

SPOOF ODD PERFECT NUMBERS

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ABSTRACT. In 1638, Descartes showed that $3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021^1$ would be an odd perfect number if 22021 were prime. We give a formal definition for such “spooft” odd perfect numbers, and construct an algorithm to find all such integers with a given number of distinct quasi-prime factors. We show that Descartes’ example is the only spooft with less than seven such factors.

INTRODUCTION

One of the oldest unsolved problems in mathematics is that of the existence of odd perfect numbers. Euclid and Euler categorized completely the even perfect numbers as those numbers of the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are primes. There are astonishingly few theoretical results for odd perfect numbers. Most results are computational. However, it is known that any odd perfect number must be of the form $n = \pi^\alpha m^2$, where π is prime, $\pi \nmid m$, and $\pi \equiv \alpha \equiv 1 \pmod{4}$. Euler proved this by considering the 2-adic information about n and $\sigma(n)$. It is also known that if n is an odd perfect number with k distinct prime factors, then $n < 2^{4^k}$ and $k \geq 9$ (see [5] and [6]).

If 22021 were prime, Descartes showed that $3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021^1$ would be an odd perfect number [2, p. 12]. By generalizing the definition of the σ -function, we give a formal definition of “spooft” odd perfect numbers. We prove the surprising fact that Descartes’ number is the only example with less than seven quasi-prime factors.

1. THE SPOOF σ -FUNCTION

Recall that for $z \in \mathbb{C}$ and for $n \in \mathbb{N}$ we can define the multiplicative (but not totally multiplicative) function

$$\sigma_z(n) = \sum_{d|n} d^z.$$

Notice that $\sigma_1(n) = \sigma(n)$ is the usual sum of divisors function, and n is a perfect number if and only if $\sigma(n) = 2n$, or equivalently $\sigma_{-1}(n) = 2$.

Definition 1.1 (Spooft σ -function). For $x, \alpha \in \mathbb{N}$ with $x \geq 2$ and $\alpha \geq 1$, the *spooft* σ -function is $\tilde{\sigma}(x, \alpha) = \frac{x^{\alpha+1}-1}{x-1}$. We also define $\tilde{\sigma}_{-1}(x, \alpha) = \frac{x^{\alpha+1}-1}{x^\alpha(x-1)} = \frac{\tilde{\sigma}(x, \alpha)}{x^\alpha}$. We write $\tilde{\sigma}_{-1}(x, \infty) = \lim_{\alpha \rightarrow \infty} \tilde{\sigma}_{-1}(x, \alpha) = \frac{x}{x-1}$.

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Throughout this paper, for convenience, we refer primarily to $\tilde{\sigma}_{-1}$, rather than $\tilde{\sigma}$. Note that for p prime, $\sigma_{-1}(p^\alpha) = \tilde{\sigma}_{-1}(p, \alpha)$. Further, note that $\tilde{\sigma}_{-1}(x, \alpha)$ is decreasing with respect to x and increasing with respect to α . In fact, the following is true:

Lemma 1.2. *For every integer $x \geq 2$, we have*

$$\tilde{\sigma}_{-1}(x + 1, \infty) = \frac{x + 1}{x} = \tilde{\sigma}_{-1}(x, 1) < \tilde{\sigma}_{-1}(x, 2) < \dots < \tilde{\sigma}_{-1}(x, \infty) = \frac{x}{x - 1}.$$

Proof. This follows immediately from elementary algebra. □

The role of the $\tilde{\sigma}$ -function is analogous to that of the σ -function on prime powers. Our next task is to model the multiplicative nature of σ .

Definition 1.3 (Quasi-prime factorization). We define a *quasi-prime factorization* to be a set of pairs $X = \{(x_1, \alpha_1), \dots, (x_k, \alpha_k)\}$, where $x_i, \alpha_i \in \mathbb{N}$, $x_i \geq 2$ (for each i). We call the x_i 's the (*quasi-prime*) *factors*, and say that this factorization has k distinct quasi-prime factors. Notice that the factors are not assumed to be relatively prime or even distinct. Throughout, we adopt the convention $1 < x_1 \leq x_2 \leq \dots \leq x_k$.

We define $\tilde{\sigma}_{-1}(X) = \prod_{i=1}^k \tilde{\sigma}_{-1}(x_i, \alpha_i)$. For simplicity, we may say that X is a quasi-prime factorization of $n = \prod_{i=1}^k x_i^{\alpha_i}$.

Definition 1.4 (Spoon perfect number). A *spoon perfect number* is a quasi-prime factorization $X = \{(x_i, \alpha_i)\}_{i \leq k}$ satisfying $\tilde{\sigma}_{-1}(X) = \prod_{i=1}^k \tilde{\sigma}_{-1}(x_i, \alpha_i) = 2$. We also say that $n = \prod_{i=1}^k x_i^{\alpha_i}$ is a spoon perfect number under the given quasi-prime factorization.

We now restate some results for odd perfect numbers that apply to spoon odd perfect numbers as well:

Lemma 1.5 (Eulerian form). *If $n = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$ is a spoon odd perfect number with these k quasi-prime factors, then there is some index i such that $x_i \equiv \alpha_i \equiv 1 \pmod{4}$, and for every index $j \neq i$, $\alpha_j \equiv 0 \pmod{2}$.*

Proof. The proof of this lemma follows, *mutatis mutandis*, from the proof of the corresponding results for odd perfect numbers, given in [3]. □

Lemma 1.6. *If n is a spoon odd perfect number with k quasi-prime factors, then $n < 2^{4^k}$.*

Proof. This follows from [5] and improvements in [7], which are straightforward generalizations of [1] and [4]. □

2. EXAMPLES OF EVEN SPOON PERFECT NUMBERS

Allowing even factors, it is an easy matter to find examples of spoon perfect numbers. We content ourselves with three infinite families of examples:

- (1) For each $n > 1$, $2^{n-1}(2^n - 1)^1$ is a spoon perfect number.
- (2) For each $n > 1$, $n^1(n + 1)^1 \dots (2n - 1)^1$ is a spoon perfect number.
- (3) More generally, for each $n > 1$ and $\alpha \geq 1$, $n^\alpha \cdot \left(\frac{n^{\alpha+1}-1}{n-1}\right)^1 \cdot \left(\frac{n^{\alpha+1}-1}{n-1} + 1\right)^1 \dots (2 \cdot n^\alpha - 1)^1$ is a spoon perfect number.

Note that the first family contains all of the even perfect numbers. Hereafter, we will deal only with spoon odd perfect numbers.

3. ABUNDANCE AND DEFICIENCY

There are a variety of techniques used in the search for odd perfect numbers. One technique that we will rely on heavily in our search is that of abundance and deficiency.

Definition 3.1 (Abundance). If $\prod_{i=1}^k \tilde{\sigma}_{-1}(x_i, \alpha_i) > 2$, we call the corresponding quasi-prime factorization *abundant*.

Definition 3.2 (Deficiency). If $\prod_{i=1}^k \tilde{\sigma}_{-1}(x_i, \alpha_i) < 2$, we call the corresponding quasi-prime factorization *deficient*.

Note that by Lemma 1.2, decreasing any of the α_i 's decreases $\prod_{i=1}^k \tilde{\sigma}_{-1}(x_i, \alpha_i)$, while increasing any of the α_i 's increases this product. Recall that $\tilde{\sigma}_{-1}(x_i, 1) = \frac{x_i+1}{x_i}$ and $\tilde{\sigma}_{-1}(x_i, \infty) = \frac{x_i}{x_i-1}$. As a consequence, we have the following useful result: If $n = \prod_{i=1}^k x_i^{\alpha_i}$ is a spoof odd perfect number, and $M \subseteq \{1, 2, \dots, k\}$, then

$$(1) \quad \prod_{i \in M} \tilde{\sigma}_{-1}(x_i, \alpha_i) \prod_{i \notin M} \frac{x_i + 1}{x_i} \leq 2 < \prod_{i \in M} \tilde{\sigma}_{-1}(x_i, \alpha_i) \prod_{i \notin M} \frac{x_i}{x_i - 1}.$$

4. INCOMPLETE FACTORIZATIONS

Our algorithm finds every spoof odd perfect number with exactly k quasi-prime factors. Hereafter, k will always be used to refer to the exact number of factors we are considering. Our algorithm is divided into two distinct parts. In the first part, we use only abundance and deficiency considerations to make restrictions on the possible factors and exponents of a spoof odd perfect number. It is natural to “build” possible factorizations from the smallest factor to the greatest, focusing first on x_1 , then on x_2 , and so forth. Using this technique, we restrict to certain sets of $k - 1$ factors. Then, in the second part, we use related techniques to further restrict the possibilities for x_k and the exponents. It is important to define precisely how these restrictions are made on the factors and exponents.

Definition 4.1 (Incomplete factorizations). We define an m -term *incomplete factorization* as an ordered m -tuple of pairs, $X = \{(x_1, \alpha_1), (x_2, \alpha_2), \dots, (x_m, \alpha_m)\}$, where the factors x_i are integers and satisfy $1 < x_1 \leq x_2 \leq \dots \leq x_m$, and the exponents satisfy $\alpha_i \in \mathbb{N} \cup \{\infty\}$. When $\alpha_i \in \mathbb{N}$ we say the i th exponent is *anchored*, while if $\alpha_i = \infty$ we say the i th exponent is *unanchored*.

For convenience, we will refer to incomplete factorizations simply as *factorizations* when it is clear from context. In the sense of our algorithm, $\alpha_i = \infty$ does *not* refer to a factor whose exponent is above some particular bound, though that is the common usage in the literature. Rather, we treat an unanchored exponent as merely representing an exponent that can vary arbitrarily. For such exponents we use $\tilde{\sigma}_{-1}(x_i, \infty)$ in upper-bound calculations, hence the terminology.

Recall that $\tilde{\sigma}_{-1}(X) = \prod_{i=1}^m \tilde{\sigma}_{-1}(x_i, \alpha_i)$. When $m = 0$, we allow $X = \emptyset$ to be considered as an incomplete factorization and we set $\tilde{\sigma}_{-1}(\emptyset) = 1$.

We impose a partial order on the incomplete factorizations. For factorizations $X = \{(x_i, \alpha_i)\}_{i \leq m}$ and $Y = \{(y_i, \beta_i)\}_{i \leq m'}$, we say that $X \leq Y$ if the following conditions are satisfied:

- (1) $m \leq m'$.
- (2) $x_i = y_i$ for all indices i with $1 \leq i \leq m$.

(3) $\alpha_i = \beta_i$ or $\alpha_i = \infty$ for all indices i with $1 \leq i \leq m$.

In other words, one factorization is greater than another if it has all of the factors of the other, and the exponents are the same except that unanchored exponents can become anchored. We write $X < Y$ if $X \leq Y$ and $X \neq Y$, and we write $X \prec Y$ if $X < Y$ and there is no Z with $X < Z < Y$. If $X < Y$ and $m = m'$, then by Lemma 1.2 it follows that $\tilde{\sigma}_{-1}(X) > \tilde{\sigma}_{-1}(Y)$.

For any given incomplete factorization X , there is a family of quasi-prime factorizations Y such that $X < Y$. We think of X as giving rise to each of these factorizations Y . For $X = \emptyset$, this family of factorizations is simply the set of non-empty incomplete factorizations. We seek to limit this set to a finite list of valid factorizations, to find all spoof odd perfect numbers in a finite-run-time algorithm.

More precisely, we desire to show that, for any given factorization X , we can find a finite set of factorizations $X \prec Y_1, Y_2, \dots, Y_s$ such that for any spoof odd perfect number Z with $X < Z$ there exists some index i with $1 \leq i \leq s$ and $Y_i \leq Z$. The following lemmas address the case when $\tilde{\sigma}_{-1}(X) > 2$.

Lemma 4.2. *Let $n = \prod_{i=1}^k x_i^{\alpha_i}$ be a spoof odd perfect number. Fix disjoint subsets $L, M \subseteq \{1, 2, \dots, k\}$, with $|M| = m \geq 1$, and suppose*

$$P = \prod_{i \in L} \tilde{\sigma}_{-1}(x_i, \alpha_i) \prod_{i \in M} \tilde{\sigma}_{-1}(x_i, \infty) > 2.$$

Then there is some index $i \in M$ such that $x_i^{\alpha_i+1} \leq mr$, where $r = \frac{P}{P-2}$.

Proof. We have $\tilde{\sigma}_{-1}(x_i, \alpha_i) = \tilde{\sigma}_{-1}(x_i, \infty) \cdot (1 - 1/x_i^{\alpha_i+1})$. Thus, we find

$$\begin{aligned} \prod_{i \in L \cup M} \frac{x_i^{\alpha_i+1} - 1}{x_i^{\alpha_i}(x_i - 1)} &= \prod_{i \in L} \tilde{\sigma}_{-1}(x_i, \alpha_i) \prod_{i \in M} \frac{x_i}{x_i - 1} \prod_{i \in M} \left(1 - \frac{1}{x_i^{\alpha_i+1}}\right) \\ &= P \prod_{i \in M} \left(1 - \frac{1}{x_i^{\alpha_i+1}}\right) \leq 2 \end{aligned}$$

with equality if and only if $|L \cup M| = k$. Dividing both sides by P gives

$$\prod_{i \in M} \left(1 - \frac{1}{x_i^{\alpha_i+1}}\right) \leq 1 - \frac{1}{r} \leq \left(1 - \frac{1}{mr}\right)^m,$$

where r is defined as in the statement of the lemma. After taking the m th root of both sides, the result follows from elementary calculations. □

Corollary 4.3. *Let $n = \prod_{i=1}^k x_i^{\alpha_i}$ be a spoof odd perfect number, $P = \prod_{i=1}^k \frac{x_i}{x_i-1} > 2$, and $r = \frac{P}{P-2}$. Then there is some index $i \in \{1, 2, \dots, k\}$ with $x_i^{\alpha_i+1} < kr$.*

Proof. Apply Lemma 4.2 with $L = \emptyset$ and $M = \{1, 2, \dots, k\}$. □

Lemma 4.4. *For any given abundant incomplete factorization X there is a finite list of factorizations $X \prec Y_1, Y_2, \dots, Y_s$, such that if Z is a spoof odd perfect number with $X < Z$, then for some index i with $1 \leq i \leq s$, we have $Y_i \leq Z$. Furthermore this list can be computed explicitly.*

Proof. Let $X = \{(x_i, \alpha_i)\}_{i \leq \ell}$ be an abundant factorization, and Z be a spoof odd perfect number, with $X < Z$. Apply Lemma 4.2 with L the set of indices such that α_i is anchored and M the set of indices i such that α_i is unanchored, and write $|M| = m$. Note that $m \geq 1$, since if $m = 0$, then $\tilde{\sigma}_{-1}(Z) \geq \tilde{\sigma}_{-1}(X) > 2$. By

Lemma 4.2 there is some $i \in M$ such that $x_i^{\alpha_i+1} < mr$. There are of course only finitely many choices for this i and this α_i . For each of these choices, we gain a new incomplete factorization Y_i , with $X \prec Y_i$. \square

This lemma encapsulates much of the algorithmic improvement over other algorithms in the literature. We anchor exponents only as needed, rather than running through a large and fixed range of exponents.

These lemmas form our arsenal for attacking abundant incomplete factorizations. We now turn to the deficient factorizations.

Lemma 4.5. *Let $m < k$. For any given m -term deficient factorization X , there is a finite list of factorizations $X \prec Y_1, Y_2, \dots, Y_s$, where the Y_i 's are obtained by adjoining some ordered pair (x_{m+1}, ∞) with $x_{m+1} \geq x_m$ such that if Z is a spoof odd perfect number with $X < Z$, then for some index i with $1 \leq i \leq s$, we have $Y_i < Z$. In fact, there is an explicitly computable bound B such that $x_{m+1} \leq B$.*

Proof. Write $P = \prod_{i=1}^m \tilde{\sigma}_{-1}(x_i, \alpha_i) < 2$. Suppose first that $x_{m+1} > 1 + \frac{1}{k-m\sqrt{2/P-1}} = B$. For any Z with $Y_i < Z$ we have by equation (1) that

$$\tilde{\sigma}_{-1}(Z) = \prod_{i=1}^k \tilde{\sigma}_{-1}(x_i, \alpha_i) < P \left(\frac{x_{m+1}}{x_{m+1} - 1} \right)^{k-m} < P \cdot \frac{2}{P} = 2,$$

which is impossible if $\tilde{\sigma}_{-1}(Z) = 2$. Thus

$$x_m \leq x_{m+1} \leq 1 + \frac{1}{k-m\sqrt{2/P-1}},$$

so there are finitely many choices for x_{m+1} , and thus finitely many choices for the Y_i 's. \square

5. ALGORITHMIC DETAILS

Our algorithm is guaranteed to consider every possible incomplete factorization that could give rise to a spoof odd perfect number. This is because, quite simply, we make no restrictions on our choices for x_i and α_i except when we are compelled to by the nature of the problem. Chains of factorizations $X_0 < X_1 < X_2 < \dots < X_\ell$ have a maximum length of $2k$, since at each step we must add at least one of the k factors or anchor one of the k exponents. All that remains is to show that in our partial ordering $<$, we only have to consider finitely many branches at each step. We do this in stages as follows:

Case 1. Suppose we have an incomplete factorization X with $t < k - 1$ factors that is deficient. In this case, by Lemma 4.5, there are finitely many choices for the next factor x_{t+1} . We consider each possibility in turn, and in each case we set $\alpha_{t+1} = \infty$; that is, we leave the exponent unanchored.

Case 2. Suppose we have an incomplete factorization X with $t < k$ factors that is abundant. We deal with this case as in the proof of Lemma 4.4. Computing r as in Lemma 4.2 and letting M be the set of indices with $\alpha_i = \infty$ (i.e., the unanchored exponents), with $|M| = m \leq t$, then for some index $i \in M$ we have $x_i^{\alpha_i+1} < mr$. In other words, we are forced to anchor one of the unanchored exponents, in a finite number of ways. We consider each possibility in turn. The only other restriction we make on the exponents is that they satisfy the conditions in Lemma 1.5.

We begin our algorithm with $X_0 = \emptyset$, $\tilde{\sigma}_{-1}(X_0) = 1$, and find ourselves at Case 1, with finitely many choices for x_1 . Each succeeding step will take us either to Case 1 or Case 2 (or give us a factorization that has all exponents anchored and is abundant, which cannot lead to a spoof odd perfect number), and thus either increase the number of factors or decrease the number of indices such that $\alpha_i = \infty$. We eventually arrive at a list of incomplete factorizations with $k - 1$ factors that are deficient. We call the set of all such $k - 1$ incomplete factorizations the “valid” factorizations.

The algorithm as outlined above finds all deficient $(k - 1)$ -term incomplete factorizations that could give rise to spoof odd perfect numbers. If we have such a factorization, there are finitely many choices for x_k . However, because the upper bound $1 + \frac{1}{\frac{k-(k-1)\sqrt{2/P-1}}{2-P}} = \frac{2}{2-P}$ from Lemma 4.5 is often quite large, it is unwise computationally to test each of these possibilities when k is bigger than, say, 4.

Instead, at this point we allow x_k to range freely between the upper and lower bounds we can compute from (1). Even without knowing x_k explicitly, we know that adjoining the ordered pair (x_k, ∞) must make the factorization abundant. We are therefore in a situation similar to Case 2 above, and we can apply Lemma 4.2 by computing the maximum possible value of r . This maximum occurs when we have $(x_k, \alpha_k) = (\lfloor \frac{2}{2-P} \rfloor, \infty)$ where $P = \tilde{\sigma}_{-1}(X)$ with X the valid $(k - 1)$ -term incomplete factorization under consideration. In other words, for this choice of x_k we have $\tilde{\sigma}_{-1}(X)\tilde{\sigma}_{-1}(x_k, \infty) > 2$, and for any large x_k this inequality fails.

As long as there are unanchored exponents, the factorization should still be abundant, which means we remain at Case 2 and can repeat the procedure described in the preceding paragraph. We do have to consider the possibility that x_k is the factor which must be anchored, in which case we have $x_k^{\alpha_k+1} \leq mr$, which gives us the bound $x_k \leq (mr)^{1/(\alpha_k+1)}$. In practice, this upper bound is better than the one given by (1), especially if α_k is large.

Our search for spoof odd perfect numbers arising from the valid factorization X will terminate whenever either there are no choices for anchorings satisfying $x_i^{\alpha_i+1} \leq mr$, or we find a spoof. Note that when all of the exponents with $1 \leq i < k$ are anchored, we must have $\tilde{\sigma}_{-1}(x_k, \alpha_k) = 2 / \prod_{i=1}^{k-1} \tilde{\sigma}_{-1}(x_i, \alpha_i)$, and so we may compute the only possible choice for x_k .

6. MAIN RESULT AND FUTURE RESEARCH

Theorem 6.1. *The only spoof odd perfect number with less than seven quasi-prime factors is Descartes’ example, $3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021^1$.*

Proof. Using our algorithm with $1 \leq k \leq 6$, we find only Descartes’ example. For the $k = 5$ case, the algorithm finds 4690 total valid deficient 4-tuples that must be tested using the method for x_k outlined above, and the algorithm runs in under 4 seconds. For the $k = 6$ case, the algorithm finds 53850049 total valid deficient 5-tuples, and the algorithm runs in about 13.3 hours. □

The tools outlined here could be turned to attack the $k = 7$ case, given sufficient computing power. However, moving from $k = 5$ to $k = 6$ increased the number of test cases by a factor of 1.15×10^4 , and we could expect for $k = 7$ at least 6.1×10^{11} total 6-tuples to test. Stronger methods would be preferable. Our method almost completely ignores issues of divisibility and factorization. While not all divisibility techniques which arise in the search for odd perfect numbers carry over to the

spoofer case, some considerations still do apply. One line of inquiry would be to use the initial technique described above for the first $k - 2$ factors, and then use more sophisticated techniques to bound the last two terms.

7. AN ODD OBSERVATION

The following surprising result describes a connection between spoof odd perfect numbers and the usual perfect number problem.

Lemma 7.1. *If $x, \alpha \in \mathbb{N}$, with $x, \alpha > 1$, then we can find a set of ordered pairs $\{(x_j, q_j)\}$, with $1 \leq j \leq r$, such that the q_j 's are prime, the x_j 's are powers of x , the equality $\tilde{\sigma}_{-1}(x, \alpha - 1) = \prod_{j=1}^r \tilde{\sigma}_{-1}(x_j, q_j - 1)$ holds, and $x^{\alpha-1} = \prod_{j=1}^r x_j^{q_j-1}$.*

Proof. If α is prime, the result is trivial, taking $r = 1$, $x_1 = x$, and $q_1 = \alpha$. We proceed by induction on $\Omega(\alpha)$, the number of (not necessarily distinct) prime factors of α . Suppose the result is true for all α such that $\Omega(\alpha) = i \in \mathbb{N}$. Fix α' with $\Omega(\alpha') = i + 1$. We can write $\alpha' = q \cdot m$, for some prime q , and $\Omega(m) = i$. But then

$$\begin{aligned} \tilde{\sigma}_{-1}(x, \alpha' - 1) &= \frac{x^{qm} - 1}{x^{qm-1}(x - 1)} = \frac{x^q - 1}{x^{q-1}(x - 1)} \cdot \frac{x^{qm} - 1}{x^{qm-q}(x^q - 1)} \\ &= \tilde{\sigma}_{-1}(x, q - 1) \cdot \tilde{\sigma}_{-1}(x^q, m - 1). \end{aligned}$$

Since $\Omega(m) = i$, we can rewrite $\tilde{\sigma}_{-1}(x^q, m - 1)$ as a product the necessary form, which completes the induction. \square

Note that we can choose primes from the factorization of α in any order. In particular, if $\alpha \equiv 2 \pmod{4}$, we can choose the sole 2 in the prime factorization of α first, so that the first term to appear will be simply $\tilde{\sigma}_{-1}(x, 1)$.

Corollary 7.2. *Every odd perfect number corresponds to a spoof odd perfect number whose exponents are all one less than a prime. More precisely, if $n = \prod_{i=1}^k p_i^{\alpha_i}$ is an odd perfect number, then n is also a spoof odd perfect number, which has a quasi-prime factorization $X = \{(x_j, q_j)\}$, with $1 \leq j \leq r$, such that the $q_j + 1$'s are prime and the x_j 's are powers of the p_i 's. We can choose the set X so that exactly one of the q_j 's will be equal to 1, and its base x_j will be prime.*

More generally:

Corollary 7.3. *For every number N there is a quasi-prime factorization $X = \{x_j, q_j\}$ with the $q_j + 1$'s prime, the x_j 's powers of the primes dividing N , and $\sigma_{-1}(N) = \sigma_{-1}(X)$.*

For $x, q \in \mathbb{N}$, with $x > 1$ and $q + 1$ prime, $\tilde{\sigma}(x, q) = \Phi_{q+1}(x)$. A stronger understanding of these cyclotomic integers may lead to a new attack on the odd perfect number problem.

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