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# ADAPTIVE FOURIER-GALERKIN METHODS

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Abstract. We study the performance of adaptive Fourier-Galerkin methods in a periodic box in $\mathbb{R}^d$ with dimension $d \geq 1$. These methods offer unlimited approximation power only restricted by solution and data regularity. They are of intrinsic interest but are also a first step towards understanding adaptivity for the *hp*-FEM. We examine two nonlinear approximation classes, one classical corresponding to algebraic decay of Fourier coefficients and another associated with exponential decay typical of spectral approximation. We investigate the natural sparsity class for the operator range and find that the exponential class is not preserved, thus in contrast with the algebraic class. This entails a striking different behavior of the feasible residuals that lead to practical algorithms, influencing the overall optimality. The sparsity degradation for the exponential class is partially compensated with coarsening. We present several feasible adaptive Fourier algorithms, prove their contraction properties, and examine the cardinality of the activated sets. The Galerkin approximations at the end of each iteration are quasi-optimal for both classes, but inner loops or intermediate approximations are sub-optimal for the exponential class.

## 1. Introduction

Adaptivity is now a fundamental tool in scientific and engineering computation. In contrast to the practice, which goes back to the 70's, the mathematical theory for multidimensional problems is rather recent. It started in 1996 with the convergence results by Dörfler [14] and Morin, Nochetto, and Siebert [19]. The first convergence rates were derived by Cohen, Dahmen, and DeVore [8] for wavelets in any dimensions $d$, and for finite element methods (AFEM) by Binev, Dahmen, and DeVore [2] for $d = 2$ and Stevenson [22] for any $d$. The most comprehensive results for AFEM are those of Cascón, Kreuzer, Nochetto, and Siebert [6] for any $d$ and $L^2$ data, and Cohen, DeVore, and Nochetto [9] for $d = 2$ and $H^{-1}$ data. We refer to the surveys [20] by Nochetto, Siebert and Veeser for AFEM and [23] by Stevenson for adaptive wavelet methods. This theory is quite satisfactory in that it shows that both AFEM and wavelets deliver a convergence rate compatible with that of the approximation classes where the solution and data belong. However, these convergence rates are limited by the approximation power of the method, which is finite and related to the polynomial degree of the basis functions, and the regularity of the solution and data. The latter is always measured in an *algebraic* approximation class.

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In contrast very little is known for methods with infinite approximation power, such as those based on Fourier analysis. We mention here the results of DeVore and Temlyakov [13] for trigonometric sums and those of Binev et al. [1] for the reduced basis method. A close relative to Fourier methods is the so-called $p$-version of the FEM (see, e.g., [21] and [5]), which uses Legendre polynomials instead of exponentials as basis functions. The purpose of this paper is to present *adaptive Fourier-Galerkin methods (ADFOUR)*, and discuss their convergence and optimality properties. We do so in the context of both *algebraic* and *exponential* approximation classes, and take advantage of the orthogonality inherent to complex exponentials. We believe that this approach can be extended to the $p$-FEM. We view this theory as a first step towards understanding adaptivity for the $hp$-FEM, which combines mesh refinement ($h$-FEM) with polynomial enrichment ($p$-FEM) and is much harder to analyze.

Our investigation reveals some striking differences between ADFOUR and AFEM or wavelet methods. The basic assumption, underlying the success of adaptivity mentioned above, is that the information read in the residual is quasi-optimal for either mesh design or choosing wavelet coefficients for the actual solution. Quasi-optimality is surely guaranteed, provided the sparsity classes of the residual and the solution coincide; this property is true for algebraic classes, but we found it to be false for exponential classes, as fully discussed in Section 5. Confronted with this unexpected fact, we have no alternative but to implement and study ADFOUR with *coarsening* for the exponential case; see Section 4.4 and Section 8. This was the original idea of Cohen et al. [8] and Binev et al. [2] for the algebraic case, but it was subsequently removed by Stevenson and co-authors [16, 22].

Now we give a brief description of the essential issues we are confronted with in designing and studying ADFOUR. To this end, we assume that we know the Fourier representation $\mathbf{v} = \{v_k\}_{k\in\mathbb{Z}}$ of a periodic function $v$, and its nonincreasing rearrangement $\mathbf{v}^* = \{v_n^*\}_{n=1}^{\infty}$, namely, $|v_{n+1}^*| \le |v_n^*|$ for all $n \ge 1$.

**Dörfler marking and best $N$-term approximation.** We recall the marking introduced by Dörfler [14], which is the only one for which there exist provable convergence rates. Given a parameter $\theta \in (0, 1)$, and a current set of Fourier frequencies or indices $\Lambda$, say the first $N$ ones according to the labeling of $\mathbf{v}$, we choose the next set $\partial\Lambda$ as the *minimal* set for which

$$\|P_{\partial\Lambda}\mathbf{r}\| \ge \theta\|\mathbf{r}\|, \tag{1.1}$$

where $\mathbf{r} := \mathbf{v} - P_\Lambda \mathbf{v}$ is the *residual* and $P_\Lambda$ is the orthogonal projection in the $\ell^2$-norm $\|\cdot\|$ onto $\Lambda$. Note that, if $\mathbf{r}_* := \mathbf{r} - P_{\partial\Lambda}\mathbf{r}$ and $\Lambda_* := \Lambda \cup \partial\Lambda$, then (1.1) can be equivalently written as

$$\|\mathbf{r}_*\| = \|\mathbf{r} - P_{\partial\Lambda}\mathbf{r}\| \le \sqrt{1-\theta^2}\|\mathbf{r}\|, \tag{1.2}$$

and that $\mathbf{r} = \mathbf{v}|_{\Lambda^c}$ where $\Lambda^c := \mathbb{N}\backslash\Lambda$ is the complement of $\Lambda$ and likewise for $\mathbf{r}_*$. This is the simplest possible scenario because the information built in $\mathbf{r}$ is exactly that of $\mathbf{v}$. Moreover, $\mathbf{v} - \mathbf{r} = \{v_n^*\}_{n=1}^{N}$ is the best $N$-term approximation of $\mathbf{v}$ in the $\ell^2$-norm and the corresponding error $E_N(v)$ is given by

$$E_N(v) = \Big(\sum_{n>N} |v_n^*|^2\Big)^{-\frac{1}{2}} = \|\mathbf{r}\|. \tag{1.3}$$

**Algebraic vs. exponential decay.** Suppose now that $\mathbf{v}$ has the precise *algebraic* decay[1]

$$|v_n^*| \simeq n^{-\frac{1}{\tau}} \quad \forall\, n \geq 1, \tag{1.4}$$

with $1/\tau = s/d + 1/2$ and $s > 0$. We denote by $\|\mathbf{v}\|_{\ell_B^s}$ the smallest constant in the upper bound in (1.4). We thus have

$$E_N(v)^2 \simeq \|\mathbf{v}\|_{\ell_w^\tau}^2 \sum_{n>N} n^{-\frac{2}{\tau}} = \|\mathbf{v}\|_{\ell_B^s}^2 \sum_{n>N} n^{-\frac{2s}{d}-1} \simeq \|\mathbf{v}\|_{\ell_B^s}^2 N^{-\frac{2s}{d}}.$$

This decay is related to certain *Besov* regularity of $v$ [13]. Note that the effect of Dörfler marking (1.2) is to reduce the residual from $\mathbf{r}$ to $\mathbf{r}_*$ by a factor $\alpha = \sqrt{1-\theta^2}$, or equivalently $E_{N_*}(v) \leq \alpha E_N(v)$, with $N_* = |\Lambda_*|$. Since the set $\Lambda_*$ is minimal, we deduce that $E_{N_*-1}(v) > \alpha E_N(v)$, whence

$$\frac{N_*}{N} \simeq \alpha^{-\frac{d}{s}}, \qquad \text{i.e.,} \qquad N_* - N \simeq (\alpha^{-\frac{d}{s}} - 1)N \tag{1.5}$$

for $\alpha$ small enough. This means that the number of degrees of freedom to be added is proportional to the current number. This simplifies considerably the complexity analysis since every step adds as many degrees of freedom as we have already accumulated.

The exponential case is quite different. Suppose that $\mathbf{v}$ has a *genuinely exponential* decay

$$|v_n^*| \simeq e^{-\eta n} \quad \forall\, n \geq 1, \tag{1.6}$$

corresponding to analytic functions [15], and let $\|\mathbf{v}\|_{\ell_G^\eta}$ be the smallest constant appearing in the upper bound in (1.6). These definitions are slight simplifications of the actual ones in Section 4.3 but enough to give insight on the main issues at stake. We thus have

$$E_N(v)^2 \simeq \|\mathbf{v}\|_{\ell_G^\eta}^2 \sum_{n>N} e^{-2\eta n} \simeq \|\mathbf{v}\|_{\ell_G^\eta}^2 e^{-2\eta N};$$

this and similar decays are related to *Gevrey* classes of $C^\infty$ functions [15]. In contrast to (1.5), Dörfler marking now yields[2]

$$N_* - N \sim \frac{1}{\eta} \log \frac{1}{\alpha}. \tag{1.7}$$

This shows that the number of additional degrees of freedom per step is fixed and independent of $N$, which makes their counting as well as their implementation a very delicate matter. Even more delicate is the situation in which not all the rearranged components of $v$ exhibit the ideal decay considered above, as in the presence of *plateaux*, where a relevant number of Fourier coefficients of $v$ are constant. Then one can show that the Dörfler condition (1.1) adds many more frequencies, which poses further difficulties in the analysis of the exponential case.

**Sparsity of the range of the operator.** In practice we do not have access to the Fourier decomposition of $v$ but rather of the residual $r(v) = f - Lv$, where $f$ is the forcing function and $L$ the differential operator. Only an operator $L$ with

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[1]Throughout the paper, $A \lesssim B$ means $A \leq c\,B$ for some constant $c > 0$ independent of the relevant parameters in the inequality; $A \simeq B$ means $B \lesssim A \lesssim B$.

[2]Throughout the paper, $A \sim B$ means $A = B + c$ for some quantity $c$ negligible with respect to $B$.

constant coefficients leads to a spectral representation with diagonal matrix $\mathbf{A}$, in which case the components of the residual $\mathbf{r} = \mathbf{f} - \mathbf{A}\mathbf{v}$ are directly those of $\mathbf{f}$ and $\mathbf{v}$. In general $\mathbf{A}$ decays away from the main diagonal with a law that depends on the regularity of the coefficients of $L$; we will examine in Section 2.4 either algebraic or exponential decay. In the much more intricate and interesting endeavor, studied in this paper, the components of $\mathbf{v}$ interact with entries of $\mathbf{A}$ to give rise to $\mathbf{r}$. The question whether $Lv$ belongs to the same approximation class of $v$ thus becomes relevant because adaptivity decisions are made with $r(v)$, and thereby on the range of $L$ rather than its domain.

As documented in Section 5.2, examples can be given which show that the action of $\mathbf{A}$ may shift the exponential class, from the one characterized by the parameter $\eta$ for $\mathbf{v}$ to the one characterized by some $\bar{\eta} < \eta$ for $\mathbf{A}\mathbf{v}$. Even more sophisticated examples illustrate the fact that the exponent $\tau = 1$ in the bound $|v_n^*| \lesssim e^{-\eta n} = e^{-\eta n^\tau}$ for $\mathbf{v}$ may deteriorate to some $\bar{\tau} < 1$ in the corresponding bound for $\mathbf{A}\mathbf{v}$. This uncovers the crucial feature that the image $\mathbf{A}\mathbf{v}$ of $\mathbf{v}$ may be substantially less sparse than $\mathbf{v}$ itself.

It is remarkable that similar constructions for the algebraic decay do not lead to a change of algebraic class, but simply to a larger norm of $\mathbf{A}\mathbf{v}$ compared to the one of $\mathbf{v}$.

**Feasible residual $\widetilde{\mathbf{r}}$.** Dörfler marking (1.1) is, however, unfeasible for pde-based problems, in that $\mathbf{r} = \mathbf{f} - \mathbf{A}\mathbf{v}$ may have infinitely many coefficients. Exploiting the knowledge of the data $\mathbf{f}$ and the matrix $\mathbf{A}$, we thus construct a feasible residual $\widetilde{\mathbf{r}}$, i.e., a finite approximation of $\mathbf{r}$, and enforce (1.1) with parameter $\widetilde{\theta}$. Therefore, it is the sparsity class of $\widetilde{\mathbf{r}}$ that determines the degrees of freedom $|\partial\Lambda|$ to be added. The same argument leading to either (1.5) or (1.7) gives

$$|\partial\Lambda| \leq \left(\frac{\|\widetilde{\mathbf{r}}\|_{\ell^s_B}}{\alpha\|\widetilde{\mathbf{r}}\|}\right)^{\frac{d}{s}} + 1 \qquad \text{or} \qquad |\partial\Lambda| \leq \frac{1}{\eta}\log\frac{\|\widetilde{\mathbf{r}}\|_{\ell^\eta_G}}{\alpha\|\widetilde{\mathbf{r}}\|} + 1,$$

for each class. We thus see that the ratios $\|\widetilde{\mathbf{r}}\|_{\ell^s_B}/\|\widetilde{\mathbf{r}}\|$ and $\|\widetilde{\mathbf{r}}\|_{\ell^\eta_G}/\|\widetilde{\mathbf{r}}\|$ control the behavior of the feasible adaptive procedures. This has already been observed and exploited by Cohen et al. [8] in the context of wavelet methods for the class $\ell^s_B$. Our estimates, discussed in Section 6, are valid for both classes, use specific decay properties of the entries of $\mathbf{A}$, and reveal that $\widetilde{\mathbf{r}}$ is uniformly less sparse than the solution $\mathbf{u}$ of $\mathbf{A}\mathbf{u} = \mathbf{f}$ for the exponential class. To cope with this difficulty we resort to coarsening [7, 23], which makes the outer loop of the feasible ADFOUR quasi-optimal but leaves the inner loop suboptimal; this issue is discussed in Section 8. Coarsening is unnecessary for the algebraic class, as in [16, 23], and is discussed in Section 7.

**Contraction constant.** It is well known that the contraction constant $\rho(\theta) = \sqrt{1 - \frac{\alpha_*}{\alpha^*}\theta^2}$ cannot be arbitrarily close to 1 for estimators whose upper and lower constants, $\alpha^* \geq \alpha_*$, do not coincide. This is, however, at odds with the philosophy of spectral methods which are expected to converge superlinearly (typically exponentially). Assuming that the decay properties of $\mathbf{A}$ are known, we can enrich Dörfler marking of $\widetilde{\mathbf{r}}$ in such a way that the contraction factor becomes

$$\widetilde{\rho}(\widetilde{\theta}) = C_*\sqrt{1 - \widetilde{\theta}^2}.$$

This leads to $\widetilde{\rho}(\widetilde{\theta})$ as close to 1 as desired and to *aggressive* versions of ADFOUR discussed in Section 3.

This paper can be viewed as a first step towards understanding adaptivity for the *hp*-FEM. However, the present results are of intrinsic interest and applicable to periodic problems with high degree of regularity and rather complex structure. One such problem is turbulence in a periodic box. Our techniques exploit periodicity and orthogonality of the complex exponentials, but many of our assertions and conclusions extend to the nonperiodic case for which the natural basis functions are Legendre polynomials; this is the case of the *p*-FEM. In any event, the study of adaptive Fourier-Galerkin methods seems to be a new paradigm in adaptivity, with many intriguing questions and surprises, some discussed in this paper. In contrast to the *h*-FEM, they exhibit unlimited approximation power which is only restricted by solution and data regularity.

We organize the paper as follows. In Section 2 we introduce the Fourier-Galerkin method, present a posteriori error estimators, and discuss properties of the underlying matrix $\mathbf{A}$ for both algebraic and exponential approximation classes. In Section 3 we deal with the ideal algorithm ADFOUR and four feasible versions, two for each class, and prove their contraction properties. We devote Section 4 to nonlinear approximation theory with an emphasis on the exponential class. In Section 5 we turn to the study of the sparsity classes for the range of the operator $L$, along the lines outlined above, which turns out to be instrumental in Section 6 to analyze the feasible residual. We discuss cardinality properties of the feasible ADFOUR algorithms for the algebraic class in Section 7 and for the exponential class in Section 8. We finally conclude in Section 9 with a comparison of results.

## 2. Fourier-Galerkin approximation

2.1. **Fourier basis and norm representation.** For $d \geq 1$, we consider $\Omega = (0, 2\pi)^d$, and the trigonometric basis $\phi_k(x) = 1/(2\pi)^{d/2} \mathrm{e}^{ik\cdot x}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, which is orthonormal in $L^2(\Omega)$. Let $v = \sum_k \hat{v}_k \phi_k$, $\hat{v}_k = (v, \phi_k)$ with $\|v\|^2_{L^2(\Omega)} = \sum_k |\hat{v}_k|^2$, be the expansion of any $v \in L^2(\Omega)$ and the representation of its norm via the Parseval identity. Let $H^1_p(\Omega)$ be the subspace of $H^1(\Omega)$ made of functions which are $2\pi$-periodic in each direction and let $H^{-1}_p(\Omega)$ be its dual. Since the trigonometric basis is orthogonal in $H^1_p(\Omega)$ as well, one has for any $v \in H^1_p(\Omega)$,

(2.1)
$$\|v\|^2_{H^1_p(\Omega)} = \sum_k (1 + |k|^2)|\hat{v}_k|^2 = \sum_k |\hat{V}_k|^2 \,, \qquad (\text{setting } \hat{V}_k := \sqrt{(1 + |k|^2)}\hat{v}_k) \;;$$

here and in the sequel, $|k|$ denotes the Euclidean norm of the multi-index $k$. On the other hand, if $f \in H^{-1}_p(\Omega)$, we set $\hat{f}_k = \langle f, \phi_k\rangle$ so that $\langle f, v\rangle = \sum_k \hat{f}_k \hat{v}_k$ $\forall v \in H^1_p(\Omega)$; the norm representation is

(2.2)
$$\|f\|^2_{H^{-1}_p(\Omega)} = \sum_k \frac{1}{(1 + |k|^2)}|\hat{f}_k|^2 = \sum_k |\hat{F}_k|^2 \,, \qquad (\text{setting } \hat{F}_k := \frac{1}{\sqrt{(1 + |k|^2)}}\hat{f}_k) \;.$$

Throughout this paper, we will use the notation $\| \,.\, \|$ to indicate both the $H^1_p(\Omega)$-norm of a function $v$, or the $H^{-1}_p(\Omega)$-norm of a linear form $f$; the specific meaning will be clear from the context.

Given any finite index set $\Lambda \subset \mathbb{Z}^d$, we define the subspace of $V := H^1_p(\Omega)$ $V_\Lambda := \operatorname{span}\{\phi_k \,|\, k \in \Lambda\}$; we set $|\Lambda| = \operatorname{card} \Lambda$, so that $\dim V_\Lambda = |\Lambda|$. A function

$w \in V_\Lambda$ will be said to have *finite support* $\Lambda$. If $g$ admits an expansion $g = \sum_k \hat{g}_k \phi_k$ (converging in an appropriate norm), then we define its projection $P_\Lambda g$ upon $V_\Lambda$ by setting $P_\Lambda g = \sum_{k \in \Lambda} \hat{g}_k \phi_k$ .

2.2. **Galerkin discretization and residual.** We now consider the elliptic problem

$$\begin{cases} Lu = -\nabla \cdot (\nu \nabla u) + \sigma u = f & \text{in } \Omega \,, \\ u \;\; 2\pi\text{-periodic in each direction} \,, \end{cases} \tag{2.3}$$

where $\nu$ and $\sigma$ are sufficiently smooth real coefficients satisfying $0 < \nu_* \leq \nu(x) \leq \nu^* < \infty$ and $0 < \sigma_* \leq \sigma(x) \leq \sigma^* < \infty$ in $\Omega$; let us set $\alpha_* = \min(\nu_*, \sigma_*)$ and $\alpha^* = \max(\nu^*, \sigma^*)$. We formulate this problem variationally as

$$u \in H^1_p(\Omega) \quad : \quad a(u,v) = \langle f, v \rangle, \qquad \forall v \in H^1_p(\Omega) \,, \tag{2.4}$$

where $a(u,v) = \int_\Omega \nu \nabla u \cdot \nabla \bar{v} + \int_\Omega \sigma u \bar{v}$ (bar indicating, as usual, complex conjugate). We denote by $|||v||| = \sqrt{a(v,v)}$ the energy norm of any $v \in H^1_p(\Omega)$, which satisfies

$$\sqrt{\alpha_*} \|v\| \leq |||v||| \leq \sqrt{\alpha^*} \|v\| \,. \tag{2.5}$$

Given any finite set $\Lambda \subset \mathbb{Z}^d$, the Galerkin approximation is defined as

$$u_\Lambda \in V_\Lambda \quad : \quad a(u_\Lambda, v_\Lambda) = \langle f, v_\Lambda \rangle, \qquad \forall v_\Lambda \in V_\Lambda \,. \tag{2.6}$$

For any $w \in V_\Lambda$, we define the residual $r(w) = f - Lw = \sum_k \hat{r}_k(w) \phi_k$ where $\hat{r}_k(w) = \langle f - Lw, \phi_k \rangle = \langle f, \phi_k \rangle - a(w, \phi_k)$. Then, the previous definition of $u_\Lambda$ is equivalent to the condition

$$P_\Lambda r(u_\Lambda) = 0 \,, \qquad \text{i.e.,} \quad \hat{r}_k(u_\Lambda) = 0, \qquad \forall k \in \Lambda \,. \tag{2.7}$$

On the other hand, by the continuity and coercivity of the bilinear form $a$, one has

$$\frac{1}{\alpha^*} \|r(u_\Lambda)\| \leq \|u - u_\Lambda\| \leq \frac{1}{\alpha_*} \|r(u_\Lambda)\| \tag{2.8}$$

or, equivalently,

$$\frac{1}{\sqrt{\alpha^*}} \|r(u_\Lambda)\| \leq |||u - u_\Lambda||| \leq \frac{1}{\sqrt{\alpha_*}} \|r(u_\Lambda)\| \,. \tag{2.9}$$

2.3. **Algebraic representations.** Let us identify the solution $u = \sum_k \hat{u}_k \phi_k$ of problem (2.4) with the vector $\mathbf{u} = (\hat{U}_k) = (c_k \hat{u}_k) \in \mathbb{C}^{\mathbb{Z}^d}$ of its $H^1_p$-normalized Fourier coefficients, where we set for convenience $c_k = \sqrt{1 + |k|^2}$. Similarly, let us identify the right-hand side $f$ with the vector $\mathbf{f} = (\hat{F}_\ell) = (c_\ell^{-1} \hat{f}_\ell) \in \mathbb{C}^{\mathbb{Z}^d}$ of its $H^{-1}_p$-normalized Fourier coefficients. Finally, let us introduce the bi-infinite, Hermitian and positive-definite matrix

$$\mathbf{A} = (a_{\ell,k}) \qquad \text{with} \qquad a_{\ell,k} = \frac{1}{c_\ell c_k} a(\phi_k, \phi_\ell) \,. \tag{2.10}$$

Then, problem (2.4) can be equivalently written as

$$\mathbf{A}\mathbf{u} = \mathbf{f} \,. \tag{2.11}$$

We observe that the orthogonality properties of the trigonometric basis implies that the matrix $\mathbf{A}$ is diagonal if and only if the coefficients $\nu$ and $\sigma$ are constant in $\Omega$.

Next, consider the Galerkin problem (2.6) and let $\mathbf{u}_\Lambda \in \mathbb{C}^{|\Lambda|}$ be the vector collecting the coefficients of $u_\Lambda$ indexed in $\Lambda$; let $\mathbf{f}_\Lambda \in \mathbb{C}^{|\Lambda|}$ be the analogous restriction for the vector of the coefficients of $f$. Finally, denote by $\mathbf{R}_\Lambda$ the matrix that restricts a bi-infinite vector to the portion indexed in $\Lambda$, so that $\mathbf{E}_\Lambda = \mathbf{R}_\Lambda^H$ is the corresponding extension matrix. Then, setting

$$\mathbf{A}_\Lambda = \mathbf{R}_\Lambda \mathbf{A} \mathbf{R}_\Lambda^H \ , \tag{2.12}$$

problem (2.6) can be equivalently written as

$$\mathbf{A}_\Lambda \mathbf{u}_\Lambda = \mathbf{f}_\Lambda \ . \tag{2.13}$$

2.4. **Properties of the stiffness matrix.** Since $L$ is an isomorphism from $H^1_p(\Omega)$ onto $H^{-1}_p(\Omega)$, and $\{\phi_k\}_{k\in\mathbb{Z}^d}$ is a Riesz basis (in fact orthogonal), we readily have the following results, whose proof can be found e.g., in [23].

**Property 2.1** (continuity and invertibility)**.** The matrix $\mathbf{A}$ is a bounded invertible operator on $\ell^2(\mathbb{Z}^d)$.

It is useful to express the elements of $\mathbf{A}$ in terms of the Fourier coefficients of the operator coefficients $\nu$ and $\sigma$. Precisely, writing $\nu = \sum_k \hat{\nu}_k \phi_k$ and $\sigma = \sum_k \hat{\sigma}_k \phi_k$ and using the orthogonality of the Fourier basis, one easily gets

$$a_{\ell,k} = \frac{1}{(2\pi)^{d/2}} \left( \frac{\ell \cdot k}{c_\ell c_k} \hat{\nu}_{\ell-k} + \frac{1}{c_\ell c_k} \hat{\sigma}_{\ell-k} \right) . \tag{2.14}$$

Note that the diagonal elements are uniformly bounded from below,

$$a_{\ell,\ell} \geq \frac{1}{(2\pi)^{d/2}} \min(\hat{\nu}_0, \hat{\sigma}_0) > 0 \ , \qquad \ell \in \mathbb{Z}^d \ , \tag{2.15}$$

whereas all elements are bounded in modulus by the elements of a *Toeplitz* matrix,

$$|a_{\ell,k}| \leq \frac{1}{(2\pi)^{d/2}} (|\hat{\nu}_{\ell-k}| + |\hat{\sigma}_{\ell-k}|) \ , \qquad \ell, k \in \mathbb{Z}^d \ , \tag{2.16}$$

which decay as $|\ell - k| \to \infty$ at a rate dictated by the smoothness of the operator coefficients. Indeed, if $\nu$ and $\sigma$ are sufficiently smooth, their Fourier coefficients decay at a suitable rate and this property is inherited by the off-diagonal elements of the matrix $\mathbf{A}$, via (2.16). To be precise, if the coefficients $\nu$ and $\sigma$ have a finite order of regularity, then the rate of decay of their Fourier coefficients is algebraic, i.e.,

$$|\hat{\nu}_k|, |\hat{\sigma}_k| \lesssim (1 + |k|)^{-\eta}, \qquad \forall k \in \mathbb{Z}^d \ , \tag{2.17}$$

for some $\eta > 0$. On the other hand, if the operator coefficients are real analytic in a neighborhood of $\Omega$, then the rate of decay of their Fourier coefficients is exponential, i.e.,

$$|\hat{\nu}_k|, |\hat{\sigma}_k| \lesssim e^{-\eta|k|}, \qquad \forall k \in \mathbb{Z}^d \ . \tag{2.18}$$

Correspondingly, the matrix $\mathbf{A}$ belongs to one of the following classes.

**Definition 2.1** (regularity classes for $\mathbf{A}$)**.** A matrix $\mathbf{A}$ is said to belong to

- the algebraic class $\mathcal{D}_a(\eta_L)$ if there exists a constant $c_L > 0$ such that its elements satisfy

$$|a_{\ell,k}| \leq c_L (1 + |\ell - k|)^{-\eta_L}, \qquad \ell, k \in \mathbb{Z}^d \ ; \tag{2.19}$$

- the exponential class $\mathcal{D}_e(\eta_L)$ if there exists a constant $c_L > 0$ such that its elements satisfy

$$|a_{\ell,k}| \leq c_L e^{-\eta_L |\ell - k|}, \qquad \ell, k \in \mathbb{Z}^d \ . \tag{2.20}$$

The following properties hold.

**Property 2.2** (inverse of $\mathbf{A}$: algebraic case)**.** If $\mathbf{A} \in \mathcal{D}_a(\eta_L)$, with $\eta_L > d$, then $\mathbf{A}^{-1} \in \mathcal{D}_a(\eta_L)$.

*Proof.* see, e.g., [17]. ☐

**Property 2.3** (inverse of $\mathbf{A}$: exponential case)**.** If $\mathbf{A} \in \mathcal{D}_e(\eta_L)$ and there exists a constant $c_L$ satisfying (2.20) such that

$$c_L < \frac{1}{2}(\mathrm{e}^{\eta_L} - 1) \min_{\ell} a_{\ell,\ell} \ , \tag{2.21}$$

then $\mathbf{A}^{-1} \in \mathcal{D}_e(\bar{\eta}_L)$ where $\bar{\eta}_L \in (0, \eta_L]$ is such that $\bar{z} = \mathrm{e}^{-\bar{\eta}_L}$ is the unique zero in the interval $(0,1)$ of the polynomial

$$z^2 - \frac{\mathrm{e}^{2\eta_L} + 2c_L + 1}{\mathrm{e}^{\eta_L}(c_L + 1)} z + 1 \ .$$

*Proof.* We follow the suggestion by Bini [3], and thus exploit the one-to-one correspondence between Toeplitz matrices and formal Laurent series (see, e.g., [4]):

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \longleftrightarrow \mathbf{T}_f = (t_{i,j}), \quad t_{i,j} = a_{i-j}.$$

We refer to the function $f(z)$ as the symbol associated to the Toeplitz matrix $\mathbf{T}_f$. We recall now a few relations between $f(z)$ and $\mathbf{T}_f$. If $f(z)$ is analytic on $\mathcal{A}_\alpha = \{z \in \mathbb{C} : \mathrm{e}^{-\alpha} < |z| < \mathrm{e}^{\alpha}\}$ with $\alpha > 0$, then $f(z) = \sum_{k=-\infty}^{+\infty} a_k z^k$ holds, where the coefficients $a_k$ have exponential decay with rate $\mathrm{e}^{-\alpha}$ in the sense that for every $0 < \rho < \mathrm{e}^{-\alpha}$ there exists a constant $\gamma > 0$ such that $|a_k| \le \gamma \rho^{|k|}$. As a consequence, the symbol $f(z)$ of the Toeplitz matrix $\mathbf{T}_f$ is analytic on $\mathcal{A}_\alpha$ for some $\alpha > 0$ if and only if the elements of $\mathbf{T}_f$ decay exponentially with rate $\mathrm{e}^{-\alpha}$. Moreover, it is known that if $f(z)$ is analytic on $\mathcal{A}_\alpha$ and it is nonzero on $\mathcal{A}_\beta \subset \mathcal{A}_\alpha$, then the function $g(z) = 1/f(z)$ is well defined and analytic on $\mathcal{A}_\beta$, the matrix $\mathbf{T}_g$ is the inverse of $\mathbf{T}_f$ and the elements of $\mathbf{T}_g$ decay exponentially with rate $\mathrm{e}^{-\beta}$.

Next we introduce the analytic functions in $\mathcal{A}_\alpha$,

$$h(z) = \sum_{k=1}^{\infty} \mathrm{e}^{-\alpha k}(z^k + z^{-k}) = \frac{z}{\mathrm{e}^{\alpha} - z} + \frac{z^{-1}}{\mathrm{e}^{\alpha} - z^{-1}}, \qquad f_c(z) = 1 - ch(z),$$

with $c > 0$. For $|z| = 1$ we deduce $|h(z)| \le 2\sum_{k=1}^{\infty} e^{-\alpha k} = 2/(e^{\alpha} - 1)$, whence $c|h(z)| < 1$ provided that $c < \frac{1}{2}(\mathrm{e}^{\alpha} - 1)$; moreover, $\|\mathbf{T}_h\| \le \|\mathbf{T}_h\|_\infty = 2/(\mathrm{e}^{\alpha} - 1)$, which is indeed a particular instance of the Schur Lemma for symmetric matrices. For this range of $c$'s, $f_c(z) \neq 0$ for $|z| = 1$ and for continuity there exists $\mathcal{A}_\beta \subset \mathcal{A}_\alpha$ on which $f_c(z)$ is nonzero. This implies that $g_c(z) := 1/f_c(z)$ is analytic on $\mathcal{A}_\beta$ and the elements of the associated Toeplitz matrix $\mathbf{T}_{g_c}$ decay exponentially with rate $\mathrm{e}^{-\beta}$. The singularities of $g_c$ correspond to zeros of $f_c$, which are in turn the roots $\zeta_1, \zeta_2$ of the polynomial

$$z^2 - \frac{\mathrm{e}^{2\alpha} + 2c + 1}{\mathrm{e}^{\alpha}(c + 1)} z + 1.$$

These roots are real provided $c < \frac{1}{2}(\mathrm{e}^{\alpha} - 1)$, in which case $e^{-\beta} = \zeta_1 = \zeta_2^{-1} < 1$.

Let $\mathbf{A} \in \mathcal{D}_e(\alpha)$, i.e., there exists a constant $c$ such that $|a_{\ell,k}| \le ce^{-\alpha|\ell-k|}$ for $\ell, k \in \mathbb{Z}^d$. By rescaling of the rows of $\mathbf{A}$, it is not restrictive to assume that the diagonal elements $\mathbf{A}$ are equal to 1. Then, it is possible to write $\mathbf{A} = \mathbf{I} - \mathbf{S}$ with $|\mathbf{S}| \leq c\mathbf{T}_h$, the absolute value and the inequality being meant element by element, and $\|\mathbf{S}\| < 1$. Since $g_c(z) = 1/(1 - ch(z)) = \sum_{k=0}^{\infty} c^k h(z)^k$ is well defined and analytic on $\mathcal{A}_\beta \subset \mathcal{A}_\alpha$, it follows that

$$\left|\sum_{k=0}^{\infty} \mathbf{S}^k\right| \leq \sum_{k=0}^{\infty} |\mathbf{S}|^k \leq \sum_{k=0}^{\infty} c^k \mathbf{T}_h^k = \mathbf{T}_{g_c}.$$

Hence, the elements of the matrix $\mathbf{T}_{g_c}$ decay exponentially with rate $\mathrm{e}^{-\beta}$. Property $\|\mathbf{S}\| < 1$ yields $\mathbf{A}^{-1} = (\mathbf{I} - \mathbf{S})^{-1} = \sum_{k=0}^{\infty} \mathbf{S}^k$ and $|\mathbf{A}^{-1}| \leq \mathbf{T}_{g_c}$, whence the coefficients of $\mathbf{A}^{-1}$ being bounded by those of $\mathbf{T}_{g_c}$ decay exponentially with rate $\mathrm{e}^{-\beta}$, i.e., $\mathbf{A}^{-1} \in \mathcal{D}_e(\beta)$ for some $\beta < \alpha$. This gives (2.21) once the row scaling of $\mathbf{A}$ is taken into account. ∎

**Example 2.1** (sharpness of (2.21))**.** The following example illustrates that (2.21) is sharp. Let $\mathbf{A}$ be

$$a_{ij} = -2^{-1-|i-j|} \quad i \neq j, \qquad a_{ii} = 1,$$

which is singular because the sum of the coefficients in every row vanishes. This $\mathbf{A}$ corresponds to $e^{\eta_L} = 2$, $c_L = \frac{1}{2}$ and $\frac{1}{2}(e^{\eta_L} - 1) = \frac{1}{2}$, which violates (2.21). ∎

For any integer $J \geq 0$, let $\mathbf{A}_J$ denote the following symmetric truncation of the matrix $\mathbf{A}$,

$$(\mathbf{A}_J)_{\ell,k} = \begin{cases} a_{\ell,k} & \text{if } |\ell - k| \leq J\ , \\ 0 & \text{elsewhere.} \end{cases} \tag{2.22}$$

Then, we have the following well-known results, whose proof is reported for completeness.

**Property 2.4** (truncation)**.** The truncated matrix $\mathbf{A}_J$ has a number of nonvanishing entries bounded by $\omega_d J^d$, where $\omega_d$ is the measure of the Euclidean unit ball in $\mathbb{R}^d$. Moreover, under the assumption of Property 2.1, there exists a constant $C_{\mathbf{A}}$ such that

$$\|\mathbf{A} - \mathbf{A}_J\| \leq \psi_{\mathbf{A}}(J, \eta) := C_{\mathbf{A}} \begin{cases} (J+1)^{-(\eta_L - d)} & \text{if } \mathbf{A} \in \mathcal{D}_a(\eta_L) \text{ (algebraic case)}, \\ (J+1)^{d-1} \mathrm{e}^{-\eta_L J} & \text{if } \mathbf{A} \in \mathcal{D}_e(\eta_L) \text{ (exponential case)}, \end{cases}$$

for all $J \geq 0$. Consequently, under the assumptions of Property 2.2 or 2.3, one has

$$\|\mathbf{A}^{-1} - (\mathbf{A}^{-1})_J\| \leq \psi_{\mathbf{A}^{-1}}(J, \bar{\eta}_L), \tag{2.23}$$

where we let $\bar{\eta}_L = \eta_L$ in the algebraic case and $\bar{\eta}_L$ be defined in Property 2.3 for the exponential case.

*Proof.* We use the Schur Lemma for symmetric matrices, $\|\mathbf{B}\| \le \|\mathbf{B}\|_\infty = \sup_\ell \sum_k |b_{\ell,k}|$ for $\mathbf{B} = \mathbf{A} - \mathbf{A}_J$. Thus, in the algebraic case,

$$\begin{aligned}\sup_\ell \sum_{k:|\ell-k|>J} |a_{\ell,k}| &\le C_L \sup_\ell \sum_{k:|\ell-k|>J} \frac{1}{(1+|\ell-k|)^{\eta_L}} \\ &\lesssim \sup_\ell \sum_{q=J+1}^{\infty} \sum_{k:|\ell-k|=q} \frac{1}{(1+q)^{\eta_L}} \\ &\lesssim \sup_\ell \sum_{q=J+1}^{\infty} \frac{q^{d-1}}{(1+q)^{\eta_L}} \lesssim (J+1)^{d-\eta_L} .\end{aligned}$$

A similar argument yields the result in the exponential case. ∎

2.5. **An equivalent formulation of the Galerkin problem.** For future reference, herafter we rewrite the Galerkin problem (2.13) in an equivalent (infinite-dimensional) way. Let $\mathbf{P}_\Lambda : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ be the projector operator defined as

$$(\mathbf{P}_\Lambda \mathbf{v})_\lambda = \begin{cases} v_\lambda & \text{if } \lambda \in \Lambda , \\ 0 & \text{if } \lambda \notin \Lambda . \end{cases}$$

Note that $\mathbf{P}_\Lambda$ can be represented as a diagonal bi-infinite matrix whose diagonal elements are 1 for indices belonging to $\Lambda$; zero otherwise. Let us set $\mathbf{Q}_\Lambda = \mathbf{I} - \mathbf{P}_\Lambda$ and we introduce the bi-infinite matrix $\widehat{\mathbf{A}}_\Lambda := \mathbf{P}_\Lambda \mathbf{A} \mathbf{P}_\Lambda + \mathbf{Q}_\Lambda$ which is equal to $\mathbf{A}_\Lambda$ for indices in $\Lambda$ and to the identity matrix, otherwise. The definitions of the projectors $\mathbf{P}_\Lambda$ and $\mathbf{Q}_\Lambda$ yield the following result.

**Property 2.5** (invertibility of $\widehat{\mathbf{A}}_\Lambda$)**.** The matrix $\widehat{\mathbf{A}}_\Lambda$ is invertible. In addition, if either $\mathbf{A} \in \mathcal{D}_a(\eta_L)$ or $\mathbf{A} \in \mathcal{D}_e(\eta_L)$, then $\widehat{\mathbf{A}}_\Lambda \in \mathcal{D}_a(\eta_L)$ or $\widehat{\mathbf{A}}_\Lambda \in \mathcal{D}_e(\eta_L)$.

Now, let us consider the following extended Galerkin problem: Find $\hat{\mathbf{u}} \in \ell^2(\mathbb{Z}^d)$ such that

$$\widehat{\mathbf{A}}_\Lambda \hat{\mathbf{u}} = \mathbf{P}_\Lambda \mathbf{f} \ . \tag{2.24}$$

Let $\mathbf{E}_\Lambda : \mathbb{C}^{|\Lambda|} \to \ell^2(\mathbb{Z}^d)$ be the extension operator defined in Section 2.3 and let $\mathbf{u}_\Lambda \in \mathbb{C}^{|\Lambda|}$ be the Galerkin solution to (2.13); then, it is easy to check that $\hat{\mathbf{u}} = \mathbf{E}_\Lambda \mathbf{u}_\Lambda$.

In the following, with an abuse of notation, the solution of (2.24) will be denoted by $\mathbf{u}_\Lambda$. We will refer to it as the (extended) Galerkin solution, meaning the infinite-dimensional representative of the finite-dimensional Galerkin solution. In case of possible confusion, we will make clear which version (infinite-dimensional or finite-dimensional) has to be considered.

## 3. Adaptive algorithms with contraction properties

Our first algorithm will be an *ideal one*; it will serve as a reference to illustrate in the simplest situation the contraction property which guarantees the convergence of the algorithm, and it will be subsequently modified to obtain more efficient versions. The ideal algorithm uses as error estimator the ideal one, i.e., the norm of the residual in $H_p^{-1}(\Omega)$; we thus set, for any $v \in H_p^1(\Omega)$,

$$\eta^2(v) = \|r(v)\|^2 = \sum_{k\in\mathbb{Z}^d} |\hat{R}_k(v)|^2 \ , \tag{3.1}$$

so that (2.8) can be rephrased as

$$\frac{1}{\alpha^*}\eta(u_\Lambda) \le \|u - u_\Lambda\| \le \frac{1}{\alpha_*}\eta(u_\Lambda) \ ; \tag{3.2}$$

recall that $\hat{R}_k(v) = (1+|k|^2)^{-1/2} r_k(v)$ according to (2.2). Obviously, this estimator is hardly computable in practice; in Section 3.2 we will introduce a feasible version, but for the moment we go through the ideal situation. Given any subset $\Lambda \subseteq \mathbb{Z}^d$, we also define the quantity

$$\eta^2(v;\Lambda) = \|P_\Lambda r(v)\|^2 = \sum_{k\in\Lambda} |\hat{R}_k(v)|^2 \ ,$$

so that $\eta(v) = \eta(v;\mathbb{Z}^d)$.

3.1. **ADFOUR: an ideal algorithm.** We now introduce the following procedures, which will enter the definition of all our adaptive algorithms.

- $u_\Lambda :=$ **GAL**$(\Lambda)$
  Given a finite subset $\Lambda \subset \mathbb{Z}^d$, the output $u_\Lambda \in V_\Lambda$ is the solution of the Galerkin problem (2.6) relative to $\Lambda$.
- $r :=$ **RES**$(v_\Lambda)$
  Given a function $v_\Lambda \in V_\Lambda$ for some finite index set $\Lambda$, the output $r$ is the residual $r(v_\Lambda) = f - Lv_\Lambda$.
- $\Lambda^* :=$ **DÖRFLER**$(r,\theta)$
  Given $\theta \in (0,1)$ and an element $r \in H_p^{-1}(\Omega)$, the ouput $\Lambda^* \subset \mathbb{Z}^d$ is a finite set such that the inequality

$$\|P_{\Lambda^*} r\| \ge \theta \|r\| \tag{3.3}$$

  is satisfied.

Note that the latter inequality is equivalent to

$$\|r - P_{\Lambda^*} r\| \le \sqrt{1-\theta^2}\|r\| \ . \tag{3.4}$$

If $r = r(u_\Lambda)$ is the residual of a Galerkin solution $u_\Lambda \in V_\Lambda$, then by (2.7) we can trivially assume that $\Lambda^*$ is contained in $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$. For such a residual, inequality (3.3) can then be stated as

$$\eta(u_\Lambda;\Lambda^*) \ge \theta\eta(u_\Lambda) \ , \tag{3.5}$$

a condition termed *Dörfler marking* in the finite element literature, or *bulk chasing* in the wavelet literature. Writing $\hat{R}_k = \hat{R}_k(u_\Lambda)$, the condition (3.5) can be equivalently stated as

$$\sum_{k\in\Lambda^*} |\hat{R}_k|^2 \ge \theta^2 \sum_{k\notin\Lambda} |\hat{R}_k|^2 \ . \tag{3.6}$$

Also note that a set $\Lambda^*$ of minimal cardinality can be immediately determined if the coefficients $\hat{R}_k$ are rearranged in nonincreasing order of modulus. Even though the subsequent convergence results would not require the property of minimality, in the sequel we will invariably make the following assumption which will become essential in the complexity analysis:

**Assumption 3.1** (Dörfler marking)**.** *The procedure* **DÖRFLER** *selects an index set* $\Lambda^*$ *of minimal cardinality among all those satisfying condition* (3.3)*.*

Given a tolerance tol $\in [0, 1)$ and a marking parameter $\theta \in (0, 1)$, we are ready to define our ideal adaptive algorithm.

**Algorithm ADFOUR**($\theta$, tol)

Set $r_0 := f$, $\Lambda_0 := \emptyset$, $n = -1$
do
  $n \leftarrow n + 1$
  $\partial\Lambda_n :=$ **DÖRFLER**$(r_n, \theta)$
  $\Lambda_{n+1} := \Lambda_n \cup \partial\Lambda_n$
  $u_{n+1} :=$ **GAL**$(\Lambda_{n+1})$
  $r_{n+1} :=$ **RES**$(u_{n+1})$
while $\|r_{n+1}\| > \text{tol}$

The following result states the convergence of this algorithm, with a guaranteed error reduction rate.

**Theorem 3.1** (convergence of **ADFOUR**)**.** *Let us set*

$$\rho = \rho(\theta) = \sqrt{1 - \frac{\alpha_*}{\alpha^*}\theta^2} \in (0, 1) \ . \tag{3.7}$$

*Let* $\{\Lambda_n, u_n\}_{n\geq 0}$ *be the sequence generated by the adaptive algorithm* **ADFOUR***. Then, the following bound holds for any* $n$*:*

$$|||u - u_{n+1}||| \leq \rho |||u - u_n||| \ .$$

*Thus, for any* tol $> 0$ *the algorithm terminates in a finite number of iterations, whereas for* tol $= 0$ *the sequence* $u_n$ *converges to* $u$ *in* $H^1_p(\Omega)$ *as* $n \to \infty$*.*

*Proof.* For convenience, we use the notation $e_n := |||u - u_n|||$ and $d_n := |||u_{n+1} - u_n|||$. As $V_{\Lambda_n} \subset V_{\Lambda_{n+1}}$, the following orthogonality property holds:

$$e^2_{n+1} = e^2_n - d^2_n. \tag{3.8}$$

On the other hand, for any $w \in H^1_p(\Omega)$, one has, in light of (2.5),

$$\|Lw\| = \sup_{v\in H^1_p(\Omega)} \frac{\langle Lw, v\rangle}{\|v\|} = \sup_{v\in H^1_p(\Omega)} \frac{a(w, v)}{\|v\|} \leq |||w||| \sup_{v\in H^1_p(\Omega)} \frac{|||v|||}{\|v\|} \leq \sqrt{\alpha^*}|||w||| \ .$$

Thus, using (3.3),

$$\begin{aligned} d^2_n \geq \frac{1}{\alpha^*}\|L(u_{n+1} - u_n)\|^2 \ &= \ \frac{1}{\alpha^*}\|r_{n+1} - r_n\|^2 \geq \frac{1}{\alpha^*}\|P_{\Lambda_{n+1}}(r_{n+1} - r_n)\|^2 \\ &= \ \frac{1}{\alpha^*}\|P_{\Lambda_{n+1}} r_n\|^2 \geq \frac{\theta^2}{\alpha^*}\|r_n\|^2 \ . \end{aligned}$$

On the other hand, the rightmost inequality in (2.9) states that $\|r_n\|^2 \geq \alpha_* e^2_n$, whence the result. ∎

3.2. **F-ADFOUR: A feasible version of ADFOUR.** The error estimator $\eta(u_\Lambda)$ based on (3.1) is not computable in practice, since the residual $r(u_\Lambda)$ contains infinitely many coefficients. We thus introduce a new estimator, defined from an approximation $\widetilde{r}(u_\Lambda)$ of such residual with finite Fourier expansion. For a fixed parameter $0 < \gamma < 1$, which eventually depends on $\theta$, and for any $v$ with a finite Fourier expansion we require that the approximation $\widetilde{r}(v)$ of $r(v)$ satisfies the following crucial inequality:

$$\|r(v) - \widetilde{r}(v)\| \le \gamma \|\widetilde{r}(v)\| \ . \tag{3.9}$$

We observe for further reference that the following inequalities easily hold:

$$(1-\gamma)\|\widetilde{r}(v)\| \le \|r(v)\| \le (1+\gamma)\|\widetilde{r}(v)\|. \tag{3.10}$$

Then, we define a new error estimator by setting

$$\widetilde{\eta}^2(u_\Lambda) = \|\widetilde{r}(u_\Lambda)\|^2 = \sum_{k \in \widetilde{\Lambda}} |\hat{\widetilde{R}}_k(u_\Lambda)|^2 \ , \tag{3.11}$$

which, in view of (3.2) and (3.10), immediately yields

$$\frac{1-\gamma}{\alpha^*}\widetilde{\eta}(u_\Lambda) \le \|u - u_\Lambda\| \le \frac{1+\gamma}{\alpha_*}\widetilde{\eta}(u_\Lambda) \ . \tag{3.12}$$

**Lemma 3.1** (feasible Dörfler marking). *Let $0 < \theta, \widetilde{\theta} < 1$ be Dörfler parameters and $\Lambda^*$ a finite set containing $\Lambda$. If*

$$\widetilde{\eta}(u_\Lambda; \Lambda^*) := \|P_{\Lambda^*}\widetilde{r}(u_\Lambda)\| \ge \widetilde{\theta}\widetilde{\eta}(u_\Lambda), \tag{3.13}$$

*and $\gamma \in (0, \widetilde{\theta})$, then*

$$\eta(u_\Lambda; \Lambda^*) \ge \theta \eta(u_\Lambda) \ , \qquad \textit{with} \quad \theta = \frac{\widetilde{\theta} - \gamma}{1+\gamma} \in (0, \widetilde{\theta}) \ . \tag{3.14}$$

*On the other hand, if*

$$\eta(u_\Lambda; \Lambda^*) \ge \theta \eta(u_\Lambda) \ , \tag{3.15}$$

*and $\gamma \in (0, \theta/(1+\theta))$, then*

$$\widetilde{\eta}(u_\Lambda; \Lambda^*) \ge \widetilde{\theta}\widetilde{\eta}(u_\Lambda) \ , \qquad \textit{with} \quad \widetilde{\theta} = \theta - \gamma(1+\theta) \ . \tag{3.16}$$

*Proof.* In view of (3.9) and (3.10), one has

$$\begin{aligned} \|P_{\Lambda^*} r(u_\Lambda)\| &\ge \|P_{\Lambda^*}\widetilde{r}(u_\Lambda)\| - \|P_{\Lambda^*}\left(r(u_\Lambda) - \widetilde{r}(u_\Lambda)\right)\| \ge \theta\|\widetilde{r}(u_\Lambda)\| - \|r(u_\Lambda) - \widetilde{r}(u_\Lambda)\| \\ &\ge (\theta - \gamma)\|\widetilde{r}(u_\Lambda)\| \ge \frac{\theta-\gamma}{1+\gamma}\|r(u_\Lambda)\| \ , \end{aligned}$$

which is the desired (3.14). The proof of (3.16) follows along the same lines. ∎

In the following, we describe a possible implementation of a procedure that leads us to build $\widetilde{r}(u_\Lambda)$ satisfying (3.9). To this end, let us introduce the following modules:

- $f_\varepsilon :=$ **F-RHS**$(\varepsilon)$. Given the right-hand side $f$ and a positive tolerance $\varepsilon > 0$, the output $f_\varepsilon$ has a finite Fourier expansion and satisfies

$$\|f - f_\varepsilon\| \le \varepsilon. \tag{3.17}$$

- $w_\varepsilon :=$ **F-APPLY**$(v, \varepsilon)$. Given $v$ and a positive tolerance $\varepsilon > 0$, the output $w_\varepsilon$ has a finite Fourier expansion and satisfies

$$\|Lv - w_\varepsilon\| \leq \varepsilon \ . \tag{3.18}$$

The first module can be implemented by a greedy procedure applied to the rearranged sequence of Fourier coefficients of $f$ (assumed to be available). The second module relies on a telescopic approximation of $Lv$ of the type (5.10) below, which in turn mimics a construction introduced in [8] (see also [7, 23]).

The algorithm **F-RES**, that builds an approximate residual $\widetilde{r}(v)$ such that (3.9) holds, reads as follows:

**Algorithm** $[\widetilde{r}(v), flag, \epsilon_1]$ :=**F-RES**$(v, \gamma, \epsilon_0, \text{tol})$

Set $flag = 0$
Set $\xi = 2\epsilon_0$
do
  $\xi \leftarrow \xi/2$
  $\widetilde{r}_\xi(v) :=$ **F-RHS**$(\xi/2) -$ **F-APPLY**$(v, \xi/2)$
  If $(\xi + \|\widetilde{r}_\xi(v)\| \leq \text{tol})$ then set $flag = 1$ and exit do-cycle
while $\xi > \gamma\|\widetilde{r}_\xi(v)\|$
$\widetilde{r}(v) := \widetilde{r}_\xi(v)$
$\epsilon_1 := \xi + \|\widetilde{r}_\xi(v)\|$

Note that given $\epsilon_0 > 0$ and tol $> 0$ the module **F-RES** is guaranteed to terminate in a finite number of iterations which is pessimistically bounded by $K_{\max}$, with $K_{\max}$ being the smallest integer satisfying

$$\frac{\epsilon_0}{2^{K_{\max}}} \leq \frac{\gamma}{1+\gamma}\text{tol};$$

indeed, this implies $\xi + \|\widetilde{r}_\xi(v)\| \leq \frac{1+\gamma}{\gamma}\xi \leq \frac{1+\gamma}{\gamma}\frac{\epsilon_0}{2^{K_{\max}}} \leq \text{tol}$. Moreover, since $r(v) = f - Lv$, we observe that if **F-RES** terminates with $flag = 0$, then

$$\|r(v) - \widetilde{r}_\xi(v)\| \leq \xi/2 + \xi/2 = \xi \leq \gamma\|\widetilde{r}_\xi(v)\|,$$

hence (3.9) is satisfied. In addition, $\epsilon_1$ is an upper bound for the norm of the true residual, i.e.,

$$\|r(v)\| \leq \|r(v) - \widetilde{r}_\xi(v)\| + \|\widetilde{r}_\xi(v)\| \leq \xi + \|\widetilde{r}_\xi(v)\| = \epsilon_1. \tag{3.19}$$

On the other hand, if $flag = 1$, then $\|r(v)\| \leq \text{tol}$ upon exit. Therefore, if tol is the prescribed accuracy of the adaptive algorithm and $v = u_\Lambda$ its Galerkin solution, then the adaptive process may stop.

Given a tolerance tol $\in [0, 1)$, a marking parameter $\widetilde{\theta} \in (0, 1)$ and two feasibility parameters $\gamma \in (0, \theta)$ and $\epsilon_0 \in (0, 1)$, we are ready to define our first feasible adaptive algorithm.

**Algorithm F-ADFOUR**$(\widetilde{\theta}, \gamma, \epsilon_0, \text{tol})$

Set $u_0 := 0$, $\Lambda_0 := \emptyset$, $n = -1$
$\widetilde{r}_0 =$ **F-RHS**$(\epsilon_0)$
do
  $n \leftarrow n + 1$
  $\partial\Lambda_n :=$ **DÖRFLER**$(\widetilde{r}_n, \widetilde{\theta})$
  $\Lambda_{n+1} := \Lambda_n \cup \partial\Lambda_n$
  $u_{n+1} :=$ **GAL**$(\Lambda_{n+1})$
  $[\widetilde{r}_{n+1}, flag, \epsilon_{n+1}] :=$ **F-RES**$(u_{n+1}, \gamma, \epsilon_n, \text{tol})$
  if $flag = 1$ then STOP
while $\|\widetilde{r}_{n+1}\| > \frac{\text{tol}}{1+\gamma}$

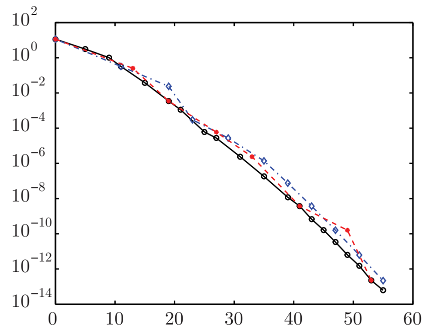


FIGURE 1. Residual norm vs. number of degrees of freedom activated by **ADFOUR**, for different choices of Dörfler parameter $\theta$; solid line: $\theta = 1 - 10^{-1}$; dash-dotted line: $\theta = 1 - 10^{-2}$; dashed line: $\theta = 1 - 10^{-3}$. The symbols (circles, diamonds, stars) identify the various **ADFOUR** iterations for the sample 1D problem (2.3) with analytic solution $u(x) = \exp(\cos 2x + \sin x)$ and coefficients with $\nu = 1 + \frac{1}{2}\sin 3x$ and $\sigma = \exp(2\cos 3x)$.

Note that whenever the algorithm stops, the condition $\|r_{n+1}\| \leq \text{tol}$ is fulfilled according to (3.10).

**Theorem 3.2** (contraction property of **F-ADFOUR**). *For the feasible variant* **F-ADFOUR** *described above, the same conclusions of Theorem* 3.1 *hold true, with the contraction factor* $\widetilde{\rho} = \rho(\theta)$ *given by* (3.7) *with* $\theta$ *is defined in* (3.14), *namely* $\widetilde{\rho} = \sqrt{1 - \frac{\alpha_*}{\alpha^*}\left(\frac{\widetilde{\theta}-\gamma}{1+\gamma}\right)^2}$. $\square$

3.3. **FA-ADFOUR: An aggressive version of F-ADFOUR.** Theorem 3.1 indicates that even if one chooses $\widetilde{\theta}$ very close to 1, the predicted error reduction rate $\widetilde{\rho}$ guaranteed by **F-ADFOUR** is always bounded from below by the quantity $\sqrt{1 - \frac{\alpha_*}{\alpha^*}\left(\frac{1-\gamma}{1+\gamma}\right)^2}$. Such a result looks overly pessimistic, particularly in the case of smooth (analytic) solutions, since a Fourier method allows for an exponential decay of the error as the number of (properly selected) active degrees of freedom is increased. Figure 1 displays the influence of the Dörfler parameter $\widetilde{\theta}$ on the decay rate and number of Galerkin solves: choosing $\widetilde{\theta}$ closer to 1 does not significantly affect the rate of decay of the error versus the number of activated degrees of freedom, but it significantly reduces the number of iterations. This in turn reduces the computational cost measured in terms of Galerkin solves.

Motivated by this observation, hereafter, we consider a variant of Algorithm **F-ADFOUR**, which—assuming either Property 2.2 or 2.3—guarantees an arbitrarily large error reduction per iteration, provided the set of new degrees of freedom detected by **DÖRFLER** is suitably enriched.

Let $\widetilde{r}_n = \widetilde{r}(u_n)$ be an approximation to $r_n$ satisfying (3.9), let

$$\partial\Lambda_n := \textbf{E-DÖRFLER}(\widetilde{r}_n, \widetilde{\theta}, J)$$

be the 2-step procedure

$$\widetilde{\partial\Lambda}_n := \textbf{DÖRFLER}(\widetilde{r}_n, \widetilde{\theta}) \,, \qquad (3.20)$$
$$\partial\Lambda_n := \textbf{ENRICH}(\widetilde{\partial\Lambda}_n, J) \,,$$

and set $\Lambda_{n+1} := \Lambda_n \cup \partial\Lambda_n$, where the latter module and the value of the integer $J$ will be specified below. We recall that the set $\widetilde{\partial\Lambda}_n$ is such that $\widetilde{g}_n := P_{\widetilde{\partial\Lambda}_n} \widetilde{r}_n$ satisfies $\|\widetilde{r}_n - \widetilde{g}_n\| \leq \sqrt{1-\widetilde{\theta}^2}\|\widetilde{r}_n\|$. Let $\widetilde{w}_n \in V$ be the solution of $L\widetilde{w}_n = \widetilde{g}_n$, which in general will have infinitely many components, and let us split it as

$$\widetilde{w}_n = P_{\Lambda_{n+1}}\widetilde{w}_n + P_{\Lambda^c_{n+1}}\widetilde{w}_n =: \widetilde{y}_n + \widetilde{z}_n \in V_{\Lambda_{n+1}} \oplus V_{\Lambda^c_{n+1}} \,.$$

Then, by the minimality property of the Galerkin solution in the energy norm and by (2.5) and (2.9), one has

$$\begin{aligned}
|\!|\!| u - u_{n+1} |\!|\!| &\leq |\!|\!| u - (u_n + \widetilde{y}_n) |\!|\!| \leq |\!|\!| u - u_n - \widetilde{w}_n + \widetilde{z}_n |\!|\!| \\
&\leq \frac{1}{\sqrt{\alpha_*}} \|L(u - u_n - \widetilde{w}_n)\| + \sqrt{\alpha^*}\|\widetilde{z}_n\| = \frac{1}{\sqrt{\alpha_*}}\|r_n - \widetilde{g}_n\| + \sqrt{\alpha^*}\|\widetilde{z}_n\| \\
&\leq \frac{1}{\sqrt{\alpha_*}}\left(\|r_n - \widetilde{r}_n\| + \|\widetilde{r}_n - \widetilde{g}_n\|\right) + \sqrt{\alpha^*}\|\widetilde{z}_n\| \,,
\end{aligned}$$

whence

$$|\!|\!| u - u_{n+1} |\!|\!| \leq \frac{1}{\sqrt{\alpha_*}}\left(\gamma + \sqrt{1-\widetilde{\theta}^2}\right)\|\widetilde{r}_n\| + \sqrt{\alpha^*}\|\widetilde{z}_n\| \,.$$

Thus, if we choose $\gamma$ such that

$$\gamma \leq \sqrt{1-\widetilde{\theta}^2} \,,$$

then we obtain

$$|\!|\!| u - u_{n+1} |\!|\!| \leq \frac{2}{\sqrt{\alpha_*}}\sqrt{1-\widetilde{\theta}^2}\,\|\widetilde{r}_n\| + \sqrt{\alpha^*}\|\widetilde{z}_n\| \,.$$

Now we can write $\widetilde{z}_n = \big(P_{\Lambda^c_{n+1}} L^{-1} P_{\widetilde{\partial\Lambda}_n}\big)\widetilde{r}_n$; hence, if $\Lambda_{n+1}$ is defined in such a way that

$$k \in \Lambda^c_{n+1} \quad \text{and} \quad \ell \in \widetilde{\partial\Lambda}_n \qquad \Rightarrow \qquad |k - \ell| > J \,,$$

then we have

$$\|P_{\Lambda^c_{n+1}} L^{-1} P_{\widetilde{\partial\Lambda}_n}\| \leq \|\mathbf{A}^{-1} - (\mathbf{A}^{-1})_J\| \leq \psi_{\mathbf{A}^{-1}}(J, \bar{\eta}_L) \,,$$

where we have used Property 2.5 of the matrix $\mathbf{A}^{-1}$. Now, $J > 0$ can be chosen to satisfy

$$\psi_{\mathbf{A}^{-1}}(J, \bar{\eta}_L) \leq \sqrt{\frac{1-\widetilde{\theta}^2}{\alpha_*\alpha^*}} \,, \qquad (3.21)$$

in such a way that

$$|\!|\!| u - u_{n+1} |\!|\!| \leq 3\frac{\sqrt{1-\widetilde{\theta}^2}}{\sqrt{\alpha_*}}\|\widetilde{r}_n\| \leq 3\left(\frac{\alpha^*}{\alpha_*}\right)^{1/2}\frac{\sqrt{1-\widetilde{\theta}^2}}{1-\gamma}|\!|\!| u - u_n |\!|\!| \,, \qquad (3.22)$$

where we have employed the variant $\|\widetilde{r}_n\| \leq \frac{\sqrt{\alpha^*}}{1-\gamma}|\!|\!| u - u_n |\!|\!|$ of (3.12).

Note that, as desired, the new error reduction rate

$$\widetilde{\rho} = 3\left(\frac{\alpha^*}{\alpha_*}\right)^{1/2}\frac{\sqrt{1-\widetilde{\theta}^2}}{1-\gamma} \qquad (3.23)$$

can be made arbitrarily small by choosing $\widetilde{\theta}$ arbitrarily close to 1. The procedure **ENRICH** is thus defined as follows:

- $\Lambda^* :=$ **ENRICH**$(\Lambda, J)$
  Given an integer $J \geq 0$ and a finite set $\Lambda \subset \mathbb{Z}^d$, the output is the set
  $$\Lambda^* := \{k \in \mathbb{Z}^d \ : \ \text{there exists } \ell \in \Lambda \text{ such that } |k - \ell| \leq J\} .$$

Note that since the procedure adds a $d$-dimensional ball of radius $J$ around each point of $\Lambda$, the cardinality of the new set $\Lambda^*$ can be estimated as

$$|\Lambda^*| \leq |\overline{B_d(0,J)} \cap \mathbb{Z}^d|\,|\Lambda| \sim \omega_d J^d |\Lambda| \,, \tag{3.24}$$

where $\omega_d$ is the measure of the $d$-dimensional Euclidean unit ball $B_d(0,1)$ centered at the origin.

Given a tolerance tol $\in [0,1)$, a marking parameter $\widetilde{\theta} \in (0,1)$ and two feasibility parameters $0 < \gamma \leq \sqrt{1-\widetilde{\theta}^2}$ and $\epsilon_0 \in (0,1)$, we choose $J \geq 1$ as the smallest integer for which (3.21) is fulfilled and define the following variant of **F-ADFOUR**, with **E-DÖRFLER** in place of **DÖRFLER**.

**Algorithm FA-ADFOUR**$(\widetilde{\theta}, \gamma, \epsilon_0, \text{tol}, J)$

```
Set u_0 := 0,  Λ_0 := ∅, n = −1
r̃_0 = F-RHS(ε_0)
do
    n ← n + 1
    ∂Λ_n := E-DÖRFLER(r̃_n, θ̃, J)
    Λ_{n+1} := Λ_n ∪ ∂Λ_n
    u_{n+1} := GAL(Λ_{n+1})
    [r̃_{n+1}, flag, ε_{n+1}] := F-RES(u_{n+1}, γ, ε_n, tol)
    if flag = 1 then STOP
while ‖r̃_{n+1}‖ > tol/(1+γ)
```

We summarize our results in the following theorem.

**Theorem 3.3** (contraction property of **FA-ADFOUR**). *Let $\widetilde{\theta}$ be such that $\widetilde{\rho} < 1$ defined in* (3.23) *and let the assumptions of either Property* 2.2 *or* 2.3 *be satisfied. Then, the same conclusions of Theorem* 3.1 *hold true for the aggressive variant* **FA-ADFOUR**, *with the contraction factor $\rho$ replaced by $\widetilde{\rho}$.* □

3.4. **FC-ADFOUR and FPC-ADFOUR: F-ADFOUR with coarsening.** The adaptive algorithm **ADFOUR** and its feasible variants introduced above are not guaranteed to be optimal in terms of cardinality. In fact, the discussion in the forthcoming Section 5 for the exponential case reveals that the residual $r(u_\Lambda)$ as well as its finite truncation $\widetilde{r}(u_\Lambda)$ may be significantly less sparse than the corresponding Galerkin solution $u_\Lambda$. In particular, we will see that indices in $\Lambda$ relatively important to represent $\widetilde{r}(u_\Lambda)$ may be insignificant to describe $u_\Lambda$; this is a striking difference between the exponential and algebraic cases. For these reasons, we propose here a new variant of algorithm **F-ADFOUR**, which incorporates a recursive coarsening step. For the ease of reading, we present our feasible algorithm under the assumption that the module **GAL** is able to compute the exact Galerkin solution. Including the inexact computation of the Galerkin solution is only a technical difficulty.

The algorithm is constructed through the procedures **GAL**, **F-RES**, **DÖRFLER** already introduced, together with the new procedure **COARSE** defined as follows:

- $\Lambda := \mathbf{COARSE}(w, \epsilon)$
  Given a function $w \in V_{\Lambda^*}$ for some finite index set $\Lambda^*$, and an accuracy $\epsilon$ which is known to satisfy $\|u - w\| \leq \epsilon$, the output $\Lambda \subseteq \Lambda^*$ is a set of minimal cardinality such that

$$\|w - P_\Lambda w\| \leq 2\epsilon \,. \tag{3.25}$$

The following result, whose proof is straightforward, will be used several times in the paper.

**Property 3.1** (coarsening)**.** The procedure **COARSE** guarantees the bounds

$$\|u - P_\Lambda w\| \leq 3\epsilon \tag{3.26}$$

and, for the Galerkin solution $u_\Lambda \in V_\Lambda$,

$$|\!|\!| u - u_\Lambda |\!|\!| \leq 3\sqrt{\alpha^*}\epsilon \,. \tag{3.27}$$

Given a tolerance tol $\in [0, 1)$, a marking parameter $\widetilde{\theta} \in (0, 1)$ and two feasibility parameters $\gamma \in (0, \widetilde{\theta})$ and $\epsilon_0 \in (0, 1)$, we define the following feasible adaptive algorithm with coarsening.

**Algorithm FC-ADFOUR**$(\widetilde{\theta}, \gamma, \epsilon_0,$ tol$)$

```
Set u_0 := 0,  Λ_0 := ∅, n = −1
r̃_0 := F-RHS(ε_0)
do
    n ← n + 1
    set Λ_{n,0} = Λ_n, r̃_{n,0} = r̃_n, ε_{n,0} = ε_n, k = −1
    do
        k ← k + 1
        ∂Λ_{n,k} := DÖRFLER(r̃_{n,k}, θ)
        Λ_{n,k+1} := Λ_{n,k} ∪ ∂Λ_{n,k}
        u_{n,k+1} := GAL(Λ_{n,k+1})
        [r̃_{n,k+1}, flag, ε_{n,k+1}] := F-RES(u_{n,k+1}, γ, ε_{n,k}, (1/(1+γ)) (√α_*/(3α^*)))
        if flag = 1 then exit do cycle
    while ‖r̃_{n,k+1}‖ > √(1 − θ²) ‖r̃_n‖
    Λ_{n+1} := COARSE( u_{n,k+1}, ((1+γ)/α_*) ‖r̃_{n,k+1}‖ )
    u_{n+1} := GAL(Λ_{n+1})
    if flag = 1 then STOP
    [r̃_{n+1}, flag, ε_{n+1}] := F-RES(u_{n+1}, γ, ε_n, tol)
    if flag = 1 then STOP
while ‖r̃_{n+1}‖ > tol/(1+γ)
```

The specific choice of accuracy $\epsilon = \frac{1+\gamma}{\alpha_*}\|\widetilde{r}_{n,k+1}\|$ in each call of **COARSE** in the algorithm above is motivated by the wish of guaranteeing a fixed reduction of the residual and error at each outer iteration. This will be clear from the subsequent theorem. Furthermore, we note that (3.10) implies that whenever the algorithm stops, the condition $\|r_{n+1}\| \leq$ tol is fulfilled.

**Theorem 3.4** (contraction property of **FC-ADFOUR**)**.** *The algorithm* **FC-ADFOUR** *satisfies:*

(i) *The number of iterations of each inner loop is finite and bounded independently of $n$.*
(ii) *The errors $u - u_n$ generated for $n \geq 0$ by the algorithm satisfy the inequality*

$$|\!|\!| u - u_{n+1} |\!|\!| \leq \widetilde{\rho} |\!|\!| u - u_n |\!|\!| \tag{3.28}$$

*for*

$$\widetilde{\rho} = 3\frac{\alpha^*}{\alpha_*}\frac{1+\gamma}{1-\gamma}\sqrt{1-\widetilde{\theta}^2}\ . \tag{3.29}$$

*In particular, if* $\widetilde{\theta}$ *is chosen in such a way that* $\widetilde{\rho} < 1$*, for any* tol $> 0$ *the algorithm terminates in a finite number of iterations, whereas for* tol $= 0$ *the sequence* $u_n$ *converges to* $u$ *in* $H^1_p(\Omega)$ *as* $n \to \infty$*.*

*Proof.* (i) For any fixed $n$, each inner iteration behaves the same as the algorithm **F-ADFOUR** considered in Section 3.2. Hence, setting again $\widetilde{\rho} = \sqrt{1 - \frac{\alpha_*}{\alpha^*}\left(\frac{\widetilde{\theta}-\gamma}{1+\gamma}\right)^2}$, we have as in Theorem 3.2,

$$|\!|\!| u - u_{n,k+1} |\!|\!| \le \widetilde{\rho}^{k+1} |\!|\!| u - u_n |\!|\!| \ , \tag{3.30}$$

which implies, by (2.9),

$$\|r_{n,k+1}\| \le \sqrt{\alpha^*} |\!|\!| u - u_{n,k+1} |\!|\!| \le \sqrt{\alpha^*}\widetilde{\rho}^{k+1} |\!|\!| u - u_n |\!|\!| \le \sqrt{\frac{\alpha^*}{\alpha_*}}\widetilde{\rho}^{k+1}\|r_n\| \ .$$

Using (3.10) yields

$$\|\widetilde{r}_{n,k+1}\| \le \frac{1+\gamma}{1-\gamma}\sqrt{\frac{\alpha^*}{\alpha_*}}\widetilde{\rho}^{k+1}\|\widetilde{r}_n\|.$$

This shows that the termination criterion

$$\|\widetilde{r}_{n,k+1}\| \le \sqrt{1-\widetilde{\theta}^2}\,\|\widetilde{r}_n\| \tag{3.31}$$

is certainly satisfied if

$$\frac{1+\gamma}{1-\gamma}\sqrt{\frac{\alpha^*}{\alpha_*}}\widetilde{\rho}^{k+1} \le \sqrt{1-\widetilde{\theta}^2}\ ,$$

i.e., as soon as

$$k+1 \ge \frac{\log\left(\frac{\alpha_*}{\alpha^*}\left(\frac{1-\gamma}{1+\gamma}\right)^2\left(1-\widetilde{\theta}^2\right)\right)}{2\log\widetilde{\rho}} > k\ .$$

We conclude that the number $K_n = k+1$ of inner iterations is bounded by $1 + \frac{\log\left(\frac{\alpha_*}{\alpha^*}\left(\frac{1-\gamma}{1+\gamma}\right)^2\left(1-\widetilde{\theta}^2\right)\right)}{2\log\widetilde{\rho}}$, which is independent of $n$.

(ii) By (2.8) and (3.10), we have

$$|\!|\!| u - u_{n,k+1} |\!|\!| \le \frac{1+\gamma}{\alpha_*}\|\widetilde{r}_{n,k+1}\| =: \delta_n\ .$$

At the exit of the inner loop, the parameter $\delta_n$ is fed to module **COARSE**; then, Property 3.1 yields

$$|\!|\!| u - u_{n+1} |\!|\!| \le 3\sqrt{\alpha^*}\delta_n\ .$$

On the other hand, the termination criterion (3.31) combined with (3.10) yields

$$\delta_n \le \frac{\sqrt{1-\widetilde{\theta}^2}}{\alpha_*}\frac{1+\gamma}{1-\gamma}\|r_n\|\ ,$$

so that

$$|\!|\!| u - u_{n+1} |\!|\!| \le 3\frac{\sqrt{\alpha^*}}{\alpha_*}\frac{1+\gamma}{1-\gamma}\sqrt{1-\widetilde{\theta}^2}\|r_n\| \le 3\frac{\alpha^*}{\alpha_*}\frac{1+\gamma}{1-\gamma}\sqrt{1-\widetilde{\theta}^2} |\!|\!| u - u_n |\!|\!|\ ,$$

according to (2.9). This concludes the proof. ∎

A coarsening step can also be inserted in the feasible aggressive algorithm **FA-ADFOUR** considered in Section 3.3; indeed, the enrichment step **ENRICH** could activate a larger number of degrees of freedom than really needed, endangering optimality. The algorithm we now propose can be viewed as a variant of **FC-ADFOUR**, in which the use of **E-DÖRFLER** instead of **DÖRFLER** allows one to take a single inner iteration; in this respect, one can consider the enrichment step as a "prediction", and the coarsening step as a "correction", of the new set of active degrees of freedom. For this reason, we call this variant the **Feasible Predictor/Corrector-ADFOUR**, or simply **FPC-ADFOUR**.

Given a tolerance tol $\in [0,1)$, a marking parameter $\widetilde{\theta} \in (0,1)$ and two feasibility parameters $0 < \gamma \le \sqrt{1-\widetilde{\theta}^2}$ and $\epsilon_0 \in (0,1)$, we choose $J \ge 1$ as the smallest integer for which (3.21) is fulfilled, and we define the following adaptive algorithm.

**Algorithm FPC-ADFOUR**$(\widetilde{\theta}, \gamma, \epsilon_0, \text{ tol}, \ J)$

Set $flag = 0$
Set $u_0 := 0, \ \Lambda_0 := \emptyset, \ n = -1$
$\widetilde{r}_0 :=$ **F-RHS**$(\epsilon_0)$
do
  $n \leftarrow n+1$
  $\widehat{\partial\Lambda}_n :=$ **E-DÖRFLER**$(\widetilde{r}_n, \widetilde{\theta}, J)$
  $\widehat{\Lambda}_{n+1} := \Lambda_n \cup \widehat{\partial\Lambda}_n$
  $\widehat{u}_{n+1} :=$ **GAL**$(\widehat{\Lambda}_{n+1})$
  $\Lambda_{n+1} :=$ **COARSE** $\left(\widehat{u}_{n+1}, \frac{3}{\alpha_*}\sqrt{1-\widetilde{\theta}^2}\, \|\widetilde{r}_n\|\right)$
  $u_{n+1} :=$ **GAL**$(\Lambda_{n+1})$
  $[\widetilde{r}_{n+1}, flag, \epsilon_{n+1}] :=$ **F-RES**$(u_{n+1}, \gamma, \epsilon_n, \text{tol})$
  if $flag = 1$ then STOP
while $\|\widetilde{r}_{n+1}\| > \frac{\text{tol}}{1+\gamma}$

**Theorem 3.5** (contraction property of **FPC-ADFOUR**). *If the assumptions of either Property* 2.2 *or* 2.3 *are satisfied, then the assertion (ii) of Theorem* 3.4 *is valid for Algorithm* **FPC-ADFOUR** *as well.*

*Proof.* The first inequalities in both (2.5) and (3.22) yield

$$\|u - \widehat{u}_{n+1}\| \le \frac{3}{\alpha_*}\sqrt{1-\widetilde{\theta}^2}\, \|\widetilde{r}_n\| = \delta_n \ .$$

Since $\delta_n$ is the parameter fed to the procedure **COARSE**, one proceeds as in the proof of Theorem 3.4. ∎

## 4. Nonlinear approximation in Fourier spaces

4.1. **Best $N$-term approximation and rearrangement.** Given any nonempty finite index set $\Lambda \subset \mathbb{Z}^d$ and the corresponding subspace $V_\Lambda \subset V = H^1_p(\Omega)$ of dimension $|\Lambda| = \text{card}\,\Lambda$, the best approximation of $v$ in $V_\Lambda$ is the orthogonal projection of $v$ upon $V_\Lambda$, i.e., the function $P_\Lambda v = \sum_{k\in\Lambda} \hat{v}_k \phi_k$, which satisfies

$$\|v - P_\Lambda v\| = \left( \sum_{k \notin \Lambda} |\hat{V}_k|^2 \right)^{1/2}$$

(we set $P_\Lambda v = 0$ if $\Lambda = \emptyset$). For any integer $N \geq 1$, we minimize this error over all possible choices of $\Lambda$ with cardinality $N$, thereby leading to the *best $N$-term approximation error*

$$E_N(v) = \inf_{\Lambda \subset \mathbb{Z}^d,\ |\Lambda|=N} \|v - P_\Lambda v\| \ .$$

A way to construct a *best $N$-term approximation* $v_N$ of $v$ consists of rearranging the coefficients of $v$ in nonincreasing order of modulus

$$|\hat{V}_{k_1}| \geq \cdots \geq |\hat{V}_{k_n}| \geq |\hat{V}_{k_{n+1}}| \geq \ldots$$

and setting $v_N = P_{\Lambda_N} v$ with $\Lambda_N = \{k_n \ : \ 1 \leq n \leq N\}$. As already mentioned in the Introduction, let us denote from now on $v_n^* = \hat{V}_{k_n}$ the rearranged and rescaled Fourier coefficients of $v$. Then,

$$E_N(v) = \left( \sum_{n>N} |v_n^*|^2 \right)^{1/2} \ .$$

Next, given a strictly decreasing function $\phi : \mathbb{N} \to \mathbb{R}_+$ such that $\phi(0) = \phi_0$ for some $\phi_0 > 0$ and $\phi(N) \to 0$ when $N \to \infty$, we introduce the corresponding *sparsity class* $\mathcal{A}_\phi$ by setting

$$\mathcal{A}_\phi = \left\{ v \in V \ : \ \|v\|_{\mathcal{A}_\phi} := \sup_{N \geq 0} \frac{E_N(v)}{\phi(N)} < +\infty \right\} . \tag{4.1}$$

We point out that in applications $\|v\|_{\mathcal{A}_\phi}$ need not be a (quasi-)norm since $\mathcal{A}_\phi$ need not be a linear space. Note, however, that $\|v\|_{\mathcal{A}_\phi}$ always controls the $V$-norm of $v$, since $\|v\| = E_0(v) \leq \phi_0 \|v\|_{\mathcal{A}_\phi}$. Observe that $v \in \mathcal{A}_\phi$ iff there exists a constant $c > 0$ such that

$$E_N(v) \leq c\phi(N) \ , \qquad \forall N \geq 0 \ . \tag{4.2}$$

The quantity $\|v\|_{\mathcal{A}_\phi}$ dictates the minimal number $N_\varepsilon$ of basis functions needed to approximate $v$ with accuracy $\varepsilon$. In fact, from the relations

$$E_{N_\varepsilon}(v) \leq \varepsilon < E_{N_\varepsilon - 1}(v) \leq \phi(N_\varepsilon - 1) \|v\|_{\mathcal{A}_\phi} \ ,$$

and the monotonicity of $\phi$, we obtain

$$N_\varepsilon \leq \phi^{-1} \left( \frac{\varepsilon}{\|v\|_{\mathcal{A}_\phi}} \right) + 1 \ . \tag{4.3}$$

The following result will be used several times throughout the paper.

**Property 4.1.** Let $v \in \mathcal{A}_\phi$ and let $w \in V$ be an element with finite support $\Lambda$, such that there exists a constant $C_*$ satisfying

$$|\Lambda| \leq \phi^{-1} \left( \frac{\|v - w\|}{C_* \|v\|_{\mathcal{A}_\phi}} \right) + 1 \ . \tag{4.4}$$

Then,

$$\|w\|_{\mathcal{A}_\phi} \leq (1 + C_*) \|v\|_{\mathcal{A}_\phi} \ . \tag{4.5}$$

*Proof.* Let us set $N = |\Lambda|$. We estimate the best $n$-term approximation error $E_n(w)$, $n \geq 0$. Since $n \geq N$ implies $E_n(w) = 0$, it suffices to consider $n < N$. Let $v_n$ be a best $n$-term approximation of $v$, which satisfies $\|v - v_n\| \leq \|v\|_{\mathcal{A}_\phi} \phi(n)$. On the other hand, (4.4) yields $\|v-w\| \le C_* \|v\|_{\mathcal{A}_\phi} \phi(N-1) \le C_* \|v\|_{\mathcal{A}_\phi} \phi(n)$ because $\phi$ is decreasing. This implies

$$E_n(w) \le \|w - v_n\| \le \|w - v\| + \|v - v_n\| \le (1 + C_*) \|v\|_{\mathcal{A}_\phi} \phi(n),$$

whence the result follows immediately. □

*Remark* 4.1 (sparsity class for $V'$). Replacing $V$ by $V'$ in (4.1) leads to the definition of a sparsity class, still denoted by $\mathcal{A}_\phi$, in the space of linear continuous forms $f$ on $H^1_p(\Omega)$. This observation applies to the subsequent definitions as well (e.g., for the class $\mathcal{A}^{\eta,t}_G$). In essence, we will treat in a unified way the nonlinear approximation of a function $v \in H^1_p(\Omega)$ and of a form $f \in H^{-1}_p(\Omega)$. □

Throughout the paper, we shall consider two main families of sparsity classes, identified by specific choices of the function $\phi$ depending upon one or more parameters. The first family is related to the best approximation in *Besov* spaces of periodic functions, thus accounting for a finite-order regularity in $\Omega$; the corresponding functions $\phi$ exhibit an algebraic decay as $N \to \infty$, which motivates our terminology of *algebraic classes*. The second family is related to the best approximation in *Gevrey* spaces of periodic functions, which are formed by infinitely-differentiable functions in $\Omega$; the associated $\phi$'s exhibit an exponential decay, and for this reason such classes will be referred to as *exponential classes*. Properties of both families are collected hereafter.

4.2. **Algebraic classes.** The following is the counterpart for Fourier approximations of by now well-known nonlinear approximation settings [12], e.g., for wavelets or nested finite elements. For this reason, we just state definitions and properties without proofs.

For $s > 0$, let us introduce the function

$$\phi(N) = \big(N+1\big)^{-s/d} \qquad \text{for } N \ge 0\,, \tag{4.6}$$

with inverse

$$\phi^{-1}(\lambda) = \lambda^{-d/s} - 1 \qquad \text{for } \lambda \le 1\,, \tag{4.7}$$

and let us consider the corresponding class $\mathcal{A}_\phi$ defined in (4.1).

**Definition 4.1** (algebraic class of functions)**.** We denote by $\mathcal{A}^s_B$ the subset of $V$ defined as

$$\mathcal{A}^s_B := \left\{ v \in V \ : \ \|v\|_{\mathcal{A}^s_B} := \sup_{N \ge 0} E_N(v) \big(N+1\big)^{s/d} < +\infty \right\}.$$

It is immediately seen that $\mathcal{A}^s_B$ contains the Sobolev space of periodic functions $H^{s+1}_p(\Omega)$. On the other hand, it is proven in [13], as a part of a more general result, that for $0 < \sigma, \tau \le \infty$, the Besov space $B^{s+1}_{\tau,\sigma}(\Omega) = B^{s+1}_\sigma(L^\tau(\Omega))$ is contained in $\mathcal{A}^{s^*}_B$ provided $s^* := s - d(1/\tau - 1/2)_+ > 0$.

Let us associate the quantity $\tau > 0$ to the parameter $s$, via the relation

$$\frac{1}{\tau} = \frac{s}{d} + \frac{1}{2}\,.$$

The condition for a function $v$ to belong to some class $\mathcal{A}^s_B$ can be equivalently stated as a condition on the vector $\boldsymbol{v} = (\hat{V}_k)_{k \in \mathbb{Z}^d}$ of its Fourier coefficients, precisely, on the rate of decay of the nonincreasing rearrangement $\boldsymbol{v}^* = (v^*_n)_{n \ge 1}$ of $\boldsymbol{v}$.

**Definition 4.2** (algebraic class of sequences)**.** Let $\ell^s_B(\mathbb{Z}^d)$ be the subset of sequences $\boldsymbol{v} \in \ell^2(\mathbb{Z}^d)$ so that

$$\|\boldsymbol{v}\|_{\ell^s_B(\mathbb{Z}^d)} := \sup_{n\geq 1} n^{1/\tau}|v^*_n| < +\infty \ .$$

Note that this space is often denoted by $\ell^\tau_w(\mathbb{Z}^d)$ in the literature, being an example of Lorentz space.

The relationship between $\mathcal{A}^s_B$ and $\ell^s_B(\mathbb{Z}^d)$ is stated in the following Proposition (see e.g. [8]).

**Proposition 4.1** (equivalence of algebraic classes)**.** *Given a function $v \in V$ and the sequence $\boldsymbol{v}$ of its Fourier coefficients, one has $v \in \mathcal{A}^s_B$ if and only if $\boldsymbol{v} \in \ell^s_B(\mathbb{Z}^d)$, with*

$$\|v\|_{\mathcal{A}^s_B} \lesssim \|\boldsymbol{v}\|_{\ell^s_B(\mathbb{Z}^d)} \lesssim \|v\|_{\mathcal{A}^s_B} \ .$$

At last, we note that the quasi-Minkowski inequality

$$\|\boldsymbol{u}+\boldsymbol{v}\|_{\ell^s_B(\mathbb{Z}^d)} \leq C_s \left(\|\boldsymbol{u}\|_{\ell^s_B(\mathbb{Z}^d)} + \|\boldsymbol{v}\|_{\ell^s_B(\mathbb{Z}^d)}\right)$$

holds in $\ell^s_B(\mathbb{Z}^d)$, yet the constant $C_s$ blows up exponentially as $s \to \infty$.

4.3. **Exponential classes.** Gevrey spaces have been employed to study the $C^\infty$ and analytic regularity of the solutions of partial differential equations (see [15]). We first recall the definition of Gevrey spaces of periodic functions in $\Omega = (0, 2\pi)^d$. Given reals $\eta > 0$, $0 < t \leq d$ and $s \geq 0$, we set

$$G^{\eta,t,s}_p(\Omega) := \left\{ v \in L^2(\Omega) \ : \ \|v\|^2_{G,\eta,t,s} = \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\eta|k|^t}(1+|k|^{2s})|\hat{v}_k|^2 < +\infty \right\} .$$

Note that $G^{\eta,t,s}_p(\Omega)$ is contained in all Sobolev spaces of periodic functions $H^r_p(\Omega)$, $r \geq 0$. Furthermore, if $t \geq 1$, $G^{\eta,t,s}_p(\Omega)$ is made of analytic functions (see e.g. [18]). From now on, we fix $s = 1$ and we normalize again the Fourier coefficients of a function $v$ with respect to the $H^1_p(\Omega)$-norm. Thus, we set

$$G^{\eta,t}_p(\Omega) = G^{\eta,t,1}_p(\Omega) = \{ v \in V \ : \ \|v\|^2_{G,\eta,t} = \sum_k \mathrm{e}^{2\eta|k|^t}|\hat{V}_k|^2 < +\infty \} \ . \tag{4.8}$$

Functions in $G^{\eta,t}_p(\Omega)$ can be approximated by the linear orthogonal projection

$$P_M v = \sum_{|k|\leq M} \hat{V}_k \phi_k \ ,$$

for which we have

$$\begin{aligned} \|v - P_M v\|^2 = & \sum_{|k|>M} |\hat{V}_k|^2 = \sum_{|k|>M} \mathrm{e}^{-2\eta|k|^t}\mathrm{e}^{2\eta|k|^t}|\hat{V}_k|^2 \\ \leq & \ \mathrm{e}^{-2\eta M^t} \sum_{|k|>M} \mathrm{e}^{2\eta|k|^t}|\hat{V}_k|^2 \leq \mathrm{e}^{-2\eta M^t}\|v\|^2_{G,\eta,t} \ . \end{aligned}$$

As already observed in Property 2.4, setting $N = \mathrm{card}\{k \ : \ |k| \leq M\}$, one has $N \sim \omega_d M^d$, so that

$$E_N(v) \leq \|v - P_M v\| \lesssim \exp\left(-\eta\omega_d^{-t/d}N^{t/d}\right)\|v\|_{G,\eta,t} \ . \tag{4.9}$$

Hence, we are led to introduce the function

$$\phi(N) = \exp\left(-\eta\omega_d^{-t/d}N^{t/d}\right) \qquad (N \geq 0) \ , \tag{4.10}$$

whose inverse is given by

$$\phi^{-1}(\lambda) = \frac{\omega_d}{\eta^{d/t}} \left( \log \frac{1}{\lambda} \right)^{d/t} \qquad (\lambda \leq 1) , \tag{4.11}$$

and to consider the corresponding class $\mathcal{A}_\phi$ defined in (4.1), which therefore contains $G_p^{\eta,t}(\Omega)$.

**Definition 4.3** (exponential class of functions)**.** We denote by $\mathcal{A}_G^{\eta,t}$ the subset of $V$ defined as

$$\mathcal{A}_G^{\eta,t} := \left\{ v \in V \ : \ \|v\|_{\mathcal{A}_G^{\eta,t}} := \sup_{N \geq 0} E_N(v) \exp\left( \eta \omega_d^{-t/d} N^{t/d} \right) < +\infty \right\} .$$

At this point, we make the subsequent notation easier by introducing the $t$-dependent function

$$\tau = \frac{t}{d} \leq 1 ;$$

the upper bound is introduced just for technical convenience and is not really restrictive, since the relevant set of all analytical functions is surely included in all such classes provided $t \leq 1$, i.e., $\tau \leq 1/d$ (see [15,18]). As in the algebraic case, the class $\mathcal{A}_G^{\eta,t}$ can be equivalently characterized in terms of the behavior of rearranged sequences of Fourier coefficients.

**Definition 4.4** (exponential class of sequences)**.** Let $\ell_G^{\eta,t}(\mathbb{Z}^d)$ be the subset of sequences $\mathbf{v} \in \ell^2(\mathbb{Z}^d)$ so that

$$\|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{Z}^d)} := \sup_{n \geq 1} \left( n^{(1-\tau)/2} \exp\left( \eta \omega_d^{-\tau} n^\tau \right) |v_n^*| \right) < +\infty ,$$

where $\mathbf{v}^* = (v_n^*)_{n=1}^\infty$ is the nonincreasing rearrangement of $\mathbf{v}$.

The relationship between $\mathcal{A}_G^{\eta,t}$ and $\ell_G^{\eta,t}(\mathbb{Z}^d)$ is stated in the following proposition.

**Proposition 4.2** (equivalence of exponential classes)**.** *Given a function $v \in V$ and the sequence $\boldsymbol{v} = (\hat{V}_k)_{k \in \mathbb{Z}^d}$ of its Fourier coefficients, one has $v \in \mathcal{A}_G^{\eta,t}$ if and only if $\mathbf{v} \in \ell_G^{\eta,t}(\mathbb{Z}^d)$, with*

$$\|v\|_{\mathcal{A}_G^{\eta,t}} \lesssim \|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{Z}^d)} \lesssim \|v\|_{\mathcal{A}_G^{\eta,t}} .$$

*Proof.* Assume first that $\mathbf{v} \in \ell_G^{\eta,t}(\mathbb{Z}^d)$. Then,

$$E_N(v)^2 = \|v - P_N(v)\|^2 = \sum_{n>N} |v_n^*|^2 \lesssim \sum_{n>N} n^{\tau-1} \exp\left( -2\eta \omega_d^{-\tau} n^\tau \right) \|\mathbf{v}\|^2_{\ell_G^{\eta,t}(\mathbb{Z}^d)} .$$

Now, setting for simplicity $\alpha = 2\eta\omega_d^{-\tau}$, one has

$$S := \sum_{n>N} n^{\tau-1} \mathrm{e}^{-\alpha n^\tau} \sim \int_N^\infty x^{\tau-1} \mathrm{e}^{-\alpha x^\tau} dx .$$

The substitution $z = x^\tau$ yields

$$S \sim \frac{d}{t} \int_{N^\tau}^\infty \mathrm{e}^{-\alpha z} dz = \frac{d}{\alpha t} \mathrm{e}^{-\alpha N^\tau} ,$$

whence $\|v\|_{\mathcal{A}_G^{\eta,t}} \lesssim \|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{Z}^d)}$. Conversely, if $v \in \mathcal{A}_G^{\eta,t}$, then we have to prove that for any $n \geq 1$, one has

$$n^{1-\tau} |v_n^*|^2 \lesssim \mathrm{e}^{-\alpha n^\tau} \|v\|^2_{\mathcal{A}_G^{\eta,t}} .$$

Let $m < n$ be the largest integer such that $n - m \geq n^{1-\tau}$, i.e., $m \sim n(1 - n^{-\tau})$, and recall that $0 \leq 1 - \tau < 1$. Then,

$$n^{1-\tau}|v_n^*|^2 \leq (n-m)|v_n^*|^2 \leq \sum_{j=m+1}^{n} |v_j^*|^2 \leq \|v - P_m(v)\|^2 \leq \mathrm{e}^{-\alpha m^\tau} \|v\|^2_{\mathcal{A}_G^{\eta,t}} .$$

Now, by Taylor expansion, $m^\tau \sim n^\tau(1-n^{-\tau})^\tau = n^\tau\left(1 - \tau n^{-\tau} + o(n^{-\tau})\right) = n^\tau - \tau + o(1)$, so that $\mathrm{e}^{-\alpha m^\tau} \lesssim \mathrm{e}^{-\alpha n^\tau}$, and $\|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{Z}^d)} \lesssim \|v\|_{\mathcal{A}_G^{\eta,t}}$ is proven. ∎

*Remark* 4.2 ($\ell_G^{\eta,t}(\mathbb{Z}^d)$ is not a vector space). It may happen that $\mathbf{u}$, $\mathbf{v}$ belong to $\ell_G^{\eta,t}(\mathbb{Z}^d)$, whereas $\mathbf{u}+\mathbf{v}$ does not. Assume for simplicity that $\tau = 1$ and consider for instance the sequences in $\ell_G^{\eta,t}(\mathbb{Z}^d)$:

$$\mathbf{u} = \left(\mathrm{e}^{-\eta}, 0, \mathrm{e}^{-2\eta}, 0, \mathrm{e}^{-3\eta}, 0, \mathrm{e}^{-4\eta}, 0, \dots\right), \quad \mathbf{v} = \left(0, \mathrm{e}^{-\eta}, 0, \mathrm{e}^{-2\eta}, 0, \mathrm{e}^{-3\eta}, 0, \mathrm{e}^{-4\eta}, \dots\right).$$

Then,

$$\mathbf{u}+\mathbf{v} = (\mathbf{u}+\mathbf{v})^* = \left(\mathrm{e}^{-\eta}, \mathrm{e}^{-\eta}, \mathrm{e}^{-2\eta}, \mathrm{e}^{-2\eta}, \mathrm{e}^{-3\eta}, \mathrm{e}^{-3\eta}, \mathrm{e}^{-4\eta}, \mathrm{e}^{-4\eta}, \dots\right) ;$$

thus, $(\mathbf{u}+\mathbf{v})^*_{2j} = \mathrm{e}^{-\eta j}$, so that $\mathrm{e}^{\eta 2j}(\mathbf{u}+\mathbf{v})^*_{2j} \to \infty$ as $j \to +\infty$, i.e., $\mathbf{u}+\mathbf{v} \notin \ell_G^{\eta,t}(\mathbb{Z}^d)$.

This simple example indicates that the exponential case is quite different from, and much more delicate than, the algebraic case for which $\ell_B^s$ is a vector space.

On the other hand, we have the following property.

**Lemma 4.1** (quasi-triangle inequality)**.** *If* $\mathbf{u}_i \in \ell_G^{\eta_i,t}(\mathbb{Z}^d)$ *for* $i = 1, 2$, *then* $\mathbf{u}_1 + \mathbf{u}_2 \in \ell_G^{\eta,t}(\mathbb{Z}^d)$ *with*

$$\|\mathbf{u}_1 + \mathbf{u}_2\|_{\ell_G^{\eta,t}} \leq \|\mathbf{u}_1\|_{\ell_G^{\eta_1,t}} + \|\mathbf{u}_2\|_{\ell_G^{\eta_2,t}}, \qquad \eta^{-\frac{1}{\tau}} = \eta_1^{-\frac{1}{\tau}} + \eta_2^{-\frac{1}{\tau}}.$$

*Proof.* We use the characterization given by Proposition 4.2, so that

$$\|u_i - P_{N_i}(u_i)\| \leq \|u_i\|_{\mathcal{A}_G^{\eta,t}} \exp\left(-\eta\omega_d^{-\tau} N_i^\tau\right) \quad i = 1, 2 .$$

Given $N \geq 1$, we seek $N_1, N_2$ so that $N = N_1 + N_2$ and $\eta_1 N_1^\tau = \eta_2 N_2^\tau$. This implies that $N = N_1 \eta_1^{\frac{1}{\tau}}\left(\eta_1^{-\frac{1}{\tau}} + \eta_2^{-\frac{1}{\tau}}\right) = N_1 \eta_1^{\frac{1}{\tau}} \eta^{-\frac{1}{\tau}}$ holds, and the assertion follows from

$$\begin{aligned}\|(u_1+u_2) - P_N(u_1+u_2)\| &\leq \|u_1 - P_{N_1}(u_1)\| + \|u_2 - P_{N_2}(u_2)\| \\ &\leq \|u_1\|_{\mathcal{A}_G^{\eta_1,t}} \exp(-\eta_1\omega_d^{-\tau} N_1^\tau) + |u_2|_{\mathcal{A}_G^{\eta_2,t}} \exp(-\eta_2\omega_d^{-\tau} N_2^\tau)) \\ &\leq \left(\|u_1\|_{\mathcal{A}_G^{\eta_1,t}} + \|u_2\|_{\mathcal{A}_G^{\eta_2,t}}\right) \exp(-\eta\omega_d^{-\tau} N^\tau).\end{aligned}$$

∎

Note that when $\eta_1 = \eta_2$ we obtain $\eta = 2^{-\tau}\eta_1 \geq 2^{-1}\eta_1$ thereby extending the previous counterexample. Lemma 4.1 reveals that the matrix-vector product $\mathbf{Av}$ will in general be less sparse than $\mathbf{v}$ even for $\mathbf{A} \in \mathcal{D}_e(\eta_L)$ with $\eta_L > d$. We will discuss this critical issue in Section 5.

4.4. **Coarsening.** The two algorithms **FC-ADFOUR** and **FPC-ADFOUR** introduced in Section 3.4 incorporate a coarsening step. In view of the analysis of their complexity, we recall the following general result (see [7, 8, 23]), which we state in the abstract framework of Section 4.1 and prove for completeness.

**Proposition 4.3** (coarsening). *Let $\varepsilon > 0$ and let $v \in \mathcal{A}_\phi$ and $w \in V$ be such that*

$$\|v - w\| \leq \varepsilon.$$

*Let $N = N(\varepsilon)$ be the smallest integer such that the best $N$-term approximation $w_N$ of $w$ satisfies*

$$\|w - w_N\| \leq 2\varepsilon.$$

*Then, $\|v - w_N\| \leq 3\varepsilon$ and*

$$N \leq \phi^{-1}\left(\frac{\varepsilon}{\|v\|_{\mathcal{A}_\phi}}\right) + 1 \, . \tag{4.12}$$

*In addition, $w_N \in \mathcal{A}_\phi$ and*

$$\|w_N\|_{\mathcal{A}_\phi} \leq 4\|v\|_{\mathcal{A}_\phi} \, . \tag{4.13}$$

*Proof.* Let $\Lambda_\varepsilon$ be the set of indices corresponding to a best approximation of $v$ with accuracy $\varepsilon$. So $\Lambda_\varepsilon$ is a minimal set with properties

$$\|v - P_{\Lambda_\varepsilon} v\| \leq \varepsilon, \qquad |\Lambda_\varepsilon| \leq \phi^{-1}\left(\frac{\varepsilon}{\|v\|_{\mathcal{A}_\phi}}\right) + 1 \, .$$

If $z = w - v$, then $\|w - P_{\Lambda_\varepsilon} w\| = \|(v - P_{\Lambda_\varepsilon} v) + (z - P_{\Lambda_\varepsilon} z)\|$, whence

$$\|w - P_{\Lambda_\varepsilon} w\| \leq \|v - P_{\Lambda_\varepsilon} v\| + \|z - P_{\Lambda_\varepsilon} z\| \leq \varepsilon + \|z\| \leq 2\varepsilon \, ,$$

because $I - P_{\Lambda_\varepsilon}$ is the projector onto $V_{\mathbb{Z}^d \setminus \Lambda_\varepsilon}$. Since $N$ is the cardinality of the smallest set satisfying the above relation, we deduce that $N \leq |\Lambda_\varepsilon|$. This concludes the proof of (4.12). In order to obtain (4.13) we use (4.12) and the monotonicity of $\phi^{-1}$ to get

$$N \leq \phi^{-1}\left(\frac{\|v - w_N\|}{3\|v\|_{\mathcal{A}_\phi}}\right) + 1 \, ,$$

and thus deduce (4.12) upon applying Property 4.1. ∎

## 5. Sparsity classes for the range of $L$

The feasible algorithms introduced in Section 3 incorporate calls to the modules **F-RHS** and **F-APPLY**, which build suitable *finite* approximations of the right-hand side $f$ of (2.3) as well as of the image $Lv$ of a function $v$ with finite Fourier expansion. Therefore, in the complexity analysis of these modules, and consequently of the module **F-RES**, it is crucial to investigate the relation between the sparsity classes of the image $Lv$ and the argument $v$. In view of the striking difference between the algebraic and the exponential case, we present a separate study below with emphasis on the latter.

5.1. **Algebraic case.** The following result shows that $f = Lu$ is in the same sparsity class of $u$.

**Proposition 5.1** (continuity of $L$ in $\mathcal{A}^s_B$). *Let $\mathbf{A} \in \mathcal{D}_a(\eta_L)$, $\eta_L > d$ and $0 < s < \eta_L - d$. If $v \in \mathcal{A}^s_B$, then $Lv \in \mathcal{A}^s_B$, with*

$$\|Lv\|_{\mathcal{A}^s_B} \lesssim \|v\|_{\mathcal{A}^s_B} \, .$$

*The constant hidden in this bound goes to infinity as $s$ approaches $\eta_L - d$.*

The proof of Proposition 5.1 is well known and thus omitted [8, 10, 11, 23]. It uses that $\mathbf{A} \in \mathcal{D}_a(\eta_L)$ is $s^*$-compressible, with $s^* = \eta_L - d$, a concept developed in the wavelet context [8], [10, Lemma 3.6]. This technique is properly modified below for proving Proposition 5.2.

5.2. **Exponential case.** In striking contrast to the previous algebraic case, the implication $v \in \mathcal{A}_G^{\eta,t} \Rightarrow Lv \in \mathcal{A}_G^{\eta,t}$ is false. The following counterexamples prove this fact, and shed light on what could be the correct implication.

**Example 5.1** (Banded matrices)**.** Fix $d = 1$ and $t = 1$ (hence, $\tau = \frac{t}{d} = 1$). Recalling the expression (2.14) for the entries of $\mathbf{A}$, let us choose $\hat{\nu}_0 = \hat{\sigma}_0 = \sqrt{2\pi}$, which gives

$$a_{\ell,\ell} = 1, \qquad \forall\, \ell \in \mathbb{Z}.$$

Next, let us choose $\hat{\sigma}_h = 0$ for all $h \neq 0$, which implies (because $d = 1$)

$$|a_{\ell,k}| = \frac{1}{\sqrt{2\pi}} \frac{|\ell|\,|k|}{c_\ell\, c_k} |\hat{\nu}_{\ell-k}|\,, \qquad \ell \neq k\,,$$

i.e.,

$$\frac{1}{2\sqrt{2\pi}}\,|\hat{\nu}_{\ell-k}| \leq |a_{\ell,k}| \leq \frac{1}{\sqrt{2\pi}}\,|\hat{\nu}_{\ell-k}|\,, \qquad \ell \neq k\,,\ \ |\ell|, |k| \geq 1\,.$$

At this point, let us fix a real $\eta_L > 0$ and an integer $p \geq 0$, and let us choose the coefficients $\hat{\nu}_h$ for $h \neq 0$ to satisfy

$$|\hat{\nu}_h| = \begin{cases} \sqrt{2\pi}\mathrm{e}^{-\eta_L|h|} & \text{if } 0 < |h| \leq p\,, \\ 0 & \text{if } |h| > p\,. \end{cases}$$

In summary, the coefficient $\nu$ of the elliptic operator $L$ is a trigonometric polynomial of degree $p$, whereas the coefficient $\sigma$ is a constant. The corresponding stiffness matrix $\mathbf{A}$ is banded with $2p + 1$ nonzero diagonals, and satisfies

$$\tag{5.1} \tfrac{1}{2}\mathrm{e}^{-\eta_L|\ell-k|} \leq |a_{\ell,k}| \leq \mathrm{e}^{-\eta_L|\ell-k|}\,, \qquad 0 \leq |\ell - k| \leq p\,,\ \ |\ell|, |k| \geq 1\,.$$

In order to define the vector $\mathbf{v}$, let us introduce the function $\iota : \mathbb{N}_* \to \mathbb{N}_*$, $\iota(n) = 2(p+1)n$. Let us fix a real $\eta > 0$ and let us define the components $(\mathbf{v})_k = \hat{v}_k$ of the vector in such a way that

$$|(\mathbf{v})_k| = \begin{cases} \mathrm{e}^{-\frac{\eta}{2}n} & \text{if } k = \iota(n) \text{ for some } n \geq 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Thus, the rearranged components $(\mathbf{v})_n^*$ satisfy $|(\mathbf{v})_n^*| = \mathrm{e}^{-\frac{\eta}{2}n}$, $n \geq 1$, whence $\mathbf{v} \in \ell_G^{\eta,1}(\mathbb{Z})$ (or, equivalently, $v \in \mathcal{A}_G^{\eta,1}$), with $\|\mathbf{v}\|_{\ell_G^{\eta,1}(\mathbb{Z})} = 1$, according to Definition 4.4.

The definition of the mapping $\iota$ and the banded structure of $\mathbf{A}$ imply that the only nonzero components of $\mathbf{Av}$ are those of indices $\iota(n) + q$ for some $n \geq 1$ and $q \in [-p, p]$. For these components one has

$$(\mathbf{Av})_{\iota(n)+q} = a_{\iota(n)+q,\iota(n)}(\mathbf{v})_{\iota(n)}\,,$$

thus, recalling (5.1), we easily obtain

$$\tag{5.2} \tfrac{1}{2}\mathrm{e}^{-\eta_L p}\mathrm{e}^{-\frac{\eta}{2}n} \leq |(\mathbf{Av})_{\iota(n)+q}| \leq \mathrm{e}^{-\frac{\eta}{2}n}\,, \qquad q \in [-p, p]\,.$$

This shows that, for any integer $N \geq 1$,

$$\#\{\ell : |(\mathbf{Av})_\ell| \geq \tfrac{1}{2}\mathrm{e}^{-\eta_L p}\mathrm{e}^{-\frac{\eta}{2}N}\} \geq (2p+1)N\,,$$

hence

$$|(\mathbf{A}\mathbf{v})^*_{(2p+1)N}|\,\mathrm{e}^{\frac{\eta}{2}(2p+1)N} \geq \tfrac{1}{2}\mathrm{e}^{-\eta_L p}\mathrm{e}^{\eta p N} \to +\infty \qquad \text{as } N \to +\infty\ ,$$

i.e., $\mathbf{A}\mathbf{v} \notin \ell^{\eta,1}_G(\mathbb{Z})$ (or, equivalently, $Lv \notin \mathcal{A}^{\eta,1}_G$) regardless of the relative values of $\eta_L$ and $\eta$.

On the other hand, let $m_p$ be the smallest integer such that $\frac{1}{2}\mathrm{e}^{-\eta_L p} > \mathrm{e}^{-\frac{\eta}{2}m_p}$. Given any $m \geq 1$, let $N \geq 1$ and $Q \in [-p,p]$ be such that $(\mathbf{A}\mathbf{v})^*_m = (\mathbf{A}\mathbf{v})_{\iota(N)+Q}$, which combined with (5.2) yields

$$\mathrm{e}^{-\frac{\eta}{2}(N+m_p)} < |(\mathbf{A}\mathbf{v})^*_m| \leq \mathrm{e}^{-\frac{\eta}{2}N}\ .$$

The rightmost inequality in (5.2), namely $|(\mathbf{A}\mathbf{v})_{\iota(N+m_p)+q}| \leq e^{-\frac{\eta}{2}(N+m_p)}$, shows that there are at most $(2p+1)(N+m_p)$ components of $\mathbf{A}\mathbf{v}$ that are larger than $\mathrm{e}^{-\frac{\eta}{2}(N+m_p)}$ in modulus. This implies $m \leq (2p+1)(N+m_p)$, whence

$$\mathrm{e}^{-\frac{\eta}{2}N} \leq \mathrm{e}^{\frac{\eta}{2}m_p}\mathrm{e}^{-\frac{\eta}{2(2p+1)}m}\ .$$

Setting $\bar\eta = \frac{\eta}{2p+1}$, we conclude that $\mathbf{A}\mathbf{v} \in \ell^{\bar\eta,1}_G(\mathbb{Z})$ (or, equivalently, $Lv \in \mathcal{A}^{\bar\eta,1}_G$), with

$$\|\mathbf{A}\mathbf{v}\|_{\ell^{\bar\eta,1}_G(\mathbb{Z})} \leq \mathrm{e}^{\frac{\eta}{2}m_p}\|\mathbf{v}\|_{\ell^{\eta,1}_G(\mathbb{Z})}\ .$$

Therefore, *the sparsity class of* $\mathbf{A}\mathbf{v}$ *deteriorates from* $\ell^{\eta,1}_G(\mathbb{Z})$ *for* $\mathbf{v}$ *to* $\ell^{\bar\eta,1}_G(\mathbb{Z})$ *with* $\bar\eta = \frac{\eta}{2p+1}$. ∎

The next counterexample shows that, when the stiffness matrix $\mathbf{A}$ is not banded, in order to have $\mathbf{A}\mathbf{v} \in \ell^{\bar\eta,\bar t}_G(\mathbb{Z})$ it is not enough to choose some $\bar\eta < \eta$ as above, but a choice of $\bar t < t$ is mandatory.

**Example 5.2** (Dense matrices)**.** Let us take again $d = t = 1$ and modify the setting of the previous example, by assuming now that the coefficients $\hat\nu_h$ satisfy

$$|\hat\nu_h| = \sqrt{2\pi}\mathrm{e}^{-\eta_L|h|}, \qquad \text{for all } |h| > 0\ ,$$

so that $\mathbf{A}$ is no longer banded, and its elements satisfy

$$\tfrac{1}{2}\mathrm{e}^{-\eta_L|\ell-k|} \leq |a_{\ell,k}| \leq \mathrm{e}^{-\eta_L|\ell-k|}, \qquad \text{for all } |\ell|, |k| \geq 1\ . \tag{5.3}$$

If $M > 0$ is an arbitrary integer, we now construct a vector $\mathbf{v}^M = \sum_{n\geq 1}\mathbf{v}^{M,n}$ with gaps of size $\lambda(M) \geq M$ between consecutive nonvanishing entries. To this end, we introduce the function $\iota_M : \mathbb{N}_* \to \mathbb{N}_*$ defined as $\iota_M(n) := \lambda(M)n$ and the vectors $\mathbf{v}^{M,n}$ with components

$$|(\mathbf{v}^{M,n})_k| = \mathrm{e}^{-\frac{\eta}{2}n}\delta_{k,\iota_M(n)}\ , \qquad k \in \mathbb{Z}\ .$$

From (5.3) and the fact that only the $\iota_M(n)$-th entry of $\mathbf{v}^{M,n}$ does not vanish, we obtain

$$\tfrac{1}{2}\mathrm{e}^{-\eta_L|\ell-\iota_M(n)|}\mathrm{e}^{-\frac{\eta}{2}n} \leq |(\mathbf{A}\mathbf{v}^{M,n})_\ell| \leq \mathrm{e}^{-\eta_L|\ell-\iota_M(n)|}\mathrm{e}^{-\frac{\eta}{2}n}\ . \tag{5.4}$$

As in Example 5.1, it is obvious that $\mathbf{v}^M \in \ell^{\eta,1}_G(\mathbb{Z})$ with $\|\mathbf{v}^M\|_{\ell^{\eta,1}_G(\mathbb{Z})} = 1$. However, we will prove below that $\|\mathbf{A}\mathbf{v}^M\|_{\ell^{\bar\eta,\bar t}_G} \lesssim \|\mathbf{v}^M\|_{\ell^{\eta,1}_G}$ *cannot hold uniformly in* $M$ *for any* $\bar\eta > 0$ *and* $\bar t > 1/2$.

We start by examining the cardinality $\#\mathcal{F}_n$ of the set

$$\mathcal{F}_n := \{\ell \in \mathbb{Z} : |(\mathbf{A}\mathbf{v}^{M,n})_\ell| > \mathrm{e}^{-\frac{\eta}{2}M}\}.$$

In view of (5.4), the condition $|(\mathbf{A}\mathbf{v}^{M,n})_\ell| > \mathrm{e}^{-\frac{\eta}{2}M}$ is satisfied by those $\ell = \iota_M(n) + m$ such that

$$0 \le |m| \le \frac{\eta}{2\eta_L}(M-n) - \frac{1}{\eta_L} \;,$$

whence $n \le M - \frac{2}{\eta} < M$ and $\#\mathcal{F}_n \ge \frac{\eta}{\eta_L}(M-n) + 1 - \frac{2}{\eta_L}$. We now claim that

$$C_M := \#\{\ell \,:\, |(\mathbf{A}\mathbf{v}^M)_\ell| \ge \mathrm{e}^{-\frac{\eta}{2}M}\} \ge \sum_{n=1}^{M} \#\mathcal{F}_n \;, \tag{5.5}$$

whose proof we postpone. Assuming (5.5) we see that

$$C_M \ge \sum_{n=1}^{M}\left(\frac{\eta}{\eta_L}(M-n) + 1 - \frac{2}{\eta_L}\right) \sim \frac{\eta}{2\eta_L}M^2$$

or, equivalently, there are about $N_M = \left\lceil \frac{\eta}{2\eta_L}M^2 \right\rceil$ coefficients of $\mathbf{v}^M$ with values at least $\mathrm{e}^{-\frac{\eta}{2}M}$. This implies that the $N_M$-th rearranged coefficient of $\mathbf{A}\mathbf{v}^M$ satisfies

$$|(\mathbf{A}\mathbf{v}^M)^*_{N_M}| \ge \mathrm{e}^{-\frac{\eta}{2}M} \ge \mathrm{e}^{-\frac{1}{2}(2\eta_L\eta)^{1/2}N_M^{1/2}}, \qquad \text{for all } M \ge 1 \;.$$

This proves that for any $\bar\eta > 0$ and $\bar t > \frac{1}{2}$, one has

$$\|\mathbf{A}\mathbf{v}^M\|_{\ell_G^{\bar\eta,\bar t}(\mathbb{Z})} \ge |(\mathbf{A}\mathbf{v}^M)^*_{N_M}|\,\mathrm{e}^{\frac{\bar\eta}{2}N_M^{\bar t}} \ge \mathrm{e}^{\frac{\bar\eta}{2}N_M^{\bar t} - \frac{1}{2}(2\eta_L\eta)^{1/2}N_M^{1/2}} \to +\infty \qquad \text{as } M \to \infty \;,$$

whence the following bound cannot be valid

$$\|\mathbf{A}\mathbf{v}\|_{\ell_G^{\bar\eta,t}(\mathbb{Z})} \lesssim \|\mathbf{v}\|_{\ell_G^{\eta,1}(\mathbb{Z})} \;, \qquad \text{for all } \mathbf{v} \in \ell_G^{\eta,1}(\mathbb{Z}) \;.$$

It remains to prove (5.5). We first note that the sets $\mathcal{F}_n$ are disjoint provided $\iota_M(n+1) - \iota_M(n) = \lambda(M) \ge \frac{\eta}{\eta_L}M$. Next, we set

$$\varepsilon_M := \min_{1\le n\le M}\,\min_{\ell\in\mathcal{F}_n} |(\mathbf{A}\mathbf{v}^{M,n})_\ell| - \mathrm{e}^{-\frac{\eta}{2}M} > 0,$$

which is a constant only dependent on $M$. We observe that for every $\ell \in \mathcal{F}_n$, it holds that

$$\begin{aligned} |(\mathbf{A}\mathbf{v}^M)_\ell| \ge&\ |(\mathbf{A}\mathbf{v}^{M,n})_\ell| - |\sum_{p\ne n}(\mathbf{A}\mathbf{v}^{M,p})_\ell| \\ \ge&\ \mathrm{e}^{-\frac{\eta}{2}M} + \varepsilon_M - \sum_{p\ne n}|(\mathbf{A}\mathbf{v}^{M,p})_\ell|. \end{aligned} \tag{5.6}$$

We write $\ell \in \mathcal{F}_n$ as $\ell = \iota_M(n) + m$, make use of (5.4) and the definition of $\iota_M(n) = \lambda(M)n$ to deduce

$$\begin{aligned} \sum_{p\ne n}|(\mathbf{A}\mathbf{v}^{M,p})_\ell| \le&\ \sum_{p\ne n} e^{-\eta_L|\ell-\iota_M(p)|}e^{-\frac{\eta}{2}p} \le \sum_{p\ne n} e^{-\eta_L|m+\lambda(M)(n-p)|} \\ \le&\ \sum_{p\ne n} e^{-\eta_L(\lambda(M)|n-p|-|m|)}. \end{aligned}$$

Since $|m| \le \frac{\eta}{2\eta_L}M$, the above inequality gives

$$\sum_{p\ne n}|(\mathbf{A}\mathbf{v}^{M,p})_\ell| \le 2e^{\eta_L|m|}\sum_{q\ge1} e^{-\eta_L\lambda(M)q} \le 2e^{\frac{\eta}{2}M}\sum_{q\ge1} e^{-\eta_L\lambda(M)q} \;. \tag{5.7}$$

Combining (5.6) and (5.7) yields

$$|(\mathbf{A}\mathbf{v}^M)_\ell| \geq \mathrm{e}^{-\frac{\eta}{2}M} + \varepsilon_M - 2e^{\frac{\eta}{2}M} \sum_{q\geq 1} e^{-\eta_L \lambda(M) q} .$$

By choosing $\lambda(M)$ sufficiently large, the last term on the right-hand side of the above inequality can be made arbitrarily small, in particular, $\leq \varepsilon_M$. We thus get $|(\mathbf{A}\mathbf{v}^M)_\ell| \geq \mathrm{e}^{-\frac{\eta}{2}M}$ and prove (5.5). ◻

Guided by Examples 5.1 and 5.2, we are ready to state the main result of this section. We define

$$\zeta(t) := \left( \frac{1+t}{2^d\,\omega_d^{1+t}} \right)^{\frac{t}{d(1+t)}}, \qquad \forall\, 0 < t \leq d. \tag{5.8}$$

**Proposition 5.2** (image of $\mathcal{A}_G^{\eta,t}$ under $L$)**.** *Let $v \in \mathcal{A}_G^{\eta,t}$ for some $\eta > 0$ and $t \in (0, d]$. Let one of the two following sets of conditions be satisfied for the stiffness matrix $\mathbf{A}$ associated to the operator $L$:*

(a) *If $\mathbf{A}$ is banded with $2p+1$ nonzero diagonals, let us set*

$$\bar\eta = \frac{\eta}{(2p+1)^\tau}, \qquad \bar t = t .$$

(b) *If $\mathbf{A} \in \mathcal{D}_e(\eta_L)$ is dense, but the coefficients $\eta_L$ and $\eta$ satisfy the inequality $\eta < \eta_L \omega_d^\tau$, let us set*

$$\bar\eta = \zeta(t)\eta , \qquad \bar t = \frac{t}{1+t} .$$

*Then, there exists a constant $C_r \geq 1$ such that $Lv \in \mathcal{A}_G^{\bar\eta,\bar t}$ with*

$$\|Lv\|_{\mathcal{A}_G^{\bar\eta,\bar t}} \leq C_r \|v\|_{\mathcal{A}_G^{\eta,t}} . \tag{5.9}$$

*Proof.* We adapt to our situation the technique introduced in [8]. Let $L_J$ $(J \geq 0)$ be the differential operator obtained by truncating the Fourier expansion of the coefficients of $L$ to the modes $k$ satisfying $|k| \leq J$. Equivalently, $L_J$ is the operator whose stiffness matrix $\mathbf{A}_J$ is defined in (2.22); thus, by Property 2.4 (exponential case) we have

$$\|L - L_J\| = \|\mathbf{A} - \mathbf{A}_J\| \leq C_{\mathbf{A}} (J+1)^{d-1} \mathrm{e}^{-\eta_L J} .$$

On the other hand, for any $j \geq 1$, let $v_j = P_j(v)$ be a best $j$-term approximation of $v$ (with $v_0 = 0$), which therefore satisfies $\|v - v_j\| \leq \mathrm{e}^{-\eta \omega_d^{-\tau} j^\tau} \|v\|_{\mathcal{A}_G^{\eta,t}}$, with $\tau = t/d$. Note that the difference $v_j - v_{j-1}$ consists of a single Fourier mode and also satisfies

$$\|v_j - v_{j-1}\| \lesssim \mathrm{e}^{-\eta \omega_d^{-\tau} j^\tau} \|v\|_{\mathcal{A}_G^{\eta,t}} .$$

Finally, let us introduce the function $\chi : \mathbb{N} \to \mathbb{N}$ defined as $\chi(j) = \lceil j^\tau \rceil$, the smallest integer larger than or equal to $j^\tau$.

For any $J \geq 1$, let $w_J$ be the approximation of $Lv$ defined as

$$w_J = \sum_{j=1}^{J} L_{\chi(J-j)} (v_j - v_{j-1}) . \tag{5.10}$$

Writing $v = v - v_J + \sum_{j=1}^J (v_j - v_{j-1})$, we obtain

$$Lv - w_J = L(v - v_J) + \sum_{j=1}^{J} (L - L_{\chi(J-j)})(v_j - v_{j-1}) \; .$$

We now assume to be in case (b). Since $L : V \to V'$ is continuous, the last equation yields

(5.11)

$$\|Lv - w_J\| \lesssim \left( \mathrm{e}^{-\eta\omega_d^{-\tau} J^\tau} + \sum_{j=1}^{J} \left( \lceil (J-j)^\tau \rceil + 1 \right)^{d-1} \mathrm{e}^{-(\eta_L \lceil (J-j)^\tau \rceil + \eta\omega_d^{-\tau} j^\tau)} \right) \|v\|_{\mathcal{A}_G^{\eta,t}} \; .$$

The exponents of the addends can be bounded from below as follows because $\tau \leq 1$:

$$\begin{aligned} \eta_L \lceil (J-j)^\tau \rceil + \eta\omega_d^{-\tau} j^\tau &= \eta_L \lceil (J-j)^\tau \rceil - \eta\omega_d^{-\tau} (J-j)^\tau + \eta\omega_d^{-\tau} ((J-j)^\tau + j^\tau) \\ &\geq \eta_L (J-j)^\tau - \eta\omega_d^{-\tau} (J-j)^\tau + \eta\omega_d^{-\tau} ((J-j) + j)^\tau \\ &= \beta (J-j)^\tau + \eta\omega_d^{-\tau} J^\tau \; , \end{aligned}$$

with $\beta = \eta_L - \eta\omega_d^{-\tau} > 0$ by assumption. Then, (5.11) yields

(5.12)

$$\|Lv - w_J\| \lesssim \left( 1 + \sum_{j=0}^{J-1} \left( \lceil j^\tau \rceil + 1 \right)^{d-1} \mathrm{e}^{-\beta j^\tau} \right) \mathrm{e}^{-\eta\omega_d^{-\tau} J^\tau} \|v\|_{\mathcal{A}_G^{\eta,t}} \lesssim \mathrm{e}^{-\eta\omega_d^{-\tau} J^\tau} \|v\|_{\mathcal{A}_G^{\eta,t}} \; .$$

On the other hand, by construction $w_J$ belongs to a finite-dimensional space $V_{\Lambda_J}$, where

(5.13)

$$|\Lambda_J| \leq \omega_d \sum_{j=1}^{J} \chi(J-j)^d = \omega_d \sum_{j=0}^{J-1} \lceil j^\tau \rceil^d \leq 2^d \omega_d \sum_{j=0}^{J-1} j^t \leq \frac{2^d \omega_d}{1+t} J^{1+t}, \qquad \forall\; J \geq 1 \; .$$

This implies

$$\|Lv - w_J\| \leq C_r \mathrm{e}^{-\bar\eta\omega_d^{-\bar\tau} |\Lambda_J|^{\bar\tau}} \|v\|_{\mathcal{A}_G^{\eta,t}} \; ,$$

with $\bar\tau = \frac{\tau}{1+d\tau} = \frac{t}{d(1+t)}$ and $\bar\eta = \left( \frac{1+d\tau}{2^d \omega_d^{1+d\tau}} \right)^{\bar\tau} \eta = \zeta(t)\eta$ as asserted.

Finally, we consider case (a). One has $L_{\chi(J-j)} = L$ if $\chi(J-j) \geq p$, whence if $j \leq J - p^{1/\tau}$, then the summation in (5.11) can be limited to those $j$ satisfying $j_p \leq j \leq J$, where $j_p = \lceil J - p^{1/\tau} \rceil$. Therefore,

$$\|Lv - w_J\| \lesssim \left( \mathrm{e}^{-\eta\omega_d^{-\tau} J^\tau} + \max_{j_p \leq j \leq J} \lceil (J-j)^\tau \rceil^{d-1} \sum_{j=j_p}^{J} \mathrm{e}^{-\eta\omega_d^{-\tau} j^\tau} \right) \|v\|_{\mathcal{A}_G^{\eta,t}} \; .$$

Now, $J - j \leq p^{1/\tau}$ if $j_p \leq j \leq J$ and $j^\tau \geq j_p^\tau \geq (J - p^{1/\tau})^\tau \geq J^\tau - p$, whence

$$\|Lv - w_J\| \lesssim \left( 1 + p^{d-1} \mathrm{e}^{\eta\omega_d^{-\tau} p} \right) \mathrm{e}^{-\eta\omega_d^{-\tau} J^\tau} \|v\|_{\mathcal{A}_G^{\eta,t}} \; .$$

We conclude by observing that $|\Lambda_J| \leq (2p+1)J$, since any matrix $\mathbf{A}_J$ has at most $2p+1$ diagonals. ∎

We remark that the factor $2^d$ appearing in (5.8), which comes from the bound in (5.13), can be easily replaced by any number $\lambda > 1$, arbitrarily close to 1, with a constant $C_r$ in (5.9) depending on $\lambda$.

Furthermore, numerical evidence indicates that $\zeta(t) \le 1$ (i.e., $\bar{\eta} \le \eta$) for all $d \le 13$ in the range $0 < t \le d$ and for all $d \le 28$ in the range $0 < t \le 1$. These limitations on $d$ stem from the fact that the measure of the unit Euclidean ball $\omega_d$ in $\mathbb{R}^d$ monotonically decreases to 0 as $d \to \infty$.

## 6. Cardinality properties of the feasible residual

All feasible algorithms we deal with require the iterative computation of finite approximate residuals $\widetilde{r}$. Therefore, elucidating the overall complexity of such algorithms must account for the cardinality of $\widetilde{r}$.

On the other hand, one viable way for estimating the cardinality of the index sets $\partial\Lambda$ activated by the Dörfler procedure, but not the only one (see Proposition 7.1 below), is to observe that $\partial\Lambda := \textbf{DÖRFLER}(\widetilde{r}, \widetilde{\theta})$ chooses a best $N$-term approximation of $\widetilde{r}$. Consequently, $\partial\Lambda$ has minimal cardinality in $\Lambda^c$ and satisfies $\|\widetilde{r} - P_{\partial\Lambda}\widetilde{r}\| \le \sqrt{1-\theta^2}\|\widetilde{r}\|$. If $\widetilde{r}$ belongs to a certain sparsity class $\mathcal{A}_{\widetilde{\phi}}$, identified by a function $\widetilde{\phi}$, then according to (4.3) it obeys the expression

$$|\partial\Lambda| \le \widetilde{\phi}^{-1}\left(\sqrt{1-\theta^2}\,\frac{\|\widetilde{r}\|}{\|\widetilde{r}\|_{\mathcal{A}_{\widetilde{\phi}}}}\right) + 1\,. \tag{6.1}$$

For the reasons above, in this section we investigate both the cardinality and the sparsity class of $\widetilde{r}$.

6.1. **Algebraic case.** In light of Proposition 5.1, $L(\mathcal{A}^s_B) \subset \mathcal{A}^s_B$ provided $\mathbf{A} \in \mathcal{D}_a(\eta_L)$ and $0 < s < \eta_L - d$. Therefore, it makes sense to assume that the solution $u$ of (2.3) and the data $f = Lu$ belong to the same sparsity class $\mathcal{A}^s_B$. This in turn implies that the residual $r(v) = f - Lv \in \mathcal{A}^s_B$ whenever $v \in \mathcal{A}^s_B$. The following Proposition 6.1 establishes that the same is true for the feasible residual $\widetilde{r}(v)$. For its proof we refer, e.g., to [23] in the wavelet context or the arguments given below for the exponential case.

**Proposition 6.1** (sparsity class of $\widetilde{r}(v)$). *Let $\mathbf{A} \in \mathcal{D}_a(\eta_L)$ and $0 < s < \eta_L - d$. If $[\widetilde{r}(v), flag, \epsilon_1] =$ **F-RES**$(v, \gamma, \epsilon_0, \mathrm{tol})$ with $flag = 0$, then the following estimate holds:*

$$|\mathrm{supp}\ \widetilde{r}(v)| \lesssim \left(\frac{\|f\|_{\mathcal{A}^s_B} + \|v\|_{\mathcal{A}^s_B}}{\|\widetilde{r}(v)\|}\right)^{d/s}\,. \tag{6.2}$$

*Moreover, the output $\widetilde{r}(v)$ satisfies $\widetilde{r}(v) \in \mathcal{A}^s_B$ along with the estimate*

$$\|\widetilde{r}(v)\|_{\mathcal{A}^s_B} \lesssim \left(\|f\|_{\mathcal{A}^s_B} + \|v\|_{\mathcal{A}^s_B}\right)\,. \tag{6.3}$$

6.2. **Exponential case.** If $u \in \mathcal{A}^{\eta,t}_G$ for some $\eta > 0$ and $t \in (0, d]$, then the generic regularity of $f = Lu$ is $f \in \mathcal{A}^{\bar{\eta},\bar{t}}_G$, where the relation between $(\eta, t)$ and $(\bar{\eta}, \bar{t})$ is given in Proposition 5.2. As in [7,8,22,23], we assume to have complete knowledge about the data $f$ and coefficients $\nu$ and $\sigma$ of $L$, in the sense that we already know or can compute their Fourier coefficients to any desired accuracy.

**Lemma 6.1** (complexity of **F-RHS**). *If $f_\varepsilon :=$ **F-RHS**$(\varepsilon)$ is implemented via a greedy procedure, then*

$$|\mathrm{supp}\ f_\varepsilon| \le \frac{\omega_d}{\bar{\eta}^{d/\bar{t}}}\left(\log\frac{\|f\|_{\mathcal{A}^{\bar{\eta},\bar{t}}_G}}{\varepsilon}\right)^{d/\bar{t}} + 1, \qquad \|f_\varepsilon\|_{\mathcal{A}^{\bar{\eta},\bar{t}}_G} \lesssim \|f\|_{\mathcal{A}^{\bar{\eta},\bar{t}}_G}\,. \tag{6.4}$$

*Proof.* Combine (4.3) and (4.5). ∎

**Lemma 6.2** (complexity of **F-APPLY**). *Let $v \in \mathcal{A}_G^{\mu,t}$ for some $0 < \mu \leq \eta$, so that $Lv \in \mathcal{A}_G^{\bar\mu,\bar t}$ with $\bar\mu, \bar t$ defined from $\mu, t$ as in Proposition* 5.2. *Let $\widehat{C}_r = e^{\eta\omega^{-\tau}} C_r$ and $\breve{C}_r = \left(1 + e^{\eta\omega^{-\tau}}\right) C_r$ with $C_r$ the constant in Proposition* 5.2. *Given $\varepsilon > 0$, let $w_\varepsilon :=$ **F-APPLY**$(v, \varepsilon)$ be the smallest finite vector $w_\varepsilon$ implemented according to* (5.10) *and satisfying $\|Lv - w_\varepsilon\| \leq \varepsilon$. Then, the following estimates are valid:*

$$|\text{supp } w_\varepsilon| \leq \frac{\omega_d}{\bar\mu^{d/\bar t}} \left( \log \frac{\widehat{C}_r \|v\|_{\mathcal{A}_G^{\mu,t}}}{\varepsilon} \right)^{d/\bar t}, \qquad \|w_\varepsilon\|_{\mathcal{A}_G^{\bar\mu,\bar t}} \leq \breve{C}_r \|v\|_{\mathcal{A}_G^{\mu,t}} . \tag{6.5}$$

*Proof.* In view of (5.12) and the definition of $w_\varepsilon = w_{J_\varepsilon}$ we deduce

$$\|Lv - w_{J_\varepsilon}\| \leq \varepsilon < \|Lv - w_{J_\varepsilon - 1}\| \leq C_r e^{-\mu \omega_d^{-t/d} (J_\varepsilon - 1)^{t/d}} \|v\|_{\mathcal{A}_G^{\mu,t}} .$$

Therefore, arguing as in (5.13), we end up with

$$\begin{aligned} |\text{supp } w_\varepsilon| \leq \frac{2^d\, \omega_d}{1+t} J_\varepsilon^{1+t} \;&\leq\; \frac{2^d\, \omega_d^{2+t}}{(1+t)\mu^{(1+t)d/t}} \left( \left( \log \frac{C_r \|v\|_{\mathcal{A}_G^{\mu,t}}}{\varepsilon} \right)^{d/t} + 1 \right)^{1+t} \\ &\leq\; \frac{\omega_d}{\bar\mu^{d/\bar t}} \left( \log \frac{\widehat{C}_r \|v\|_{\mathcal{A}_G^{\mu,t}}}{\varepsilon} \right)^{d/\bar t} \end{aligned}$$

as asserted. The second estimate in (6.5) follows from the proof of Property 4.1 and (5.9). ∎

We are now ready to estimate the support and sparsity class for the feasible residual $\widetilde{r}$ of **F-RES**.

**Proposition 6.2** (sparsity class of $\widetilde{r}(v)$). *Let $0 < \mu \leq \eta$, $0 < t \leq d$ and let $0 < \bar\mu \leq \bar\eta, \bar t$ be defined as in Proposition* 5.2. *Let $f \in \mathcal{A}_G^{\bar\eta,\bar t}$, $v \in \mathcal{A}_G^{\mu,t}$, and let* **A** *satisfy either assumption* (a) *or* (b) *in Proposition* 5.2. *If $[\widetilde{r}(v), flag, \epsilon_1] =$* **F-RES**$(v, \gamma, \epsilon_0, \text{tol})$ *with $flag = 0$, then the following estimate holds:*

$$|\text{supp } \widetilde{r}(v)| \leq \frac{\omega_d}{\left(\frac{\bar\mu}{2}\right)^{d/\bar t}} \left( \log \frac{C_* \left( \|f\|_{\mathcal{A}_G^{\bar\eta,\bar t}} + \widehat{C}_r \|v\|_{\mathcal{A}_G^{\mu,t}} \right)}{\|\widetilde{r}(v)\|} \right)^{d/\bar t} + 1 , \tag{6.6}$$

*with a constant $C_*$ depending on $\gamma$ and $\widehat{C}_r$ the constant in Lemma* 6.2. *Moreover, $\widetilde{r}(v) \in \mathcal{A}_G^{\widetilde\mu,\bar t}$ with $\widetilde\mu$ satisfying $\widetilde\mu^{-d/\bar t} = \bar\mu^{-d/\bar t} + \bar\eta^{-d/\bar t}$, and*

$$\|\widetilde{r}(v)\|_{\mathcal{A}_G^{\widetilde\mu,\bar t}} \lesssim \left( \|f\|_{\mathcal{A}_G^{\bar\eta,\bar t}} + \|v\|_{\mathcal{A}_G^{\mu,t}} \right) . \tag{6.7}$$

*Proof.* Let $\xi$ be the final tolerance of **F-RES** satisfying the exit condition $\xi \leq \gamma \|\widetilde{r}_\xi(v)\|$. Setting $\varepsilon := \xi/2$ we have by construction $\widetilde{r}(v) = \widetilde{r}_\xi(v)$ and $|\text{supp } \widetilde{r}(v)| \leq |\text{supp } f_\varepsilon| + |\text{supp } w_\varepsilon|$, whence

$$|\text{supp } \widetilde{r}(v)| \leq \frac{\omega_d}{\bar\eta^{d/\bar t}} \left( \log \frac{\|f\|_{\mathcal{A}_G^{\bar\eta,\bar t}}}{\varepsilon} \right)^{d/\bar t} + \frac{\omega_d}{\bar\mu^{d/\bar t}} \left( \log \frac{\widehat{C}_r \|v\|_{\mathcal{A}_G^{\mu,t}}}{\varepsilon} \right)^{d/\bar t} + 1 ,$$

in accordance with (6.4) and (6.5). Since $\bar\mu \le \bar\eta$ and $a^s + b^s \le (a+b)^s$ for $a, b \ge 0$, provided $s \ge 1$, we obtain with $s = d/\bar t \ge 1$,

$$
\begin{aligned}
|\mathrm{supp}\ \widetilde{r}(v)| \ &\le\ \frac{\omega_d}{\bar\mu^{d/\bar t}} \left( \log \frac{\|f\|_{\mathcal{A}_G^{\bar\eta,\bar t}} \widehat{C}_r \|v\|_{\mathcal{A}_G^{\mu,t}}}{\varepsilon^2} \right)^{d/\bar t} + 1 \\
&\le\ \frac{\omega_d}{\left(\frac{\bar\mu}{2}\right)^{d/\bar t}} \left( \log \frac{\|f\|_{\mathcal{A}_G^{\bar\eta,\bar t}} + \widehat{C}_r \|v\|_{\mathcal{A}_G^{\mu,t}}}{\varepsilon} \right)^{d/\bar t} + 1\ .
\end{aligned}
$$

To relate $\varepsilon$ and $\|\widetilde{r}(v)\|$, we note that $\gamma\|\widetilde{r}_{2\xi}(v)\| < 2\xi$ because $2\xi$ is not the final tolerance of **F-RES**. Thus,

$$
\|r(v)\| \le \|r(v) - \widetilde{r}_{2\xi}(v)\| + \|\widetilde{r}_{2\xi}(v)\| \le 2\xi + \|\widetilde{r}_{2\xi}(v)\| \le (1 + 1/\gamma)2\xi,
$$

i.e., $\|r(v)\| \le 4(1 + 1/\gamma)\varepsilon$ from which (6.6) follows after using (3.10).

Finally, to derive (6.7) we resort to (6.4) and (6.5) as well as Lemma 4.1. ∎

It is instructive to observe that (6.6) and (6.7) entail different decay constants $\frac{\bar\mu}{2} \le 2^{\frac{\bar t}{d}-1}\widetilde{\mu} \le \widetilde{\mu}$. However, optimizing the choice of $\widetilde{\mu}$ in the current application of Lemma 4.1 would require knowledge of the sparsity classes of $f$ and $Lv$ in the design of both **F-RES** and **F-APPLY**, which is not practical and thus not assumed.

## 7. Cardinality of adaptive algorithms: algebraic case

The rest of the paper is devoted to investigating cardinality of the various active sets of indices created by the four feasible algorithms developed in Section 3, and their relation to corresponding sparsity classes.

We first deal with the algebraic case which, like wavelets [16, 23] and finite elements [2, 6, 22], does not require coarsening to achieve optimality. We extend this approach below to the Fourier setting.

### 7.1. F-ADFOUR: feasible ADFOUR with moderate Dörfler marking.

The two following lemmas, well known in the wavelet and finite element frameworks, will be useful in the subsequent analysis.

**Lemma 7.1** (localized a posteriori upper bound)**.** *Let $\Lambda \subset \Lambda_* \subset \mathbb{Z}^d$ be nonempty subsets of indices. Let $u_\Lambda \in V_\Lambda$ and $u_{\Lambda_*} \in V_{\Lambda_*}$ be the Galerkin approximations of Problem* (2.4)*. Then*

$$
|\!|\!| u_{\Lambda_*} - u_\Lambda |\!|\!|^2 \le \frac{1}{\alpha_*} \sum_{k \in \Lambda_* \setminus \Lambda} |\hat{R}_k(u_\Lambda)|^2 = \frac{1}{\alpha_*} \eta^2(u_\Lambda, \Lambda_*)\ .
$$

*Proof.* Use that the support of $u_{\Lambda_*} - u_\Lambda$ is precisely $\Lambda_* \setminus \Lambda$. ∎

**Lemma 7.2** (feasible Dörfler property)**.** *Let $\Lambda \subset \Lambda_* \subset \mathbb{Z}^d$ be nonempty subsets of indices. Let $u_\Lambda \in V_\Lambda$ and $u_{\Lambda_*} \in V_{\Lambda_*}$ be the Galerkin approximations of Problem* (2.4)*. Let the feasibility parameter $\gamma$ and marking parameter $\widetilde{\theta}$ satisfy*

$$
0 < \gamma < \gamma_* := \frac{\sqrt{\frac{\alpha_*}{\alpha^*}}}{1 + \sqrt{\frac{\alpha_*}{\alpha^*}}}, \qquad 0 < \widetilde{\theta} < \widetilde{\theta}_* := \sqrt{\frac{\alpha_*}{\alpha^*}}(1 - \gamma) - \gamma.
$$

*If* $0 < \mu \le \mu_* := 1 - \big(\frac{\widetilde{\theta}+\gamma}{1-\gamma}\big)^2 \frac{\alpha^*}{\alpha_*}$ *and* $\|u - u_{\Lambda_*}\|^2 \le \mu \|u - u_\Lambda\|^2$*, then* $\Lambda_*$ *fulfills the feasible Dörfer's condition* (3.16)*, i.e.,*

$$\widetilde{\eta}(u_\Lambda, \Lambda_*) \ge \widetilde{\theta}\widetilde{\eta}(u_\Lambda) .$$

*Proof.* We first point out that the definition of $\mu_*$ and condition $\mu_* > 0$ is consistent with the definitions of $\widetilde{\theta}_*, \gamma_*$ and conditions on $\widetilde{\theta}, \gamma$. Since $u - u_{\Lambda_*} \perp u_\Lambda - u_{\Lambda_*}$ in the energy norm because of Pythagoras equality, the assumption $\|u - u_{\Lambda_*}\|^2 \le \mu\|u - u_\Lambda\|^2$ yields

$$\|u - u_\Lambda\|^2 = \|u - u_{\Lambda_*}\|^2 + \|u_{\Lambda_*} - u_\Lambda\|^2 \le \mu\|u - u_\Lambda\|^2 + \|u_{\Lambda_*} - u_\Lambda\|^2 .$$

Invoking the lower bound in (2.9) gives $\|u_{\Lambda_*} - u_\Lambda\|^2 \ge (1-\mu)\frac{1}{\alpha^*}\eta^2(u_\Lambda)$. Recalling that $\mu \le \mu_*$ and applying Lemma 7.1 implies $\eta^2(u_\Lambda, \Lambda_*) \ge \theta^2\eta^2(u_\Lambda)$ with $\theta^2 = (1-\mu_*)\frac{\alpha_*}{\alpha^*} = \big(\widetilde{\theta}+\gamma\big)^2/\big(1-\gamma\big)^2$. This is (3.15) and $\widetilde{\theta} > 0$ is equivalent to $\gamma < \theta/(1+\theta)$. We thus deduce the assertion from Lemma 3.1. ∎

We are ready to estimate the growth of degrees of freedom generated by **F-ADFOUR**. We deal with the abstract framework of Section 4.1, with only Theorem 7.1 being specific to the algebraic case.

**Proposition 7.1** (cardinality of $\Lambda_n$: general case)**.** *Let* $\gamma \in (0, \gamma_*)$, $\widetilde{\theta} \in (0, \widetilde{\theta}_*)$ *and* $\widetilde{\mu} \in (0, \mu_*]$ *be fixed parameters, where* $\gamma_*$, $\widetilde{\theta}_*$ *and* $\mu_*$ *are defined in Lemma 7.2. Let* $\{\Lambda_n,\, u_n\}_{n\ge 0}$ *be the sequence generated by the adaptive algorithm* **F-ADFOUR***. If the solution* $u$ *belongs to the sparsity class* $\mathcal{A}_\phi$*, then*

$$|\partial\Lambda_n| = |\Lambda_{n+1}| - |\Lambda_n| \le \phi^{-1}\left(\frac{\sqrt{\mu}\|u - u_n\|}{\|u\|_{\mathcal{A}_\phi}}\right) + 1 , \qquad \forall\, n \ge 0 . \tag{7.1}$$

*Moreover, if* $\widetilde{\rho} = \sqrt{1 - \frac{\alpha_*}{\alpha^*}\big(\frac{\widetilde{\theta}-\gamma}{1+\gamma}\big)^2}$*, then*

$$|\Lambda_n| \le n + \sum_{k=0}^{n-1} \phi^{-1}\left(\widetilde{\rho}^{\,k-n}\,\frac{\sqrt{\mu}\|u - u_n\|}{\|u\|_{\mathcal{A}_\phi}}\right) , \qquad \forall\, n \ge 0 . \tag{7.2}$$

*Proof.* Let $\varepsilon = \sqrt{\mu}\|u - u_n\|$ and make use of (4.3) for $u \in \mathcal{A}_\phi$; there exists $\Lambda_\varepsilon$ and $w_\varepsilon \in V_{\Lambda_\varepsilon}$ such that

$$\|u - w_\varepsilon\|^2 \le \varepsilon^2 \qquad \text{and} \qquad |\Lambda_\varepsilon| \le \phi^{-1}\left(\frac{\varepsilon}{\|u\|_{\mathcal{A}_\phi}}\right) + 1.$$

Let $\Lambda_* = \Lambda_n \cup \Lambda_\epsilon$ and $u_* \in V_{\Lambda_*}$ be the Galerkin approximation of problem (2.4). Since $V_{\Lambda_\varepsilon} \subseteq V_{\Lambda_*}$, we have $\|u - u_*\|^2 \le \|u - w_\varepsilon\|^2 \le \mu\|u - u_n\|^2$ . We are thus entitled to apply Lemma 7.2 to $\Lambda_n$ and $\Lambda_*$, thereby yielding $\widetilde{\eta}(u_n, \Lambda^*) \ge \widetilde{\theta}\widetilde{\eta}(u_n)$. By the minimality property of $|\Lambda_{n+1}|$ among all sets satisfying the feasible Dörfler property for $u_n$ (Assumption 3.1), we deduce $|\Lambda_{n+1}| \le |\Lambda_*| \le |\Lambda_n| + |\Lambda_\epsilon|$ along with (7.1). Estimate (7.2) follows by adding $|\partial\Lambda_k|$, starting with $|\Lambda_0| = 0$, and using $\|u - u_n\| \le \widetilde{\rho}^{\,n-k}\|u - u_k\|$ which results from Theorem 3.2. ∎

**Lemma 7.3** (sparsity of $u_n$ and $\widetilde{r}(u_n)$: algebraic case)**.** *Let* $\mathbf{A} \in \mathcal{D}_a(\eta_L)$ *with* $\eta_L > d$*, and let* $u \in \mathcal{A}^s_B$ *for some* $0 < s < \eta_L - d$*. If* $u_n \in V_{\Lambda_n}$ *is the Galerkin solution and* $\widetilde{r}(u_n)$ *is the output of* **F-RES**$(u_n, \gamma, \epsilon_0, \mathrm{tol})$*, then it holds that*

$$\|u_n\|_{\mathcal{A}^s_B} \lesssim \|u\|_{\mathcal{A}^s_B}, \qquad \|\widetilde{r}(u_n)\|_{\mathcal{A}^s_B} \lesssim \|u\|_{\mathcal{A}^s_B} .$$

*Proof.* In order to bound $\|u_n\|_{\mathcal{A}^s_B}$, we invoke the equivalent formulation of the Galerkin problem given by (2.24), which yields $\widehat{\mathbf{u}}_n = (\widehat{\mathbf{A}}_{\Lambda_n})^{-1}(\mathbf{P}_{\Lambda_n}\mathbf{f})$. Using $\mathbf{A} \in \mathcal{D}_a(\eta_L)$, and combining Properties 2.5 and 2.2, we obtain $(\widehat{\mathbf{A}}_{\Lambda_n})^{-1} \in \mathcal{D}_a(\eta_L)$ uniformly in $n$. Applying Proposition 5.1 to $(\widehat{\mathbf{A}}_{\Lambda_n})^{-1}$ we thus get

$$\|\mathbf{u}_n\|_{\ell^s_B} = \|\widehat{\mathbf{u}}_n\|_{\ell^s_B} = \|(\widehat{\mathbf{A}}_{\Lambda_n})^{-1}(\mathbf{P}_{\Lambda_n}\mathbf{f})\|_{\ell^s_B} \lesssim \|\mathbf{P}_{\Lambda_n}\mathbf{f}\|_{\ell^s_B} \leq \|\mathbf{f}\|_{\ell^s_B} \lesssim \|\mathbf{u}\|_{\ell^s_B} ,$$

whence the first asserted estimate follows. The second one results from the bound $\|\widetilde{r}(u_n)\|_{\mathcal{A}^s_B} \lesssim \|u\|_{\mathcal{A}^s_B} + \|u_n\|_{\mathcal{A}^s_B}$ which is a consequence of Propositions 6.1 and 5.1. ∎

**Theorem 7.1** (cardinality of $\Lambda_n$ and supp $\widetilde{r}(u_n)$: algebraic case)**.** *Let $\gamma, \widetilde{\theta}, \mu$ and $\widetilde{\rho}$ be as in Proposition 7.1. Let $\mathbf{A} \in \mathcal{D}_a(\eta_L)$ with $\eta_L > d$, and let $u \in \mathcal{A}^s_B$ with $0 < s < \eta_L - d$. The active degrees of freedom produced by* **F-ADFOUR** *grow as follows:*

$$|\Lambda_n| \lesssim \|u - u_n\|^{-d/s}\|u\|^{d/s}_{\mathcal{A}^s_B} , \qquad \forall\, n \geq 0 ,$$

*where the hidden constant depends only on $\alpha_*$, $\mu$ and $\widetilde{\rho}$. Likewise, the feasible residual satisfies*

$$|\operatorname{supp} \widetilde{r}(u_n)| \lesssim \|u - u_n\|^{-d/s}\|u\|^{d/s}_{\mathcal{A}^s_B} , \qquad \forall\, n \geq 0 .$$

*Proof.* It suffices to use that $\phi^{-1}(\lambda) = \lambda^{-d/s} - 1$ in (7.2) to obtain

$$|\Lambda_n| \leq \mu^{-d/2s}\|\!|u - u_n|\!\|^{-d/s}\|u\|^{d/s}_{\mathcal{A}^s_B} \sum_{k=0}^{n-1} \left(\widetilde{\rho}^{d/s}\right)^{n-k} , \qquad \forall\, n \geq 0 .$$

Summing up the geometric series and employing (2.5), we arrive at the estimate for $|\Lambda_n|$. The remaining bound is a consequence of Proposition 6.1, Lemma 7.3 and (3.12). ∎

This result is *quasi-optimal* in that the number of active degrees of freedom is governed, up to a multiplicative constant, by the same law (4.3)–(4.6) as for the best approximation of $u$. The optimality of this result is related to the "sufficiently fast" growth of the active degrees of freedom: its increment at each interation is comparable to the total number of previously activated indices (geometric growth).

7.2. **FA-ADFOUR: the aggressive version of F-ADFOUR.** We now examine Algorithm **FA-ADFOUR**, defined in Section 3.3, which allows for the choice of the parameter $\widetilde{\theta}$ as close to 1 as desired. Such a feature is in the spirit of high regularity, or equivalently a large value of $s$ for $u \in \mathcal{A}^s_B$. This is a novel approach which combines the contraction property in Theorem 3.3 and the key property of uniform boundedness of the feasible residuals established in Lemma 7.3.

**Theorem 7.2** (cardinality of $\Lambda_n$ for **FA-ADFOUR**)**.** *Let $\mathbf{A} \in \mathcal{D}_a(\eta_L)$ with $\eta_L > d$, and let $\widetilde{\theta} < 1$ be such that $\widetilde{\rho} < 1$ in (3.23). If $u \in \mathcal{A}^s_B$ for some $0 < s < \eta_L - d$, then the active degrees of freedom produced by* **FA-ADFOUR** *grow as follows:*

$$|\Lambda_n| \leq C_*\, J^d\, \|u - u_n\|^{-d/s}\|u\|^{d/s}_{\mathcal{A}^s_B} , \qquad \forall\, n \geq 0 .$$

*Here, $J$ is the ($\widetilde{\theta}$-dependent) input parameter of* **ENRICH***, whereas the constant $C_*$ is independent of $\widetilde{\theta}$. Moreover, the feasible residual $\widetilde{r}(u_n)$ satisfies*

$$|\operatorname{supp}\, \widetilde{r}(u_n)| \leq C_*\, J^d\, \|u - u_n\|^{-d/s}\|u\|_{\mathcal{A}^s_B}, \quad \forall\, n \geq 0.$$

*Proof.* Since the set $\widetilde{\partial\Lambda}_n$ selected by **DÖRFLER** is minimal, (3.4), (4.3) and (4.7) imply $|\widetilde{\partial\Lambda}_n| \leq \left(1-\widetilde{\theta}^2\right)^{-d/2s} \|\widetilde{r}(u_n)\|^{-d/s}\|\widetilde{r}(u_n)\|^{d/s}_{\mathcal{A}^s_B} + 1$ . Relating $\widetilde{r}(u_n)$ with $u$ and $u_n$ via (2.9), (3.10) and Lemma 7.3, this bound becomes

$$|\widetilde{\partial\Lambda}_n| \lesssim \left(1-\widetilde{\theta}^2\right)^{-d/2s} \|\!|u-u_n|\!\|^{-d/s}\|u\|^{d/s}_{\mathcal{A}^s_B} .$$

On the other hand, (3.24) yields $|\partial\Lambda_n| \lesssim J^d \left(1-\widetilde{\theta}^2\right)^{-d/2s} \|\!|u-u_n|\!\|^{-d/s}\|u\|^{d/s}_{\mathcal{A}^s_B}$, whence

$$\begin{aligned} |\Lambda_n| &\lesssim J^d\,(1-\widetilde{\theta}^2)^{-d/2s}\|u\|^{d/s}_{\mathcal{A}^s_B} \sum_{k=0}^{n-1} \|\!|u-u_k|\!\|^{-d/s} \\ &\leq J^d\,(1-\widetilde{\theta}^2)^{-d/2s}\|u\|^{d/s}_{\mathcal{A}^s_B}\|\!|u-u_n|\!\|^{-d/s} \sum_{k=0}^{n-1} \widetilde{\rho}^{\frac{d}{s}(n-k)} . \end{aligned}$$

We finally conclude the proof by arguing as in Theorem 7.1 and using that $\widetilde{\rho} = C_0\sqrt{1-\widetilde{\theta}^2}$. ∎

## 8. Cardinality of adaptive algorithms: exponential case

We now assume that $u \in \mathcal{A}^{\eta,t}_G$ for some $\eta > 0$ and $t \in (0,d]$ and study the cardinality properties of the outer and inner loops for both **FC-ADFOUR** and **FPC-ADFOUR**.

**Theorem 8.1** (cardinality of the outer loop of **F-ADFOUR** with coarsening)**.** *If $u \in \mathcal{A}^{\eta,t}_G$ for some $\eta > 0$ and $t \in (0,d]$, then the cardinality of the set $\Lambda_n$ of active degrees of freedom produced by either* **FC-ADFOUR** *or* **FPC-ADFOUR** *satisfies the following quasi-optimal bound with $C_* = 3\sqrt{\frac{\alpha^*}{\alpha_*}}$:*

$$|\Lambda_n| \leq \frac{\omega_d}{\eta^{d/t}}\left(\log \frac{C_*\|u\|_{\mathcal{A}^{\eta,t}_G}}{\|u-u_n\|}\right)^{d/t} + 1\,, \qquad \forall\; n\geq 0. \tag{8.1}$$

*Proof.* Let us start with **FC-ADFOUR**. Since each Galerkin approximation $u_{n+1}$ comes just after a call $\Lambda_{n+1} = \textbf{COARSE}(u_{n,k+1}, \varepsilon_n)$ with threshold $\varepsilon_n = \alpha_*^{-1}(1+\gamma)\|\widetilde{r}_{n,k+1}\|$, Proposition 4.3 yields

$$|\Lambda_{n+1}| \leq \frac{\omega_d}{\eta^{d/t}}\left(\log \frac{\|u\|_{\mathcal{A}^{\eta,t}_G}}{\varepsilon_n}\right)^{d/t} + 1\,.$$

On the other hand, (3.12) implies $\|u-u_{n,k+1}\| \leq \varepsilon_n$, whereas (2.5) and Property 3.1 yield

$$\|u-u_{n+1}\| \leq \alpha_*^{-1/2}\|\!|u-u_{n+1}|\!\| \leq 3(\alpha^*/\alpha_*)^{1/2}\varepsilon_n.$$

Since $n \geq -1$, this gives the result, up to a shift in the index.

For **FPC-ADFOUR**, we argue in the same manner as before, now with $\Lambda_{n+1} = \textbf{COARSE}(\widehat{u}_n, \widehat{\varepsilon}_n)$ and $\widehat{\varepsilon}_n = 3\alpha_*^{-1}\sqrt{1-\widetilde{\theta}^2}\|\widetilde{r}_n\|$ as well as $\widehat{\varepsilon}_n \geq 3^{-1}\left(\alpha_*/\alpha^*\right)^{1/2}\|u-u_{n+1}\|$. ∎

We observe that the degrees of freedom in (8.1) are asymptotically the same as the optimal ones

$$|\Lambda_n^{\text{best}}| \le \frac{\omega_d}{\eta^{d/t}} \left( \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_n\|} \right)^{d/t} + 1,$$

dictated by (4.3) for the exponential class. This is due to the coarsening step.

However, the cardinality of the inner loops in both **FC-ADFOUR** and **FPC-ADFOUR** is governed by the sparsity properties of the feasible residual $\widetilde{r}(u_n)$, which is manifestly less sparse than the solution $u$, as discussed in Proposition 6.2. We examine this crucial but novel issue next.

8.1. **FC-ADFOUR: feasible ADFOUR with coarsening.** We already know from Theorem 3.4 that the number $K_n$ of inner iterations is bounded independently of $n$ for **FC-ADFOUR**, which uses moderate Dörfler marking. So, we just estimate the number of active indices $|\Lambda_{n,k}|$ at step $k$, with $0 \le k \le K_n$.

**Theorem 8.2** (cardinality of $\Lambda_{n,k}$ for **FC-ADFOUR**). *Let $u \in \mathcal{A}_G^{\eta,t}$ for some $\eta > 0$ and $t \in (0,d]$, and let* **A** *satisfy either assumption* (a) *or* (b) *in Proposition* 5.2. *Let the parameters $\gamma$, $\widetilde{\theta}$, $\mu$ as well as the contraction factor $\widetilde{\rho}$ be as in Proposition* 7.1. *Then, there exists a constant $C_* > 1$ such that, for all $n \ge 0$ and all $k = 0, \dots, K_n \lesssim 1$, one has*

$$|\Lambda_{n,k}| \le (1+k)^{d/t} \frac{\omega_d}{\eta^{d/t}} \left( \log \frac{C_* \|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_{n+1}\|} \right)^{d/t} .$$

*Moreover, if $\bar{\eta}, \bar{t}$ are the parameters defined in Proposition* 5.2, *setting $\widetilde{\eta} := 2^{-\bar{t}/d}\bar{\eta}$, the feasible residual $\widetilde{r}(u_{n,k})$ satisfies*

$$|\text{supp } \widetilde{r}(u_{n,k})| \le (1+k)^{d/\bar{t}} \frac{\omega_d}{\widetilde{\eta}^{d/\bar{t}}} \left( \log \frac{C_* \|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_{n+1}\|} \right)^{d/\bar{t}} +2 . \tag{8.2}$$

*Proof.* We first notice that $\Lambda_{n,0} = \Lambda_n$, whence we deduce

$$|\Lambda_{n,0}| \le \frac{\omega_d}{\eta^{d/t}} \big( \log \frac{C_* \|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_n\|} \big)^{d/t} + 1$$

from (8.1). Since each inner loop of **FC-ADFOUR** can be viewed as a truncated version of **F-ADFOUR**, Proposition 7.1 applies to $\phi^{-1}(\lambda) = \frac{\omega_d}{\eta^{d/t}} \big( \log \frac{1}{\lambda} \big)^{d/t}$. The fact that $|\Lambda_{n,k}|^{t/d} \le |\Lambda_n|^{t/d} + \sum_{j=0}^{k-1} |\partial \Lambda_{n,j}|^{t/d}$ because $t/d \le 1$, combined with (7.2), gives

$$|\Lambda_{n,k}|^{t/d} \le \frac{\omega_d^{t/d}}{\eta} \log \frac{C_* \|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_n\|} + \frac{\omega_d^{t/d}}{\eta} \sum_{j=0}^{k-1} \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\widetilde{\rho}^{k-j} \sqrt{\mu} \|u-u_{n,k}\|} + k .$$

We next observe that $\Lambda_{n,0} \subset \Lambda_{n,k}$ implies $\|u-u_{n,k}\| \le \sqrt{\frac{\alpha^*}{\alpha_*}} \|u-u_n\|$, along with

$$\sum_{j=0}^{k-1} \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\widetilde{\rho}^{k-j} \sqrt{\mu} \|u-u_{n,k}\|} \le k \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_{n,k}\|} + \frac{k}{2} |\log(\mu \widetilde{\rho}^{k+1})|.$$

This yields

$$|\Lambda_{n,k}|^{t/d} \le (1+k)\frac{\omega_d^{t/d}}{\eta} \log \frac{C_* \|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_{n,k}\|} \,, \tag{8.3}$$

with a constant $C_*$ proportional to $\widetilde{\rho}^{-k-1}$, hence uniformly bounded with respect to $k \le K_n$, because the term $\frac{k}{2}|\log(\mu\widetilde{\rho}^{k+1})|$ can be absorbed into $C_*$. The asserted bound on $|\Lambda_{n,k}|$ follows from $\|u-u_{n+1}\| \le 3\big(\frac{\alpha^*}{\alpha_*}\big)^{3/2}\frac{1+\gamma}{1-\gamma}\|u-u_{n,K_n}\|$, which is just a consequence of Property 3.1 together with (3.12) and (2.5), and the fact that the norms $|\!|\!|u-u_{n,k}|\!|\!|$ contract according to Theorem 3.2.

To prove the estimate for $|\text{supp}\ \widetilde{r}(u_{n,k})|$ we observe that, in view of Proposition 6.2,

$$\|u_{n,k}\|_{\mathcal{A}_G^{\eta_k,t}} \lesssim \|u\|_{\mathcal{A}_G^{\eta,t}} \,, \quad 0 \le k \le K_n,$$

where $\eta_k = \eta/(1+k)$; this follows from Property 4.1, using (8.3) and the trivial bound $\|u\|_{\mathcal{A}_G^{\eta_k,t}} \le \|u\|_{\mathcal{A}_G^{\eta,t}}$. Then (6.6) in conjunction with (5.9) yields

$$|\text{supp}\ \widetilde{r}(u_{n,k})| \le (1+k)^{d/\bar{t}}\frac{\omega_d}{\bar{\eta}^{d/\bar{t}}}\left(2\log\frac{C_*\|u\|_{\mathcal{A}_G^{\mu,t}}}{\|\widetilde{r}(u_{n,k})\|}\right)^{d/\bar{t}} + 2\,.$$

We finally proceed as before upon using that $\|u-u_{n,k}\| \simeq \|\widetilde{r}(u_{n,k})\|$ according to (3.12). ∎

We note that the appearance of the powers of $(1+k)$ in the estimates of the above theorem reflects the fact that in the exponential case the crude bound (7.2) is no longer absorbed by the summation of a geometric series as in the algebraic case.

8.2. **FPC-ADFOUR: feasible PC-ADFOUR.** At last, we discuss the cardinality of the sets $\widehat{\Lambda}_{n+1}$ preceding the call to **COARSE** of algorithm **FPC-ADFOUR** of Section 3.4. In contrast to **FC-ADFOUR**, these sets are generated in one single step within **E-DÖRFLER**.

**Theorem 8.3** (cardinality of $\widehat{\Lambda}_{n+1}$ of **FPC-ADFOUR**). *Let $u \in \mathcal{A}_G^{\eta,t}$ for some $\eta > 0$ and $t \in (0,d]$, and let the marking parameter $\widetilde{\theta} \in (0,1)$ and the feasibility parameter $\gamma$ satisfy $\gamma \in (0,\widetilde{\theta})$. Then, there exist a constant $\kappa \ge 1$, arbitrarily close to 1, and a constant $C_* > 1$ proportional to $\big(1-\widetilde{\theta}^2\big)^{-1/2}$ such that the cardinality of the intermediate sets $\widehat{\Lambda}_{n+1}$ activated in the predictor step of* **FPC-ADFOUR** *is bounded by*

$$|\widehat{\Lambda}_{n+1}| \le \kappa\frac{\omega_d}{\hat{\eta}^{d/\bar{t}}}\left(\log\frac{C_*\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_n\|}\right)^{d/\bar{t}} \,, \qquad \forall\, n \ge 0\,,$$

*where $J$ is the input parameter of* **ENRICH**, *$\bar{\eta}, \bar{t}$ are the parameters defined in Proposition 5.2, and $\hat{\eta}$ is defined by the relation $\hat{\eta}^{-d/\bar{t}} = \eta^{-d/t} + J^d\omega_d\bar{\eta}^{-d/\bar{t}}$. Moreover, setting $\widetilde{\eta} = 2^{-\bar{t}/d}\bar{\eta}$, the feasible residual $\widetilde{r}(u_n)$ satisfies*

$$|\text{supp}\ \widetilde{r}(u_n)| \le \frac{\omega_d}{\widetilde{\eta}^{d/\bar{t}}}\left(\log\frac{C_*\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_n\|}\right)^{d/\bar{t}} \,, \qquad \forall\, n \ge 0\,. \tag{8.4}$$

*Proof.* Invoking Property 4.1 we infer from (8.1) that $\|u_n\|_{\mathcal{A}_G^{\eta,t}} \lesssim \|u\|_{\mathcal{A}_G^{\eta,t}}$. Next, we apply Proposition 6.2 in conjunction with Proposition 5.2 to deduce that

$$\|\widetilde{r}(u_n)\|_{\mathcal{A}_G^{\widetilde{\eta},\widetilde{t}}} \lesssim \|u\|_{\mathcal{A}_G^{\eta,t}},$$

with $\widetilde{\eta}$ as asserted and $\widetilde{t} = \bar{t}$. We recall that $\|\widetilde{r}(u_n) - P_{\widetilde{\partial\Lambda_n}}\widetilde{r}(u_n)\| \leq (1 - \widetilde{\theta}^2)^{1/2}\|\widetilde{r}(u_n)\|$ for each iteration $n$, which combined with the minimality of the set $\widetilde{\partial\Lambda}_n$ selected by **DÖRFLER**, gives

$$|\widetilde{\partial\Lambda}_n| \leq \frac{\omega_d}{\widetilde{\eta}^{d/\widetilde{t}}}\left(\log \frac{\|\widetilde{r}(u_n)\|_{\mathcal{A}_G^{\widetilde{\eta},\widetilde{t}}}}{\sqrt{1-\theta^2}\|\widetilde{r}(u_n)\|}\right)^{d/\widetilde{t}} + 1\ .$$

Estimate (3.24) for **ENRICH** yields

$$|\partial\Lambda_n| \leq \kappa J^d \frac{\omega_d^2}{\widetilde{\eta}^{d/\widetilde{t}}}\left(\log \frac{\|\widetilde{r}(u_n)\|_{\mathcal{A}_G^{\widetilde{\eta},\widetilde{t}}}}{\sqrt{1-\theta^2}\|\widetilde{r}(u_n)\|}\right)^{d/\widetilde{t}} + \kappa\omega_d J^d\ ,$$

with $\kappa \sim 1$; this combined with the previous estimate for $\|\widetilde{r}(u_n)\|_{\mathcal{A}_G^{\widetilde{\eta},\widetilde{t}}}$ and (3.12) gives

$$|\widehat{\Lambda}_{n+1}| \leq |\Lambda_n| + \kappa J^d \frac{\omega_d^2}{\widetilde{\eta}^{d/\widetilde{t}}}\left(\log \frac{C_*\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u-u_n\|}\right)^{d/\widetilde{t}} + \kappa\omega_d J^d\ , \qquad \forall\, n \geq 0\ .$$

Using (8.1) and the inequality $\tilde{t} < t \leq d$, we obtain (8.4), observing that the terms $\kappa\omega_d J^d$ and 1 from (8.1) can be absorbed within the constant $C_*$.

Finally, (8.4) is a consequence of (6.6) and (5.9) as in proof of Theorem 8.2. □

## 9. Final discussion and complexity

• **Sparsity for the algebraic class:** Assuming that $\mathbf{A} \in \mathcal{D}_a(\eta_L)$ and $u \in \mathcal{A}_B^s$ for $0 < s < \eta_L - d$, we show in Lemma 7.3 that $\widetilde{r}(u_n) \in \mathcal{A}_B^s$ where $u_n$ is the Galerkin solution and uniformly in $n$

$$\|\widetilde{r}(u_n)\|_{\mathcal{A}_B^s} \lesssim \|u\|_{\mathcal{A}_B^s}. \tag{9.1}$$

Theorem 7.1 establishes that the supports of both $u_n$ and $\widetilde{r}(u_n)$ have quasi-optimal comparable growth

$$|\Lambda_n|,\ |\text{supp}\ \widetilde{r}(u_n)| \leq C_*\|u-u_n\|^{-d/s}\|u\|_{\mathcal{A}_B^s}^{d/s}. \tag{9.2}$$

The proof of the bound for $|\Lambda_n|$ resorts to Lemma 7.1, Lemma 7.2 and Proposition 7.1, namely to Stevenson's argument [16, 22] which does not need $\mathbf{A} \in \mathcal{D}_a(\eta_L)$. This assumption and Lemma 7.3 are used to prove the bound on $|\text{supp}\ \widetilde{r}(u_n)|$. We now claim that the second bound in (9.2) implies (9.1) uniformly in $n$. To see this, it suffices to consider $N < |\text{supp}\ \widetilde{r}(u_n)|$ for otherwise $E_N(\widetilde{r}(u_n)) = 0$. If $\widetilde{r}_N$ is a best $N$-term approximation of $\widetilde{r}(u_n)$, then

$$\begin{aligned} E_N(\widetilde{r}(u_n)) &= \|\widetilde{r}(u_n) - \widetilde{r}_N\| \leq \|\widetilde{r}(u_n)\| \lesssim \|u-u_n\| \\ &\lesssim \|u\|_{\mathcal{A}_B^s}|\text{supp}\ \widetilde{r}(u_n)|^{-s/d} \leq \|u\|_{\mathcal{A}_B^s}N^{-s/d}, \end{aligned}$$

whence (9.1) follows. Since the total operation count includes constructing the feasible residual $\widetilde{r}(u_n)$, both bounds in (9.2) are essential to assert quasi-optimal complexity of both feasible algorithms **F-ADFOUR** and **FA-ADFOUR**, of course, provided that the linear algebraic systems can be solved optimally. The statement (9.1) about the sparsity of $\widetilde{r}(u_n)$ thus appears to be necessary and sufficient.

• **Sparsity for the exponential class:** If we were able to prove a similar estimate to Theorem 7.1,

$$|\Lambda_n|,\ |\text{supp } \widetilde{r}(u_n)| \leq C_1 \frac{\omega_d}{\eta^{d/t}} \left( \log \frac{C_2 \|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u - u_n\|} \right)^{d/t} \tag{9.3}$$

with $C_1, C_2 \geq 1$, the same argument above would yield $\widetilde{r}(u_n) \in \mathcal{A}_G^{\widetilde{\eta},t}$ with $\widetilde{\eta} = C_1^{-t/d}\eta \leq \eta$ and

$$\|\widetilde{r}(u_n)\|_{\mathcal{A}_G^{\widetilde{\eta},t}} \leq \frac{\alpha^* C_2}{1-\gamma} \|u\|_{\mathcal{A}_G^{\eta,t}}.$$

We thus see that for the feasible residual $\widetilde{r}(u_n)$ to belong to the same sparsity class of the solution, we need $C_1 = 1$ which is a daunting task. In fact, Examples 5.1 and 5.2 indicate that in general the sparsity class of $Lu_n$, and thus of the residual $r(u_n) = f - L(u_n)$, is worse than that of $u$. The feasible residual $\widetilde{r}(u_n)$, being a finite approximation of $r(u_n)$, cannot be expected to be less sparse uniformly in $n$; this is proved in Proposition 6.2.

In Theorem 8.1 we show the estimate (9.3) for $|\Lambda_n|$ with $C_1 = 1$ as a by-product of coarsening. This implies that the *outer loop* of both **FC-ADFOUR** and **FPC-ADFOUR** have quasi-optimal complexity provided the linear solver of **GAL** does.

• **Feasible residual computation:** The cardinalities $|\text{supp } \widetilde{r}(u_{n,k})|$ in Theorem 8.2 for **FC-ADFOUR** and $|\text{supp } \widetilde{r}(u_n)|$ in Theorem 8.3 for **FPC-ADFOUR** reflect the cost of the computation of the feasible residuals. By a comparison between (8.2) and (8.4) it emerges that in both cases the cardinality of the supports depends on the same parameters $\widetilde{\eta}, \bar{t}$ which define the sparsity class of the residuals. On the other hand, the presence of the factor $(1+k)^{d/\bar{t}}$ in (8.2) but not in (8.4) indicates that **FC-ADFOUR** may require the computation of feasible residuals with much larger supports than for **FPC-ADFOUR**.

• **Galerkin solves:** The cardinalities $|\Lambda_{n,k}|$ in Theorem 8.2 for **FC-ADFOUR** and $|\widehat{\Lambda}_n|$ in Theorem 8.3 for **FPC-ADFOUR** dictate the complexity of the linear solver of **GAL**. Theorem 8.2 establishes (9.3) for $|\Lambda_{n,k}|$ with $C_1 = (1+k)^{d/t}$ and parameters $\eta, t$, with $k \leq K_n$ being the inner iteration counter. In contrast, Theorem 8.3 gives (9.3) for $\widehat{\Lambda}_{n+1}$ with $C_1 = \kappa$, a universal constant $\sim 1$, and with parameters $\hat{\eta}$ (surely strictly less than $\eta$ for moderate values of $d$, including the relevant cases $d \leq 3$) and $\bar{t} \leq t$. Therefore, the apparent advantage of **FC-ADFOUR** of having optimal parameters $\eta, t$ is overshadowed by as many as $k \leq K_n$ calls to **GAL**, whereas **FPC-ADFOUR** calls **GAL** just once in the inner loop.

A further observation in favor of **FPC-ADFOUR** is that the bounds in Theorems 8.3 and 8.2 depend on $\|u - u_n\|$ and $\|u - u_{n+1}\|$, respectively, at the same $n$-th outer iteration. This may be significant since $\|u - u_{n+1}\|$ might be several orders of magnitude smaller than $\|u - u_n\|$ for spectral convergence.

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