invertible, we have $A_{14} = 0$.

Now, we get that $Y \in A\{1,2,3\}$ if and only if

$$Y = \begin{bmatrix} Y_{11} & 0 & 0 \\ A_{12}^{-1} & 0 & 0 \\ Y_{31} & 0 & 0 \\ Y_{41} & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}, \tag{3.9}$$

where Y_{11}, Y_{31}, Y_{41} are arbitrary. It is evident that for arbitrary $X \in B\{1,2,3\}$ and $Y \in A\{1,2,3\}$ there exists $Z \in (AB)\{1,2,3\}$ such that $XY = Z$, i.e.,

$$B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}.$$

(2) When $\mathcal{N}(AB) = \mathcal{N}(B)$, the operators A and B are represented by (2.6) and (2.7), respectively, and the proof is analogous to case (1). □

Remark 3.2. 1° If $AB = 0$, then $(AB)\{1,2,3\} = \{0\}$. In the case when $A = 0$ or $B = 0$, evidently $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$. If it is not the case, we have that $AB = 0 \Leftrightarrow \mathcal{H}_2 = \{0\} \Leftrightarrow \mathcal{H}_1 = \mathcal{R}(B) \Leftrightarrow \mathcal{L}_2 = \{0\} \Leftrightarrow \mathcal{L}_1 = \mathcal{R}(B^*) \Leftrightarrow \mathcal{K}_1 = \{0\} \Leftrightarrow \mathcal{K}_2 = \mathcal{R}(A)$. Also, A and B are represented by (2.8) and (2.9), respectively, so arbitrary $X \in B\{1,2,3\}$ and $Y \in A\{1,2,3\}$ are represented by

$$X = \begin{bmatrix} B_{11}^{-1} & 0 & 0 \\ F_1 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{N}(B) \end{bmatrix}$$

and

$$Y = \begin{bmatrix} F_2 & 0 \\ F_3 & 0 \\ A_{24}^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

for some operators F_1, F_2 and F_3.

By simple computation, we observe that

$$XY = \begin{bmatrix} B_{11}^{-1}F_2 & 0 \\ F_1F_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \neq 0,$$

i.e., $B\{1,2,3\}A\{1,2,3\} \neq \{0\}$.

Hence,

$$AB = 0, A \neq 0, B \neq 0 \Rightarrow B\{1,2,3\}A\{1,2,3\} \nsubseteq (AB)\{1,2,3\}.$$

2° From Theorem 3.1 we conclude that the condition

$$(ABB^\dagger)^\dagger ABB^\dagger = BB^\dagger \quad \text{or} \quad (AB)(AB)^\dagger = AA^\dagger$$

from [2, Theorem 3.3] can be replaced by the sole condition $(AB)(AB)^\dagger = AA^\dagger$, i.e., $\mathcal{R}(AB) = \mathcal{R}(A)$.

A similar result in the case $K = \{1,2,4\}$ follows from Theorem 3.1 by reversal of products:

Theorem 3.2. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

(i) $B\{1,2,4\}A\{1,2,4\} \subseteq (AB)\{1,2,4\}$.

(ii) $\mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus^\perp [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$, $\mathcal{N}(AB) = \mathcal{N}(B)$.

Remark 3.3. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed, $AB \neq 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$. We have that operator B is represented by (2.5), so

$$B^\dagger = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & B_{22}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}. \tag{3.10}$$

Also, if we suppose that the operator A is represented by (2.5), by Remark 3.1 we get that $A^\dagger = Y$ is represented by (3.4), where Y_{ij} satisfy (3.5). Now, since YA is an orthogonal projection on $\mathcal{N}(A)^\perp$, we get that

$$YA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

which implies that $Y_{21}A_{12} = I_{\mathcal{H}_2}$, $Y_{41}A_{12} = 0$, $Y_{21}A_{14} + Y_{22}A_{24} = 0, Y_{41}A_{14} + Y_{42}A_{24} = I_{\mathcal{H}_4}$. Now by $YAY = 0$ we get that $Y_{11} = 0$, $Y_{12} = 0$, $Y_{31} = 0$ and $Y_{32} = 0$. Hence,

$$A^\dagger = \begin{bmatrix} 0 & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ 0 & Y_{32} & 0 \\ Y_{41} & Y_{42} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \tag{3.11}$$

where

$$\begin{cases} Y_{i2}A_{24}Y_{42} = Y_{i2}, i = \overline{1,4}, \\ A_{12}Y_{21} + A_{14}Y_{41} = I_{\mathcal{K}_1}, \\ A_{12}Y_{22} + A_{14}Y_{42} = 0, \\ A_{24}Y_{42} = I_{\mathcal{K}_2}, A_{24}Y_{41} = 0, \\ Y_{21}A_{14} + Y_{22}A_{24} = 0, \\ Y_{41}A_{14} + Y_{42}A_{24} = I_{\mathcal{H}_4}, \\ Y_{21}A_{12} = I_{\mathcal{H}_2}, Y_{41}A_{12} = 0. \end{cases} \tag{3.12}$$

Simple computation shows that

$$\begin{aligned} B^\dagger A^\dagger &= \begin{bmatrix} -B_{11}^{-1}B_{12}B_{22}^{-1}Y_{21} & M_3 & 0 \\ B_{22}^{-1}Y_{21} & B_{22}^{-1}Y_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &: \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}, \end{aligned} \tag{3.13}$$

where $M_3 = B_{11}^{-1}Y_{12} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{22}$.

Using the previous remark, we obtain the following result:

Theorem 3.3. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

(i) $B^\dagger A^\dagger \in (AB)\{1,2,3\}$.
(ii) $B\{1,2,3\}A^\dagger \subseteq (AB)\{1,2,3\}$.
(iii) $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$.

Proof. We will use the same decompositions of spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ as in Theorem 3.1. We distinguish two cases:

(1) Let $\mathcal{N}(AB) \neq \mathcal{N}(B)$.

(i)⇒(iii) If $B^\dagger A^\dagger \in (AB)\{1,2,3\}$, then there exists an operator $Z \in (AB)\{1,2,3\}$ such that $B^\dagger A^\dagger = Z$, where Z is represented by (3.7). Comparing (3.7) with (3.13), we obtain $Y_{21} = A_{12}^{-1}$, $Y_{22} = 0$, $Y_{12} = 0$.

We have that (3.12) implies $Y_{21} = A_{12}^{-1}$ only if $A_{14} = 0$ which implies the invertibility of A_{24}. Hence

$$A = \begin{bmatrix} 0 & A_{12} & 0 & 0 \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix}.$$

It is easy to get $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$.

(iii)⇒(i) Since $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ is equivalent to $A_{14} = 0$, we obtain from (3.12) that $Y_{21} = A_{12}^{-1}$, $Y_{22} = 0$, $Y_{12} = 0$. Hence, $B^\dagger A^\dagger \in (AB)\{1,2,3\}$.

(i)⇔(ii) Using the representation of an arbitrary $X \in B\{1,2,3\}$ given by (3.2), we get that $XA^\dagger \in (AB)\{1,2,3\}$ if and only if $B^\dagger A^\dagger \in (AB)\{1,2,3\}$.

(2) If $\mathcal{N}(AB) = \mathcal{N}(B)$, the proof is analogous to case (1). □

The case $K = \{1,2,4\}$ is treated completely analogously, and the corresponding result follows by taking adjoints, or by reversal of products:

Theorem 3.4. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

(i) $B^\dagger A^\dagger \in (AB)\{1,2,4\}$.
(ii) $B^\dagger A\{1,2,4\} \subseteq (AB)\{1,2,4\}$.
(iii) $\mathcal{R}(BB^*A^*) = \mathcal{R}(A^*) \ominus^\perp [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$.

From the above two theorems, we get the following equivalent condition for the reverse order law for the Moore-Penrose inverse.

Theorem 3.5. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

(i) $(AB)^\dagger = B^\dagger A^\dagger$.
(ii) $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ *and* $\mathcal{R}(BB^*A^*) = \mathcal{R}(A^*) \ominus^\perp [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$.

Remark 3.4. The conditions *(ii)* from Theorem 3.5 are equivalent to the conditions $R(A^*AB) \subseteq \mathcal{R}(B)$ and $R(BB^*A^*) \subseteq \mathcal{R}(A^*)$ given in the paper by Greville [8] for matrices. Also, they are equivalent to those given in [5, Theorem 2.2 (c)] in the case of bounded linear operators on Hilbert space.

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