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DIVERGENCE-CONFORMING HDG METHODS FOR STOKES FLOWS

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Abstract. In this paper, we show that by sending the normal stabilization function to infinity in the hybridizable discontinuous Galerkin methods previously proposed in [Comput. Methods Appl. Mech. Engrg. 199 (2010), 582–597], for Stokes flows, a new class of divergence-conforming methods is obtained which maintains the convergence properties of the original methods. Thus, all the components of the approximate solution, which use polynomial spaces of degree k, converge with the optimal order of $k+1$ in L^2 for any $k \geq 0$. Moreover, the postprocessed velocity approximation is also divergence-conforming, exactly divergence-free and converges with order $k+2$ for $k \geq 1$ and with order 1 for $k = 0$. The novelty of the analysis is that it proceeds by taking the limit when the normal stabilization goes to infinity in the error estimates recently obtained in [Math. Comp., 80 (2011) 723–760].

1. Introduction

The aim of this paper is to introduce and analyze new hybridizable discontinuous Galerkin (HDG) methods for the numerical solution of the following formulation of the Stokes problem:

$$\mathrm{L} - \nabla \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$-\boldsymbol{\nabla}\cdot(\nu \mathrm{L}) + \boldsymbol{\nabla} p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1b}$$

$$\nabla\cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \partial\Omega, \tag{1.1d}$$

$$\overline{p} = 0, \tag{1.1e}$$

where Ω is a polygon in $\mathbb{R}^2$ or a Lipschitz polyhedron in $\mathbb{R}^3$, $\overline{p} = \frac{1}{|\Omega|}\int_\Omega p$, and the boundary data satisfies

$$\int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n} = 0, \tag{1.2}$$

which is a necessary and sufficient condition for existence of a solution to (1.1).

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To better describe and convey the relevance of our results, let us begin with a short historical overview. Up to the beginning of this century, not very many finite element, even non-conforming, methods for the Stokes equations (see [16] and the references therein) had been proposed which could provide exactly divergence-free velocities. For example, in the 2002 essay on the inf-sup condition, [17], the only family of finite element methods with such property is noted to be the one developed back in 1985 in [22] for the two-dimensional case and for velocities that are piecewise-polynomials of degree bigger or equal to four.

Although discontinuous Galerkin (DG) methods using piecewise divergence-free velocities were proposed first in 1990 [1] and later in 2008 [18], none of these methods could provide an exactly divergence-free approximate velocity. A few years later, two new approaches to generate exactly divergence-free velocities appeared: by *hybridization* and by *postprocessing*; see [3].

In the first approach, the method is formulated by using spaces of divergence-free velocities. To avoid the almost impossible task of actually computing the (non-local) bases of these spaces, the method is rewritten in such a way that its new formulation uses spaces of completely discontinuous velocities. This was done for the first time in 2006 [2] for a DG method and then in 2005 [5, 6] for a mixed method.

In the second approach, the approximate solution provided by a DG method is postprocessed in an element-by-element fashion to give rise to a new divergence-free approximate velocity. This was done for the first time in 2005 [12] in the framework of the Navier-Stokes equations; see also [14]. Therein it was also shown that, in some cases, the postprocessing becomes the identity and so the DG method automatically provides a solenoidal velocity. This idea was further developed in 2007 in [13] and in [23].

Recently, by exploiting the superconvergence properties of the hybridizable discontinuous Galerkin (HDG) methods for the Stokes equations based on a Gradient-Velocity-Pressure formulation, [9, 20], a new postprocessing was introduced which provides a solenoidal approximate velocity converging with *an order more* than the original approximate velocity. (HDG methods based on different formulations, like the one based on a Vorticity-Velocity-Pressure formulation in [7], do not seem to have this property; see a comparison in [19].) Similar results hold for HDG methods for diffusion problems; see [4, 8, 10, 11].

The HDG methods considered in this paper can be thought of as a *limit* of the above-mentioned HDG methods. The limit is that obtained by sending *to infinity* the stabilization function associated to the *jumps of the normal component of the velocity* across interelement boundaries. This results in divergence-conforming methods providing divergence-free approximate velocities. Here we show that this passage to the limit *does not degrade* any of the convergence properties of the HDG methods introduced and studied in [9, 20]. Thus, all the components of the approximate solution, which use polynomial spaces of degree k, converge with the optimal order of $k + 1$ in L^2 for any $k \geq 0$. Moreover, the postprocessed velocity approximation is also divergence-conforming, exactly divergence-free and converges with order $k + 2$ for $k \geq 1$ and with order 1 for $k = 0$. This holds for a really wide range of values of the stabilization function associated with the jumps in the tangential component of the velocities. Indeed, the tangential stabilization function can be *any* uniformly bounded non-negative function.

The novelty in the analysis we present here is that it is carried out by a *passage to the limit* in the estimates obtained in [9]: as a by-product, we obtain a better understanding on the effects of normal and tangential stabilization in the HDG methods analyzed in that paper. In particular, we show that the normal stabilization function can be sent to infinity and the tangential one to zero without altering the convergence properties of the method. This generates a new mixed method that can be rewritten solely in terms of a Gradient-Velocity formulation whose hybridized version is nothing but our divergence-conforming HDG method. Last but not least, our technique makes very strong use of a new projection of velocity fields which we prove *coincides* with the Raviart–Thomas projection when the velocity is divergence-free. To establish this result, we use in an essential manner a new *orthogonal decomposition* for the space of discontinuous piecewise polynomial functions on the boundary of a triangle or tetrahedron which is interesting in itself. This decomposition is used in an *essential* manner in several steps of our analysis.

The organization of the paper is as follows. In Section 2, we introduce the divergence-conforming HDG methods. We then note that, when tangential stabilization is set to zero, a new mixed method is obtained which uses velocity spaces of divergence-free functions. We briefly describe how to hybridize these HDG methods and end by commenting on the postprocessed velocity. In Section 3, we state and discuss all our error estimates. In Section 4, we prove that the methods are well defined and in Section 5 we provide full details about their hybridization. The remaining sections are devoted to proving the error estimates of Section 3. In Section 6, we study the projection of the velocity. We relate it to several known projections including the Raviart–Thomas projection. In Section 7, we introduce the projection we are going to use for our analysis, prove that it is well defined and show that it is the limit, as the normal stabilization is sent to infinity, of the projection used to analyze the HDG methods in [9,20]. In Section 8, we show that the divergence-conforming HDG methods are the limit, as the normal stabilization is sent to infinity, of the HDG methods considered in [9,20] and use his fact to use the estimates already obtained in [9] to obtain ours. Finally, in Section 9, we end by briefly commenting on the extension of our results to other boundary conditions.

Notational foreword. We will extensively use the most common Sobolev spaces $H^r(\mathcal{O})$ for non-negative integer r. The spaces of vector- and matrix-valued functions with all the components in $H^r(\mathcal{O})$ will be respectively denoted $\boldsymbol{H}^r(\mathcal{O})$ and $\mathrm{H}^r(\mathcal{O})$. The same convention of using boldface (such as $\boldsymbol{u}$) or Roman fonts (such as L) will be used for vector- and matrix-valued functions and spaces of L^2 type.

2. Numerical approximation with divergence-conforming HDG methods

2.1. **Notation.** Let $\mathcal{T}_h$ be a family of shape-regular simplicial triangulations of Ω. For simplicity, we take conforming triangulations. Given an element (triangle/tetrahedron) $K \in \mathcal{T}_h$, $\mathcal{E}(K)$ will denote the set of its edges in the two-dimensional case and of its faces in the three-dimensional case. Elements of $\mathcal{E}(K)$ will be generally referred to as faces, independently of the dimension, and denoted F. The set of all (interior) faces of the triangulation will be denoted $\mathcal{E}_h$ ($\mathcal{E}_h^o$). We will distinguish functions defined on the faces of the triangulation (the skeleton) by saying that they are defined on $\mathcal{E}_h$ from functions defined on the boundaries of the elements (and therefore having the ability to display two different values on interior faces) by saying that they are defined on $\partial\mathcal{T}_h$. Hence the spaces $L^2(\mathcal{E}_h)$ and $L^2(\partial\mathcal{T}_h)$ have different meanings.

Integration over Ω of functions that are only piecewise defined (like the gradient of a piecewise smooth function) will be written as $(u,v)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}\int_K u\,v = \sum_{K\in\mathcal{T}_h}(u,v)_K$. Integrals on $\partial\mathcal{T}_h$ will be denoted as $\langle u,v\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}\int_{\partial K} u\,v = \sum_{K\in\mathcal{T}_h}\langle u,v\rangle_{\partial K}$. The same symbols will be used for vector-valued functions, namely, $(\boldsymbol{u},\boldsymbol{v})_{\mathcal{T}_h} = \sum_{i=1}^d (u_i,v_i)_{\mathcal{T}_h}$, $\langle\boldsymbol{u},\boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = \sum_{i=1}^d\langle u_i,v_i\rangle_{\partial\mathcal{T}_h}$, as well as for matrix-valued functions $(\mathrm{L},\mathrm{G})_{\mathcal{T}_h} = \sum_{i,j=1}^d (L_{ij},G_{ij})_{\mathcal{T}_h}$. On ∂K, the normal vector field $\boldsymbol{n}$ will always be taken pointing outwards.

The set of polynomials of total degree not larger than $k\ge 0$ defined on K will be denoted $\mathcal{P}_k(K)$. We will also denote

$$\boldsymbol{\mathcal{P}}_k(K) := (\mathcal{P}_k(K))^d, \qquad \mathrm{P}_k(K) := (\mathcal{P}_k(K))^{d\times d},$$

for the sets of vector- and matrix-valued polynomials, respectively. The following particular subset will be of interest:

$$\mathcal{P}_k(K)^\perp := \{p\in\mathcal{P}_k(K)\,:\,(p,q)_K = 0 \quad \forall q\in\mathcal{P}_{k-1}(K)\}.$$

If $F\in\mathcal{E}_h$, then $\mathcal{P}_k(F)$ and $\boldsymbol{\mathcal{P}}_k(F)$ will be the corresponding polynomial spaces on F. Finally, we will need the space

$$\mathcal{R}_k(\partial K) := \{\delta\in L^2(\partial K)\,:\,\delta|_F\in\mathcal{P}_k(F) \quad F\in\mathcal{E}(K)\} \cong \prod_{F\in\mathcal{E}(K)}\mathcal{P}_k(F).$$

The approximation spaces that will be used for the method (respectively for the Gradient, Velocity, Pressure and two Multipliers) are:

$$\mathrm{G}_h := \{\mathrm{G}\in\mathrm{L}^2(\mathcal{T}_h): \quad \mathrm{G}|_K\in\mathrm{P}_k(K) \quad \forall\, K\in\mathcal{T}_h\}, \tag{2.1a}$$

$$\boldsymbol{V}_h := \{\boldsymbol{v}\in\boldsymbol{L}^2(\mathcal{T}_h): \quad \boldsymbol{v}|_K\in\boldsymbol{\mathcal{P}}_k(K) \quad \forall\, K\in\mathcal{T}_h\}, \tag{2.1b}$$

$$P_h := \{q\in L^2(\mathcal{T}_h): \quad q|_K\in\mathcal{P}_k(K) \quad \forall\, K\in\mathcal{T}_h\}, \tag{2.1c}$$

$$\boldsymbol{M}_h := \{\boldsymbol{\mu}\in L^2(\mathcal{E}_h): \quad \boldsymbol{\mu}|_F\in\boldsymbol{\mathcal{P}}_k(F) \quad \forall\, F\in\mathcal{E}_h\}, \tag{2.1d}$$

$$M_h^\partial := \{\mu\in L^2(\partial\mathcal{T}_h): \quad \mu|_{\partial K}\in\mathcal{R}_k(\partial K) \quad \forall\, K\in\mathcal{T}_h\}. \tag{2.1e}$$

Let us emphasize that functions in M_h^∂ are allowed to have two different values of each interior face F.

In its role as test space, the discrete space for the pressure P_h will be decomposed as the orthogonal sum

$$P_h = P_h^{k-1}\oplus P_h^\perp, \tag{2.2a}$$

$$P_h^{k-1} := \{q\in L^2(\mathcal{T}_h): \quad q|_K\in\mathcal{P}_{k-1}(K) \quad \forall\, K\in\mathcal{T}_h\}, \tag{2.2b}$$

$$P_h^\perp := \{q\in L^2(\mathcal{T}_h): \quad q|_K\in\mathcal{P}_k(K)^\perp \quad \forall\, K\in\mathcal{T}_h\}. \tag{2.2c}$$

2.2. **Motivation: The method as a limit.** In this section, we motivate the introduction of our divergence-conforming HDG method by formally taking the limit, as the stabilization of the normal component of the velocity, τ_n, goes to infinity in the HDG methods introduced in [20].

2.2.1. *The HDG method in* [9, 20]. The following discontinuous Galerkin method for the Stokes problem was introduced in [20] and analyzed in [9]. We look for

$$(\mathrm{L}_h^{\tau_n}, \boldsymbol{u}_h^{\tau_n}, p_h^{\tau_n}, \widehat{\boldsymbol{u}}_h^{\tau_n}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$$

satisfying

$$(\mathrm{L}_h^{\tau_n}, \mathrm{G})_{\mathcal{T}_h} + (\boldsymbol{u}_h^{\tau_n}, \boldsymbol{\nabla\cdot}\, \mathrm{G})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h^{\tau_n}, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.3a}$$

$$(-\boldsymbol{\nabla\cdot}(\nu \mathrm{L}_h^{\tau_n}) + \boldsymbol{\nabla}\, p_h^{\tau_n}, \boldsymbol{v})_{\mathcal{T}_h} + \langle \mathrm{S}(\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n}), \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.3b}$$

$$-(\boldsymbol{u}_h^{\tau_n}, \boldsymbol{\nabla}\, q)_{\mathcal{T}_h} + \langle \widehat{\boldsymbol{u}}_h^{\tau_n} \cdot \boldsymbol{n}, q\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.3c}$$

$$\langle \widehat{\boldsymbol{u}}_h^{\tau_n}, \boldsymbol{\mu}\rangle_{\partial\Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega}, \tag{2.3d}$$

$$\langle \nu \mathrm{L}_h^{\tau_n}\boldsymbol{n} - p_h^{\tau_n}\boldsymbol{n} - \mathrm{S}(\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n}), \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{2.3e}$$

$$(p_h^{\tau_n}, 1)_\Omega = 0, \tag{2.3f}$$

for all $(\mathrm{G}, \boldsymbol{v}, q, \boldsymbol{\mu}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h$, where S is given by

$$\mathrm{S}\boldsymbol{u} = \nu\tau_t \boldsymbol{u}_t + \nu\tau_n(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}, \tag{2.3g}$$

where τ_n is a constant quantity that is not dependent of the element K and τ_t is a non-negative piecewise constant function on $\mathcal{E}(K)$. The subscript t on a vector-valued quantity will denote its tangential component, namely, $\boldsymbol{u}_t := \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$.

The case studied in [9] is more general in the treatment of τ_n, but here we are going to take the limit as this quantity goes to infinity. For this purpose taking τ_n to be a constant is enough. The dependence on the parameter τ_n is made explicit in the notation since we are interested in its limiting behavior with respect to this particular parameter. Note that the unique solvability of (2.3) is a consequence of the analysis of [9].

Let us note that the method, for two space dimensions, proposed in [15] has strong similarities with these HDG methods. It uses exactly the same formulation but sets the stabilization function S to be identically zero. The spaces it uses are ours with $k = 0$ except for the space $\mathrm{G}|_K$ whose rows are nothing but the Raviart–Thomas space of lowest index.

2.2.2. *The expanded HDG method.* As a stepping stone towards letting τ_n go to infinity in the previous formulation, we present an *expanded* version of this method by means of the introduction of the *key* quantity

$$\delta_h^{\tau_n} := \tau_n(\boldsymbol{u}_h^{\tau_n}\cdot\boldsymbol{n} - \widehat{\boldsymbol{u}}_h^{\tau_n}\cdot\boldsymbol{n}) \in M_h^{\partial} \tag{2.4}$$

as an unknown in the system.

Also, we break the set of equations (2.3c) in two, one corresponding to $q \in \mathcal{P}_{k-1}(K)$, which notices the presence of the divergence $\nabla\cdot \boldsymbol{u}_h$, and another one corresponding to $q \in \mathcal{P}_k(K)^\perp$, which does not. This induces the *same* orthogonal decomposition $P_h = P_h^{k-1} \oplus P_h^\perp$ given in (2.2a).

We then look for

$$(\mathrm{L}_h^{\tau_n}, \boldsymbol{u}_h^{\tau_n}, p_h^{\tau_n}, \widehat{u}_h^{\tau_n}, \delta_h^{\tau_n}) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h \times M_h^{\partial} =: \mathbb{P}_h,$$

such that

$$(\mathrm{L}_h^{\tau_n}, \mathrm{G})_{\mathcal{T}_h} + (\boldsymbol{u}_h^{\tau_n}, \boldsymbol{\nabla\cdot}\, \mathrm{G})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h^{\tau_n}, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.5a}$$

$$(-\boldsymbol{\nabla\cdot}(\nu \mathrm{L}_h^{\tau_n}) + \boldsymbol{\nabla}\, p_h^{\tau_n}, \boldsymbol{v})_{\mathcal{T}_h} + \langle \nu\tau_t(\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n})_t + \nu\delta_h^{\tau_n}\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.5b}$$

$$(\nabla\cdot \boldsymbol{u}_h^{\tau_n}, q)_{\mathcal{T}_h} - \langle \tau_n^{-1}\delta_h^{\tau_n}, q\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.5c}$$

$$\langle \delta_h^{\tau_n}, q^\perp \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.5d}$$

$$\langle \widehat{\boldsymbol{u}}_h^{\tau_n}, \boldsymbol{\mu} \rangle_{\partial\Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial\Omega}, \tag{2.5e}$$

$$\langle \nu \mathrm{L}_h^{\tau_n} \boldsymbol{n} - p_h^{\tau_n} \boldsymbol{n} - \nu\tau_t (\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n})_t - \nu \delta_h^{\tau_n} \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0, \tag{2.5f}$$

$$\langle \tau_n^{-1} \delta_h^{\tau_n} - (\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n}) \cdot \boldsymbol{n}, \eta \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.5g}$$

$$(p_h^{\tau_n}, 1)_\Omega = 0, \tag{2.5h}$$

for all $(\mathrm{G}, \boldsymbol{v}, q, q^\perp, \boldsymbol{\mu}, \eta) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h^{k-1} \times P_h^\perp \times \boldsymbol{M}_h \times M_h^\partial$. Note that, because (2.5g) is equivalent to (2.4), it follows that the systems (2.3) and (2.5) are equivalent.

The divergence-conforming HDG method (2.6) arises when we *formally* set $\tau_n^{-1} = 0$, in equations (2.5c) and (2.5g).

2.3. **The method.** Thus, the method consists in looking for

$$(\mathrm{L}_h, \boldsymbol{u}_h, p_h, \widehat{u}_h, \delta_h) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h \times \boldsymbol{M}_h \times M_h^\partial$$

such that

$$(\mathrm{L}_h, \mathrm{G})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \boldsymbol{\nabla\cdot}\, \mathrm{G})_{\mathcal{T}_h} - \langle \widehat{\boldsymbol{u}}_h, \mathrm{G}\boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.6a}$$

$$(-\boldsymbol{\nabla\cdot}(\nu \mathrm{L}_h) + \boldsymbol{\nabla}\, p_h, \boldsymbol{v})_{\mathcal{T}_h} + \langle \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t + \nu\delta_h \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial \mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.6b}$$

$$(\nabla\cdot \boldsymbol{u}_h, q)_{\mathcal{T}_h} = 0, \tag{2.6c}$$

$$\langle \delta_h, q^\perp \rangle_{\partial \mathcal{T}_h} = 0 \tag{2.6d}$$

$$\langle \widehat{\boldsymbol{u}}_h, \boldsymbol{\mu} \rangle_{\partial\Omega} = \langle \boldsymbol{g}, \boldsymbol{\mu} \rangle_{\partial\Omega}, \tag{2.6e}$$

$$\langle \nu \mathrm{L}_h \boldsymbol{n} - p_h \boldsymbol{n} - \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t - \nu\delta_h \boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial \mathcal{T}_h \setminus \partial\Omega} = 0, \tag{2.6f}$$

$$\langle (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \cdot \boldsymbol{n}, \eta \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.6g}$$

$$(p_h, 1)_\Omega = 0, \tag{2.6h}$$

for all $(\mathrm{G}, \boldsymbol{v}, q, q^\perp, \boldsymbol{\mu}, \eta) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h^{k-1} \times P_h^\perp \times \boldsymbol{M}_h \times M_h^\partial$. The parameter τ_t that controls the tangential penalization on the velocity field is taken to be

$$\tau_t \in \prod_{K \in \mathcal{T}_h} \mathcal{P}_0(\partial K), \qquad \tau_t \geq 0.$$

Unique solvability of the discrete equations (2.6) *will be proved in Section* 2 *(Proposition* 4.2*).*

Note that $\widehat{\boldsymbol{u}}_h$ is single-valued on each interior face F. Then, (2.6g) implies that $\boldsymbol{u}_h \cdot \boldsymbol{n}$ is continuous across element interfaces and therefore $\boldsymbol{u}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$. As a consequence, (2.6c) implies that $\nabla\cdot \boldsymbol{u}_h = 0$ in Ω (and not only element by element).

2.4. **A new mixed method using spaces of divergence-free velocities.** It is interesting to note that when we set $\tau_t = 0$, the HDG method under consideration gives rise to a new mixed method that can be expressed only in terms of the Gradient and the Velocity. We set

$$\widetilde{\mathrm{G}}_h := \{\mathrm{G} \in \mathrm{G}_h : \quad [\![(\mathrm{G}\boldsymbol{n})_t]\!]|_{\mathcal{E}_h^\circ} = 0\}, \tag{2.7a}$$

$$\widetilde{\boldsymbol{V}}_h(\boldsymbol{g}) := \{\boldsymbol{v} \in \boldsymbol{V}_h : \quad [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]|_{\mathcal{E}_h^\circ} = 0, \quad \langle (\boldsymbol{v} - \boldsymbol{g}) \cdot \boldsymbol{n}, \eta \rangle_{\partial\Omega} = 0 \quad \forall \eta \in M_h^\partial, \quad \nabla\cdot \boldsymbol{v} = 0\}, \tag{2.7b}$$

where the double square bracket $[\![\cdot]\!]$ is used to denote the jump across interelement faces. We have the following result.

Theorem 2.1. *The component* $(\mathrm{L}_h, \boldsymbol{u}_h)$ *of the approximation provided by the HDG method with* $\tau_t = 0$ *is the only solution in* $\widetilde{\mathrm{G}}_h \times \widetilde{\boldsymbol{V}}_h(\boldsymbol{g})$ *of the equations*

$$(\mathrm{L}_h, \mathrm{G})_{\mathcal{T}_h} + (\boldsymbol{u}_h, \boldsymbol{\nabla\cdot}\,\mathrm{G})_{\mathcal{T}_h} - \langle \boldsymbol{u}_h \cdot \boldsymbol{n}, (\mathrm{G}\boldsymbol{n}) \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.8a}$$

$$-(\boldsymbol{\nabla\cdot}(\nu \mathrm{L}_h), \boldsymbol{v})_{\mathcal{T}_h} + \langle \nu(\mathrm{L}_h\boldsymbol{n}) \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}, \tag{2.8b}$$

for all $(\mathrm{G}, \boldsymbol{v}) \in \widetilde{\mathrm{G}}_h \times \widetilde{\boldsymbol{V}}_h(\mathbf{0})$.

Note that, as pointed out in the Introduction, there are DG and mixed methods that use the Vorticity-Velocity formulation but, to the knowledge of the authors, there is no other mixed method that uses a Gradient-Velocity formulation and divergence-free approximate velocities.

2.5. **Hybridized form.** The main idea for hybridization of the system (2.6) consists in eliminating from the system all the unknowns that are discontinuous (essentially L_h, $\boldsymbol{u}_h$, p_h and δ_h) and writing an equivalent system in the coupling variable $\widehat{\boldsymbol{u}}_h$. Because of the lack of uniqueness of pressure in the Stokes problem, we will have to keep

$$\rho_h \in \mathcal{P}_0(\mathcal{T}_h) := \{\rho_h \in L^2(\mathcal{T}_h) \,:\, \rho_h|_K \in \mathcal{P}_0(K) \quad \forall K \in \mathcal{T}_h\} \cong \mathbb{R}^{\#\mathcal{T}_h}$$

as a separate unknown. The value of ρ_h on the element K will be nothing but the average of the pressure p_h in that element.

Given $\boldsymbol{\lambda} \in \boldsymbol{L}^2(\partial\mathcal{T}_h)$ and $\boldsymbol{f} \in \boldsymbol{L}^2(\mathcal{T}_h)$, we consider the solution to the set of local problems in each $K \in \mathcal{T}_h$: find

$$(\mathrm{L}_h, \boldsymbol{u}_h, p_h, \delta_h) \in \mathrm{P}_k(K) \times \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{R}_k(\partial K)$$

such that

$$(\mathrm{L}_h, \mathrm{G})_K + (\boldsymbol{u}_h, \boldsymbol{\nabla\cdot}\,\mathrm{G})_K = \langle \boldsymbol{\lambda}, \mathrm{G}\boldsymbol{n}\rangle_{\partial K}, \tag{2.9a}$$

$$\begin{aligned}(-\boldsymbol{\nabla\cdot}(\nu \mathrm{L}_h) + \boldsymbol{\nabla}\, p_h, \boldsymbol{v})_K &+ \langle \nu\tau_t \boldsymbol{u}_{h,t} + \nu\delta_h\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial K} \\ &= (\boldsymbol{f}, \boldsymbol{v})_K + \langle \nu\tau_t\boldsymbol{\lambda}_t, \boldsymbol{v}_t\rangle_{\partial K},\end{aligned} \tag{2.9b}$$

$$(\nabla\cdot \boldsymbol{u}_h, q - \overline{q}_K)_K = 0, \tag{2.9c}$$

$$\langle \delta_h, q^\perp\rangle_{\partial K} = 0, \tag{2.9d}$$

$$\langle \boldsymbol{u}_h \cdot \boldsymbol{n}, \eta\rangle_{\partial K} = \langle \boldsymbol{\lambda} \cdot \boldsymbol{n}, \eta\rangle_{\partial K}, \tag{2.9e}$$

$$(p_h, 1)_K = (\rho_h, 1)_K, \tag{2.9f}$$

for all $(\mathrm{G}, \boldsymbol{v}, q, q^\perp, \eta) \in \mathrm{P}_k(K) \times \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_{k-1}(K) \times \mathcal{P}_k(K)^\perp \times \mathcal{R}_k(\partial K)$. Here $\overline{q}_K := \frac{1}{|K|}\int_K q$.

Unique solvability of this problem will be a simple consequence of the unique solvability for the general equations (2.6), by considering a domain consisting of only one element. We will deal with this in Section 5.

The solution to (2.9) can be written as

$$(\mathrm{L}_h, \boldsymbol{u}_h, p_h, \delta_h) = (\mathrm{L}_h^{\boldsymbol{\lambda}}, \boldsymbol{u}_h^{\boldsymbol{\lambda}}, p_h^{\boldsymbol{\lambda}}, \delta_h^{\boldsymbol{\lambda}}) + (\mathrm{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{f}}, p_h^{\boldsymbol{f}}, \delta_h^{\boldsymbol{f}}) + (0, \mathbf{0}, \rho_h, 0)$$

by considering separately the influence of $\boldsymbol{f}$ and $\boldsymbol{\lambda}$ in the solution. For example, $(\mathrm{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{f}}, p_h^{\boldsymbol{f}}, \delta_h^{\boldsymbol{f}})$ is the solution of (2.9) when $\boldsymbol{\lambda} = \mathbf{0}$.

We now consider the affine manifold

$$\begin{aligned}\boldsymbol{M}_h(\boldsymbol{g}) \;&:=\; \{\boldsymbol{\lambda} \in \boldsymbol{M}_h \,:\, \langle \boldsymbol{\lambda} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_{\partial\Omega} = 0 \quad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h\} \\ &=\; \{\boldsymbol{\lambda} \in \boldsymbol{M}_h \,:\, \langle \boldsymbol{\lambda} - \boldsymbol{g}, \boldsymbol{\mu}\rangle_F = 0 \quad \forall \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}_k(F) \quad \text{s.t. } F \in \mathcal{E}_h,\, F \subset \partial\Omega\}.\end{aligned}$$

To introduce the hybridized form of (2.6) we need to use two bilinear forms that use the local solvers defined by (2.9):

$$\begin{aligned} a_h(\boldsymbol{\lambda},\boldsymbol{\mu}) &:= (\nu \mathrm{L}_h^{\boldsymbol{\lambda}}, \mathrm{L}_h^{\boldsymbol{\mu}})_{\mathcal{T}_h} + \langle \nu\tau_t(\boldsymbol{u}_h^{\boldsymbol{\lambda}} - \boldsymbol{\lambda})_t, (\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu})_t\rangle_{\partial\mathcal{T}_h},\\ b_h(\boldsymbol{\lambda},\rho) &:= -\langle \rho, \boldsymbol{\lambda}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

The hybrid problem has a general mixed form with side conditions: we look for

$$(\boldsymbol{\lambda}_h, \rho_h) \in \boldsymbol{M}_h(\boldsymbol{g}) \times \mathcal{P}_0(\mathcal{T}_h)$$

satisfying

$$a_h(\boldsymbol{\lambda}_h,\boldsymbol{\mu}) + b_h(\boldsymbol{\mu},\rho_h) = (\boldsymbol{f}, \boldsymbol{u}_h^{\boldsymbol{\mu}})_{\mathcal{T}_h} \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\boldsymbol{0}), \tag{2.10a}$$

$$b_h(\boldsymbol{\lambda}_h, \psi) = 0 \qquad \forall \psi \in \mathcal{P}_0(\mathcal{T}_h), \tag{2.10b}$$

$$(\rho_h, 1)_\Omega = 0. \tag{2.10c}$$

The following result relates the hybrid equations (2.10) with the HDG method (2.6). Its proof will be given in Section 5.

Theorem 2.2. *The hybrid system* (2.10) *has a unique solution. Moreover,*

$$(\mathrm{L}_h^{\boldsymbol{\lambda}_h} + \mathrm{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{\lambda}_h} + \boldsymbol{u}_h^{\boldsymbol{f}}, p_h^{\boldsymbol{\lambda}_h} + p_h^{\boldsymbol{f}} + \rho_h, \boldsymbol{\lambda}_h, \delta_h^{\boldsymbol{\lambda}_h} + \delta_h^{\boldsymbol{f}}) \tag{2.11}$$

is the unique solution of (2.6).

When $k = 0$, we automatically have that $p_h^{\boldsymbol{f}} = p_h^{\boldsymbol{\lambda}_h} = 0$ so $p_h = \rho_h$. Otherwise $\rho_h|_K = \frac{1}{|K|}\int_K p_h$, which means that ρ_h is the $L^2(\Omega)$-best piecewise constant approximation to p_h.

2.6. **Postprocessing.** Once equations (2.6) (or their hybrid version (2.10) with the reconstruction of the local variables) have been solved, we can obtain a new approximation of the velocity

$$\boldsymbol{u}_h^\star \in \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{T}_h) \,:\, \boldsymbol{u}|_K \in \mathcal{P}_{k+1}(K) \quad \forall K \in \mathcal{T}_h\},$$

by using exactly the same method that was described in [9, Section 2.1]. This is a local element-by-element postprocessing of the velocity field that uses $(\mathrm{L}_h, \boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ as data. The resulting velocity field is also divergence-conforming and divergence-free.

3. Convergence estimates

Following the pattern established in [10] in the context of diffusion problems and extended in [9] for a wide class of HDG methods for the Stokes problem, the error estimates for this method are given by comparison with a certain discrete projection of the exact solution.

3.1. **The projection.** Given $(\mathrm{L}, \boldsymbol{u}, p) \in \mathrm{H}^1(\mathcal{T}_h) \times \boldsymbol{H}^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h)$, we construct, element by element, the function

$$(\Pi\,\mathrm{L}, \boldsymbol{\Pi}\boldsymbol{u}, \Pi p, \Pi^\partial\delta) \in \mathrm{P}_k(K) \times \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{R}_k(\partial K)$$

satisfying

$$(\Pi\,\mathrm{L}, \mathrm{G})_K = (\mathrm{L}, \mathrm{G})_K \quad \forall\, \mathrm{G} \in \mathrm{P}_{k-1}(K), \tag{3.1a}$$

$$(\boldsymbol{\Pi}\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K \quad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K), \tag{3.1b}$$

$$(\Pi p, q)_K = (p, q)_K \quad \forall\, q \in \mathcal{P}_{k-1}(K), \tag{3.1c}$$

$$(\text{tr}\,\Pi\,\mathrm{L}, q)_K = (\text{tr}\,\mathrm{L}, q)_K \quad \forall\, q \in \mathcal{P}_k(K), \tag{3.1d}$$

$$\langle(\nu\Pi\,\mathrm{L} - \mathit{\Pi} p\,\mathrm{I})\boldsymbol{n} - \nu \mathit{\Pi}^\partial \delta\,\boldsymbol{n}, \boldsymbol{\mu}\rangle_F$$
$$-\langle\nu\tau_t(\boldsymbol{\Pi u})_t, \boldsymbol{\mu}\rangle_F = \langle(\nu\mathrm{L} - p\,\mathrm{I})\,\boldsymbol{n} - \nu\boldsymbol{u}_t, \boldsymbol{\mu}\rangle_F \quad \forall\, \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}_k(F), \forall F \in \mathcal{E}(K), \tag{3.1e}$$

$$\langle \mathit{\Pi}^\partial \delta, q\rangle_{\partial K} = 0 \quad \forall\, q \in \mathcal{P}_k(K)^\perp \tag{3.1f}$$

$$\langle \boldsymbol{\Pi u}\cdot\boldsymbol{n}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = \langle \boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} \quad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^\perp. \tag{3.1g}$$

In (3.1g) we have used the space $\boldsymbol{\mathcal{P}}_k(K)^\perp := (\mathcal{P}_k(K)^\perp)^d$. *Existence and uniqueness of solutions to this system will be proved in Section* 7.

The regularity assumption on $(\mathrm{L}, \boldsymbol{u}, p)$ can be relaxed, as long as the local traces on the skeleton make sense. Although all four components of the discrete solution depend on the three variables, we abuse the notation by writing $(\Pi\,\mathrm{L}, \boldsymbol{\Pi u}, \mathit{\Pi} p, \mathit{\Pi}^\partial\delta)$ as if they were separate projections. We will actually prove (Proposition 7.1) that $\boldsymbol{\Pi u}$ depends only on $\boldsymbol{u}$ through equations (3.1b) and (3.1g), and that the system (3.1) can be staggered so that the dependence of the discrete functions on the continuous ones is not fully coupled.

As will be apparent from Proposition 7.1 below, the quantity $\mathit{\Pi}^\partial\delta \in M_h^\partial$ is a local residual of some normal flux and is not approximating any particular quantity but converging to zero. For simplicity, we will refer to the operator that is defined by the equations (3.1) as a projection. Indeed, if $(\mathrm{L}, \boldsymbol{u}, p) \in \mathrm{G}_h \times \boldsymbol{V}_h \times P_h$, then $(\Pi\,\mathrm{L}, \boldsymbol{\Pi u}, \mathit{\Pi} p, \mathit{\Pi}^\partial\delta) = (\mathrm{L}, \boldsymbol{u}, p, 0)$, so the operator is actually a projection if we drop the residual quantity $\mathit{\Pi}^\partial\delta$ after its computation.

The following theorem gives a convergence estimate for this projection in the particular case when $\nabla\cdot\boldsymbol{u} = \text{tr}\,\mathrm{L} = 0$. Its proof will be given by means of a limiting argument in Section 7, based upon estimates of a similar projection defined in [9]. The estimate for $\boldsymbol{u}$ will be proven independently, with different techniques, in Section 6 without using the hypothesis $\nabla\cdot\boldsymbol{u} = 0$.

Theorem 3.1. *Assume that* $\nabla\cdot\boldsymbol{u} = \text{tr}\,\mathrm{L} = 0$ *and that*

$$(\mathrm{L}, \boldsymbol{u}, p) \in \mathrm{H}^{\ell_\mathrm{L}+1}(K) \times \boldsymbol{H}^{\ell_{\boldsymbol{u}}+1}(K) \times H^{\ell_p+1}(K)$$

for $\ell_\mathrm{L}, \ell_{\boldsymbol{u}}, \ell_p \in [0, k]$. *Then*

$$\begin{aligned}
\|\boldsymbol{\Pi u} - \boldsymbol{u}\|_K &\le C h^{\ell_{\boldsymbol{u}}+1} |\boldsymbol{u}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}}+1}(K)}\\
\|\nu\Pi\,\mathrm{L} - \nu\mathrm{L}\|_K &\le C h_K^{\ell_\mathrm{L}+1} |\nu\mathrm{L}|_{\mathrm{H}^{\ell_\mathrm{L}+1}(K)} + C\nu\tau_t\Big(\|\boldsymbol{\Pi u} - \boldsymbol{u}\|_K + h^{\ell_{\boldsymbol{u}}+1}|\boldsymbol{u}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}}+1}(K)}\Big)\\
\|\mathit{\Pi} p - p\|_K &\le C h_K^{\ell_p+1} |p|_{H^{\ell_p+1}(K)} + \|\nu\Pi\,\mathrm{L} - \nu\mathrm{L}\|_K + C h_K^{\ell_\mathrm{L}+1} |\nu\mathrm{L}|_{\mathrm{H}^{\ell_\mathrm{L}+1}(K)}.
\end{aligned}$$

3.2. **Estimates of the projection of the errors.** Convergence of L_h and p_h towards the corresponding components of the projection is optimal. This is expressed in the following result, whose proof will be given by a limiting argument in Section 8.

Theorem 3.2.

$$\|\mathrm{L}_h - \Pi\,\mathrm{L}\|_\Omega \le \|\mathrm{L} - \Pi\,\mathrm{L}\|_\Omega, \tag{3.2}$$

$$\|p_h - \mathit{\Pi} p\|_\Omega \le |\overline{(p - \mathit{\Pi} p)}|\,|\Omega|^{1/2} + C\nu C_t \|\mathrm{L} - \Pi\,\mathrm{L}\|_\Omega, \tag{3.3}$$

where

$$C_t := \max\{1, \max_{K\in\mathcal{T}_h} (\tau_t h_K)^{1/2}\}. \tag{3.4}$$

It is an easy consequence of (3.1c) that

$$\overline{p - \mathit{\Pi} p} = 0 \qquad \text{if } k \geq 1,$$

and this term disappears from the right-hand side of the estimate (3.3). When $k = 0$ this is not the case but we have the estimate

$$|\overline{p - \mathit{\Pi} p}| \leq |\Omega|^{1/2} \|p - \mathit{\Pi} p\|_\Omega.$$

The variables $\boldsymbol{u}_h$ and $\widehat{\boldsymbol{u}}_h$ can be shown to superconverge to some projections if $k \geq 1$ and some regularity result for the Stokes problem is assumed. This superconvergence property will be inherited by the postprocessed velocity $\boldsymbol{u}^\star$. The hypothesis is the following: *for every* $\boldsymbol{\theta} \in \boldsymbol{L}^2(\Omega)$*, the solution* $(\Phi, \boldsymbol{\phi}, \phi)$ *to the problem*

$$\Phi + \nabla \boldsymbol{\phi} = 0 \quad \textit{in } \Omega, \tag{3.5a}$$

$$\boldsymbol{\nabla} \cdot (\nu \, \Phi) - \boldsymbol{\nabla} \phi = \boldsymbol{\theta} \quad \textit{in } \Omega, \tag{3.5b}$$

$$-\nabla \cdot \boldsymbol{\phi} = 0 \quad \textit{in } \Omega, \tag{3.5c}$$

$$\boldsymbol{\phi} = \boldsymbol{0} \quad \textit{on } \partial\Omega, \tag{3.5d}$$

is in $\mathrm{H}^1(\Omega) \times \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$*. Therefore, there exists* C_{reg} *such that*

$$\nu \|\Phi\|_{\mathrm{H}^1(\Omega)} + \nu \|\boldsymbol{\phi}\|_{\boldsymbol{H}^2(\Omega)} + \|\phi\|_{H^1(\Omega)} \leq C_{\mathrm{reg}} \|\boldsymbol{\theta}\|_\Omega. \tag{3.6}$$

For the variable $\widehat{\boldsymbol{u}}_h$, convergence will be expressed with respect to the following norm in $\boldsymbol{L}^2(\partial\mathcal{T}_h)$:

$$\|\boldsymbol{\mu}\|_h := \left(\sum_{K \in \mathcal{T}_h} h_K \langle \boldsymbol{\mu}, \boldsymbol{\mu} \rangle_{\partial K} \right)^{1/2}. \tag{3.7}$$

The variable $\widehat{\boldsymbol{u}}_h$ will superconverge to $\boldsymbol{P}_M \boldsymbol{u}$, where $\boldsymbol{P}_M : \boldsymbol{L}^2(\partial\mathcal{T}_h) \to \boldsymbol{M}_h$ is the orthogonal projector onto $\boldsymbol{M}_h$. Because the range of this operator is the space $\boldsymbol{M}_h$, which is separately defined on faces, it is immaterial whether we choose the usual inner product in $\boldsymbol{L}^2(\partial\mathcal{T}_h)$ or the one given by the weighted norm (3.7).

Theorem 3.3. *If the regularity hypothesis of inequality* (3.6) *holds, then*

$$\|\boldsymbol{u}_h - \boldsymbol{\mathit{\Pi}} \boldsymbol{u}\|_\Omega + \|\widehat{\boldsymbol{u}}_h - \boldsymbol{P}_M \boldsymbol{u}\|_h \leq C_1 h^{\min\{k,1\}} (1 + C_t^2) \|\mathrm{L} - \Pi \, \mathrm{L}\|_\Omega, \tag{3.8}$$

where C_t *is defined in* (3.4)*. Moreover, for the postprocessed velocity we have the estimate*

$$\|\boldsymbol{u}_h^\star - \boldsymbol{u}\|_\Omega \leq C_2 h^{\ell_{\boldsymbol{u}}+2} |\boldsymbol{u}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}}+2}(\Omega)} + C_3 h^{\min\{k,1\}} (1 + C_t^2) \|\mathrm{L} - \Pi \, \mathrm{L}\|_\Omega \tag{3.9}$$

for $\ell_{\boldsymbol{u}} \in [0, k]$*.*

It is interesting to remark that the fact that the influence of the tangential parameter τ_t in all the bounds for the projection of the errors is through the quantity C_t *only.* So, we can take $\tau_t = 0$, $\tau_t = 1$ or $\tau_t = h_K^{-1}$ (which forces the approximation of the velocity field to be "more" continuous in the tangential direction) without disturbing those bounds.

Note, however, that the approximation errors of the projection given in Theorem 3.1 behave *differently* with respect to τ_t. Indeed, although the approximation error in the velocity is actually *independent* of τ_t, the upper bounds for the errors in the gradient and pressure are affine functions of τ_t.

It is not difficult to see that whenever τ_t is uniformly bounded on $\partial\mathcal{T}_h$, *both* bounds behave in an optimal manner. Thus, all the components of the approximate solution converge with the optimal order of $k+1$ in L^2 for any $k \geq 0$. Moreover, the postprocessed velocity $\boldsymbol{u}_h^\star$ is divergence-conforming, exactly divergence-free and converges with order $k+2$ for $k \geq 1$ and with order 1 for $k=0$. These are exactly the same convergence properties as those of the HDG methods analyzed in [9].

When we take $\tau_t = h_K^{-1}$, the upper bounds for the gradient, pressure and postprocessed velocity *lose* a full order of convergence because this is what happens to the approximation errors of the projection; see Theorem 3.1. This is in perfect agreement with the numerical results obtained in [9, 20]. Therefore, taking the tangential stabilization τ_t large is not a good idea. Moreover, sending τ_t to infinity leads to a H^1-conforming method that would not be convergent.

3.3. **Conclusion: Estimates of the errors.** We end by summarising our estimates in the case of very smooth exact solutions. The following result follows directly from the estimates of the projection of the errors just obtained and from the approximation properties of the projection in Theorem 3.1.

Theorem 3.4. *Assume that, for any $K \in \mathcal{T}_h$,*

$$(\mathrm{L}, \boldsymbol{u}, p) \in \mathrm{H}^{k+1}(K) \times \boldsymbol{H}^{k+1}(K) \times H^{k+1}(K)$$

where $k \geq 0$. Assume also that $C_t = \max\{1, \max_{K\in\mathcal{T}_h}(\tau_t h_K)^{1/2}\}$ is uniformly bounded. Then we have

$$\begin{aligned}
\|\mathrm{L}_h - \mathrm{L}\|_\Omega &\leq C_\mathrm{L}\, h^{k+1},\\
\|p_h - p\|_\Omega &\leq C_p\, h^{k+1},\\
\|\boldsymbol{u}_h - \boldsymbol{u}\|_\Omega &\leq C_{\boldsymbol{u}}\ h^{k+1}.
\end{aligned}$$

Moreover, if $\boldsymbol{u} \in \boldsymbol{H}^{k+1+\min\{1,k\}}(K)$ for all $K \in \mathcal{T}_h$, then

$$\|\boldsymbol{u}_h^\star - \boldsymbol{u}\|_\Omega \leq C_{\boldsymbol{u}^\star}\ h^{k+1+\min\{1,k\}}.$$

Here,

$$\begin{aligned}
C_\mathrm{L} &:= C\,|\mathrm{L}|_{\mathrm{H}^{k+1}(\mathcal{T}_h)} + C\,\nu\tau_t\,|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(\mathcal{T}_h)},\\
C_p &:= |p|_{H^{k+1}(\mathcal{T}_h)} + C_\mathrm{L},\\
C_{\boldsymbol{u}} &:= |\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(\mathcal{T}_h)} + C_\mathrm{L},\\
C_{\boldsymbol{u}^\star} &:= |\boldsymbol{u}|_{\boldsymbol{H}^{k+1+\min\{1,k\}}(\mathcal{T}_h)} + C_\mathrm{L}.
\end{aligned}$$

This result states that, when the solution is very smooth, all three variables converge with the optimal order of $k+1$, for $k \geq 0$. Moreover, for $k \geq 1$, the postprocessed velocity converges with order $k+2$, as claimed. Finally, note also that in the case $k=0$ the HDG method can be also viewed as a finite volume method. Remarkably enough, it provides approximations to the three variables each converging with order one.

4. Unique solvability of the HDG method

In this section, we prove that our divergence-conforming HDG method is well defined. To do that, be begin by obtaining a new characterization of the space of traces $\mathcal{R}_k(\partial K)$. We then obtain an energy identity for the method and use it, in combination with the result on $\mathcal{R}_k(\partial K)$, to conclude.

4.1. **An orthogonal decomposition of $\mathcal{R}_k(\partial K)$.** The following result is not only relevant in the proof of the solvability of the HDG method under consideration but plays a *key* role in several steps of our error analysis.

Lemma 4.1. *The following decomposition is orthogonal in $L^2(\partial K)$:*

$$\mathcal{R}_k(\partial K) = \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} \,:\, \boldsymbol{v}\in\boldsymbol{\mathcal{P}}_k(K)^\perp\} \oplus \{q|_{\partial K} \,:\, q\in\mathcal{P}_k(K)^\perp\}.$$

To prove this lemma, we are going to use the following auxiliary result.

Lemma 4.2. *If $\boldsymbol{v}\in\boldsymbol{\mathcal{P}}_k(K)^\perp$ and $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on ∂K, then $\boldsymbol{v}=\boldsymbol{0}$.*

Proof. Note that if $F\in\mathcal{E}(K)$, then

$$q\in\mathcal{P}_k(K)^\perp:\ q=0 \text{ on } F \quad\Longrightarrow\quad q=0, \tag{4.1}$$

as can be shown with a simple argument. Now, fix a face F and consider the polynomial $q:=\boldsymbol{v}\cdot\boldsymbol{n}_F$. Then $q\in\mathcal{P}_k(K)^\perp$ and $q=0$ on F, and by (4.1), $q=0$. Since this process can be applied to each of the faces, then $\boldsymbol{v}\cdot\boldsymbol{n}_F=0$ for every F and that proves that $\boldsymbol{v}=\boldsymbol{0}$. ☐

We are now ready to prove Lemma 4.1.

Proof. If $\boldsymbol{v}\in\boldsymbol{\mathcal{P}}_k(K)^\perp$ and $q\in\mathcal{P}_k(K)^\perp$, then

$$\langle\boldsymbol{v}\cdot\boldsymbol{n}, q\rangle_{\partial K} = (\nabla\cdot\boldsymbol{v}, q)_K + (\boldsymbol{v}, \boldsymbol{\nabla} q)_K = 0,$$

which proves that the sets in the statement of the lemma are orthogonal. Consider now the operators

$$\begin{array}{llll} T_1: & \boldsymbol{\mathcal{P}}_k(K)^\perp & \longrightarrow & \mathcal{R}_k(\partial K) \\ & \boldsymbol{v} & \longmapsto & \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} \end{array} \qquad \begin{array}{llll} T_2: & \mathcal{P}_k(K)^\perp & \longrightarrow & \mathcal{R}_k(\partial K) \\ & q & \longmapsto & q|_{\partial K}. \end{array}$$

Lemma 4.2 proves that T_1 is injective and (4.1) shows that so is T_2. Finally, by the orthogonality of the ranges of these two operators, $\mathcal{R}(T_1)$ and $\mathcal{R}(T_2)$, and the injectivity of these operators, it follows that

$$\begin{aligned} \dim(\mathcal{R}(T_1)\oplus\mathcal{R}(T_2)) &= \dim\mathcal{R}(T_1)+\dim\mathcal{R}(T_2) \\ &= \dim\boldsymbol{\mathcal{P}}_k(K)^\perp + \dim\mathcal{P}_k(K)^\perp \\ &= (d+1)\dim\mathcal{P}_k(F) = \dim\mathcal{R}_k(\partial K). \end{aligned}$$

This proves the result. ☐

4.2. **An energy argument.** Next, we obtain an identity by means of a standard energy argument.

Proposition 4.1. *Any solution of* (2.6) *satisfies*

$$\begin{aligned} &\nu\|\mathrm{L}_h\|^2_\Omega + \langle\nu\tau_t(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)_t, (\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)_t\rangle_{\partial\mathcal{T}_h} \\ &\qquad = (\boldsymbol{f},\boldsymbol{u}_h)_{\mathcal{T}_h} + \langle\boldsymbol{g}, \nu\mathrm{L}_h\boldsymbol{n} - p_h\boldsymbol{n} - \nu\tau_t(\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h)_t - \nu\delta_h\boldsymbol{n}\rangle_{\partial\Omega}. \end{aligned}$$

Proof. We first use (2.6a) with $\mathrm{G}=\nu\mathrm{L}_h$ and apply (2.6e) with $\boldsymbol{\mu}=\nu\mathrm{L}_h\boldsymbol{n}|_{\partial\Omega}$, to obtain

$$\nu\|\mathrm{L}_h\|^2_\Omega + \nu(\boldsymbol{u}_h, \boldsymbol{\nabla\cdot}\,\mathrm{L}_h)_{\mathcal{T}_h} - \langle\widehat{\boldsymbol{u}}_h, \nu\mathrm{L}_h\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} - \langle\boldsymbol{g}, \nu\mathrm{L}_h\boldsymbol{n}\rangle_{\partial\Omega} = 0. \tag{4.2}$$

Testing (2.6b) with $\boldsymbol{u}_h$ and using that $\nabla\cdot \boldsymbol{u}_h = 0$, we obtain

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{u}_h)_{\mathcal{T}_h} &= (-\nabla\cdot(\nu \mathrm{L}_h), \boldsymbol{u}_h)_{\mathcal{T}_h} + \langle p_h, \boldsymbol{u}_h \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t, \boldsymbol{u}_h\rangle_{\partial\mathcal{T}_h} + \langle \nu\delta_h, \boldsymbol{u}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&= -\nu(\boldsymbol{u}_h, \boldsymbol{\nabla\cdot}\, \mathrm{L}_h)_{\mathcal{T}_h} + \langle \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t, \boldsymbol{u}_h\rangle_{\partial\mathcal{T}_h} + \langle p_h + \nu\delta_h, \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \\
&= -\nu(\boldsymbol{u}_h, \boldsymbol{\nabla\cdot}\, \mathrm{L}_h)_{\mathcal{T}_h} + \langle \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t, \boldsymbol{u}_h\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle p_h + \nu\delta_h, \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} + \langle \boldsymbol{g}, (p_h+\nu\delta_h)\boldsymbol{n}\rangle_{\partial\Omega},
\end{aligned}
\tag{4.3}
$$

where we have applied (2.6g) and (2.6e). Testing (2.6f) with $\boldsymbol{\mu} = \widehat{\boldsymbol{u}}_h$ and rearranging terms yields

$$
\langle \nu\mathrm{L}_h\boldsymbol{n}, \widehat{\boldsymbol{u}}_h\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} - \langle \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t, \widehat{\boldsymbol{u}}_h\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} - \langle p_h + \nu\delta_h, \widehat{\boldsymbol{u}}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0. \tag{4.4}
$$

Addition of (4.2), (4.3) and (4.4) and an application of (2.6e) gives

$$
\begin{aligned}
(\boldsymbol{f}, \boldsymbol{u}_h)_{\mathcal{T}_h} &= \nu\|\mathrm{L}_h\|^2_\Omega + \langle \boldsymbol{g}, -\nu\mathrm{L}_h\boldsymbol{n} + (p_h+\nu\delta_h)\boldsymbol{n}\rangle_{\partial\Omega} \\
&\quad + \langle \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t, \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\rangle_{\partial\mathcal{T}_h} + \langle \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t, \widehat{\boldsymbol{u}}_h\rangle_{\partial\Omega} \\
&= \nu\|\mathrm{L}_h\|^2_\Omega + \langle \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t, (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle \boldsymbol{g}, -\nu\mathrm{L}_h\boldsymbol{n} + (p_h+\nu\delta_h)\boldsymbol{n} + \nu\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t\rangle_{\partial\Omega},
\end{aligned}
$$

which is the identity we wanted to prove. □

4.3. **Existence and uniqueness of the approximate solution.** We are now ready to state and prove our unique solvability result.

Proposition 4.2. *Equations* (2.6) *are uniquely solvable.*

Proof. As already mentioned, we can interpret (2.6) as a square system of linear equations, so we only need to prove uniqueness of solution for the homogeneous case ($\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{g} = \boldsymbol{0}$). By Proposition 4.1, $\mathrm{L}_h = 0$ and $\tau_t(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)_t = \boldsymbol{0}$.

We next prove that

$$
-(\nabla\boldsymbol{u}_h, \mathrm{G})_{\mathcal{T}_h} + \langle \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0 \qquad \forall\, \mathrm{G}\in \mathrm{G}_h \tag{4.5}
$$

together with

$$
(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\mathcal{T}_h \qquad \text{and} \qquad \widehat{\boldsymbol{u}}_h = \boldsymbol{0} \quad \text{on } \partial\Omega \tag{4.6}
$$

imply that $\boldsymbol{u}_h = \boldsymbol{0}$. To see this, let us first choose an element $K\in\mathcal{T}_h$ and

$$
\mathrm{G} = \begin{cases} \boldsymbol{e}_i\otimes\boldsymbol{v} & \text{in } K \\ 0 & \text{otherwise} \end{cases} \qquad \text{with } \boldsymbol{v}\in\boldsymbol{\mathcal{P}}_k(K)^\perp,
$$

where $\boldsymbol{e}_i$ denotes the i-th vector of the canonical basis of $\mathbb{R}^d$. Then (4.5) gives

$$
\langle (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{e}_i, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = 0 \qquad \forall \boldsymbol{v}\in\boldsymbol{\mathcal{P}}_k(K)^\perp
$$

and by Lemma 4.1, it follows that $(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)\cdot\boldsymbol{e}_i = q_i|_{\partial K}$, for some $q_i\in\mathcal{P}_k(K)^\perp$. Defining $\boldsymbol{q} = \sum_i q_i\boldsymbol{e}_i$, it follows that

$$
(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)|_{\partial K} = \boldsymbol{q}|_{\partial K}, \qquad \boldsymbol{q}\in\boldsymbol{\mathcal{P}}_k(K)^\perp
$$

and $\boldsymbol{q}\cdot\boldsymbol{n}|_{\partial K} = 0$ by (4.6). Lemma 4.2 proves then that $\boldsymbol{q} = \boldsymbol{0}$. Since we can prove this element by element, we deduce that $\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h = \boldsymbol{0}$ on $\partial\mathcal{T}_h$. Given the fact that $\widehat{\boldsymbol{u}}_h$ is single-valued, we have proved that $\boldsymbol{u}_h\in\boldsymbol{H}^1(\Omega)$. Substituting in (4.5) we prove that $\nabla\boldsymbol{u}_h = \boldsymbol{0}$, which together with the condition $\boldsymbol{u}_h|_{\partial\Omega} = \widehat{\boldsymbol{u}}_h|_{\partial\Omega} = \boldsymbol{0}$, implies that $\boldsymbol{u}_h = \boldsymbol{0}$.

At the same time, equations (2.6b) and (2.6f) have been reduced to their simpler form

(4.7a) $$(\boldsymbol{\nabla} p_h, \boldsymbol{v})_{\mathcal{T}_h} + \langle \nu\delta_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0 \qquad \forall \boldsymbol{v}\in\boldsymbol{V}_h,$$

(4.7b) $$\langle p_h + \nu\delta_h, \boldsymbol{\mu}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0 \qquad \forall \boldsymbol{\mu}\in\boldsymbol{M}_h.$$

On the other hand, Lemma 4.1 and equation (2.6d) imply that there exists $\boldsymbol{v}\in\boldsymbol{V}_h$ such that

$$\boldsymbol{v}|_K \in \boldsymbol{\mathcal{P}}_k(K)^\perp \qquad \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} = \delta_h|_{\partial K} \qquad \forall K\in\mathcal{T}_h.$$

Using this function as a test in (4.7a), it follows that $\delta_h = 0$. Since $\delta_h = 0$, then (4.7b) proves that both possible values of p_h on an interior face coincide, so $p_h \in H^1(\Omega)$. Now (4.7a) can be tested with $\boldsymbol{v} = \boldsymbol{\nabla} p_h$ and because $\delta_h = 0$ and $p_h\in H^1(\Omega)$, it follows that p_h is constant in Ω. Finally, the condition (2.6h) proves that this constant value has to be zero. This finishes the proof of uniqueness. ☐

5. Hybridization

This section is devoted to the study of the hybridization of the divergence-free HDG method. We first show that the local solvers (2.9) are well defined. We next prove that the hybrid system (2.10) is uniquely solvable. Finally, we show that the solution of the hybrid system can be used to reconstruct the (unique) solution of the original HDG system (2.6).

5.1. **The local solvers.**

We begin by studying the local solvers.

Proposition 5.1. *The equations* (2.9) *defining the local solvers are uniquely solvable.*

Proof. Note that (2.9) is a square system of linear equations, since the test $q\equiv 1$ in (2.9c) does not produce any equation, which compensates for the normalization equation for the pressure (2.9f). If $(\mathrm{L}_h, \boldsymbol{u}_h, p_h, \delta_h)$ solves this system with homogeneous right-hand side ($\boldsymbol{f} = \boldsymbol{0}, \boldsymbol{\lambda} = \boldsymbol{0}$), then a simple argument shows that $(\boldsymbol{\nabla\cdot} \boldsymbol{u}_h, q)_K = 0$ for all $q\in\mathcal{P}_{k-1}(K)$. If we consider a domain $\Omega = K$, formed by a single element, then $(\mathrm{L}_h, \boldsymbol{u}_h, p_h, \boldsymbol{0}, \delta_h)$ is a solution to the global system (2.6) with homogeneous right-hand side (in this case, $\partial\mathcal{T}_h\setminus\partial\Omega = \emptyset$ and equation (2.6f) is void) and is therefore zero by Proposition 4.2. ☐

5.2. **Unique solvability of the hybrid method.**

To prove that the equations (2.10) are uniquely solvable, we are going to use the following auxiliary result.

Proposition 5.2. *Let $\boldsymbol{\lambda}\in\boldsymbol{L}^2(\partial K)$. Then the following statements are equivalent:*

(a) $\boldsymbol{P}_{\partial K}\boldsymbol{\lambda} = \boldsymbol{c}$*, where $\boldsymbol{c}$ is a constant vector.*
(b) $(\mathrm{L}_h^{\boldsymbol{\lambda}}, \boldsymbol{u}_h^{\boldsymbol{\lambda}}, p_h^{\boldsymbol{\lambda}}, \delta_h^{\boldsymbol{\lambda}}) = (0, \boldsymbol{c}, 0, 0)$ *with $\boldsymbol{c}$ constant.*
(c) $\mathrm{L}_h^{\boldsymbol{\lambda}} = 0$ *and* $\tau_t(\boldsymbol{u}_h^{\boldsymbol{\lambda}} - \boldsymbol{P}_{\partial K}\boldsymbol{\lambda})_t = \boldsymbol{0}$.

The constant vector $\boldsymbol{c}$ in (a) *and* (b) *is the same.*

Proof. Note that the actual data for the local solver (2.9) (with $\boldsymbol{f} = \boldsymbol{0}$) is $\boldsymbol{P}_{\partial K}\boldsymbol{\lambda}\in\boldsymbol{R}_k(\partial K)$. Therefore, if (a) holds, it is simple to show that the solution proposed in (b) satisfies the local equations (2.9) with the same $\boldsymbol{c}$. Reciprocally, if (b) holds, then

$$\langle \boldsymbol{c} - \boldsymbol{P}_{\partial K}\boldsymbol{\lambda}, \mathrm{G}\boldsymbol{n}\rangle_{\partial K} = 0 \qquad \forall \mathrm{G}\in\mathrm{P}_k(K),$$
$$\langle (\boldsymbol{c} - \boldsymbol{P}_{\partial K}\boldsymbol{\lambda})\cdot\boldsymbol{n}, \eta\rangle_{\partial K} = 0 \qquad \forall \eta\in\mathcal{R}_k(\partial K).$$

Arguing as in the proof of Proposition 4.2, we can show that $\boldsymbol{c} - \boldsymbol{P}_{\partial K}\boldsymbol{\lambda} = \boldsymbol{0}$.

It is now clear that (a) and (b) imply each other with the same constant vector $\boldsymbol{c}$, so each of these conditions imply (c). Finally, we can prove that (c) implies both (a) and (b) with the same constant vector by following step by step the proof of Proposition 4.2. □

Proposition 5.3. *The equations* (2.10) *of the hybrid method are uniquely solvable.*

Proof. As usual, we only have to prove uniqueness of solution for the homogeneous problem. Assume then that $(\boldsymbol{\lambda}_h, \rho_h) \in \boldsymbol{M}_h(\mathbf{0}) \times \mathcal{P}_0(\mathcal{T}_h)$ satisfies

$$a_h(\boldsymbol{\lambda}_h, \boldsymbol{\mu}) + b_h(\boldsymbol{\mu}, \rho_h) + b_h(\boldsymbol{\lambda}_h, \psi) = 0 \qquad \forall(\boldsymbol{\mu}, \psi) \in \boldsymbol{M}_h(\mathbf{0}) \times \mathcal{P}_0(\mathcal{T}_h). \tag{5.1}$$

Taking $(\boldsymbol{\mu}, \psi) = (\boldsymbol{\lambda}_h, -\rho_h)$ and using the definition of a_h it follows that

$$\mathrm{L}_h^{\boldsymbol{\lambda}_h} = 0 \qquad \tau_t(\boldsymbol{u}_h^{\boldsymbol{\lambda}_h} - \boldsymbol{\lambda}_h)_t = 0.$$

By Proposition 5.2 it follows that $\boldsymbol{\lambda}_h|_{\partial K} = \boldsymbol{c}_K$ for each K, but because $\boldsymbol{\lambda}_h$ is single-valued on interior faces, this means that $\boldsymbol{\lambda}_h \equiv \boldsymbol{c}$ and this constant has to vanish because of the boundary condition that is hidden in the fact that $\boldsymbol{\lambda}_h \in \boldsymbol{M}_h(\mathbf{0})$.

By (5.1), we now know that

$$b_h(\boldsymbol{\mu}, \rho_h) = 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}_h(\mathbf{0}). \tag{5.2}$$

Let $F = K_1 \cap K_2$ be an interior face and take the choice of direction of the normal vector $\boldsymbol{n}_F$ that goes from K_1 to K_2. Choosing $\boldsymbol{\mu}|_F \equiv \boldsymbol{n}_F$ and $\boldsymbol{\mu}|_{\mathcal{E}_h \setminus F} \equiv \mathbf{0}$, the condition (5.2) implies that $(\rho_h|_{K_1} - \rho_h|_{K_2})|F| = 0$. Going from element to element, this fact implies that $\rho_h \equiv c$. Finally, the normalization condition $(\rho_h, 1)_\Omega = 0$ shows that $\rho_h = 0$. □

5.3. **Reconstruction of the approximate solution.** We have two options to show that solutions of the hybrid system (2.10) can be used to reconstruct the solution of (2.6) with the formula (2.11). The first one starts in the hybridized form of the method (2.3) (given in [20]), expands and modifies the local solvers so that they can cope with the limiting process as $\tau_n \to \infty$ and then uses the same argument that we have used both for the projection and the method. The second option is more direct and this is why we use it next.

We start with some identities of almost straightforward verification:

Lemma 5.1. *For every $\boldsymbol{f} \in \boldsymbol{L}^2(K)$ and $\boldsymbol{\mu} \in \boldsymbol{L}^2(\partial K)$,*

$$(\boldsymbol{u}_h^{\boldsymbol{\mu}}, \nabla q)_K = \langle q\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial K} \qquad \forall q \in \mathcal{P}_k(K), \quad s.t. \quad (q,1)_K = 0, \tag{5.3}$$

$$(\boldsymbol{u}_h^{\boldsymbol{f}}, \nabla q)_K = 0 \qquad \forall q \in \mathcal{P}_k(K), \tag{5.4}$$

and

$$(\nabla\cdot \mathrm{L}_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{\mu}})_K - (\boldsymbol{u}_h^{\boldsymbol{f}}, \nabla\cdot \mathrm{L}_h^{\boldsymbol{\mu}})_K = \langle \mathrm{L}_h^{\boldsymbol{f}}\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial K}. \tag{5.5}$$

The following two identities are consequences of the previous ones.

Lemma 5.2. *For all $\boldsymbol{f} \in \boldsymbol{L}^2(K)$ and $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{L}^2(\partial K)$,*

$$\langle \nu\tau_t(\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu})_t, \boldsymbol{u}_h^{\boldsymbol{f}}\rangle_{\partial K} = (\nabla\cdot(\nu\mathrm{L}_h^{\boldsymbol{f}}), \boldsymbol{u}_h^{\boldsymbol{\mu}})_K - \langle \nu\mathrm{L}_h^{\boldsymbol{f}}\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial K}, \tag{5.6}$$

$$\langle \nu\tau_t(\boldsymbol{u}_h^{\boldsymbol{\lambda}} - \boldsymbol{\lambda})_t, \boldsymbol{u}_h^{\boldsymbol{\mu}}\rangle_{\partial K} = (\nabla\cdot(\nu\mathrm{L}_h^{\boldsymbol{\lambda}}), \boldsymbol{u}_h^{\boldsymbol{\mu}})_K - \langle p_h^{\boldsymbol{\lambda}} + \nu\delta_h^{\boldsymbol{\lambda}}\boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial K}. \tag{5.7}$$

Proof. Starting from the second equation for the local solver (2.9b) and applying (5.4), by the fact that $\boldsymbol{u}_h^{\boldsymbol{f}} \cdot \boldsymbol{n} = 0$ and (5.5) we prove

$$\begin{aligned}\langle \nu\tau_t \boldsymbol{\mu}_t, \boldsymbol{u}_h^{\boldsymbol{f}}\rangle_{\partial K} &= (-\nabla\cdot(\nu \mathrm{L}_h^{\boldsymbol{\mu}}), \boldsymbol{u}_h^{\boldsymbol{f}})_K + (\nabla p_h^{\boldsymbol{\mu}}, \boldsymbol{u}_h^{\boldsymbol{f}})_K \\ &\quad + \langle \nu\tau_t \boldsymbol{u}_{h,t}^{\boldsymbol{\mu}}, \boldsymbol{u}_h^{\boldsymbol{f}}\rangle_{\partial K} + \langle \nu\delta_h^{\boldsymbol{\mu}}, \boldsymbol{u}_h^{\boldsymbol{f}} \cdot \boldsymbol{n}\rangle_{\partial K} \\ &= (-\nabla\cdot(\nu \mathrm{L}_h^{\boldsymbol{f}}), \boldsymbol{u}_h^{\boldsymbol{\mu}})_K + \langle \nu \mathrm{L}_h^{\boldsymbol{f}} \boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial K} + \langle \nu\tau_t \boldsymbol{u}_{h,t}^{\boldsymbol{\mu}}, \boldsymbol{u}_h^{\boldsymbol{f}}\rangle_{\partial K},\end{aligned}$$

i.e., (5.6). Similarly, using now (5.3) and the fact that $(p_h^{\boldsymbol{\lambda}}, 1)_K = 0$ as well as $\langle (\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu}) \cdot \boldsymbol{n}, \delta_h^{\boldsymbol{\lambda}}\rangle_{\partial K} = 0$, we prove (5.7). □

The third and last set of identities relate the linear form in the right-hand side of (2.10a) and the bilinear form a_h with numerical traces.

Lemma 5.3. *Let* $\boldsymbol{f} \in \boldsymbol{L}^2(K)$ *and* $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \boldsymbol{L}^2(\partial K)$. *Then*

$$(\boldsymbol{f}, \boldsymbol{u}_h^{\boldsymbol{\mu}})_K = \langle \widehat{\boldsymbol{\phi}}_h^{\boldsymbol{f}}, \boldsymbol{\mu}\rangle_{\partial K}, \tag{5.8}$$

$$(\nu \mathrm{L}_h^{\boldsymbol{\lambda}}, \mathrm{L}_h^{\boldsymbol{\mu}})_K + \langle \nu\tau_t (\boldsymbol{u}_h^{\boldsymbol{\lambda}} - \boldsymbol{\lambda})_t, (\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu})_t\rangle_{\partial K} = -\langle \widehat{\boldsymbol{\phi}}_h^{\boldsymbol{\lambda}}, \boldsymbol{\mu}\rangle_{\partial K}, \tag{5.9}$$

where

$$\begin{aligned}\widehat{\boldsymbol{\phi}}_h^{\boldsymbol{f}} &:= -\nu \mathrm{L}_h^{\boldsymbol{f}} \boldsymbol{n} + p_h^{\boldsymbol{f}} \boldsymbol{n} + \nu\tau_t \boldsymbol{u}_{h,t}^{\boldsymbol{f}} + \nu\delta_h^{\boldsymbol{f}} \boldsymbol{n}, \\ \widehat{\boldsymbol{\phi}}_h^{\boldsymbol{\lambda}} &:= -\nu \mathrm{L}_h^{\boldsymbol{\lambda}} \boldsymbol{n} + p_h^{\boldsymbol{\lambda}} \boldsymbol{n} + \nu\tau_t (\boldsymbol{u}_h^{\boldsymbol{\lambda}} - \boldsymbol{\lambda})_t + \nu\delta_h^{\boldsymbol{\lambda}} \boldsymbol{n}.\end{aligned}$$

Proof. In order to prove (5.8) we start with (2.9b) and show that

$$\begin{aligned}(\boldsymbol{f}, \boldsymbol{u}_h^{\boldsymbol{\mu}})_K &= (-\nabla\cdot(\nu \mathrm{L}_h^{\boldsymbol{f}}), \boldsymbol{u}_h^{\boldsymbol{\mu}})_K + (\boldsymbol{\nabla} p_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{f}})_K \\ &\quad + \langle \nu\tau_t \boldsymbol{u}_{h,t}^{\boldsymbol{f}}, \boldsymbol{u}_{h,t}^{\boldsymbol{\mu}}\rangle_{\partial K} + \langle \nu\delta_h^{\boldsymbol{f}}, \boldsymbol{u}_h^{\boldsymbol{\mu}} \cdot \boldsymbol{n}\rangle_{\partial K} \\ &= (-\nabla\cdot(\nu \mathrm{L}_h^{\boldsymbol{f}}), \boldsymbol{u}_h^{\boldsymbol{\mu}})_K + \langle p_h^{\boldsymbol{f}} \boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial K} + \langle \nu\tau_t \boldsymbol{u}_{h,t}^{\boldsymbol{\mu}}, \boldsymbol{u}_h^{\boldsymbol{f}}\rangle_{\partial K} + \langle \nu\delta_h^{\boldsymbol{f}}, \boldsymbol{\mu} \cdot \boldsymbol{n}\rangle_{\partial K}\end{aligned}$$

by (5.3) (note that $(p_h^{\boldsymbol{f}}, 1)_K = 0$) and (2.9e). Then the result is a simple consequence of (5.6). Using (5.7) and the equality

$$(\nu \mathrm{L}_h^{\boldsymbol{\mu}}, \mathrm{L}_h^{\boldsymbol{\lambda}})_K + (\boldsymbol{u}_h^{\boldsymbol{\mu}}, \nabla\cdot \mathrm{L}_h^{\boldsymbol{\lambda}})_K = \langle \boldsymbol{\mu}, \nu \mathrm{L}_h^{\boldsymbol{\lambda}} \boldsymbol{n}\rangle_{\partial K},$$

we can easily prove (5.9). □

Proposition 5.4. *If* $(\boldsymbol{\lambda}_h, \rho_h) \in \boldsymbol{M}_h(\boldsymbol{g}) \times \mathcal{P}_0(\mathcal{T}_h)$ *is the solution of* (2.10), *then the reconstruction formula* (2.11) *gives the solution of the divergence-conforming HDG equations* (2.6).

Proof. It is clear from the definition of the local solvers that equations (2.6a), (2.6b), (2.6d) and (2.6g) are satisfied.

The discrete boundary condition (2.6e) is equivalent to the fact that $\widehat{\boldsymbol{u}}_h = \boldsymbol{\lambda}_h \in \boldsymbol{M}_h(\boldsymbol{g})$. Also, from the definition of the local solvers it follows that $(p_h^{\boldsymbol{\lambda}_h} + p_h^{\boldsymbol{f}} + \rho_h, 1)_\Omega = (\rho_h, 1)_\Omega$, and (2.6h) is equivalent to (2.10c).

Localizing the restriction (2.10b), we show that

$$(\boldsymbol{\nabla\cdot}(\boldsymbol{u}_h^{\boldsymbol{\lambda}_h} + \boldsymbol{u}_h^{\boldsymbol{f}}), 1)_K = \langle (\boldsymbol{u}_h^{\boldsymbol{\lambda}_h} + \boldsymbol{u}_h^{\boldsymbol{f}}) \cdot \boldsymbol{n}, 1\rangle_{\partial K} = \langle \boldsymbol{\lambda}_h \cdot \boldsymbol{n}, 1\rangle_{\partial K} = 0,$$

which allows us to eliminate the term $\overline{q}_K$ in the local equation (2.9c) and prove (2.6c). Finally, using the notation and the results of Lemma 5.3, we have

$$\begin{aligned}-(\nu \mathrm{L}_h^{\boldsymbol{\lambda}}, \mathrm{L}_h^{\boldsymbol{\mu}})_K &- \langle \nu\tau_t (\boldsymbol{u}_h^{\boldsymbol{\lambda}} - \boldsymbol{\lambda})_t, (\boldsymbol{u}_h^{\boldsymbol{\mu}} - \boldsymbol{\mu})_t\rangle_{\partial K} \\ &+ \langle \rho_h, \boldsymbol{\mu} \cdot \boldsymbol{n}\rangle_{\partial K} + (\boldsymbol{f}, \boldsymbol{u}_h^{\boldsymbol{\mu}})_K = \langle \widehat{\boldsymbol{\phi}}_h^{\boldsymbol{\lambda}} + \widehat{\boldsymbol{\phi}}_h^{\boldsymbol{f}} + \rho_h \boldsymbol{n}, \boldsymbol{\mu}\rangle_{\partial K}.\end{aligned}$$

This identity shows that the remaining equation (2.6f) is just (2.10a) written in terms of local solvers. □

Propositions 5.3 and 5.4 provide a proof of Theorem 2.2.

6. The projection of the velocity

In this section, we begin our convergence analysis of the divergence-conforming HDG methods by studying the projection of the velocity $\boldsymbol{\Pi u}$. First, we show that it is well defined in terms of $\boldsymbol{u}$ only. Then we relate it with two L^2-projections and finally we show that it coincides with the Raviart–Thomas projection when $\boldsymbol{u}$ is divergence-free.

6.1. Definition of the velocity projection. We begin by considering the operator $\boldsymbol{\Pi} : \boldsymbol{H}^1(K) \to \boldsymbol{\mathcal{P}}_k(K)$ that associates $\boldsymbol{\Pi u} \in \boldsymbol{\mathcal{P}}_k(K)$ such that

$$(\boldsymbol{\Pi u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K), \tag{6.1a}$$

$$\langle \boldsymbol{\Pi u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = \langle \boldsymbol{u} \cdot \boldsymbol{n}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}. \tag{6.1b}$$

A simple inspection of equations (3.1b) and (3.1g) shows that $\boldsymbol{\Pi u}$ is the corresponding component of the projection defined in (3.1) no matter what the other input variables are. The fact that this operator is well defined is a straightforward consequence of Lemma 4.1.

6.2. Preliminaries: Some auxiliary L^2-projections. We next introduce the orthogonal projections $P_k : L^2(K) \to \mathcal{P}_k(K)$ and $P_{\partial K} : L^2(\partial K) \to \mathcal{R}_k(\partial K)$. Also, we recall the one-faced projection from [4]: given $F \in \mathcal{E}(K)$, $P_F u \in \mathcal{P}_k(K)$ is defined as the only element that satisfies

$$(P_F u, v)_K = (u, v)_K \qquad \forall v \in \mathcal{P}_{k-1}(K), \tag{6.2a}$$

$$\langle P_F u, \mu \rangle_F = \langle u, \mu \rangle_F \qquad \forall \mu \in \mathcal{P}_k(F). \tag{6.2b}$$

This projection is well defined because it is given by a square linear system that is uniquely solvable by (4.1). Note that

$$(P_F u)\Big|_F = (P_{\partial K} u|_{\partial K})\Big|_F \qquad \forall u \in H^1(K), \tag{6.3}$$

and that

$$\|P_F u\|_F \le \|u\|_F \qquad \forall u \in H^1(K). \tag{6.4}$$

A simple scaling argument and (4.1) show the following two inequalities:

$$\|p\|_{\partial K} \le C h_K^{-1/2} \|p\|_K \qquad \forall p \in \mathcal{P}_k(K), \tag{6.5}$$

$$\|p\|_K \le C h_K^{1/2} \|p\|_F \qquad \forall p \in \mathcal{P}_k(K)^{\perp}, \quad \forall F \in \mathcal{E}(K). \tag{6.6}$$

Moreover, we can easily show that P_k and P_F are *close*.

Proposition 6.1. *For $\ell \in [0, k]$,*

$$\|P_k u - P_F u\|_K \le C h_K^{\ell+1} |u|_{H^{\ell+1}(K)} \qquad \forall u \in H^{\ell+1}(K).$$

6.3. **Comparison with some L^2-projections.** We are now ready to compare $\boldsymbol{\Pi u}$ with the projection $\boldsymbol{P}_k$ which is the operator that applies P_k to each of the components of a vector-valued function. We then have the following result that shows quasi-optimality for $\boldsymbol{\Pi}$. Note that this result proves the first inequality of Theorem 3.1.

Proposition 6.2. *For $\ell \in [0, k]$,*

$$\|\boldsymbol{P}_k \boldsymbol{u} - \boldsymbol{\Pi u}\|_K \le C h_K^{\ell+1} |\boldsymbol{u}|_{\boldsymbol{H}^{\ell+1}(K)} \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^{\ell+1}(K).$$

Proof. Now, note that $\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u} \in \boldsymbol{\mathcal{P}}_k(K)^\perp$ which implies that

$$\begin{aligned} \|(\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}) \cdot \boldsymbol{n}\|_{\partial K}^2 &= \langle \boldsymbol{\Pi}(\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}, (\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}) \cdot \boldsymbol{n}\rangle_{\partial K} \\ &= \langle (\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}, (\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}) \cdot \boldsymbol{n}\rangle_{\partial K} \end{aligned}$$

and therefore

$$\begin{aligned} \|(\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}) \cdot \boldsymbol{n}\|_{\partial K} \le\ & \|(\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{n}\|_{\partial K} \\ \le\ & \|\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{u}\|_{\partial K} \le h_K^{1/2+\ell} |\boldsymbol{u}|_{\boldsymbol{H}^{\ell+1}(K)} \end{aligned} \tag{6.7}$$

by using the trace inequality applied componentwise.

Now, choose one face $F_0 \in \mathcal{E}(K)$ and let $\mathcal{E}_0 := \mathcal{E}(K) \setminus \{F_0\}$. Since $\{\boldsymbol{n}_F\}_{F\in\mathcal{E}_0}$ is a basis of $\mathbb{R}^d$, we can compute its dual (biorthogonal) basis $\{\boldsymbol{n}_F^*\}_{F\in\mathcal{E}_0}$, that satisfies

$$\boldsymbol{n}_F^* \cdot \boldsymbol{n}_{F'} = \delta_{F,F'} \qquad \forall F, F' \in \mathcal{E}_0.$$

Then, there exists a constant that depends only on shape regularity constants of the triangulation such that

$$|\boldsymbol{n}_F^*| \le C_{\mathrm{sh}} \qquad \forall F \in \mathcal{E}_0,$$

and therefore

$$\|\boldsymbol{w}\|_K = \Big\| \sum_{F\in\mathcal{E}_0} (\boldsymbol{w}\cdot\boldsymbol{n}_F)\boldsymbol{n}_F^* \Big\|_K \le C_{\mathrm{sh}} \sum_{F\in\mathcal{E}_0} \|\boldsymbol{w}\cdot\boldsymbol{n}_F\|_K \quad \forall \boldsymbol{w} \in \boldsymbol{L}^2(K). \tag{6.8}$$

Since for all F, $(\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}) \cdot \boldsymbol{n}_F \in \mathcal{P}_k(K)^\perp$, we can bound

$$\begin{aligned} \|\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}\|_K &\le C_{\mathrm{sh}} \sum_{F\in\mathcal{E}_0} \|(\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}) \cdot \boldsymbol{n}_F\|_K \quad \text{by (6.8)}, \\ &\le C h_K^{1/2} \sum_{F\in\mathcal{E}_0} \|(\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}) \cdot \boldsymbol{n}_F\|_F \quad \text{by 6.6}, \\ &\le C' h_K^{1/2} \|(\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}) \cdot \boldsymbol{n}\|_{\partial K} \le C'' h_K^{\ell+1} |\boldsymbol{u}|_{\boldsymbol{H}^{\ell+1}(K)} \end{aligned}$$

by (6.7). This completes the proof. □

Next, we compare operator $\boldsymbol{\Pi}$ with the projection $\boldsymbol{P}_{\partial K}$ that arises by componentwise application of $P_{\partial K}$.

Proposition 6.3. *Let $\ell \in [0, k]$. Then*

$$\|\boldsymbol{P}_{\partial K}\boldsymbol{u} - \boldsymbol{\Pi u}\|_{\partial K} \le C h_K^{1/2+\ell} |\boldsymbol{u}|_{\boldsymbol{H}^{\ell+1}(K)} \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^{\ell+1}(K).$$

Proof. Let $\boldsymbol{P}_F$ be the vector-valued version of P_F and note that $\boldsymbol{P}_F\boldsymbol{u} - \boldsymbol{\Pi u} \in \boldsymbol{\mathcal{P}}_k(K)^\perp$ by the definition of both operators. Using (6.3) and (6.5), it follows that for all F,

$$\begin{aligned} \|\boldsymbol{P}_{\partial K}\boldsymbol{u} - \boldsymbol{\Pi u}\|_F &= \|\boldsymbol{P}_F\boldsymbol{u} - \boldsymbol{\Pi u}\|_F \le Ch_K^{-1/2}\|\boldsymbol{P}_F\boldsymbol{u} - \boldsymbol{\Pi u}\|_K \\ &\le Ch_K^{-1/2}\Big(\|\boldsymbol{P}_F\boldsymbol{u} - \boldsymbol{P}_k\boldsymbol{u}\|_K + \|\boldsymbol{P}_k\boldsymbol{u} - \boldsymbol{\Pi u}\|_K\Big) \end{aligned}$$

and the result is a direct consequence of Propositions 6.1 and 6.2. □

6.4. **Comparison with the Raviart–Thomas projection.** Although the next result was not used explicitly in our analysis, we believe it is quite interesting in its own right.

Proposition 6.4. *Assume that $\boldsymbol{u} \in \boldsymbol{H}^1(K)$ is divergence-free. Then $\boldsymbol{\Pi u}$ is nothing but the Raviart–Thomas projection of $\boldsymbol{u}$.*

Proof. It is well known that the Raviart–Thomas projection of $\boldsymbol{u}$ on the simplex K is the only element $\boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u} \in \boldsymbol{\mathcal{P}}_k(K) \oplus \boldsymbol{x}\mathcal{P}_k(K)$ such that

$$\begin{aligned} (\boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u}, \boldsymbol{v})_K &= (\boldsymbol{u}, \boldsymbol{v})_K & \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K),\\ \langle \boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u}\cdot\boldsymbol{n}, \delta\rangle_{\partial K} &= \langle \boldsymbol{u}\cdot\boldsymbol{n}, \delta\rangle_{\partial K} & \forall \delta \in \mathcal{R}_k(K). \end{aligned}$$

By the orthogonal decomposition of $\mathcal{R}_k(K)$ of Lemma 4.1, this is equivalent to

$$\begin{aligned} (\boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u}, \boldsymbol{v})_K &= (\boldsymbol{u}, \boldsymbol{v})_K & \forall \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K),\\ \langle \boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= \langle \boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} & \forall \boldsymbol{v} \in \boldsymbol{P}_k^\perp(K),\\ \langle \boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u}\cdot\boldsymbol{n}, q\rangle_{\partial K} &= \langle \boldsymbol{u}\cdot\boldsymbol{n}, q\rangle_{\partial K} & \forall q \in \mathcal{P}_k^\perp(K). \end{aligned}$$

Since, by using the first of the above equations, we can replace the third equation by

$$(\nabla\cdot\boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u}, q)_K = (\nabla\cdot\boldsymbol{u}, q)_K \qquad \forall q \in \mathcal{P}_k^\perp(K),$$

we see that when $\nabla\cdot\boldsymbol{u} = 0$ then $\boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u}$ belongs to $\boldsymbol{\mathcal{P}}_k(K)$. A glance to the equations defining $\boldsymbol{\Pi u}$, (6.1), allows us to see that $\boldsymbol{\Pi}^{\mathrm{RT}}\boldsymbol{u}$ is identical to $\boldsymbol{\Pi u}$ in this case. This completes the proof. □

7. The projection as a limit

In this section, we study our main tool of analysis, namely, the projection defined by equations (3.1). To do that, we begin by showing that it can be defined in a staggered manner and that it is well defined. Then we show that it can actually be obtained as the limit as τ_n goes to infinity of the projection used to analyze the HDG methods in [9].

7.1. **A characterization of the projection defined by** (3.1)**.** We have the following result.

Proposition 7.1. *The projection defined by equations* (3.1) *is well defined and can be computed progressively in each of the variables as the solution of the following* *four uniquely solvable problems as follows:* $\boldsymbol{\Pi u}$ *is defined by the equations* (6.1), $\Pi\,\mathrm{L}$ *is then defined by*

$$(\Pi\,\mathrm{L},\mathrm{G})_K = (\mathrm{L},\mathrm{G})_K \qquad \forall\, \mathrm{G}\in \mathrm{P}_{k-1}(K), \tag{7.1a}$$

$$(\mathrm{tr}\,\Pi\,\mathrm{L}, q)_K = (\mathrm{tr}\,\mathrm{L}, q)_K \qquad \forall\, q \in \mathcal{P}_k(K), \tag{7.1b}$$

$$\langle \nu \Pi\,\mathrm{L}\boldsymbol{n}, \boldsymbol{\mu}_t\rangle_F = \langle \nu \mathrm{L}\,\boldsymbol{n} + \nu\tau_t(\boldsymbol{\Pi u}-\boldsymbol{u})_t, \boldsymbol{\mu}_t\rangle_F \qquad \forall\, \boldsymbol{\mu}\in \boldsymbol{\mathcal{P}}_k(F),\ \forall F, \tag{7.1c}$$

Πp *is defined by*

$$(\Pi p, q)_K = (p,q)_K \qquad \forall\, q\in \mathcal{P}_{k-1}(K), \tag{7.2a}$$

$$\langle \Pi p, q\rangle_{\partial K} = \langle p,q\rangle_{\partial K} - \langle \nu(\mathrm{L}\boldsymbol{n} - \Pi\,\mathrm{L}\boldsymbol{n}, q\boldsymbol{n}\rangle_{\partial K} - \langle \nu\tau_t(\boldsymbol{\Pi u}-\boldsymbol{u})_t, q\rangle_{\partial K} \qquad \forall\, q\in \mathcal{P}_k(K)^\perp, \tag{7.2b}$$

and finally $\Pi^\partial\delta$ *is the solution of*

$$\nu\langle \Pi^\partial\delta, \mu\rangle_F = \langle (\nu\Pi\,\mathrm{L}\boldsymbol{n} - \Pi p\boldsymbol{n}) - (\nu\mathrm{L}\boldsymbol{n} - p\boldsymbol{n}), \mu\boldsymbol{n}\rangle_F \qquad \forall\, \mu \in \mathcal{P}_k(F),\ \forall F. \tag{7.3}$$

Proof. We begin by showing that the system for $\Pi\,\mathrm{L}$, (7.1), is uniquely solvable. Note first that equation (7.1b) can be reduced to tests in $\mathcal{P}_k(K)^\perp$ and that on each of the faces in (7.1c) we can test with elements that have no normal component. Therefore, since we have $d+1$ faces and $d-1$ copies of the space $\mathcal{P}_k(F)$ on each face,

$$\begin{aligned}\#\ \text{equations} &= d^2\dim\mathcal{P}_{k-1}(K) + \dim\mathcal{P}_k(K)^\perp + (d-1)(d+1)\dim\mathcal{P}_k(F)\\ &= d^2\dim\mathcal{P}_{k-1}(K) + d^2\,\mathcal{P}_k(K)^\perp = d^2\dim\mathcal{P}_k(K) = \dim\mathrm{P}_k(K).\end{aligned}$$

If we set the right-hand side of (7.1) equal to zero and consider $\boldsymbol{\sigma} := \Pi\,\mathrm{L}\boldsymbol{n}_F$, it follows that $\boldsymbol{\sigma}\in\boldsymbol{\mathcal{P}}_k(K)^\perp$ and that $\boldsymbol{\sigma}_t = 0$ on F. With this and (4.1), we prove that there exists $p_F\in\mathcal{P}_k(K)$ such that $\boldsymbol{\sigma} = p_F\,\boldsymbol{n}_F$. Next, we fix one of the faces, that we call F', and note that

$$\mathbf{0} = \Pi\,\mathrm{L}(\sum_F |F|\boldsymbol{n}_F) = \sum_F p_F|F|\boldsymbol{n}_F = \sum_{F\neq F'}(p_F - p_{F'})|F|\boldsymbol{n}_F,$$

so $p_F = p_{F'}$ for all F. Therefore, $\Pi\,\mathrm{L} = p_{F'}\mathrm{I}$ and using (7.1b) (that is satisfied with homogeneous right-hand side) necessarily $p_{F'} = 0$. The equations (7.1) are satisfied by $\Pi\,\mathrm{L}$, as can be easily seen by taking $\boldsymbol{\mu}$ with no normal component in (3.1e).

We now turn our attention to the equations (7.2). It is clear that they are uniquely solvable. To see that Πp satisfies (7.2b), take $q\in\mathcal{P}_k(K)^\perp$, pick $\boldsymbol{\mu} = q\boldsymbol{n}$ in (3.1e) and sum over all the faces, applying (3.1f) to cancel the occurrence of $\Pi^\partial\delta$ in the equation.

Finally, the equations (7.3) arise from taking $\boldsymbol{\mu} = \mu\boldsymbol{n}$ in (3.1e). It is straightforward to prove that they have a unique solution. □

An immediate consequence of this result is that the projection (3.1) is well defined independently of how τ_t is chosen: this function can vanish or even be negative. Note also that, when $\tau_t = 0$, the projection $\boldsymbol{\Pi u}$ is completely decoupled from the rest of the set of equations.

7.2. **The projection of the HDG methods in** [9]**.** Next, we show that the projection we just considered, that is, the one defined by equations (3.1), is in fact the limit, as τ_n goes to infinity, of the projection used to analyze a class of HDG methods for the Stokes problem in [9].

Such a projection is defined as follows: we look for

$$(\Pi_{\tau_n}\mathrm{L}, \boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}, \mathit{\Pi}_{\tau_n}p) \in \mathrm{P}_k(K) \times \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$$

such that

$$(\Pi_{\tau_n}\mathrm{L}, \mathrm{G})_K = (\mathrm{L}, \mathrm{G})_K \qquad \forall\, \mathrm{G} \in \mathrm{P}_{k-1}(K), \tag{7.4a}$$

$$(\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K), \tag{7.4b}$$

$$(\mathit{\Pi}_{\tau_n}p, q)_K = (p, q)_K \qquad \forall\, q \in \mathcal{P}_{k-1}(K), \tag{7.4c}$$

$$(\mathrm{tr}\ \Pi_{\tau_n}\mathrm{L}, q)_K = (\mathrm{tr}\ \mathrm{L}, q)_K \qquad \forall\, q \in \mathcal{P}_k(K), \tag{7.4d}$$

$$\langle \nu\Pi_{\tau_n}\mathrm{L}\boldsymbol{n} - \mathit{\Pi}_{\tau_n}p\,\boldsymbol{n} - \mathrm{S}\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}, \boldsymbol{\mu}\rangle_F = \langle \nu\mathrm{L}\boldsymbol{n} - p\,\boldsymbol{n} - \mathrm{S}\boldsymbol{u}, \boldsymbol{\mu}\rangle_F \qquad \forall\, \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}_k(F),\ \forall F \in \mathcal{E}(K), \tag{7.4e}$$

where the stabilization tensor S is given by the formula (2.3g).

Note that (7.4d) contains redundant equations that were already included in (7.4a) by choosing $\mathrm{G} = q\,\mathrm{I}$ with $q \in \mathcal{P}_{k-1}(K)$ and I the identity matrix. Instead, we can restrict the tests in (7.4d) to $q \in \mathcal{P}_k(K)^{\perp}$. Also, we can gather equations (7.4a) and (7.4d) in the single equation

$$(\Pi_{\tau_n}\mathrm{L}, \mathrm{G})_K = (\mathrm{L}, \mathrm{G})_K \qquad \forall\, \mathrm{G} \in \mathrm{P}_{k-1}(K) \oplus \{q\,\mathrm{I} \,:\, q \in \mathcal{P}_k(K)^{\perp}\}.$$

7.3. **The expanded version of the projection.** To be able to actually let τ_n go to infinity, we need to introduce an *expanded* version of the projection defined by the equations (7.4) by including $\tau_n(\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}\cdot\boldsymbol{n} - P_{\partial K}(\boldsymbol{u}\cdot\boldsymbol{n})) \in \mathcal{R}_k(\partial K)$ as an additional unknown in the system. Recall that $P_{\partial K} : L^2(\partial K) \to \mathcal{R}_k(\partial K)$ is the orthogonal projection onto $\mathcal{R}_k(\partial K)$.

We now look for

$$(\Pi_e\mathrm{L}, \boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}, \mathit{\Pi}_e p, \mathit{\Pi}_e^{\partial}\delta) \in \mathrm{P}_k(K) \times \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K) \times \mathcal{R}_k(\partial K) =: \mathbb{P}_k(K)$$

as the solution of the equations

$$(\Pi_e\mathrm{L}, \mathrm{G})_K = (\mathrm{L}, \mathrm{G})_K \qquad \forall\, \mathrm{G} \in \mathrm{P}_{k-1}(K), \tag{7.5a}$$

$$(\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}, \boldsymbol{v})_K = (\boldsymbol{u}, \boldsymbol{v})_K \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K), \tag{7.5b}$$

$$(\mathit{\Pi}_e p, q)_K = (p, q)_K \qquad \forall\, q \in \mathcal{P}_{k-1}(K), \tag{7.5c}$$

$$(\mathrm{tr}\ \Pi_e\mathrm{L}, q)_K = (\mathrm{tr}\ \mathrm{L}, q)_K \qquad \forall\, q \in \mathcal{P}_k(K), \tag{7.5d}$$

$$\begin{aligned}&\langle \nu\Pi_e\mathrm{L}\boldsymbol{n} - \mathit{\Pi}_e p\,\boldsymbol{n} - \nu\mathit{\Pi}_e^{\partial}\delta\,\boldsymbol{n}, \boldsymbol{\mu}\rangle_F \\ &- \langle \nu\tau_t(\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u})_t, \boldsymbol{\mu}\rangle_F = \langle \nu\mathrm{L}\boldsymbol{n} - p\,\boldsymbol{n} - \nu\tau_t\boldsymbol{u}_t, \boldsymbol{\mu}\rangle_F \qquad \forall\, \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}_k(F),\ \forall F \in \mathcal{E}(K),\end{aligned} \tag{7.5e}$$

$$\langle \mathit{\Pi}_e^{\partial}\delta, q\rangle_{\partial K} = -\tau_n(\nabla\cdot\boldsymbol{u}, q)_K \qquad \forall\, q \in \mathcal{P}_k(K)^{\perp} \tag{7.5f}$$

$$\langle \tau_n^{-1}\mathit{\Pi}_e^{\partial}\delta - \boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = -\langle \boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}. \tag{7.5g}$$

Note that, if $\nabla\cdot\boldsymbol{u} = 0$, all the dependence on τ_n appears in the left-hand side of (7.5g). Note also that, if we formally set $\tau_n^{-1} = 0$, we obtain the equations that define the projection (3.1); this justifies the introduction of the extended version of the projection.

The forthcoming arguments will show that the projection (7.5) is well defined and that (7.5) is essentially the projection defined by (7.4).

Proposition 7.2. *The operator defined by the equations* (7.5) *is well defined.*

Proof. Using Lemma 4.1, it follows that the number of equations in (7.5f) and (7.5g) is equal to the number of unknowns introduced by $\mathit{\Pi}_e^{\partial}\delta$. Since (7.4) is equivalent to a square linear system (see [9, Section 4] for a detailed proof), then so is (7.5) and we only need to prove that setting the right-hand side of (7.5) to zero, the only solution is the trivial one.

We are first going to show that $\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u} = \mathbf{0}$. Note that (7.5b) (with zero right-hand side) implies that $\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}$. Also

$$\langle \Pi_e\mathrm{L}\,\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial K} = (\boldsymbol{\nabla\cdot}\,\Pi_e\mathrm{L}, \boldsymbol{v})_K + (\Pi_e\mathrm{L}, \nabla\cdot\boldsymbol{v})_K = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}, \tag{7.6}$$

by (7.5a). Similarly,

$$\langle \mathit{\Pi}_e p\,\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial K} = (\boldsymbol{\nabla}\,\mathit{\Pi}_e p, \boldsymbol{v})_K + (\mathit{\Pi}_e p, \nabla\cdot\boldsymbol{v})_K = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}, \tag{7.7}$$

where we have now used (7.5c). If we insert these two equalities into (7.5e), after summing over all faces, it follows that

$$\langle \tau_t(\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u})_t + \mathit{\Pi}_e^{\partial}\delta\,\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial K} = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}.$$

Substituting this equation in (7.5g) it follows that

$$\langle \tau_t(\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u})_t + \tau_n(\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}\cdot\boldsymbol{n})\,\boldsymbol{n}, \boldsymbol{v}\rangle_{\partial K} = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}.$$

Taking $\boldsymbol{v} = \boldsymbol{\mathit{\Pi}}_e\boldsymbol{u} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}$ in the previous expression, we obtain that

$$\langle \tau_t(\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u})_t, (\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u})_t\rangle_{\partial K} + \langle \tau_n\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}\cdot\boldsymbol{n}, \boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\partial K} = 0.$$

Using the fact that $\tau_n > 0$ and $\tau_t \geq 0$ we have proved that $\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on ∂K. Applying Lemma 4.2, it follows that $\boldsymbol{\mathit{\Pi}}_e\boldsymbol{u} = \mathbf{0}$.

We can now use this in (7.5f) and (7.5g) to prove that

$$\begin{aligned} \langle \mathit{\Pi}_e^{\partial}\delta, q\rangle_{\partial K} &= 0 \qquad && \forall\, q \in \mathcal{P}_k(K)^{\perp}, \\ \langle \mathit{\Pi}_e^{\partial}\delta, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0 \qquad && \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}, \end{aligned}$$

which implies that $\mathit{\Pi}_e^{\partial}\delta = 0$ by Lemma 4.1.

We now fix a face F and define

$$\boldsymbol{\sigma} := -\nu\Pi_e\mathrm{L}\,\boldsymbol{n}_F + \mathit{\Pi}_e p\,\boldsymbol{n}_F.$$

From (7.5a) and (7.5c) it follows that $\boldsymbol{\sigma} \in \boldsymbol{\mathcal{P}}_k(K)^{\perp}$ and from (7.5e) it follows that

$$\langle \boldsymbol{\sigma}, \boldsymbol{\mu}\rangle_F = 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}_k(F),$$

so $\boldsymbol{\sigma} = \mathbf{0}$ on F. A direct application of (4.1) to each of the components of $\boldsymbol{\sigma}$ proves that $\boldsymbol{\sigma} = \mathbf{0}$. Hence, $-\nu\Pi_e\mathrm{L}\,\boldsymbol{n}_F + \mathit{\Pi}_e p\,\boldsymbol{n}_F = \mathbf{0}$ for every face F and therefore $-\nu\Pi_e\mathrm{L} + \mathit{\Pi}_e p\,\mathrm{I} = 0$. Using (7.5d) it follows that

$$\begin{aligned} 0 &= (-\nu\Pi_e\mathrm{L} + \mathit{\Pi}_e p\,\mathrm{I}, \mathit{\Pi}_e p\,\mathrm{I})_K \\ &= (-\nu\mathrm{tr}\,\Pi_e\mathrm{L}, \mathit{\Pi}_e p)_K + d\,(\mathit{\Pi}_e p, \mathit{\Pi}_e p)_K = d\,(\mathit{\Pi}_e p, \mathit{\Pi}_e p)_K, \end{aligned}$$

which proves that $\mathit{\Pi}_e p = 0$ and therefore $\Pi_e\mathrm{L} = 0$. □

7.4. **The relation between the projection and its extension.** We have the following result.

Proposition 7.3. *For all* $(\mathrm{L}, \boldsymbol{u}, p)$,

$$(\Pi_e\mathrm{L}, \boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}, \mathit{\Pi}_e p, \mathit{\Pi}_e^{\partial}\delta) = \Big(\Pi_{\tau_n}\mathrm{L}, \boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}, \mathit{\Pi}_{\tau_n}p, \tau_n(\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}\cdot\boldsymbol{n} - P_{\partial K}(\boldsymbol{u}\cdot\boldsymbol{n}))\Big).$$

Proof. We have already established that both sets of equations (7.4) and (7.5) are uniquely solvable, so we only need to prove that the solution of (7.4), adding $\mathit{\Pi}_e^\partial \delta := \tau_n(\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}\cdot\boldsymbol{n} - P_{\partial K}(\boldsymbol{u}\cdot\boldsymbol{n}))$ as a fourth component, solves (7.5). It is clear that we only need to deal with equations (7.5e), (7.5f) and (7.5g).

Note that

$$\langle \tau_n(P_{\partial K}(\boldsymbol{u}\cdot\boldsymbol{n}) - \boldsymbol{u}\cdot\boldsymbol{n}), \boldsymbol{\mu}\rangle_F = 0 \qquad \forall\, \boldsymbol{\mu}\in \boldsymbol{\mathcal{P}}_k(F) \quad \forall F\in \mathcal{E}(K).$$

Hence, (7.5e) follows from (7.4e), the definition of S in (2.3g) and the definition of $\mathit{\Pi}_e^\partial\delta$ above. Equation (7.5g) is a straightforward consequence of the definition of $\mathit{\Pi}_e^\partial\delta$ as a function of $\boldsymbol{u}$, of the fact that τ_n is constant and of the definition of $P_{\partial K}$. Finally,

$$\begin{aligned}\langle \tau_n^{-1}\mathit{\Pi}_e^\partial\delta, q\rangle_{\partial K} &= \langle (\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u} - \boldsymbol{u})\cdot\boldsymbol{n}, q\rangle_{\partial K}\\ &= (\nabla\cdot(\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}-\boldsymbol{u}), q)_K + (\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}-\boldsymbol{u}, \boldsymbol{\nabla} q)_K\\ &= (\nabla\cdot\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}, q)_K - (\nabla\cdot\boldsymbol{u}, q)_K = -(\nabla\cdot\boldsymbol{u}, q)_K \qquad \forall\, q\in\mathcal{P}_k(K)^\perp,\end{aligned}$$

and the proof is thus complete. ☐

7.5. **The limiting argument.** We are now ready to show that when $\tau_n\to\infty$, then the solution of (7.5) converges to the one of (3.1).

Proposition 7.4. *If* $\nabla\cdot\boldsymbol{u}=0$*, then*

$$(\Pi_e\mathrm{L}, \boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}, \mathit{\Pi}_e p, \mathit{\Pi}_e^\partial\delta) \overset{\tau_n\to\infty}{\longrightarrow} (\Pi\,\mathrm{L}, \boldsymbol{\mathit{\Pi}}\boldsymbol{u}, \mathit{\Pi}p, \mathit{\Pi}^\partial\delta),$$

as elements of the finite dimensional space $\mathbb{P}_k(K)$.

Proof. Let $s := \tau_n^{-1}$ and denote $\Pi(s) := (\Pi_e\mathrm{L}, \boldsymbol{\mathit{\Pi}}_e\boldsymbol{u}, \mathit{\Pi}_e, \mathit{\Pi}_e^\partial\delta)\in\mathbb{P}_k(K)$. The equations that define this element, i.e., (7.5), can be written in the form

$$\mathbb{A}(s)\Pi(s) = \Theta,$$

where $\mathbb{A}(s)$ is a linear operator from $\mathbb{P}_k(K)$ to its dual space $\mathbb{P}_k(K)'$ and $\Theta\in\mathbb{P}_k(K)'$ is independent of s if $\nabla\cdot\boldsymbol{u}=0$. Note that $\mathbb{A}(s)$ is an affine function of s and that $\mathbb{A}(s)$ is invertible for every $s\ge 0$ (these are Propositions 7.1 and 7.2 for $s=0$ and $s>0$ respectively). Therefore $\lim_{s\to 0}\mathbb{A}(s)^{-1} = \mathbb{A}(0)^{-1}$ and

$$\lim_{s\to 0}\Pi(s) = \mathbb{A}(0)^{-1}\Theta = (\Pi\,\mathrm{L}, \boldsymbol{\mathit{\Pi}}\boldsymbol{u}, \mathit{\Pi}p, \mathit{\Pi}^\partial\delta),$$

which proves the result. ☐

7.6. **Proof of Theorem 3.1.** The result is a direct consequence of [9, Theorem 2.3] for the error of the projection (7.4) (explicit dependence of every constant on τ_t and τ_n is shown there), of the identification of this projection with the expanded projection (7.5) (Proposition 7.3) and of the limit property in Proposition 7.4. This completes the proof of Theorem 3.1.

8. Convergence analysis

Our convergence analysis of the HDG method (2.6) will consist in taking the limit $\tau_n\to\infty$ in the error bounds of the method (2.3), a process that we rigorously justify here. The error estimates for (2.3) were obtained in [9] dealing with general stabilization tensors S. Here we restrict our attention to S given by (2.3g), when $\tau_n>0$ is constant and $\tau_t\ge 0$ is constant on each ∂K. The error estimates in [9] are obtained in terms of the following quantities:

$$\begin{aligned}\mathrm{E}^{\mathrm{L}} &:= \Pi_{\tau_n}\mathrm{L} - \mathrm{L}_h^{\tau_n}, &\qquad \boldsymbol{\varepsilon}^{\boldsymbol{u}} &:= \boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u} - \boldsymbol{u}_h^{\tau_n}\\ \varepsilon^p &:= \mathit{\Pi}_{\tau_n}p - p_h^{\tau_n}, &\qquad \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}} &:= \boldsymbol{P}_M\boldsymbol{u} - \widehat{\boldsymbol{u}}_h^{\tau_n}.\end{aligned}$$

We will have to refine some of the estimates of [9] to obtain very precise bounds in terms of the stabilization parameters τ_n and τ_t.

8.1. **The divergence-conforming HDG method as a limit.** We are now ready to characterize our divergence-conforming HDG method as the limit of HDG methods as the normal stabilization goes to infinity.

Proposition 8.1. *Let* $(\mathrm{L}_h^{\tau_n}, \boldsymbol{u}_h^{\tau_n}, p_h^{\tau_n}, \widehat{\boldsymbol{u}}_h^{\tau_n})$ *be the solution of* (2.3) *and* $\delta_h^{\tau_n} := \tau_n(\boldsymbol{u}_h^{\tau_n}\cdot\boldsymbol{n} - \widehat{\boldsymbol{u}}_h^{\tau_n}\cdot\boldsymbol{n})$. *Then*

$$(\mathrm{L}_h^{\tau_n}, \boldsymbol{u}_h^{\tau_n}, p_h^{\tau_n}, \widehat{\boldsymbol{u}}_h^{\tau_n}, \delta_h^{\tau_n}) \stackrel{\tau_n\to\infty}{\longrightarrow} (\mathrm{L}_h, \boldsymbol{u}_h, p_h, \widehat{\boldsymbol{u}}_h, \delta_h),$$

where the last vector is the solution of (2.6). *Convergence occurs in the finite dimensional space* $\mathbb{P}_h$.

Proof. The argument is exactly the same as the one of the proof of Proposition 7.4. □

8.2. **Estimates for gradient and pressure.** The starting point of the error analysis in [9] is the proof of the error equations (Lemma 3.1 in [9]):

$$(\mathrm{E}^{\mathrm{L}}, \mathrm{G})_{\mathcal{T}_h} + (\boldsymbol{\varepsilon}^{\boldsymbol{u}}, \boldsymbol{\nabla\cdot}\,\mathrm{G})_{\mathcal{T}_h} - \langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}, \mathrm{G}\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = (\Pi_{\tau_n}\mathrm{L} - \mathrm{L}, \mathrm{G})_{\mathcal{T}_h}, \tag{8.1a}$$

$$(-\,\boldsymbol{\nabla\cdot}(\nu\mathrm{E}^{\mathrm{L}}) + \boldsymbol{\nabla}\,\varepsilon^{p}, \boldsymbol{v})_{\mathcal{T}_h} + \langle \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{v}\rangle_{\partial\mathcal{T}_h} = 0, \tag{8.1b}$$

$$-(\boldsymbol{\varepsilon}^{\boldsymbol{u}}, \boldsymbol{\nabla}\,q)_{\mathcal{T}_h} + \langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}, q\,\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{8.1c}$$

$$\langle \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}, \boldsymbol{\mu}\rangle_{\partial\Omega} = 0, \tag{8.1d}$$

$$\langle \nu\mathrm{E}^{\mathrm{L}}\boldsymbol{n} - \varepsilon^{p}\boldsymbol{n} - \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{8.1e}$$

$$(\varepsilon^{p}, 1)_{\Omega} = (\mathit{\Pi}_{\tau_n}p - p, 1)_{\Omega}. \tag{8.1f}$$

From these equations, it is possible to prove [9, Proposition 3.2] that

$$\begin{aligned} &\|\mathrm{E}^{\mathrm{L}}\|_{\Omega}^2 + \langle \tau_t(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t, (\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t\rangle_{\partial\mathcal{T}_h} \\ &\quad + \langle \tau_n(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})\cdot\boldsymbol{n}, (\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \;=\; (\Pi_{\tau_n}\mathrm{L} - \mathrm{L}, \mathrm{E}^{\mathrm{L}})_{\mathcal{T}_h}. \end{aligned} \tag{8.2}$$

Therefore,

$$\|\mathrm{E}^{\mathrm{L}}\|_{\Omega} = \|\Pi_{\tau_n}\mathrm{L} - \mathrm{L}_h^{\tau_n}\|_{\Omega} \le \|\Pi_{\tau_n}\mathrm{L} - \mathrm{L}\|_{\Omega} \tag{8.3}$$

and taking the limit as $\tau_n \to \infty$ (by Proposition 7.4 for the projection and Proposition 8.1 for the method) we obtain (3.2) directly.

Lemma 8.1. *With the stabilization tensor* S *given by* (2.3g) *it follows that*

$$|\langle \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{\mathit{\Pi}}\boldsymbol{w} - \boldsymbol{P}_M\boldsymbol{w}\rangle_{\partial\mathcal{T}_h}| \le C\nu\max_K(\tau_t h_K)^{1/2}\|\,\boldsymbol{\nabla}\,\boldsymbol{w}\|_{\Omega}\|\Pi_{\tau_n}\mathrm{L} - \mathrm{L}\|_{\Omega}$$

for all $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$.

Proof. We start by observing that on ∂K,

$$\begin{aligned} \tau_n(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})\cdot\boldsymbol{n} &= \tau_n(\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u} - \boldsymbol{P}_M\boldsymbol{u})\cdot\boldsymbol{n} - \tau_n(\boldsymbol{u}_h^{\tau_n} - \widehat{\boldsymbol{u}}_h^{\tau_n})\cdot\boldsymbol{n} \\ &= \tau_n(\boldsymbol{\mathit{\Pi}}_{\tau_n}\boldsymbol{u}\cdot\boldsymbol{n} - P_{\partial K}(\boldsymbol{u}\cdot\boldsymbol{n})) - \delta_h^{\tau_n} = \mathit{\Pi}_e^{\partial}\delta - \delta_h^{\tau_n}, \end{aligned}$$

by (2.4) and Proposition 7.3. Using (7.5f) (note that $\boldsymbol{\nabla\cdot}\,\boldsymbol{u} = 0$) and (2.5d) we prove that

$$\langle \mathit{\Pi}_e^{\partial}\delta - \delta_h^{\tau_n}, q\rangle_{\partial K} = 0 \qquad \forall q \in \mathcal{P}_k(K)^{\perp}.$$

Lemma 4.1 now implies that there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k(K)^\perp$ such that

$$\Pi_e^\partial \delta - \delta_h^{\tau_n} = \boldsymbol{v} \cdot \boldsymbol{n} \quad \text{on } \partial K.$$

Therefore,

$$\langle \tau_n(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}) \cdot \boldsymbol{n}, (\boldsymbol{\Pi w} - \boldsymbol{P}_M \boldsymbol{w}) \cdot \boldsymbol{n}\rangle_{\partial K} = \langle \boldsymbol{v} \cdot \boldsymbol{n}, (\boldsymbol{\Pi w} - \boldsymbol{w}) \cdot \boldsymbol{n}\rangle_{\partial K} = 0. \tag{8.4}$$

On the other hand,

$$\begin{aligned}
&|\langle \tau_t(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t, (\boldsymbol{\Pi w} - \boldsymbol{P}_M \boldsymbol{w})_t\rangle_{\partial K}| \\
&\qquad \le \tau_t^{1/2} \|\boldsymbol{\Pi w} - \boldsymbol{P}_{\partial K} \boldsymbol{w}\|_{\partial K} \langle \tau_t(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t, (\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t\rangle_{\partial K}^{1/2} \\
&\qquad \le (\tau_t h_K)^{1/2} \|\boldsymbol{\nabla} \boldsymbol{w}\|_K \langle \tau_t(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t, (\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t\rangle_{\partial K}^{1/2}
\end{aligned}$$

by Proposition 6.3 (with $\ell = 0$). Therefore,

$$\begin{aligned}
&|\langle \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{\Pi w} - \boldsymbol{P}_M \boldsymbol{w}\rangle_{\partial \mathcal{T}_h}| \\
&\qquad = \nu |\langle \tau_t(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t, (\boldsymbol{\Pi w} - \boldsymbol{P}_M \boldsymbol{w})_t\rangle_{\partial \mathcal{T}_h}| \\
&\qquad \le \nu \max_{K \in \mathcal{T}_h} (\tau_t h_K)^{1/2} \|\boldsymbol{\nabla} \boldsymbol{w}\|_\Omega \langle \tau_t(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t, (\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}})_t\rangle_{\partial \mathcal{T}_h}^{1/2} \\
&\qquad \le \nu \max_{K \in \mathcal{T}_h} (\tau_t h_K)^{1/2} \|\boldsymbol{\nabla} \boldsymbol{w}\|_\Omega \|\Pi_{\tau_n} \mathrm{L} - \mathrm{L}\|_\Omega
\end{aligned}$$

by (8.2) and (8.3). □

Proposition 8.2.

$$\|\Pi_{\tau_n} p - p_h\|_\Omega \le |\Omega| |\overline{\Pi_{\tau_n} p - p}| + C \nu C_t \|\Pi_{\tau_n} \mathrm{L} - \mathrm{L}\|_\Omega.$$

Proof. This is just a slight modification of the proof of [9, Proposition 3.4], which benefits from the introduction of the operator $\boldsymbol{\Pi}$ and of the previous lemma. We will just sketch the process. First, we bound

$$\|\varepsilon^p - \overline{(\Pi_{\tau_n} p - p)}\|_\Omega\| = \|\varepsilon^p - \overline{\varepsilon^p}\|_\Omega \le C_0 \sup_{\boldsymbol{0} \ne \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)} \frac{(\varepsilon^p, \nabla \cdot \boldsymbol{w})_\Omega}{\|\boldsymbol{\nabla} \boldsymbol{w}\|_\Omega},$$

thanks to a well-known result on the surjectivity of the divergence operator (cf. [16, Section I.5.1]. Using two of the error equations ((8.1b) and (8.1e)) we can write

$$(\varepsilon^p, \boldsymbol{\nabla \cdot w})_{\mathcal{T}_h} = (\nu \mathrm{E}^{\mathrm{L}}, \boldsymbol{\nabla} \boldsymbol{w})_{\mathcal{T}_h} + \langle \mathrm{S}(\boldsymbol{\varepsilon}^{\boldsymbol{u}} - \boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}), \boldsymbol{\Pi w} - \boldsymbol{P}_M \boldsymbol{w}\rangle_{\partial \mathcal{T}_h}$$

and the result is just a direct consequence of (8.3) and Lemma 8.1. □

Proof of Theorem 3.2: estimates for the gradient and pressure. As already mentioned, (3.2) is a straightforward consequence of (8.3) and of the fact that $\Pi_{\tau_n} \mathrm{L} \to \mathrm{L}$ (Proposition 7.4) and $L_h^{\tau_n} \to L_h$ (Proposition 8.1). Similarly, (3.3) is a limit in $\tau_n \to \infty$ of Proposition 8.2. We emphasize that the existing bound in [9, Theorem 2.4] was not finely tuned in the parameters τ_t and τ_n and that τ_n made an appearance in the right-hand side of the bound for ε^p. The introduction of the operator $\boldsymbol{\Pi}$ allows for a finer bound, where τ_n is not present in any of the constants of the right-hand side of the bound in Proposition 8.2.

8.3. **Estimates for the velocity.** As we have done with the estimate for the pressure, the estimate for $\boldsymbol{\varepsilon}^{\boldsymbol{u}}$ and $\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}$ in [9, Theorem 2.4] can be improved in terms of their dependence of the parameter τ_n.

Lemma 8.2. *Let $\boldsymbol{\theta} \in \boldsymbol{L}^2(\Omega)$ and Φ be the matrix-valued component of the solution of* (3.5)*. Assume that the regularity estimate* (3.6) *holds and that $\tau_n > \tau_t$. Then*

$$\nu \|\Pi_{\tau_n}\Phi - \Phi\|_\Omega \le C h^{\min\{k,1\}} \Big(1 + C_t^2 + \max_{K\in\mathcal{T}_h} \frac{\tau_t}{\tau_n}\Big) \|\boldsymbol{\theta}\|_\Omega.$$

Proof. In order to lighten some forthcoming notations, we will momentarily set $\underline{k} := \min\{k,1\}$. Let $(\Phi, \boldsymbol{\phi}, \phi)$ be the solution of (3.5). We now apply the convergence estimate for the projection (7.4) given in [9, Theorem 2.3] to bound

$$\begin{aligned}
\nu\|\Pi_{\tau_n}\Phi - \Phi\|_K &\le C h_K |\nu\Phi|_{\mathrm{H}^1(K)} + C\nu\tau_t \Big(\|\boldsymbol{\Pi}_{\tau_n}\boldsymbol{\phi} - \boldsymbol{\phi}\|_K + h_K^{1+\underline{k}} |\boldsymbol{\phi}|_{\boldsymbol{H}^{1+\underline{k}}(K)} \Big) \\
&\le C h_K |\nu\Phi|_{\mathrm{H}^1(K)} + C\nu\tau_t h_K^{1+\underline{k}} |\boldsymbol{\phi}|_{\boldsymbol{H}^{1+\underline{k}}(K)} \\
&\quad + C'\tau_t \frac{h_K}{\max\{\tau_n, \tau_t\}} \|\nabla\cdot(\nu\Phi - \phi \mathrm{I})\|_K \\
&\le C'' h^{\underline{k}} \Big(|\nu\Phi|_{\mathrm{H}^1(K)} + \tau_t h_K |\nu\boldsymbol{\phi}|_{\boldsymbol{H}^{1+\underline{k}}(K)} + \frac{\tau_t}{\tau_n} \|\boldsymbol{\theta}\|_K \Big).
\end{aligned}$$

Adding the contributions of all the elements $K \in \mathcal{T}_h$ and using (3.6), the result follows readily. ☐

Proof of Theorem 3.2: estimates of the velocity. Corollary 3.8 of [9] shows that

$$\begin{aligned}
\|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_\Omega + \|\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}\|_h &\le C \max\Big\{ h, \nu \sup_{\boldsymbol{0}\neq\boldsymbol{\theta}\in\boldsymbol{L}^2(\Omega)} \frac{\|\Pi_{\tau_n}\Phi - \Phi\|_\Omega + \|\mathrm{P}_{k-1}\Phi - \Phi\|_\Omega}{\|\boldsymbol{\theta}\|_\Omega} \Big\} \\
&\quad \times \|\Pi_{\tau_n}\mathrm{L} - \mathrm{L}\|_\Omega,
\end{aligned}$$

where Φ is the matrix-valued component of the solution of (3.5). Here P_{k-1} is the $\mathrm{L}^2(\Omega)$ orthogonal projection onto the space of discontinuous piecewise $\mathrm{P}_{k-1}(K)$ functions. (When $k = 0$ we just take $\mathrm{P}_{k-1} = 0$.) Therefore, using (3.6), it follows that

$$\nu\|\mathrm{P}_{k-1}\Phi - \Phi\|_\Omega \le C h^{\min\{k,1\}} \nu |\Phi|_{\mathrm{H}^{\min\{k,1\}}(\Omega)} \le C' h^{\min\{k,1\}} \|\boldsymbol{\theta}\|_\Omega.$$

Using this and Lemma 8.2, we prove that

$$\|\boldsymbol{\varepsilon}^{\boldsymbol{u}}\|_\Omega + \|\boldsymbol{\varepsilon}^{\widehat{\boldsymbol{u}}}\|_h \le C h^{\min\{k,1\}} \Big(1 + C_t^2 + \frac{1}{\tau_n} \max_{K\in\mathcal{T}_h} \tau_t\Big) \|\Pi_{\tau_n}\mathrm{L} - \mathrm{L}\|_\Omega$$

and we can now take the limit as $\tau_n \to \infty$ using Propositions 7.4 and 8.1 in order to prove (3.8).

The proof of superconvergence for the postprocessed velocity inherits the estimate (3.8), since the postprocessing method does not depend on the particular method but on the quality of that estimate. Therefore, the proof of the corresponding result [9, Theorem 2.5] does not need any adaptation to be used in our new situation.

9. Extension to other boundary conditions

Let us end this paper by pointing out that all our results can be easily extended to the case of Neumann-like boundary conditions

$$\nu \mathrm{L}\boldsymbol{n} + p\,\boldsymbol{n} = \boldsymbol{g} \quad \text{ on } \partial\Omega.$$

On the other hand, the error analysis carried out in [9] does *not* seem to work for the case in which the boundary data is

$$\nu \frac{1}{2}(\mathrm{L} + \mathrm{L}^t)\boldsymbol{n} + p\,\boldsymbol{n} = \boldsymbol{g} \quad \text{ on } \partial\Omega,$$

where L^t is the transpose of L. In [21], a variation of the method is used and the numerical experiments provided suggest that the order of convergence of all the variables remains optimal. However, the superconvergence of the velocity seems to be lost. It is reasonable to believe that a similar result would hold in our case. How to deal with the last boundary conditions remains an open problem and constitutes the subject of ongoing research.

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