MÖBIUS INVERSION FORMULAE FOR APOSTOL-BERNOULLI TYPE POLYNOMIALS AND NUMBERS

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Abstract. In this paper, we establish Möbius inversion formulae for the Fourier expansions of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. As an application, by specializing our formulae at some special values we obtain interesting number-theoretical relations. We derive explicit formulae for Apostol-Bernoulli numbers. These formulae involve Stirling numbers of the second kind and powers of cotangent. Our proofs are very simple.

1. Introduction

For \( \omega \in \mathbb{C} \setminus \{0\} \) and \( x \) variable, the \( n \)th Apostol-Bernoulli polynomial \( B_n(x; \omega) \) is defined by the generating function

\[
\sum_{n \geq 0} B_n(x; \omega) \frac{t^n}{n!} = \frac{te^{xt}}{\omega e^t - 1}, \quad (|t + \log(\omega)| < 2\pi),
\]

where

\[
\omega = |\omega|e^{i\theta}, \quad -\pi \leq \theta < \pi \text{ and } \log(\omega) = \log |\omega| + i\theta.
\]

For \( \omega = 1 \), \( B_n(x; 1) = B_n(x) \) is the \( n \)th Bernoulli polynomial and \( B_n(0; 1) = B_n \) is the \( n \)th Bernoulli number.

The Möbius inversion formula for the Bernoulli polynomials \( B_n(x) \) has been studied in [5] for \( n \geq 2 \). Hence, the purpose of this paper is to investigate the case \( \omega \neq 1 \).

The Möbius inversion formula for the Apostol-Bernoulli polynomials \( B_n(x; \omega) \) has the following Fourier expansion,

\[
-\frac{\omega^x}{n!} B_n(x; \omega) = \sum_{k \in \mathbb{Z}} \frac{e^{2i\pi kx}}{(2i\pi k - \log(\omega))^n},
\]

for \( 0 < x < 1 \) if \( n = 1 \), \( 0 \leq x \leq 1 \) if \( n \geq 2 \).

Remark 1. The convergence, of the Fourier series (3), is absolute and uniform in \([0, 1]\) except for \( B_1(x; \omega) \). In this paper, the case \( n = 1 \) is not treated.

The paper is organized as follows. In section 2 we establish a Möbius inversion type formula for Apostol-Bernoulli polynomials. For that, we shall slightly modify
the expression (3). Similarly, in section 3 we obtain Möbius inversion formulae for Apostol-Euler and Apostol-Genocchi polynomials.

2. MöBIUS INVERSION FOR APOLST-BERNOULLI POLYNOMIALS

With a view towards stating our main results, we transform the Fourier expansion (3) in sum over positive integers. Let us define a 1-periodic, in the variable $x$, function by

$$F_n(x; \omega) = -\frac{x^n B_n(x; \omega)}{n!} - \frac{1}{(-\log(\omega))^n},$$

for $0 < x < 1$ if $n = 1$, and $0 \leq x \leq 1$ if $n \geq 2$.

We set

$$f_n(k; x; \omega) = e^{2i\pi kx} \left(\frac{e^{2i\pi kx}}{(2i\pi k - \log(\omega))^n} + \frac{e^{-2i\pi kx}}{(-2i\pi k - \log(\omega))^n}\right),$$

then from (3) we observe that

$$F_n(x; \omega) = \sum_{k=1}^{\infty} f_n(k; x; \omega),$$

and its Möbius inversion is given by the following interesting and useful theorem.

**Theorem 1.** For any positive integer $n \geq 2$, real $x$ and complex $\omega \in \mathbb{C} \setminus \{0, 1\}$, we have

$$f_n(1; x; \omega) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^n} F_n(mx; \omega^{1/m}),$$

where $\mu$ is the arithmetical Möbius function.

**Proof.** We first note that the function $f_n$ satisfies the remarkable property,

$$f_n(k; mx; \omega^{1/m}) = \frac{e^{2i\pi mkx}}{(2i\pi k - \log(\omega)/m)^n} + \frac{e^{-2i\pi mkx}}{(-2i\pi k - \log(\omega)/m)^n} = m^n f_n(mk; x; \omega).$$

Hence for any integer $n \geq 2$, real $x$ and complex $\omega \in \mathbb{C} \setminus \{0, 1\}$, we have

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} |\mu(m)| m^n f_n(k; mx; \omega^{1/m}) | \leq \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} |\mu(m)| \left(\frac{1}{|2imk\pi - \log(\omega)|^n} + \frac{1}{|2imk\pi + \log(\omega)|^n}\right).$$

Using the elementary inequalities,

$$|2imk\pi \pm \log(\omega)| \geq 2mk\pi - |\arg(\omega)| \geq 2mk\pi - \pi \geq mk\pi,$$

(7) $|\mu(m)| \leq 1$, we obtain

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} |\mu(m)| m^n f_n(k; mx; \omega^{1/m}) | \leq \frac{2}{\pi^n} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{m^nk^n} \leq \frac{2}{\pi^n} \zeta^2(n).$$
The convergence is then absolute and uniform. Substituting the expansion (4) of \( F_n \) in the right-hand of (5), we obtain
\[
\sum_{m=1}^{\infty} \frac{\mu(m)}{m^n} F_n(mx; \omega^{1/m}) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mu(m)}{m^n} f_n(k; mx; \omega^{1/m}) = \sum_{m,k=1}^{\infty} \mu(m) f_n(mk; x; \omega)
\]
\[
= \sum_{m=1}^{\infty} \left( \sum_{d|m} \mu(d) \right) f_n(m; x; \omega).
\]

To end the proof of the theorem we use the following analytical-number property: for every positive integer \( m \) we have
\[
\sum_{d|m} \mu(d) = \begin{cases} 
1 & \text{if } m = 1, \\
0 & \text{if } m \geq 2.
\end{cases}
\]

Remark 2. Theorem 1 seems to fit into the general framework for the Möbius inversion outlined in [3]. In fact, we observe that Fourier series (3) has the form
\[
F_n(x; \omega) = \sum_{k \geq 1} \alpha(k) f_n(\phi(k; x, \omega)),
\]
where the \( \alpha(k) = k^{-n} \) is a completely multiplicative arithmetical function, and \( \phi: (k; x, \omega) \mapsto (kx, \omega^k) \) is a flow, i.e., \( \phi(m; \phi(k; x, \omega)) = \phi(mk; x, \omega) \).

Now we can rewrite the Theorem 1 in terms of Apostol-Bernoulli polynomials.

**Theorem 2.** Let \( \omega \in \mathbb{C} \setminus \{0, 1\} \), \( x \) a real number and \( n \geq 2 \) an integer. Then we have
\[
\frac{e^{2i\pi x}}{(2i\pi - \log(\omega))^n} + \frac{e^{-2i\pi x}}{(-2i\pi - \log(\omega))^n}
\]
\[
= -\sum_{m=1}^{\infty} \frac{\mu(m)}{m^n} \left( \frac{\omega^{(mx)}}{n!} B_n\{mx\}; \omega^{1/m} \right) + \frac{m^n}{(-\log(\omega))^n},
\]
where \( \{mx\} \) denote the fractional part of \( mx \).

### 3. The Apostol-Euler and Apostol-Genocchi polynomials

For \( \omega \in \mathbb{C} \setminus \{0\} \) and \( x \) variable, the Apostol-Euler polynomial \( E_n(x; \omega) \) and the Apostol-Genocchi polynomial \( G_n(x; \omega) \) are given by the generating functions
\[
\sum_{n \geq 0} E_n(x; \omega) \frac{t^n}{n!} = \frac{2e^{xt}}{\omega e^t + 1}, \quad (|t + \log(-\omega)| < 2\pi),
\]
\[
\sum_{n \geq 0} G_n(x; \omega) \frac{t^n}{n!} = \frac{2te^{xt}}{\omega e^t + 1}, \quad (|t + \log(-\omega)| < 2\pi).
\]

They also have Fourier type expansions (see [2]). In fact, for any complex \( \omega \in \mathbb{C} \setminus \{-1, 0\} \), we have
\[
\frac{\omega^x}{n!} E_n(x; \omega) = 2 \sum_{k \in \mathbb{Z}} \frac{e^{2i\pi (k + \frac{1}{2})x}}{(2i\pi (k + \frac{1}{2}) - \log(\omega))^{n+1}},
\]
for $0 < x < 1$ if $n = 0$ and $0 \leq x \leq 1$ if $n \geq 1$, and
\begin{equation}
\frac{\omega^x}{n!} G_n(x; \omega) = 2 \sum_{k \in \mathbb{Z}} \frac{e^{2i\pi (k + \frac{1}{2}) x}}{(2i\pi (k + \frac{1}{2}) - \log(\omega))^n},
\end{equation}
for $0 < x < 1$ if $n = 1$ and $0 \leq x \leq 1$ if $n \geq 2$. We note that
\[ E_n(x; \omega) = -\frac{2}{n+1} B_{n+1}(x; -\omega) \quad \text{and} \quad G_{n+1}(x; \omega) = (n+1)E_n(x; \omega). \]

Then we get immediately, from the Theorem 1, the following identities:

**Theorem 3.** Let $\omega \in \mathbb{C} \setminus \{0, -1\}$, and $x$ a real number. Then, for any $n \geq 1$ we have
\[ \frac{e^{2i\pi x}}{(2i\pi - \log(-\omega))^{n+1}} + \frac{e^{-2i\pi x}}{(-2i\pi - \log(-\omega))^{n+1}} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{n+1}} \left( \frac{(-\omega)(\frac{m+1}{m})}{2(n!)} E_n(\{mx\}; (-\omega)^{1/m}) - \frac{m^{n+1}}{(-\log(-\omega))^{n+1}} \right), \]
and for any $n \geq 2$ we have
\[ \frac{e^{2i\pi x}}{(2i\pi - \log(-\omega))^n} + \frac{e^{-2i\pi x}}{(-2i\pi - \log(-\omega))^n} = \sum_{m=1}^{\infty} \frac{\mu(p)}{m^n} \left( \frac{(-\omega)(\frac{m+1}{m})}{2(n!)} G_n(\{mx\}; (-\omega)^{1/m}) - \frac{m^n}{(-\log(-\omega))^n} \right). \]

**Remark 3.** Note that we can write the last theorem in terms of $\log(\omega)$, thanks to the equalities $\log(-\omega) = \log(\omega + i\pi)$ if $-\pi \leq \text{arg} \omega < 0$ and $\log(-\omega) = \log(\omega - i\pi)$ if $0 \leq \text{arg} \omega < \pi$.

4. **Apostol-Bernoulli numbers and applications**

4.1. **Apostol-Bernoulli numbers.** Let $n$ be an integer $\geq 1$ and $\omega \neq 1$. It is well known that
\[ B_n(x; \omega) = \sum_{k=1}^{n} \binom{n}{k} B_k(0; \omega) x^{n-k}, \]
with
\[ B_k(0; \omega) = \frac{1}{\omega} \sum_{j=0}^{k-1} (-1)^j j! S(k-1, j) \left( \frac{\omega}{\omega - 1} \right)^{j+1} \quad \text{for } k \geq 1 \quad \text{(see [1])}, \]
where $S(k, j)$ are Stirling numbers of the second kind defined by $S(k, j) = \frac{\Delta^{(j)}0^k}{j!}$, with
\[ \Delta^{(j)}0^n = (\Delta^{(j)}x^n)_{x=0}, \quad \Delta^{(j)}0^n = 0 \text{ if } j > n, \quad \Delta^{(0)}0^0 = 1, \]
in the usual notation of finite differences. We call the numbers $B_k(0; \omega)$ the Apostol-Bernoulli numbers.

Now, we set $\omega = e^{2i\pi z}$ with $z \notin \mathbb{Z}$. Then, for $k \geq 1$, we have
\[ B_k(0; e^{2i\pi z}) = \frac{1}{e^{2i\pi z} - 1} \sum_{j=0}^{k-1} (-1)^j j! S(k-1, j) e^{-2i\pi z} \left( \frac{e^{2i\pi z}}{e^{2i\pi z} - 1} \right)^{j+1}. \]
On the other hand, an elementary computation allows us to write

\[ e^{-2i\pi z} \left( \frac{e^{2i\pi z}}{e^{2i\pi z} - 1} \right)^{j+1} = \frac{1}{(2i)^{j+1}} (1 + \cot(\pi z)^2) (i + \cot(\pi z))^{j-1}. \]

Finally, we get the formula:

**Theorem 4.** For any integer \( k \geq 1 \), we have

\[ B_k(0; e^{2i\pi z}) = k \sum_{j=0}^{k-1} (-1)^j j! S(k-1, j) \frac{1}{(2i)^{j+1}} (1 + \cot(\pi z)^2) (i + \cot(\pi z))^{j-1}. \]

The formula (14) shows that, for \( k \geq 1 \), the \( k \)th Apostol-Bernoulli number \( B_k(0; e^{2i\pi z}) \) is a polynomial in terms of \( \cot(\pi z) \) of degree \( k \). Let us give the first values of \( B_k(0; e^{2i\pi z}) \):

\[ B_1(0; e^{2i\pi z}) = \frac{1}{2i} (\cot(\pi z) - i), \quad B_2(0; e^{2i\pi z}) = \frac{1}{2} \left( 1 + \cot(\pi z)^2 \right), \quad B_3(0; e^{2i\pi z}) = \frac{3i}{4} \cot(\pi z) (1 + \cot^2(\pi z)). \]

From Theorem 2 and Theorem 4, with \( x = 0 \) and \( \omega = e^{2i\pi z} \), we obtain the following M"obius inversion formula:

**Proposition 5.** Let \( n \geq 2 \) be an integer and \( z \notin \mathbb{Z} \) with \(-1/2 < \Re(z) < 1/2\). Then, we have

\[ \frac{z^n}{(1-z)^n} + \frac{(-z)^n}{(1+z)^n} = -\sum_{m=1}^{\infty} \mu(m) \left( 1 + (\pi z/m)^n \sum_{j=1}^{n-1} A_{(n-1,j)} \left( \frac{d}{dz} (i + \cot)^j \right) (\pi z/m) \right) \]

where \( A_{(n,j)} = (-1)^{n-j} \frac{(-2i)^{n-j-1}}{n!} S(n, j) \).

4.2. **Euler’s formulas for the zeta function at the even positive integers.**

Recall that the Riemann zeta function is given by

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\Re(s) > 1). \]

As an application of this section, we prove Euler’s formulas for the zeta function at the even positive integers

\[ \zeta(2k) = \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} \pi^{2k} B_{2k}. \]

From the Proposition 5, we obtain for \( n = 2 \) the formula

\[ \frac{z^2}{(1-z)^2} + \frac{z^2}{(1+z)^2} = -\sum_{m=1}^{\infty} \mu(m) \left( 1 + (\pi z/m)^2 \cot'(\pi z/m) \right). \]

On the other hand, for \( |z| < 1 \), we have the expansions

\[ \frac{z^2}{(1-z)^2} + \frac{z^2}{(1+z)^2} = 2 \sum_{k=1}^{\infty} (2k-1) z^{2k} \]
and
\[ z^2 \cot'(z) = \sum_{k \geq 0} (2k-1)\beta_{2k}z^{2k}, \quad \text{where} \quad \beta_{2k} = (-1)^k \frac{2^{2k}}{(2k)!} B_{2k}. \]

Hence, from (18), after switching the order of summation in the double series, we have
\[ \sum_{m=1}^{\infty} \mu(m) \left(1 + (\pi z/m)^2 \cot\left(\pi z/m\right)\right) = \sum_{k=1}^{\infty} (2k-1)\pi^{2k} \beta_{2k} \left(\sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2k}}\right) z^{2k}. \]

Using the equalities (16), (17) and (19) we obtain
\[ 2 \sum_{k=1}^{\infty} (2k-1)z^{2k} = -\sum_{k=1}^{\infty} (2k-1)\pi^{2k} \beta_{2k} \left(\sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2k}}\right) z^{2k}. \]

By comparing the coefficients of \( z^{2k} \) in the right and left sides of formula (20), we obtain
\[ -2 = \pi^{2k} \beta_{2k} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2k}}. \]

Now, using the classical Möbius inversion
\[ \frac{1}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}, \quad (\Re(s) > 1), \]
we recover, from (21) and (18), Euler’s formulae (15).

ACKNOWLEDGMENTS

We are very grateful to the anonymous referee for many helpful comments and constructive remarks on a previous version of this work.

REFERENCES


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