

SERIES REPRESENTATION OF THE RIEMANN ZETA FUNCTION AND OTHER RESULTS: COMPLEMENTS TO A PAPER OF CRANDALL

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ABSTRACT. We supplement a very recent paper of R. Crandall concerned with the multiprecision computation of several important special functions and numbers. We show an alternative series representation for the Riemann and Hurwitz zeta functions providing analytic continuation throughout the whole complex plane. Additionally, we demonstrate some series representations for the initial Stieltjes constants appearing in the Laurent expansion of the Hurwitz zeta function. A particular point of elaboration in these developments is the hypergeometric form and its equivalents for certain derivatives of the incomplete Gamma function. Finally, we evaluate certain integrals including $\int_{\text{Re } s=c} \frac{\zeta(s)}{s} ds$ and $\int_{\text{Re } s=c} \frac{\eta(s)}{s} ds$, with ζ the Riemann zeta function and η its alternating form.

1. INTRODUCTION AND STATEMENT OF RESULTS

Very recently R. Crandall has described series-based algorithms for computing multiprecision values of several important functions appearing in analytic number theory and the theory of special functions [9]. These functions include the Lerch transcendent $\Phi(z, s, a) = \sum_{n=0}^{\infty} z^n / (n+a)^s$, and many of its special cases, and Epstein zeta functions.

In the following we let $B_n(x)$ and $E_n(x)$ be the Bernoulli and Euler polynomials, respectively, Γ the Gamma function, and $\Gamma(x, y)$ the incomplete Gamma function (e.g., [1, 2, 13]). An example series representation from [9, (27)] is that for the Hurwitz zeta function $\zeta(s, a)$,

$$(1.1) \quad \zeta(s, a) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{\Gamma[s, \lambda(n+a)]}{(n+a)^s} + \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} (-1)^m \frac{B_m(a)}{m!} \frac{\lambda^{m+s-1}}{(m+s-1)},$$

with free parameter $\lambda \in [0, 2\pi)$.

We complement the presentation of [9] with further representation of the Riemann zeta function $\zeta(s) = \zeta(s, 1)$ and some results on the Stieltjes constants. The latter constants $\gamma_n(a)$ (e.g., [4–7]) appear in the Laurent expansion

$$(1.2) \quad \zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a) (s-1)^n,$$

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where $\gamma_0(a) = -\psi(a)$, the Euler constant $\gamma_0 = \gamma = -\psi(1)$, and by convention one takes $\gamma_k = \gamma_k(1)$. Here $\psi = \Gamma'/\Gamma$ denotes the digamma function (e.g., [1, 2, 13]). The Stieltjes constants may be expressed via the limit formula

$$\gamma_k(a) = \lim_{N \rightarrow \infty} \left[\sum_{n=0}^N \frac{\ln^k(n+a)}{n+a} - \frac{\ln^{k+1}(N+a)}{k+1} \right].$$

We let $\eta(s) = (1-2^{1-s})\zeta(s)$ denote the alternating Riemann zeta function, given for $\text{Re } s > 0$ by $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$. The following series representation provides an analytic continuation of the Riemann zeta function to the whole complex plane.

Proposition 1. *Let $|\lambda| < \pi$. Then*

$$(1.3) \quad \Gamma(s)\eta(s) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{\Gamma(s, m\lambda)}{m^s} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{E_n(0)}{n!} \frac{\lambda^{n+s}}{(n+s)}.$$

This representation holds in all of \mathbb{C} . In particular, it delivers the special values $\eta(-j) = (-1)^j E_j(0)/2$ for $j \geq 0$. An alternative expression for the values $E_n(0)$ is [1, p. 805] $E_n(0) = -2(n+1)^{-1}(2^{n+1} - 1)B_{n+1}$ for $n \geq 1$ in terms of Bernoulli numbers $B_n = B_n(0)$.

Remark. The values $E_n(0) = -E_n(1)$ for $n \geq 2$ even are zero while $E_0(0) = 1$. The signs of two adjacent values of odd indices are opposite.

Proposition 1 has many implications. For example, for $\lambda = 1$ we have the following, with Li_s denoting the polylogarithm function.

Corollary 1.

$$(1.4a) \quad \ln 2 = \ln(1+e) - 1 + \frac{1}{2} \sum_{n=0}^{\infty} \frac{E_n(0)}{(n+1)!}$$

and

$$(1.4b) \quad \frac{1}{2}\zeta(2) = \ln(1+e) - 1 - \text{Li}_2\left(-\frac{1}{e}\right) + \frac{1}{2} \sum_{n=0}^{\infty} \frac{E_n(0)}{n!(n+2)}.$$

We apply the representation (1.1) for a later proposition. First, we verify some well-known properties of the Hurwitz zeta function from (1.1). Their proofs are given so as to elucidate *how* they follow from that series representation.

Corollary 2. (a)

$$(1.5) \quad \zeta(s, a) = \zeta(s, a+1) + a^{-s},$$

(b)

$$(1.6) \quad \partial_a \zeta(s, a) = -s\zeta(s+1, a),$$

(c) *for integers $q \geq 2$,*

$$(1.7) \quad \sum_{r=1}^{q-1} \zeta\left(s, \frac{r}{q}\right) = (q^s - 1)\zeta(s),$$

(d) *for $\text{Re } s < 1$,*

$$(1.8) \quad \int_0^1 \zeta(s, a) da = 0.$$

By induction, (1.6) gives $\partial_a^j \zeta(s, a) = (-1)^j (s)_j \zeta(s + j, a)$, where $(z)_n = \Gamma(z + n)/\Gamma(z)$ is the Pochhammer symbol.

We let ${}_pF_q$ denote the generalized hypergeometric function (e.g., [1, 2, 13]). Derivatives of the incomplete Gamma function with respect to the first argument may be expressed in hypergeometric form. An example is given in the following.

Proposition 2.

$$(1.9) \quad \Gamma^{(1)}(s, x) \equiv \frac{\partial}{\partial s} \Gamma(s, x) = \int_x^\infty t^{s-1} \ln t e^{-t} dt$$

$$= \frac{x^s}{s^2} {}_2F_2(s, s; s + 1, s + 1; -x) + \Gamma(s)[- \ln x + \psi(s)] + \ln x \Gamma(s, x).$$

Later we provide a separate discussion for such derivatives.

Proposition 3. *Let $Re a > 0$. Then*

(a)

$$(1.10) \quad -\gamma - \psi(a) = \gamma_0(a) - \gamma = e^{-a} \Phi\left(\frac{1}{e}, 1, a\right) + \sum_{m=1}^\infty \frac{(-1)^m B_m(a)}{m! m},$$

(b)

$$(1.11) \quad \frac{\gamma^2}{2} + \gamma\psi(a) + \frac{\zeta(2)}{2} - \gamma_1(a) = \sum_{n=0}^\infty \frac{\Gamma(0, n+a)}{n+a} - \sum_{m=1}^\infty \frac{(-1)^m B_m(a)}{m! m^2},$$

(c)

$$(1.12) \quad -\frac{\gamma^3}{6} - \gamma \frac{\zeta(2)}{2} - \left(\frac{\gamma^2}{2} + \frac{\zeta(2)}{2}\right) \psi(a) + \gamma\gamma_1(a) - \frac{\zeta(3)}{3} + \frac{\gamma_2(a)}{2}$$

$$= \sum_{n=0}^\infty \left\{ -{}_3F_3(1, 1, 1; 2, 2, 2; -n-a) + \frac{1}{(a+n)} \left[\gamma \ln(n+a) + \frac{\gamma^2}{2} + \frac{\zeta(2)}{2} + \frac{1}{2} \ln^2(n+a) \right] \right\}$$

$$+ \sum_{m=1}^\infty \frac{(-1)^m B_m(a)}{m! m^3}.$$

Corollary 3. *For $\lambda \in [0, 2\pi)$,*

$$\ln \Gamma(x) = \gamma(1-x) - \ln \lambda(x-1) + \sum_{n=0}^\infty \{ \Gamma[0, \lambda(n+x)] - \Gamma[0, \lambda(n+1)] \}$$

$$+ \sum_{m=2}^\infty \frac{(-1)^m}{(m-1)m!} [B_m(x) - B_m] \lambda^{m-1}.$$

Alternative forms of the Bernoulli polynomial sums of Proposition 3 and of another sum in (1.11) are given in the following.

Lemma 1. *Let $Re a > 0$. Then*

(a)

$$(1.13) \quad \sum_{m=1}^\infty \frac{(-1)^m B_m(a)}{m! m} = -\frac{1}{a} [e^{-a} {}_2F_1(1, a; a+1; e^{-1}) + a(\gamma + \psi(a))].$$

In particular,

$$(1.14) \quad -\sum_{m=1}^{\infty} \frac{B_m}{m!} \frac{1}{m} = \sum_{n=1}^{\infty} \frac{e^{-n}}{n} = 1 - \ln(e-1).$$

(b)

$$(1.15) \quad \sum_{m=1}^{\infty} \frac{B_m(a)}{m!} \frac{t^m}{m} = -\frac{1}{a} [e^{at} {}_2F_1(1, a; a+1; e^t) + a(\gamma + \ln(-t) + \psi(a))].$$

(c)

$$(1.16) \quad \sum_{m=1}^{\infty} \frac{B_m(a)}{m!} \frac{z^m}{m^2} = -\frac{1}{a} \int_0^z [e^{at} {}_2F_1(1, a; a+1; e^t) + a(\gamma + \ln(-t) + \psi(a))] \frac{dt}{t}.$$

(d)

$$(1.17) \quad \sum_{n=0}^{\infty} \frac{\Gamma(0, n+a)}{n+a} = -\frac{1}{a} \int_0^{1/e} \frac{u^{a-1}}{\ln u} {}_2F_1(1, a; 1+a; u) du.$$

2. PROOF OF PROPOSITIONS

Proposition 1. We apply the integration technique Crandall refers to as “Bernoulli splitting” and use the integral representation for $\operatorname{Re} s > 0$ (e.g., [1, p. 807]):

$$(2.1) \quad \Gamma(s)\eta(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t + 1} dt.$$

Splitting the integral at λ , we use the generating function

$$(2.2) \quad \frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}, \quad |z| < \pi,$$

for the integration on $[0, \lambda)$. For the other integration we use a standard integral representation for the incomplete Gamma function, together with a geometric series expansion:

$$(2.3) \quad \begin{aligned} \int_{\lambda}^{\infty} \frac{t^{s-1}}{e^t + 1} dt &= \sum_{m=0}^{\infty} (-1)^m \int_{\lambda}^{\infty} e^{-(m+1)t} t^{s-1} dt \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^s} \Gamma[s, \lambda(m+1)]. \end{aligned}$$

The sum in (1.3) converges uniformly away from poles so it provides an analytic continuation to $\operatorname{Re} s \leq 0$.

For Corollary 1 we recall that for $n \geq 0$ [13, p. 941],

$$(2.4) \quad \Gamma(n+1, x) = n! e^{-x} \sum_{m=0}^n \frac{x^m}{m!},$$

and $\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$.

Remarks. We have supplied details for Crandall's Algorithm 1 for the η function. For the values $E_n(0)$, their asymptotic form is a case of the following [11, 24.11.6],

$$(-1)^{\lfloor (n+1)/2 \rfloor} \frac{\pi^{n+1}}{4(n!)} E_n(x) \rightarrow \begin{cases} \sin(\pi x), & n \text{ even,} \\ \cos(\pi x), & n \text{ odd,} \end{cases}$$

and one has the bounds for $0 < x < 1/2$ [11, 24.9.5],

$$\frac{4(2n-1)! 2^{2n} - 1}{\pi^{2n} 2^{2n} - 2} > (-1)^n E_{2n-1}(x) > 0.$$

A similar treatment can be made of

$$(2.5) \quad \int_0^\infty \frac{t^{s-1} e^{-(a-1)t}}{e^t + 1} dt = 2^{-s} \Gamma(s) \left[\zeta\left(s, \frac{a}{2}\right) - \zeta\left(s, \frac{a+1}{2}\right) \right] \\ = \Gamma(s) \sum_{n=0}^\infty \frac{(-1)^n}{(n+a)^s},$$

where $\text{Re } a > 0$ and $\text{Re } s > 0$. This yields, for $|\lambda| < \pi$,

$$(2.6) \quad 2^{-s} \Gamma(s) \left[\zeta\left(s, \frac{a}{2}\right) - \zeta\left(s, \frac{a+1}{2}\right) \right] = \sum_{m=0}^\infty (-1)^m \frac{\Gamma[s, \lambda(m+a)]}{(m+a)^s} \\ + \frac{1}{2} \sum_{n=0}^\infty \frac{E_n(1-a)}{n!} \frac{\lambda^{n+s}}{(n+s)},$$

again providing analytic continuation to all of \mathbb{C} . Here the polar part is absent, and this representation could be used to develop expressions for the differences of Stieltjes constants $\gamma_k(a/2) - \gamma_k[(a+1)/2]$.

As an illustration of Proposition 1, Figure 1 plots the natural logarithm of the absolute value of the relative error in the value of $\eta(-3/2)$, $\ln[|\eta_{\text{num}}(-3/2) - \eta(-3/2)|/|\eta(-3/2)|]$, versus the truncation size N of the sums in (1.3), wherein $\eta_{\text{num}}(-3/2)$ is calculated from that equation, and the reference value of $\eta(-3/2)$ has been computed from Mathematica[®] V8. The values of λ in this figure are 1, 1.75, and 2.5, with increasing accuracy obtained for decreasing λ . For $N = 40$ and $\lambda = 1$, the value of $\eta(-3/2)$ from (1.3) is correct to 19 decimal digits. As $\Gamma(s, y) \sim e^{-y} y^{s-1}$ as $y \rightarrow \infty$, the rate of convergence is governed by the second sum in (1.3) with the Euler numbers $E_n(0)$.

Corollary 2. For part (a), use the property $B_m(a+1) = B_m(a) + ma^{m-1}$ and the sum (by [13, p. 941]),

$$(2.7) \quad \sum_{m=0}^\infty (-1)^{m+1} \frac{a^m}{m!} \frac{\lambda^{m+s}}{(m+s)} = a^{-s} [\Gamma(s, a\lambda) - \Gamma(s)].$$

For part (b), use the derivative $d\Gamma(\alpha, x)/dx = -x^{\alpha-1} e^{-x}$ and the functional relations $\alpha\Gamma(\alpha, x) = \Gamma(\alpha+1, x) - x^\alpha e^{-x}$ and $\Gamma(s) = \Gamma(s+1)/s$. For part (c), use $\sum_{r=1}^{q-1} B_m\left(\frac{r}{q}\right) = (q^{1-m} - 1)B_m$ and $(-1)^m B_m(1) = B_m$. The following decomposition, wherein the generating function (2.24) is employed, can then be used to verify

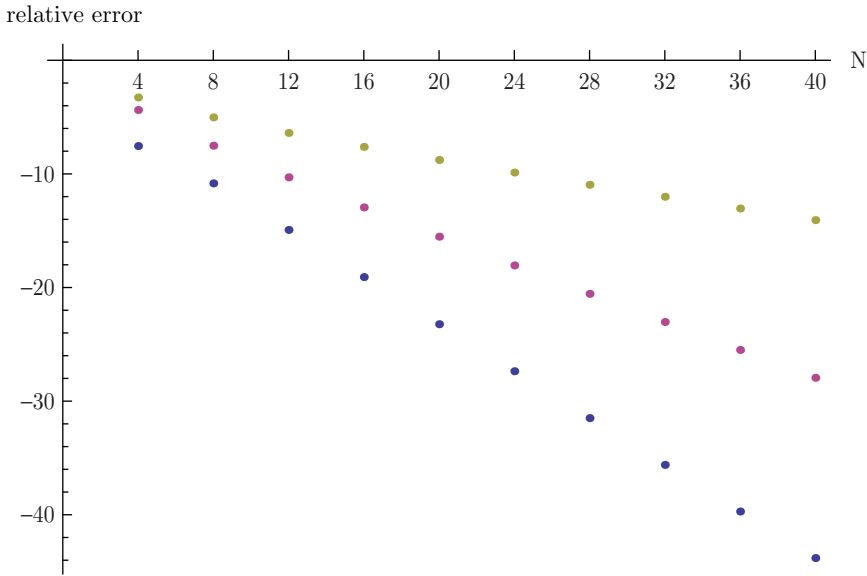


FIGURE 1. Plot of the natural logarithm of the absolute value of the relative error in $\eta(-3/2)$ computed from (1.3).

the stated property:

$$\begin{aligned}
 q^s \zeta(s) &= \frac{q^s}{\Gamma(s)} \int_0^\infty \frac{y^{s-1}}{e^y - 1} dy \\
 &= \frac{q^s}{\Gamma(s)} \left[\int_0^{\lambda/q} \frac{y^{s-1}}{e^y - 1} dy + \int_{\lambda/q}^\infty \frac{y^{s-1}}{e^y - 1} dy \right] \\
 (2.8) \quad &= \frac{1}{\Gamma(s)} \sum_{m=0}^\infty q^{1-m} \frac{B_m}{m!} \frac{\lambda^{m+s-1}}{(m+s-1)} \\
 &\quad + \frac{q^s}{\Gamma(s)} \sum_{n=0}^\infty \int_{\lambda/q}^\infty \frac{e^{-(n+1)qy} (e^{qy} - 1) y^{s-1}}{e^y - 1} dy.
 \end{aligned}$$

For part (d), we first note that $\int_0^1 B_m(a) da = 0$ for all $m \geq 1$, giving from (1.1),

$$(2.9) \quad \Gamma(s) \int_0^1 \zeta(s, a) da = \sum_{n=0}^\infty \int_0^1 \frac{\Gamma(s, \lambda(n+a))}{(n+a)^s} da + \frac{\lambda^{s-1}}{s-1}.$$

Integrating by parts,

$$\begin{aligned}
 \int_0^1 \frac{\Gamma(s, \lambda(n+a))}{(n+a)^s} da &= -\frac{1}{(s-1)} \int_0^1 \Gamma(s, \lambda(n+a)) \left(\frac{d}{da} \frac{1}{(n+a)^{s-1}} \right) da \\
 (2.10) \quad &= \frac{1}{s-1} \left[-\lambda^{s-1} e^{-\lambda n} + \lambda^{s-1} e^{-\lambda(n+1)} + n^{1-s} \Gamma(s, \lambda n) \right. \\
 &\quad \left. - (n+1)^{1-s} \Gamma(s, \lambda(n+1)) \right].
 \end{aligned}$$

Then the sum in (2.9) telescopes to $-\lambda^{s-1}/(s-1)$ and (1.8) follows. □

Remark. More generally than part (c), we have the reciprocity relation ([7, Lemma 1]),

$$(2.11) \quad \sum_{r=1}^q \zeta \left(s, \frac{pr}{q} - b \right) = \left(\frac{q}{p} \right)^s \sum_{\ell=0}^{p-1} \zeta \left(s, \frac{\ell q + p}{p} - \frac{qb}{p} \right).$$

Here $p \geq 1$ and $q \geq 1$ are integers, $b \geq 0$, and $\min(p/q, q/p) > b$. From (2.11) follows ([7, Corollary 3]),

$$(2.12) \quad \sum_{r=1}^q B_m \left(\frac{pr}{q} - b \right) = \left(\frac{q}{p} \right)^{1-m} \sum_{\ell=0}^{p-1} B_m \left[1 + (\ell - b) \frac{q}{p} \right].$$

Proposition 2. First integrating by parts we have

$$(2.13) \quad \begin{aligned} \Gamma^{(1)}(s, x) &\equiv \frac{\partial}{\partial s} \Gamma(s, x) = \int_x^\infty t^{s-1} \ln t e^{-t} dt \\ &= - \int_x^\infty \ln t \frac{\partial}{\partial t} \Gamma(s, t) dt = \int_x^\infty \frac{\Gamma(s, t)}{t} dt + \ln x \Gamma(s, x). \end{aligned}$$

We next recall a hypergeometric form of $\Gamma(x, y)$,

$$(2.14) \quad \begin{aligned} \Gamma(\alpha, x) &= \Gamma(\alpha) - \frac{x^\alpha}{\alpha} {}_1F_1(\alpha, 1 + \alpha; -x) \\ &= \Gamma(\alpha) - \frac{x^\alpha}{\alpha} \sum_{j=0}^\infty \frac{\alpha}{(\alpha + j)} \frac{(-x)^j}{j!}, \end{aligned}$$

with ${}_1F_1$ the confluent hypergeometric function. This expression may be integrated on a finite interval. For an improper integral extending to infinity we need some asymptotic information contained in the following.

Lemma 2 (Asymptotic form of special ${}_pF_p$ functions). *As $x \rightarrow \infty$*

(a)

$$(2.15) \quad {}_2F_2(s, s; s + 1, s + 1; -x) \sim e^{-x} \frac{s^2}{x^2} + x^{-s} s^2 \Gamma(s) [\ln x - \psi(s)],$$

(b)

$$(2.16) \quad \begin{aligned} {}_3F_3(s, s, s; s + 1, s + 1, s + 1; -x) &\sim -e^{-x} \frac{s^3}{x^3} \\ &+ x^{-s} \frac{s^3}{2} \Gamma(s) [\ln^2 x - 2 \ln x \psi(s) + \psi^2(s) + \psi'(s)]. \end{aligned}$$

Here ψ' is the trigamma function.

(a) We use a general procedure based upon the Barnes integral representation of ${}_pF_q$ ([15, Section 2.3]). We have

$$(2.17) \quad {}_2F_2(s, s; s + 1, s + 1; -x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(-y) \Gamma(y + 1) g(y) x^y dy,$$

where the path of integration is a Barnes contour, indented to the left of the origin but staying to the right of $-s$, and

$$(2.18) \quad \Gamma(y + 1) g(y) = \frac{\Gamma^2(y + s)}{\Gamma^2(y + s + 1)} \frac{\Gamma^2(s + 1)}{\Gamma^2(s)} = \frac{s^2}{(y + s)^2}.$$

The contour can be thought of as closed in the right half-plane, over a semicircle of infinite radius. We then move the contour to the left of $y = -s$, picking up the residue there. We find

$$(2.19) \quad s^2 \operatorname{Res}_{y=-s} \frac{\Gamma(-y)x^y}{(y+s)^2} = \frac{s^2}{x^s} \Gamma(s) [\ln x - \psi(s)].$$

This gives the algebraic part of the asymptotic form of the ${}_2F_2$ function, that is, the leading portion. The higher order terms come from the exponential expansion of these functions, and they are infinite in number. The latter expansion may be determined according to ([15, Section 2.3, p. 57]).

(b) proceeds similarly, where now

$$(2.20) \quad s^3 \operatorname{Res}_{y=-s} \frac{\Gamma(-y)x^y}{(y+s)^3} = \frac{s^3}{2x^s} \Gamma(s) [\ln^2 x - 2 \ln x \psi(s) + \psi^2(s) + \psi'(s)].$$

We may integrate in (2.13) from x to b using (2.14),

$$(2.21) \quad \int_x^b \frac{\Gamma(s,t)}{t} dt = \frac{x^s}{s^2} {}_2F_2(s, s; s+1, s+1; -x) - \frac{b^s}{s^2} {}_2F_2(s, s; s+1, s+1; -b) + \Gamma(s)(\ln b - \ln x).$$

Then with the lemma in hand, taking $b \rightarrow \infty$ completes the proposition.

Remark. The result (1.9) can be obtained directly from (2.14). For, since $\Gamma'(\alpha) = \Gamma(\alpha)\psi(\alpha)$, we have

$$\begin{aligned} \frac{\partial}{\partial \alpha} \Gamma(\alpha, x) &= \Gamma(\alpha)\psi(\alpha) + \ln x [\Gamma(\alpha, x) - \Gamma(\alpha)] + x^\alpha \sum_{j=0}^\infty \frac{1}{(\alpha+j)^2} \frac{(-x)^j}{j!} \\ &= \frac{\partial}{\partial \alpha} \Gamma(\alpha, x) = \Gamma(\alpha)\psi(\alpha) + \ln x [\Gamma(\alpha, x) - \Gamma(\alpha)] + \frac{x^\alpha}{\alpha^2} \sum_{j=0}^\infty \frac{(\alpha)_j^2}{(\alpha+1)_j^2} \frac{(-x)^j}{j!} \\ &= \frac{\partial}{\partial \alpha} \Gamma(\alpha, x) = \Gamma(\alpha)\psi(\alpha) + \ln x [\Gamma(\alpha, x) - \Gamma(\alpha)] + \frac{x^\alpha}{\alpha^2} {}_2F_2(\alpha, \alpha; \alpha+1, \alpha+1; -x). \end{aligned}$$

Proposition 3. We take $\lambda = 1$ in (1.1) and, making use of Proposition 2 and related considerations, expand the function

$$(2.22) \quad \Gamma(s)\zeta(s, a) - \frac{1}{s-1} = \sum_{n=0}^\infty \frac{\Gamma(s, n+a)}{(n+a)^s} + \sum_{m=1}^\infty \frac{(-1)^m}{m!} \frac{B_m(a)}{(m+s-1)}$$

about $s = 1$, and apply the defining series (1.2).

Corollary 3. Maintaining the free parameter λ , this follows from $-\int_1^x [\gamma + \psi(a)] da = \gamma(1-x) - \ln \Gamma(x)$, the corresponding expression similar to (1.10),

$$(2.23) \quad -\ln \lambda - \gamma - \psi(a) = e^{-\lambda a} \Phi \left(\frac{1}{e^\lambda}, 1, a \right) + \sum_{m=1}^\infty \frac{(-1)^m}{m!} \frac{B_m(a)}{m} \lambda^m,$$

and the property $\int B_m(a) da = B_{m+1}(a)/(m+1)$. □

To illustrate Corollary 3, Figure 2 shows a plot of the natural logarithm of the absolute value of the relative error in the value of $\ln \Gamma(7/4)$ versus N , the truncation size of the sums therein. Now $\lambda = \pi$ and $\pi/2$, and the reference value of $\ln \Gamma(7/4)$ is computed from Mathematica[®] V8.

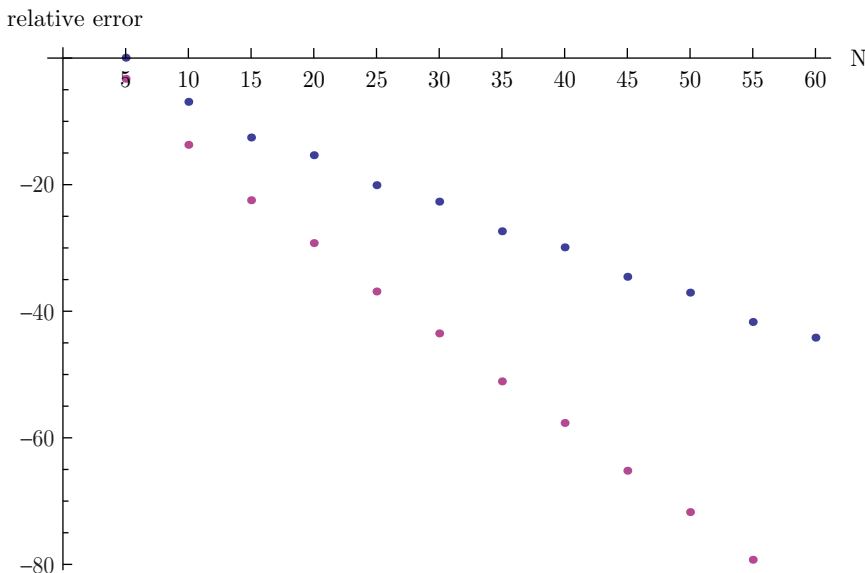


FIGURE 2. Plot of the natural logarithm of the absolute value of the relative error in $\ln \Gamma(7/4)$ computed from Corollary 3.

Lemma 1. For (a)–(c), repeatedly integrate the generating function

$$(2.24) \quad \sum_{n=1}^{\infty} \frac{B_n(a)}{n!} z^n = \frac{ze^{az}}{e^z - 1} - 1, \quad |z| < 2\pi.$$

For (d), write

$$(2.25) \quad \sum_{n=0}^{\infty} \frac{\Gamma(0, n+a)}{n+a} = \sum_{n=0}^{\infty} \frac{1}{n+a} \int_{n+a}^{\infty} e^{-t} \frac{dt}{t} = \sum_{n=0}^{\infty} \frac{1}{n+a} \int_1^{\infty} e^{-(n+a)v} \frac{dv}{v}.$$

Then interchange summation and integration and perform another change of variable. □

Remark. As $\Gamma(0, x) = -\text{Ei}(-x)$, where Ei is the exponential integral having many integral representations, there are a multitude of ways of obtaining (1.17).

3. DISCUSSION: DERIVATIVES OF THE INCOMPLETE GAMMA FUNCTION

Derivatives of the incomplete Gamma function are important for the computational methods of [9], and they provide a starting point for many useful integrals. Geddes et al. [12] investigated these derivatives, introducing a function $T(m, a, z)$, with $\Gamma(a, z) = zT(2, a, z)$, $m \geq 1$ an integer and initially $|z| < 1$, such that

$$(3.1a) \quad \Gamma^{(1)}(a, x) = xT(3, a, x) + \ln x \Gamma(a, x),$$

$$(3.1b) \quad \Gamma^{(2)}(a, x) = \ln^2 x \Gamma(a, x) + 2x[\ln x T(3, a, x) + T(4, a, x)],$$

and more generally, with $P_j^i = i!/(i - j)!$,

$$(3.2) \quad \frac{d^m \Gamma(a, x)}{da^m} \equiv \Gamma^{(m)}(a, x) = \ln^m x \Gamma(a, x) + mx \sum_{i=0}^{m-1} P_i^{m-1} \ln^{m-i-1} x T(3 + i, a, x).$$

The function T satisfies the derivative relations

$$(3.3a) \quad \frac{dT(m, a, z)}{da} = \ln z T(m, a, z) + (m - 1)T(m + 1, a, z)$$

and

$$(3.3b) \quad \frac{dT(m, a, z)}{dz} = -\frac{1}{z} [T(m - 1, a, z) + T(m, a, z)].$$

One form of T is [12, (37)],

$$(3.4) \quad T(m, a, z) = -\text{Res}_{s=-1} \left[\left(-\frac{1}{s+1} \right)^{m-1} \Gamma(a-1-s) z^s \right] + \sum_{i=0}^{\infty} \frac{(-1)^i z^{a+i-1}}{i!(-a-i)^{m-1}},$$

where a is neither zero or a negative integer. This can be obtained by expressing T in terms of the Meijer G -function and then using a contour integral for the latter function. Using [12, (38)], and the relation $(a)_i/(a + 1)_i = a/(a + i)$, as we have previously done, we may write

$$(3.5) \quad T(m, a, z) = \frac{(-1)^{m-1}}{(m-2)!} \left(\frac{d}{dt} \right)^{m-2} [\Gamma(a-t) z^{t-1}]_{t=0} + (-1)^{m-1} \frac{z^{a-1}}{a^{m-1}} {}_{m-1}F_{m-1}(a, a, \dots, a; a+1, a+1, \dots, a+1; -z).$$

Proposition 2 and like considerations are in accord with this result.

We note that Milgram [14] studied derivatives with respect to the order s of the exponential integral function $E_s(z) = z^{s-1} \Gamma(1 - s, z)$ by using the G -function and contour integral representation.

4. CERTAIN ZETA FUNCTIONS AND OTHER INTEGRALS

Elsewhere we have considered second moment integrals of the Riemann zeta and other functions [8]. The following concerns the integrals

$$(4.1) \quad I(c) \equiv \int_{-\infty}^{\infty} \frac{\zeta(c+it)}{c+it} dt = -i \int_{\text{Res}=c} \frac{\zeta(s)}{s} ds,$$

for integer $p \geq 2$,

$$(4.2) \quad I_p(c) \equiv \int_{-\infty}^{\infty} \frac{\zeta(c+it)}{(c+it)^p} dt = -i \int_{\text{Res}=c} \frac{\zeta(s)}{s^p} ds,$$

$$(4.3) \quad I_a(c) \equiv \int_{-\infty}^{\infty} \frac{\eta(c+it)}{c+it} dt = -i \int_{\text{Res}=c} \frac{\eta(s)}{s} ds,$$

and

$$(4.4) \quad I_L(c) \equiv \int_{-\infty}^{\infty} \frac{\text{Li}_{c+it}(x)}{c+it} dt = -i \int_{\text{Res}=c} \frac{\text{Li}_s(x)}{s} ds.$$

Proposition 4. *We have $I(c) = -\pi$ for $0 < c < 1$, and $I(c) = \pi$ for $c > 1$, $I_p(c) = 0$ for $c > 1$ and $I_p(c) = -2\pi$ for $0 < c < 1$, $I_a(c = 0) = 2\pi$ and $I_a(c) = \pi$ for $c > 0$, and $I_L(c) = \pi x$ for $|x| < 1$, $x \in \mathbb{R}$ and $c > 0$ and $I_L(c) = -\pi x(1+x)/(1-x)$ for $c < 0$.*

Proof. These results may be developed with the aid of Perron’s formula ([3, p. 245], or [15, p. 95]) one form of which is [10, p. 147]

$$(4.5) \quad \frac{1}{2\pi i} \int_{\text{Res}=\sigma} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \theta(x-n),$$

where $\theta(x)$ is the step function taking the values 1, 1/2, and 0 as $x > 0$, $x = 0$, and $x < 0$, respectively. The line of integration may be moved by including the poles of the integrand. This may be seen by integrating over a rectangular contour with corners at $c \pm iT$ and $b \pm iT$. For the Hurwitz zeta function one has the growth estimates [16, p. 276], with $s = \sigma + it$ and $1/2 > \delta > 0$,

$$(4.6) \quad \begin{aligned} \zeta(s, a) &= O(1) \text{ for } \sigma > 1 + \delta, \\ &= O(|t|^{1/2-\sigma}) \text{ for } \sigma \leq \delta, \\ &= O(|t|^{1/2}) \text{ for } \delta \leq \sigma \leq 1 - \delta, \\ &= O(|t|^{1-\sigma} \ln |t|) \text{ for } 1 - \delta \leq \sigma \leq 1 + \delta, \\ &= O(|t|^{1/2} \ln |t|) \text{ for } -\delta \leq \sigma \leq \delta. \end{aligned}$$

Further inequalities for $\zeta_a(1-s) = \sum_{n=1}^{\infty} e^{2n\pi ia}/n^{1-s}$ are also given in [16]. Cases of these may be used to show that the “top” and “bottom” contributions of the rectangular contour of integration vanish as $T \rightarrow \infty$. Then the Cauchy residue theorem is applied. Example estimations along horizontal line segments include the following, wherein we note the independence of the parameter a , and c_1 and c_2 are positive constants,

$$\begin{aligned} &\left| \int_{-\delta}^{\delta} \frac{\zeta(\sigma \pm iT)}{s} d\sigma \right| \leq \int_{-\delta}^{\delta} \frac{|\zeta(\sigma \pm iT)|}{s} d\sigma \\ &\leq \frac{1}{T} \int_{-\delta}^{\delta} |\zeta(\sigma \pm iT)| d\sigma \leq \frac{c_1}{T} \int_{-\delta}^{\delta} |T|^{1/2} \ln |T| d\sigma = 2\delta \frac{c_1}{T^{1/2}} \ln |T| \end{aligned}$$

and

$$\begin{aligned} &\left| \int_{1-\delta}^{1+\delta} \frac{\zeta(\sigma \pm iT)}{s} d\sigma \right| \leq \int_{1-\delta}^{1+\delta} \frac{|\zeta(\sigma \pm iT)|}{s} d\sigma \\ &\leq \frac{1}{T} \int_{1-\delta}^{1+\delta} |\zeta(\sigma \pm iT)| d\sigma \leq \frac{c_2}{T} \int_{1-\delta}^{1+\delta} |T|^{1-\sigma} \ln |T| d\sigma \\ &\leq \frac{c_2 \ln T}{T} \int_{1-\delta}^{1+\delta} |T|^{1-\sigma} d\sigma = \frac{c_2}{T} (T^\delta - T^{-\delta}). \end{aligned}$$

The interval $[c, b]$ can then be appropriately decomposed with such contributions.

The relevant residues are the following. For $\eta(s)/s$, 1/2 at $s = 0$. For $\zeta(s)/s$, $-1/2$ at $s = 0$, being a case of $\zeta(0, a) = 1/2 - a$, and 1 at $s = 1$. For $\text{Li}_s(x)/s$, $x/(1-x)$ at $s = 0$. For the latter we recall the series $\text{Li}_0(x) = \sum_{k=1}^{\infty} x^k$ for

$|x| < 1$, or alternatively that $\text{Li}_1(x) = -\ln(1 - x)$ and the derivative property $d\text{Li}_s(x)/dx = \text{Li}_{s-1}(x)/x$.

So, for instance, for (4.1) with $c > 1$, the sum on n in (4.5) may be performed,

$$(4.7) \quad \frac{1}{2\pi i} \int_{\text{Res}=c} \zeta(s) \frac{ds}{s} = \sum_{n=1}^{\infty} \theta(1 - n) = \frac{1}{2}.$$

This gives $\int_{\text{Res}=c} \zeta(s)(ds/s) = i\pi$. For $c < 1$ the contribution from the pole at $s = 1$ is subtracted, $\pi i - 2\pi i = -i\pi$.

Similarly, for (4.4), when $c < 0$, the stated formula for $I_L(c)$ is obtained from

$$\pi x - \frac{2\pi x}{1 - x} = \pi x \left(\frac{x + 1}{x - 1} \right).$$

For (4.2), we repeatedly divide (4.5) by x and integrate with respect to x . This gives

$$(4.8) \quad \frac{1}{2\pi i} \int_{\text{Res}=\sigma} \frac{x^s}{n^s} \frac{ds}{s^{m+1}} = \frac{\theta(x - n)}{m!} \ln^m \left(\frac{x}{n} \right).$$

This relation follows by induction, and the base case is given by integration by parts. In detail for that case,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\text{Res}=\sigma} \frac{x^s}{n^s} \frac{ds}{s^2} &= \int \left[\frac{d}{dx} \theta(x - n) \ln x - \delta(x - n) \ln x \right] dx \\ &= \theta(x - n) \ln(x/n), \end{aligned}$$

wherein $\delta(x) = d\theta(x)/dx$ is the Dirac delta function. Summing on n and then putting $x = 1$ then yields, for $c > 1$,

$$\frac{1}{2\pi i} \int_{\text{Res}=c} \zeta(s) \frac{ds}{s^{m+1}} = 0.$$

Since $\text{Res}[\zeta(s)/s^{m+1}]_{s=1} = 1$, the stated result for $0 < c < 1$ follows.

Remarks. In [15, p. 145], the ‘‘astonishing results’’ for $n \in \mathbb{N}^+$,

$$(4.9a) \quad \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s)n^s}{s(s+1)} ds = -i\pi, \quad 0 < c < 1$$

and

$$(4.9b) \quad \int_{-c-i\infty}^{-c+i\infty} \frac{\zeta(s)n^s}{s(s+1)} ds = 0, \quad 0 < c < 1,$$

are given. These integrals, as in Proposition 4, follow from Perron’s formula. For instance, integrating (4.5) with respect to x gives

$$(4.10) \quad \frac{1}{2\pi i} \int_{\text{Res}=\sigma} x \left(\frac{x}{n} \right)^s \frac{ds}{s(s+1)} = (x - n)\theta(x - n).$$

The integration of the right side may be readily verified with the identity $x\delta(x) = 0$. When $x = 1$, summation over n in (4.10) then yields

$$\frac{1}{2\pi i} \int_{\text{Res}=c} \frac{\zeta(s)ds}{s(s+1)} = 0, \quad c > 1.$$

The line of integration may then be moved successively to the left of $c = 1$ and $c = 0$, by accounting for the simple poles, to give the $n = 1$ cases of (4.9).

This line of development may be continued. For instance, an integration of (4.10) with respect to x now gives

$$(4.11) \quad \frac{1}{2\pi i} \int_{\text{Res}=\sigma} \frac{x^{s+2}}{n^s} \frac{ds}{s(s+1)(s+2)} = \frac{1}{2}(x-n)^2\theta(x-n).$$

More generally, we have

$$(4.12) \quad \frac{1}{2\pi i} \int_{\text{Res}=\sigma} \frac{x^{s+m}}{n^s} \frac{ds}{s(s+1)\cdots(s+m)} = \frac{1}{m!}(x-n)^m\theta(x-n),$$

provable by induction on nonnegative integer m . This result may then be applied in a number of ways. \square

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