ON THE NUMBER OF PRIME FACTORS OF AN ODD PERFECT NUMBER

PASCAL OCHEM AND MICHAËL RAO

Abstract. Let $\Omega(n)$ and $\omega(n)$ denote, respectively, the total number of prime factors and the number of distinct prime factors of the integer $n$. Euler proved that an odd perfect number $N$ is of the form $N = p^e m^2$ where $p \equiv e \equiv 1 \pmod{4}$, $p$ is prime, and $p \nmid m$. This implies that $\Omega(N) \geq 2\omega(N) - 1$. We prove that $\Omega(N) \geq (18\omega(N) - 31)/7$ and $\Omega(N) \geq 2\omega(N) + 51$.

1. Introduction

A natural number $N$ is said to be perfect if it is equal to the sum of its positive divisors (excluding $N$). It is well known that an even natural number $N$ is perfect if and only if $N = 2^k - 1(2^k - 1)$ for an integer $k$ such that $2^k - 1$ is a Mersenne prime. On the other hand, it is a long-standing open question whether an odd perfect number exists.

In order to investigate this question, several authors gave necessary conditions for the existence of an odd perfect number $N$. Let $\Omega(n)$ and $\omega(n)$ denote, respectively, the total number of prime factors and the number of distinct prime factors of the integer $n$. Euler proved that $N = p^e m^2$ for a prime $p$, with $p \equiv e \equiv 1 \pmod{4}$, $p$ is prime, and $p \nmid m$. Moreover, recent results showed that $N > 10^{1500}$ [4], $\omega(N) \geq 9$ [3], and $\Omega(N) \geq 101$ [4].

In this paper, we study the relationship between $\Omega(N)$ and $\omega(N)$. By Euler’s result, we have $\Omega(N) \geq 2\omega(N) - 1$. Steuerwald [6] proved that $m$ is not square-free, that is, the exponents of the non-special primes cannot be all equal to 2. This implies that $\Omega(N) \geq 2\omega(N) + 1$. We improve this inequality in two ways:

Theorem 1. If $N$ is an odd perfect number, then $\Omega(N) \geq (18\omega(N) - 31)/7$.

Theorem 2. If $N$ is an odd perfect number, then $\Omega(N) \geq 2\omega(N) + 51$.

We prove Theorem 1 in Section 3 using standard arguments. We prove Theorem 2 in Section 4 via computations using the general method in [4].

To summarize the known results for $\Omega(N)$, we have

$$\Omega(N) \geq \max \{101, 2\omega(N) + 51, (18\omega(N) - 31)/7\}.$$ 

2. Preliminaries

Let $n$ be a natural number. Let $\sigma(n)$ denote the sum of the positive divisors of $n$, and let $\sigma_{-1}(n) = \frac{\sigma(n)}{n}$ be the abundancy of $n$. Clearly, $n$ is perfect if and only if $\sigma_{-1}(n) = 2$. We first recall some easy results on the functions $\sigma$ and $\sigma_{-1}$. If $p$ is
prime, $\sigma(p^q) = \frac{p^{q+1} - 1}{p-1}$, and $\sigma_1(p^\infty) = \lim_{q \rightarrow +\infty} \sigma_1(p^q) = \frac{p}{p-1}$. If $\gcd(a, b) = 1$, then $\sigma(ab) = \sigma(a)\sigma(b)$ and $\sigma_1(ab) = \sigma_1(a)\sigma_1(b)$.

Euler proved that if an odd perfect number $N$ exists, then it is of the form $N = p^e m^2$ where $p \equiv e \equiv 1 \pmod{4}$, $p$ is prime, and $p \nmid m$. The prime $p$ is said to be the special prime.

3. Proof of $\Omega(N) \geq \frac{(18\omega(N) - 31)}{7}$

We want to obtain a result of the form $\Omega(N) \geq a\omega(N) - c$ for some $a > 2$ using the following idea. If $a$ is close to 2, then $N$ has a large number of prime factors $p$ such that both $p^2 \parallel N$ and $p \parallel \sigma(q^2)$ where $q^2 \parallel N$. It is well known (see [5]) that for primes $t$, $r$, and $s$ such that $t \mid \sigma(p^{s-1})$, either $t = s$ or $t \equiv 1 \pmod{s}$. In particular, this gives $p \equiv 1 \pmod{3}$ and thus $3 \mid \sigma(p^2)$. The exponent of the prime 3 is then large, so that $\Omega(N)$ is significantly greater than $2\omega(N)$.

Now we detail the number of certain types of factors of $N$ and obtain the results by contradiction with the involved quantities.

- $p = \omega(N)$: number of distinct prime factors,
- $f = \Omega(N)$: total number of prime factors,
- $p_2$: number of distinct prime factors with exponent 2, distinct from 3,
- $p_{2,1}$: number of distinct prime factors with exponent 2 congruent to 1 mod 3,
- $p_4$: number of distinct prime factors with exponent at least 4, distinct from 3 and the special prime,
- $f_4$: total number of prime factors with exponent at least 4, distinct from 3 and the special prime,
- $e$: exponent of the special prime,
- $f_3$: exponent of the prime 3.

Now we obtain useful inequalities among these quantities. The special exponent is at least 1:

(1) \[ 1 \leq e. \]

By detailing the total number of prime factors, we have

(2) \[ e + f_3 + 2p_2 + f_4 = f. \]

By considering the prime factors (distinct from 3 and the special prime) with exponent at least 4, we have

(3) \[ 4p_4 \leq f_4. \]

As already mentioned, if $p \equiv 1 \pmod{3}$ and $p^2 \parallel N$, then $3 \mid \sigma(p^2)$, so that

(4) \[ p_{2,1} \leq f_3. \]

Let us consider the number of distinct prime factors. We have the special prime, the primes from $p_2$ and $p_4$, and maybe the prime 3. So it is $1 + p_2 + p_4$ if $f_3 = 0$ and $2 + p_2 + p_4$ if $f_3 \geq 2$. Thus, we have

(5) \[ p \leq f_3/2 + 1 + p_2 + p_4 \]

and

(6) \[ p \leq 2 + p_2 + p_4. \]
For the sake of contradiction, we suppose that

\[(7) \quad 7f \leq 18p - 32.\]

The following lemma is useful to obtain one last inequality:

**Lemma 3.** Let \( p, q, \) and \( r \) be positive integers. If \( p^2 + p + 1 = r \) and \( q^2 + q + 1 = 3r \), then \( p \) is not an odd prime.

**Proof.** Since \( q^2 + q + 1 \equiv 0 \mod 3 \), then \( q \equiv 1 \mod 3 \) and we set \( q = 3s + 1 \). The equality \( q^2 + q + 1 = 3(p^2 + p + 1) \) reduces to \( 3s(s+1) = p(p+1) \). Notice that \( p \) divides \( 3s(s+1) \), so that if \( p \) is an odd prime, then either \( p \mid 3 \), \( p \mid s \), or \( p \mid (s+1) \).

We have \( p = 3 \) in the first case, which gives no solution. We have \( s \geq p - 1 \) in the other two cases, so that \( p(p+1) = 3s(s+1) \geq 3(p-1)p \). This gives \( p+1 \geq 3(p-1) \), so that \( p \leq 2 \), which is a contradiction. \( \square \)

Let \( K \) be the multiset of all the primes distinct from 3 produced by all the components \( \sigma(p^2) \) of \( N \). The primes in \( K \) are 1 mod 3, so \( |K| \leq e + 2p_{2,1} + f_4 \). For a prime \( u > 3 \), let \( \alpha(u) \) be such that \( \alpha(u) = \sigma(u^2) \) if \( u \equiv 2 \mod 3 \) and \( \alpha(u) = \sigma(u^2)/3 \) if \( u \equiv 1 \mod 3 \). By Lemma 3, \( \alpha(u) = \alpha(v) \) implies \( u = v \). So all primes from \( p_2 \) produce at least two prime factors, except for at most one per distinct prime from \( K \). That is, \( 2p_2 - 1 - p_{2,1} - p_4 \leq |K| \). Thus, we have \( 2p_2 - 1 - p_{2,1} - p_4 \leq e + 2p_{2,1} + f_4 \), which gives

\[(8) \quad 2p_2 \leq 1 + e + 3p_{2,1} + p_4 + f_4.\]

The combination \( 5 \times (1) + 7 \times (2) + 5 \times (3) + 6 \times (4) + 2 \times (5) + 16 \times (6) + (7) + 2 \times (8) \) gives \( 1 \leq 0 \), a contradiction. This means that for assumption (7) that \( 7f \leq 18p - 32 \) is false, and thus \( \Omega(N) \geq (18\omega(N) - 31)/7 \).

4. **Proof of** \( \Omega(N) \geq 2\omega(N) + 51 \)

We use the general method and the computer program discussed in [4].

We use the following contradictions:

- The abundancy of the current number is strictly greater than 2.
- The current number \( n \) satisfies \( \Omega(n) \geq 2\omega(n) + 51 \).

We forbid the factors in \( S = \{3, 5, 7, 11, 13, 17, 19\} \), in this order. We branch on the smallest available prime congruent to 1 mod 3. If there is no such prime, we branch on the smallest available prime congruent to 2 mod 3. We still use a combination of exact branchings and standard branchings, as in [4]. We use exact branchings only for the special components \( p^1 \) and for all the even powers \( 3^{2e} \) of 3.

**By-passing roadblocks.** A roadblock is a situation such that there is no contradiction and no possibility to branch on a prime. This happens when we have already made suppositions for the multiplicity of all the known primes and the other numbers are composites.

Given a roadblock \( M \), we check that the composites involved are not divisible by an already considered prime, are not perfect powers, have no factor less than \( 10^{10} \), and are pairwise coprime. Then we compute the following quantities:

- **F**: It is a lower bound on the number of distinct prime factors of \( M \). We count the number of known prime factors of \( M \) plus two primes per composite number.
• **A**: It is an upper bound on the abundancy of $M$. For the abundancy of a component $p^k$, we use $\sigma_1(p^k)$ for an exact branching and $\sigma_1(p^\infty) = p/(p - 1)$ for a standard branching.

  For a composite $C$, we know that $C$ has at most \(\left\lfloor \frac{\ln C}{10\ln 10} \right\rfloor\) prime factors since $C$ has no factor less than $10^{10}$. So, the abundancy due to $C$ is at most $(1 + 10^{-10})^{\left\lfloor \frac{\ln C}{10\ln 10} \right\rfloor}$.

• **T**: It is the target lower bound on $\Omega(N) - 2\omega(N)$, thus an odd integer. We use $T = 51$ in the proof of Theorem 2.

For the sake of contradiction, we suppose that $\Omega(N) - 2\omega(N) \leq T - 2$. By Theorem 1, we have $\Omega(N) \geq (18\omega(N) - 31)/7$. So $(18\omega(N) - 31)/7 - 2\omega(N) \leq \Omega(N) - 2\omega(N) \leq T - 2$, which gives $\omega(N) \leq (7T + 17)/4$. Thus, $N$ has at most $\omega(N) \leq (7T + 17)/4 - F$ prime factors that do not divide $M$. Let $p$ be the smallest of these extra factors. We see that if

\[(9) \quad A(p/(p - 1))^{(7T + 17)/4 - F} < 2,
\]

then $N$ cannot reach abundancy 2. This gives an upper bound on $p$. To get around the roadblock, we branch on every prime number $p$ (except those that divide $M$ or are already forbidden) in increasing order until (9) is satisfied.

**Example.**

\[3^4 \implies 11^2\]

\[11^{18} \implies 6115909044841454629\]

\[6115909044841454629^{16} \implies \sigma(6115909044841454629^{16}) \quad \text{Roadblock 1}\]

\[5^1 \implies 2 \times 3 \quad \text{Roadblock 2}\]

We first branch on the components $3^4$, $11^{18}$, and $\sigma(11^{18})^{16}$ and hit a first roadblock, as no factors of $C_1 = \sigma(\sigma(11^{18})^{16})$ are known. When trying to get around this roadblock, we first branch on $5^1$ and hit a second roadblock. Consider this second roadblock:

• **F** = 6: We have the four primes 3, 5, 11, $\sigma(11^{18})$, and at least two primes from $C_1$.

• $A = \sigma_1\left(3^4 \times 5 \times 11^\infty \times \sigma(11^{18})^\infty\right) \times (1 + 10^{-10})^{\left\lfloor \frac{\ln C}{10\ln 10} \right\rfloor} = 1.9718518 \ldots$.

• **T** = 51.

Equation (9) is satisfied for $p \geq 6174$, so to circumvent $M$, we branch on every prime $p$ between 7 and 6173, except 11.

**When $N$ has no factors in $S$.** If $N$ has no factor in $S$, then it must have at least 115 distinct prime factors. We obtain this by considering the product

\[\prod_{23 \leq p \leq 673} \frac{p^2 - 1}{p - 1} = 1.99807632 \ldots \]

over the first 114 primes $p$ greater than 19, which is an upper bound on the abundancy and is smaller than 2.

Using Theorem 1, we obtain

\[\Omega(N) - 2\omega(N) \geq (18\omega(N) - 31)/7 - 2\omega(N)\]

\[= (4\omega(N) - 31)/7\]

\[\geq (4 \times 115 - 31)/7\]

\[= 61 + 2/7.\]

So, we have $\Omega(N) \geq 2\omega(N) + 62$, which concludes the proof of Theorem 2.
ACKNOWLEDGMENT

We thank Robert Gerbicz for a much simpler proof of Lemma [3]

REFERENCES


CNRS, LIRMM, Université Montpellier 2, 161 rue Ada, 34095 Montpellier Cedex 5, France
E-mail address: ochem@lirmm.fr

CNRS, LIP, ENS Lyon, 15 parvis R. Descartes BP 7000, 69342 Lyon Cedex 07, France
E-mail address: michael.rao@ens-lyon.fr