ON THE NUMBER OF PRIME FACTORS OF AN ODD PERFECT NUMBER

PASCAL OCHEM AND MICHAËL RAO

Abstract. Let \( \Omega(n) \) and \( \omega(n) \) denote, respectively, the total number of prime factors and the number of distinct prime factors of the integer \( n \). Euler proved that an odd perfect number \( N \) is of the form \( N = p^e m^2 \) where \( p \equiv e \equiv 1 \pmod{4}, \) \( p \) is prime, and \( p \nmid m \). This implies that \( \Omega(N) \geq 2\omega(N) - 1 \). We prove that \( \Omega(N) \geq (18\omega(N) - 31)/7 \) and \( \Omega(N) \geq 2\omega(N) + 51 \).

1. Introduction

A natural number \( N \) is said to be perfect if it is equal to the sum of its positive divisors (excluding \( N \)). It is well known that an even natural number \( N \) is perfect if and only if \( N = 2^{k-1}(2^k - 1) \) for an integer \( k \) such that \( 2^k - 1 \) is a Mersenne prime. On the other hand, it is a long-standing open question whether an odd perfect number exists.

In order to investigate this question, several authors gave necessary conditions for the existence of an odd perfect number \( N \). Let \( \Omega(n) \) and \( \omega(n) \) denote, respectively, the total number of prime factors and the number of distinct prime factors of the integer \( n \). Euler proved that \( N = p^e m^2 \) for a prime \( p \), with \( p \equiv e \equiv 1 \pmod{4} \), \( p \) is prime, and \( p \nmid m \). Moreover, recent results showed that \( N > 10^{1500} \) \cite{4}, \( \omega(N) \geq 9 \) \cite{3}, and \( \Omega(N) \geq 101 \) \cite{4}.

In this paper, we study the relationship between \( \Omega(N) \) and \( \omega(N) \). By Euler’s result, we have \( \Omega(N) \geq 2\omega(N) - 1 \). Steuerwald \cite{6} proved that \( m \) is not square-free, that is, the exponents of the non-special primes cannot be all equal to 2. This implies that \( \Omega(N) \geq 2\omega(N) + 1 \). We improve this inequality in two ways:

Theorem 1. If \( N \) is an odd perfect number, then \( \Omega(N) \geq (18\omega(N) - 31)/7 \).

Theorem 2. If \( N \) is an odd perfect number, then \( \Omega(N) \geq 2\omega(N) + 51 \).

We prove Theorem 1 in Section 3 using standard arguments. We prove Theorem 2 in Section 4 via computations using the general method in \cite{4}.

To summarize the known results for \( \Omega(N) \), we have

\[
\Omega(N) \geq \max \{101, 2\omega(N) + 51, (18\omega(N) - 31)/7\}
\]

2. Preliminaries

Let \( n \) be a natural number. Let \( \sigma(n) \) denote the sum of the positive divisors of \( n \), and let \( \sigma_{-1}(n) = \frac{\sigma(n)}{n} \) be the abundancy of \( n \). Clearly, \( n \) is perfect if and only if \( \sigma_{-1}(n) = 2 \). We first recall some easy results on the functions \( \sigma \) and \( \sigma_{-1} \). If \( p \) is
prime, \( \sigma(p^q) = \frac{q^{q+1} - 1}{p^q - 1} \), and \( \sigma_1(p^\infty) = \lim_{q \to +\infty} \sigma_1(p^q) = \frac{p}{p-1} \). If \( \gcd(a, b) = 1 \), then \( \sigma(ab) = \sigma(a)\sigma(b) \) and \( \sigma_1(ab) = \sigma_1(a)\sigma_1(b) \).

Euler proved that if an odd perfect number \( N \) exists, then it is of the form \( N = p^e m^2 \) where \( p \equiv 1 \pmod{4} \), \( p \) is prime, and \( p \nmid m \). The prime \( p \) is said to be the special prime.

3. **Proof of** \( \Omega(N) \geq (18\omega(N) - 31)/7 \)

We want to obtain a result of the form \( \Omega(N) \geq a\omega(N) - c \) for some \( a > 2 \) using the following idea. If \( a \) is close to 2, then \( N \) has a large number of prime factors \( p \) such that both \( p^2 \parallel N \) and \( p \parallel \sigma(q^2) \) where \( q^2 \parallel N \). It is well known (see [5]) that for primes \( t, r, \) and \( s \) such that \( t \mid \sigma(r^{s-1}) \), either \( t = s \) or \( t \equiv 1 \pmod{s} \). In particular, this gives \( p \equiv 1 \pmod{3} \) and thus \( 3 \mid \sigma(p^2) \). The exponent of the prime 3 is then large, so that \( \Omega(N) \) is significantly greater than \( 2\omega(N) \).

Now we detail the number of certain types of factors of \( N \) and obtain the results by contradiction with the involved quantities.

- \( p = \omega(N) \): number of distinct prime factors,
- \( f = \Omega(N) \): total number of prime factors,
- \( p_2 \): number of distinct prime factors with exponent 2, distinct from 3,
- \( p_{2,1} \): number of distinct prime factors with exponent 2 congruent to 1 mod 3,
- \( p_4 \): number of distinct prime factors with exponent at least 4, distinct from 3 and the special prime,
- \( f_4 \): total number of prime factors with exponent at least 4, distinct from 3 and the special prime,
- \( e \): exponent of the special prime,
- \( f_3 \): exponent of the prime 3.

Now we obtain useful inequalities among these quantities. The special exponent is at least 1:

1. \( 1 \leq e \).

By detailing the total number of prime factors, we have

2. \( e + f_3 + 2p_2 + f_4 = f \).

By considering the prime factors (distinct from 3 and the special prime) with exponent at least 4, we have

3. \( 4p_4 \leq f_4 \).

As already mentioned, if \( p \equiv 1 \pmod{3} \) and \( p^2 \parallel N \), then \( 3 \mid \sigma(p^2) \), so that

4. \( p_{2,1} \leq f_3 \).

Let us consider the number of distinct prime factors. We have the special prime, the primes from \( p_2 \) and \( p_4 \), and maybe the prime 3. So it is \( 1 + p_2 + p_4 \) if \( f_3 = 0 \) and \( 2 + p_2 + p_4 \) if \( f_3 \geq 2 \). Thus, we have

5. \( p \leq f_3/2 + 1 + p_2 + p_4 \)

and

6. \( p \leq 2 + p_2 + p_4 \).
For the sake of contradiction, we suppose that

(7) \[ 7f \leq 18p - 32. \]

The following lemma is useful to obtain one last inequality:

**Lemma 3.** Let \( p \), \( q \), and \( r \) be positive integers. If \( p^2 + p + 1 = r \) and \( q^2 + q + 1 = 3r \), then \( p \) is not an odd prime.

**Proof.** Since \( q^2 + q + 1 \equiv 0 \mod 3 \), then \( q \equiv 1 \mod 3 \) and we set \( q = 3s + 1 \). The equality \( q^2 + q + 1 = 3(p^2 + p + 1) \) reduces to \( 3s(s + 1) = p(p + 1) \). Notice that \( p \) divides \( 3s(s + 1) \), so that if \( p \) is an odd prime, then either \( p \mid 3 \), \( p \mid s \), or \( p \mid (s + 1) \). We have \( p = 3 \) in the first case, which gives no solution. We have \( s \geq p - 1 \) in the other two cases, so that \( p(p + 1) = 3s(s + 1) \geq 3(p - 1)p \). This gives \( p + 1 \geq 3(p - 1) \), so that \( p \leq 2 \), which is a contradiction. \( \square \)

Let \( K \) be the multiset of all the primes distinct from 3 produced by all the components \( \sigma(p^2) \) of \( N \). The primes in \( K \) are 1 mod 3, so \( |K| \leq e + 2p_{2,1} + f_4 \). For a prime \( u > 3 \), let \( \alpha(u) \) be such that \( \alpha(u) = \sigma(u^2) \) if \( u \equiv 2 \mod 3 \) and \( \alpha(u) = \sigma(u^2)/3 \) if \( u \equiv 1 \mod 3 \). By Lemma 3 \( \alpha(u) = \alpha(v) \) implies \( u = v \). So all primes from \( p_2 \) produce at least two prime factors, except for at most one per distinct prime from \( K \). That is, \( 2p_2 - 1 - p_{2,1} - p_4 \leq |K| \). Thus, we have \( 2p_2 - 1 - p_{2,1} - p_4 \leq e + 2p_{2,1} + f_4 \), which gives

(8) \[ 2p_2 \leq 1 + e + 3p_{2,1} + p_4 + f_4. \]

The combination \( 5 \times \) \( \boxed{1} \) \( + 7 \times \) \( \boxed{2} \) \( + 5 \times \) \( \boxed{3} \) \( + 6 \times \) \( \boxed{4} \) \( + 2 \times \) \( \boxed{5} \) \( + 16 \times \) \( \boxed{6} \) \( + \) \( \boxed{7} \) \( + 2 \times \) \( \boxed{8} \) gives \( 1 \leq 0 \), a contradiction. This means that for assumption \( \boxed{7} \) that \( 7f \leq 18p - 32 \) is false, and thus \( \Omega(N) \geq (18\omega(N) - 31)/7 \).

4. **Proof of** \( \Omega(N) \geq 2\omega(N) + 51 \)

We use the general method and the computer program discussed in [4].

We use the following contradictions:

- The abundancy of the current number is strictly greater than 2.
- The current number \( n \) satisfies \( \Omega(n) \geq 2\omega(n) + 51 \).

We forbid the factors in \( S = \{3, 5, 7, 11, 13, 17, 19\} \), in this order. We branch on the smallest available prime congruent to 1 mod 3. If there is no such prime, we branch on the smallest available prime congruent to 2 mod 3. We still use a combination of exact branchings and standard branchings, as in [4]. We use exact branchings only for the special components \( p^4 \) and for all the even powers \( 3^{2e} \) of 3.

**By-passing roadblocks.** A *roadblock* is a situation such that there is no contradiction and no possibility to branch on a prime. This happens when we have already made suppositions for the multiplicity of all the known primes and the other numbers are composites.

Given a roadblock \( M \), we check that the composites involved are not divisible by an already considered prime, are not perfect powers, have no factor less than 10^{10}, and are pairwise coprime. Then we compute the following quantities:

- \( F \): It is a lower bound on the number of distinct prime factors of \( M \).
- \( M \): It is a lower bound on the number of distinct prime factors of \( M \) plus two primes per composite number.
\(A\): It is an upper bound on the abundancy of \(M\). For the abundancy of a component \(p^k\), we use \(\sigma_1(p^k)\) for an exact branching and \(\sigma_1(p^\infty) = p/(p - 1)\) for a standard branching.

For a composite \(C\), we know that \(C\) has at most \(\lfloor \frac{\ln C}{10\ln 10} \rfloor\) prime factors since \(C\) has no factor less than \(10^{10}\). So, the abundancy due to \(C\) is at most \(\lceil (1 + 10^{-10})^{\ln C/10\ln 10} \rceil\).

\(T\): It is the target lower bound on \(\Omega(N) - 2\omega(N)\), thus an odd integer. We use \(T = 51\) in the proof of Theorem 2.

For the sake of contradiction, we suppose that \(\Omega(N) - 2\omega(N) \leq T - 2\). By Theorem 1, we have \(\Omega(N) \geq (18\omega(N) - 31)/7\). So \((18\omega(N) - 31)/7 - 2\omega(N) \leq \Omega(N) - 2\omega(N) \leq T - 2\), which gives \(\omega(N) \leq (7T + 17)/4\). Thus, \(N\) has at most \(\omega(N) \leq (7T + 17)/4\) prime factors that do not divide \(M\). Let \(p\) be the smallest of these extra factors. We see that if
\[
A(p/(p - 1))^{(7T+17)/4-F} < 2,
\]
then \(N\) cannot reach abundancy 2. This gives an upper bound on \(p\). To get around the roadblock, we branch on every prime number \(p\) (except those that divide \(M\) or are already forbidden) in increasing order until (9) is satisfied.

**Example.**
\[
3^4 \Rightarrow 11^2 \\
11^{18} \Rightarrow 611509044841454629 \\
611509044841454629^{16} \Rightarrow \sigma(611509044841454629^{16}) \quad \text{Roadblock 1} \\
5^1 \Rightarrow 2 \times 3 \quad \text{Roadblock 2}
\]

We first branch on the components \(3^4, 11^{18}\), and \(\sigma(11^{18})^{16}\) and hit a first roadblock, as no factors of \(C_1 = \sigma(\sigma(11^{18})^{16})\) are known. When trying to get around this roadblock, we first branch on \(5^1\) and hit a second roadblock. Consider this second roadblock:

- \(F = 6\): We have the four primes 3, 5, 11, \(\sigma(11^{18})\), and at least two primes from \(C_1\).
- \(A = \sigma_1(3^4 \times 5 \times 11^{\infty} \times \sigma(11^{18})^{\infty}) \times (1 + 10^{-10})^{\ln C/10\ln 10} = 1.9718518\ldots\).
- \(T = 51\).

Equation (9) is satisfied for \(p \geq 6174\), so to circumvent \(M\), we branch on every prime \(p\) between 7 and 6173, except 11.

**When \(N\) has no factors in \(S\).** If \(N\) has no factor in \(S\), then it must have at least 115 distinct prime factors. We obtain this by considering the product \(\Pi_{23 \leq p \leq 673} \frac{p-1}{p}= 1.99807632\ldots\) over the first 114 primes \(p\) greater than 19, which is an upper bound on the abundancy and is smaller than 2.

Using Theorem 1, we obtain
\[
\Omega(N) - 2\omega(N) \geq (18\omega(N) - 31)/7 - 2\omega(N) \\
= (4\omega(N) - 31)/7 \\
\geq (4 \times 115 - 31)/7 \\
= 61 + 2/7.
\]

So, we have \(\Omega(N) \geq 2\omega(N) + 62\), which concludes the proof of Theorem 2.
ACKNOWLEDGMENT

We thank Robert Gerbicz for a much simpler proof of Lemma 3.

REFERENCES


CNRS, LIRMM, Université Montpellier 2, 161 rue Ada, 34095 Montpellier Cedex 5, France
E-mail address: ochem@lirmm.fr

CNRS, LIP, ENS Lyon, 15 parvis R. Descartes BP 7000, 69342 Lyon Cedex 07, France
E-mail address: michael.rao@ens-lyon.fr